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EXTENDED HARDY-LITTLEWOOD INEQUALITIES AND APPLICATIONS TO THE CALCULUS OF VARIATIONS

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Abstract. An extension of the Hardy-Littlewood inequality for rearrangements is established. It is used for giving several conditions of existence of a minimum for nonweakly-lower-semicontinuous functionals of the form $J(v)=\int_0^1 f(x,v(x),v'(x))dx$ with constraints on v and v'.

1. Introduction. Let us consider the functional

$$J(v) = \int_0^1 f(x, v(x), v'(x))\, dx,$$

where the function v belongs to Sobolev space $V = W^{1,p}((0,1);\mathbb{R}^N) + u_0$. The problem we study is

$$\inf[J(v) : v \in V]. \qquad (\mathcal{P})$$

It is well known that without the lower semicontinuity of J, the minimizing sequences of $(\mathcal{P})$ can develop oscillations and $(\mathcal{P})$ can have no solution. Such a phenomenon occurs in the following classical example of Bolza:

$$\inf\Big[\int_0^1 (1-(v')^2)^2 dx + \int_0^1 v^2 dx : v \in W_0^{1,4}((0,1))\Big]$$

for which there is a minimizing sequence u_n satisfying

$$\forall n \geq 0,\ u_n'(x) \in \{-1,1\} \text{ a.e. } x \in [0,1], \quad u_n \to 0 \text{ uniformly in } [0,1].$$

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When oscillations in problem $(\mathcal{P})$ appear, the direct method of calculus of variations does not apply. To solve problem $(\mathcal{P})$, there are some methods of overcoming this difficulty. All these methods are based on the study of the relaxed problem of $(\mathcal{P})$:

$$\inf\Big[J^{**}(v) = \int_0^1 f^{**}(x,v,v')\,dx : v \in V\Big], \qquad (\mathcal{PR})$$

where $f^{**}(x,\eta,\xi)$ is the convex envelope of $f(x,\eta,\xi)$ with respect to ξ. These are also based on the link between problems $(\mathcal{P})$ and $(\mathcal{PR})$.

In this work, we propose a new approach to solve certain constrained problems. These constraints are of local or of global nature. For instance, if $u = (u_1,\ldots,u_N)$, the constraints can be of the form

$$\begin{array}{llll} i) & \displaystyle\int_0^1 \phi(x,u(x))\,dx \le 0, & ii) & \varphi(x,u(x)) \le 0 \text{ on } [0,1] \\ iii) & \|u'\|_{L^p(0,1)} = 1, & iv) & u'(x) \in K \text{ for a.e. } \ x \in [0,1] \end{array}$$

with suitable ϕ, φ and K.

Let us point out that, to our knowledge, there is no existence result with constraints of the form (i), (ii) or (iii). Constraints of the form (iv) can be found in [8]. Our method is based on the rearrangement of function in the sense of Hardy and Littlewood. This work is divided in two parts. In the first one, we establish some extension of Hardy-Littlewood inequalities given in [11], [25] and give a positive answer to the conjecture of [25] concerning the exact conditions under which these inequalities are valid. The main result of this part is the following:

Theorem 1.1. *Let $f : [0,1]\times\mathbb{R}^N\times\mathbb{R}^N \to \mathbb{R}$ be $\mathcal{C}^2$. We assume that f satisfies the following conditions:*

C1) *If $N > 1$, $\forall (x,\eta,\xi) \in [0,1]\times\mathbb{R}^N\times\mathbb{R}^N$, $\forall i \neq j$, $\frac{\partial^2 f}{\partial\xi_i\partial\xi_j}(x,\eta,\xi) \le 0$.*

C2) *$\forall (x,\eta) \in [0,1]\times\mathbb{R}^N$, $\forall \alpha,\beta \in \mathbb{R}^N$ with $\alpha \le \beta$,*

$$\Big\langle \beta-\alpha; \frac{\partial f}{\partial\eta}(x,\eta,\alpha) + \frac{\partial f}{\partial\eta}(x,\eta,\beta)\Big\rangle \\ -\Big\langle \beta-\alpha; \int_0^1 \Big[\frac{\partial f}{\partial\eta} + \frac{\partial^2 f}{\partial x\partial\xi} + \frac{\partial}{\partial\eta}\big(\frac{\partial f}{\partial\xi}\big).\xi\Big](x,\eta,\alpha+t(\beta-\alpha))\,dt\Big\rangle \ \ge 0,$$

where we have set

$$\frac{\partial}{\partial \eta}\Big(\frac{\partial f}{\partial \xi}\Big) \cdot \xi \ = \ \sum_{i=1}^{N} \frac{\partial^2 f}{\partial \eta_i \partial \xi} \cdot \xi_i .$$

Then, for any $u \in W^{1,\infty}((0,1);\mathbb{R}^N)$, the following symmetrization inequality holds true:

$$\int_0^1 f(x,u(x),u'(x))\,dx \geq \int_0^1 f(x,v(x),v'(x))\,dx \tag{1}$$

with $v'(x) = (u')_*(x) = ((u'_1)_*(x),\ldots,(u'_N)_*(x))$, where $(u'_i)_*(x)$ denotes the nondecreasing unidimensional rearrangement of the map $u_i(x)$, and $v(x) = u(0) + \int_0^x v'(t)\,dt$.

In the case $f(x,\xi)$, independent of η, our theorem (see also Theorem 2.7 below) entails the Hardy-Littlewood inequality obtained by [25] (see also [11]). Namely, we prove that, if $f : \mathbb{R}^+ \times \mathbb{R}^N \to \mathbb{R}$ is such that

$$\forall i \neq j, \quad \frac{\partial^2 f}{\partial \xi_i \partial \xi_j} \leq 0 \ , \frac{\partial^2 f}{\partial |x| \partial \xi_i} \geq 0 \text{ and } \frac{\partial f}{\partial |x|} \geq 0 \ ,$$

then

$$\int_\Omega f(|x|,u(x))\,dx \geq \int_{\widetilde{\Omega}} f(|x|,\widetilde{u}(x))\,dx$$

where $\widetilde{u}$ is the Schwartz rearrangement of u and $\widetilde{\Omega}$ is the ball such that $|\Omega| = |\widetilde{\Omega}|$. This generalizes the Hardy-Littlewood inequality, which is the case where $N = 2$ and $f(\xi_1,\xi_2) = -\xi_1\xi_2$.

Global condition (**C2**) is a consequence of the dependence of f with respect to η. It will be shown (Proposition 2.2) that hypothesis (**C2**) entails the following local condition:

$$\frac{\partial f}{\partial \eta}(x,\eta,\xi) - \frac{\partial^2 f}{\partial x \partial \xi}(x,\eta,\xi) - \frac{\partial}{\partial \eta}\Big(\frac{\partial f}{\partial \xi}\Big) \cdot \xi \ \geq \ 0$$

for any $(x,\eta,\xi) \in [0,1] \times \mathbb{R}^N \times \mathbb{R}^N$. In addition, we will prove that (**C1**) and (**C2**) are optimal in some sense to be explained later.

The second part of this paper is devoted to the study of the minimization of functionals which are not sequentially weakly lower semicontinuous. We are essentially concerned with two different situations:

1) when f satisfies conditions (**C1**) and (**C2**), then problem $(\mathcal{P})$, under the constraints described above, has solutions. The use of Theorem 1.1 is crucial here. Unfortunately, assumption (**C2**) is of global nature and it might be difficult to check. For this reason, we have weakened this assumption in the following:

2) when f^{**} satisfies the following conditions:

($\mathbf{C1}^{**}$) $\forall (x,\eta) \in [0,1] \times \mathbb{R}^N$, $\forall \alpha, \beta \in \mathbb{R}^N$, we have

$$f^{**}(x,\eta,\alpha) + f^{**}(x,\eta,\beta) \geq f^{**}(x,\eta,\max(\alpha,\beta)) + f^{**}(x,\eta,\min(\alpha,\beta))$$

(this condition is nothing but a discretization of (**C1**), because f^{**} is not twice differentiable),

($\mathbf{C2}^{**}$) $\forall (x,\eta,\xi) \in [0,1] \times \mathbb{R}^N \times \mathbb{R}^N$,

$$\frac{\partial f^{**}}{\partial \eta}(x,\eta,\xi) - \frac{\partial^2 f^{**}}{\partial x \partial \xi}(x,\eta,\xi) - \frac{\partial}{\partial \eta}\big(\frac{\partial f^{**}}{\partial \xi}(x,\eta,\xi)\big) \cdot \xi \ \geq \ 0.$$

In this setting, problem $(\mathcal{P})$—without constraints—has solutions. We cannot consider constraints here because, for this result, the use of the Euler-Lagrange equation is essential.

The link between case 1) and case 2) is the following: in some sense, condition ($\mathbf{C2}^{**}$) implies "locally" condition (**C2**). Therefore, the use of a local version of Theorem 1.1 cannot be avoided: this is Theorem 2.1. Conditions ($\mathbf{C1}^{**}$) and ($\mathbf{C2}^{**}$) are deeply related to the convexity of each component of the solutions of $(\mathcal{P})$. The key point is that we can modify (locally) a minimizing sequence in such a way that it becomes (at least locally) convex. This convexity prevents the oscillation of the minimizing sequence, which therefore converges (strongly) in V.

This property of convexity has been used, although with a completely different approach, in [1] when $N = 1$, i.e., in the scalar case. When $N = 1$, only condition (**C2**) or ($\mathbf{C2}^{**}$) remains. This later condition can already be found in Aubert and Tahraoui [1] for f of the form $f(x,\eta,\xi) = g(x,\eta) + h(x,\xi)$ and in Raymond [20] in the general case. Raymond's paper is a generalization of Ekeland's [13], and both works contain a geometric interpretation of such conditions.

Still when $N = 1$, qualitative properties of the relaxed problem are obtained in [22], [23]. From these properties, existence results for the initial problem $(\mathcal{P})$ are established. These works use intensively the Euler-Lagrange equation for the relaxed problem $(\mathcal{PR})$.

In [8], where $N \geq 1$, the author uses the Lyapunov lemma for building a solution of the initial problem $(\mathcal{P})$ from a solution of problem $(\mathcal{PR})$ (see also [9]). In this paper, no regularity condition is required on f, and this analysis also works for problems with constraints on the derivative $u'(x)$ (these constraints have to be of local nature). Another approach to the problem can also be found in [18].

Without claiming to exhaustiveness, we can also point out several papers based more or less on the study of the relaxed problem associated with the initial problem. In general, the difficulty comes in building a solution of problem $(\mathcal{P})$ while avoiding passages to the limit on functionals which are not weakly lower semicontinuous. Up to now, constrained problems are not treated in this framework: See [3], [2], [6], [7], [12], [15], [19], [24].

2. Symmetrization inequalities.

2.1. Rearrangements. For the reader's convenience, we briefly recall the definition and the basic properties of the rearrangements used in this paper (for the proofs and a general survey of the results, see for instance [4, 10, 17]).

Let us first define the nondecreasing unidimensional rearrangement of a map $u \in L^\infty([0,1],\mathbb{R})$.

i) If $u : [0,1] \to \mathbb{R}$ is a finite-valued function $u(x) = \sum_{i=1}^k \alpha_i \mathbf{1}_{A_i}(x)$, where $\alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_k$ and the A_i are measurable subsets of $[0,1]$ ($\mathbf{1}_{A_i}$ denoting the usual characteristic function), then the following function, $u_*(x) = \sum_{i=1}^k \alpha_i \mathbf{1}_{[a_i,a_{i+1})}(x)$, where $a_1 = 0$ and $a_i = \sum_{l=1}^{i-1} |A_l|$ if $i \in \{2,\dots,k+1\}$, is called the nondecreasing rearrangement of u.

ii) If $u \in L^\infty([0,1],\mathbb{R})$ and if u^n is a sequence of finite-valued functions such that $\|u-u^n\|_1 \to 0$, then the limit of u_*^n exists and does not depend on the approximating sequence u^n. We call this limit the *nondecreasing rearrangement* of u and it is also denoted u_*.

It can easily be checked that u_* is indeed a nondecreasing function. A crucial property of rearrangement is the equimeasurability: namely, if $f : \mathbb{R} \to \mathbb{R}$ is continuous, and if $u(\cdot) \in L^\infty([0,1],\mathbb{R})$, then

$$\int_0^1 f(u(x))\,dx = \int_0^1 f(u_*(x))\,dx$$

The nonincreasing unidimensional rearrangement of a map $u \in L^\infty([0,1],\mathbb{R})$, denoted u^*, is defined by $\forall x \in [0,1]$, $u^*(x) = -(-u)_*(x)$. It is easy to check that u^* is a nonincreasing function. The equimeasurability property also holds true for u^*.

Finally, if Ω is a bounded, open subset of $\mathbb{R}^P$, we define the Schwartz symmetrization of a map $u \in L^\infty(\Omega,\mathbb{R})$ in the following way:

If $u : \Omega \to \mathbb{R}$ is a finite-valued function: $u(x) = \sum_{i=1}^k \alpha_i \mathbf{1}_{A_i}(x)$, with $\alpha_1 \le \alpha_2 \le \cdots \le \alpha_k$ and the A_i are measurable subsets of Ω, then

$$\widetilde{u}(x) = \sum_{i=1}^{k} \alpha_i \mathbf{1}_{[B(0,a_{i+1})\setminus B(0,a_i)]}(x),$$

where $B(0,a_i)$ is the open ball centered at zero and of radius a_i with

$$a_1 = 0 \ \text{ and } \ |B(0,a_i)| = \sum_{l=1}^{i-1} |A_l| \ \text{ if } \ i \in \{2,\dots,k+1\}.$$

Note that now $\widetilde{u} : \widetilde{\Omega} \to \mathbb{R}$, where $\widetilde{\Omega} = B(0,|\Omega|)$. If $u \in L^\infty(\Omega,\mathbb{R})$ and if u^n is a sequence of finite-valued functions such that $\|u-u^n\|_1 \to 0$, then the limit of $\widetilde{u^n}$ exists and does not depend on the approximating sequence u^n. This limit is called the *Schwartz symmetrization* of u and it is denoted $\widetilde{u}$.

The map $\tilde{u}$ belongs to $L^\infty(\widetilde{\Omega},\mathbb{R})$ and is radial and nonincreasing along the radius. It also enjoys the equimeasurability property.

Finally, if $u \in L^p(\Omega,\mathbb{R})$ $(p \ge 1)$ and if u^n is a sequence of finite-valued functions such that $\|u-u^n\|_p \to 0$, then the limit of $\widetilde{u^n}$ exists and does not depend on the approximating sequence u^n. This limit is also called the *Schwartz symmetrization* of u and it is denoted $\widetilde{u}$.

2.2. A local rearrangement result. Let us consider a map $f \in \mathcal{C}^2([0,1]\times\mathbb{R}^N\times\mathbb{R}^N;\mathbb{R})$. We also fix $[\sigma,\tau]$, Q and C closed, convex subsets of respectively $[0,1]$, $\mathbb{R}^N$ and $\mathbb{R}^N$.

We introduce the following notation:

Notation. Let $\xi_1 = (\xi_{1,1},\dots,\xi_{1,N})$ and $\xi_2 = (\xi_{2,1},\dots,\xi_{2,N})$ belong to $\mathbb{R}^N$. We denote by $\max(\xi_1,\xi_2)$ (respectively $\min(\xi_1,\xi_2)$) the vector $\xi = (\xi_1,\dots,\xi_N)$ of $\mathbb{R}^N$ such that

$$\forall i \in \{1,\dots,N\},\ \xi_i = \max(\xi_{1,i},\xi_{2,i}) \ \text{ (respectively } \xi_i = \min(\xi_{1,i},\xi_{2,i})).$$

We assume that the set C satisfies the following symmetry condition:

C0) If $\xi_1,\xi_2 \in C$, then $\max(\xi_1,\xi_2) \in C$ and $\min(\xi_1,\xi_2) \in C$.

For instance, if C is the segment $[\xi_1, \xi_2]$ (with $\xi_1, \xi_2 \in \mathbb{R}^N$), then C satisfies condition (**C0**) if and only if $\xi_1 \le \xi_2$ or $\xi_2 \le \xi_1$. Another example is given by C of the form $\Pi_{i=1}^N [a_i, b_i]$.

We also assume that f satisfies the following assumptions:

C1′) If $N > 1$, $\forall (x, \eta, \xi) \in [\sigma, \tau] \times Q \times C$, $\forall \alpha, \beta \in C$,

$$\left\langle \frac{\partial^2 f(x, \eta, \xi)}{\partial \xi^2} (\max(\alpha, \beta) - \alpha); (\alpha - \min(\alpha, \beta)) \right\rangle \le 0.$$

where $< \cdot ; \cdot >$ denotes the scalar product of $\mathbb{R}^N$. Note that the previous inequality involves only the crossed derivatives $\dfrac{\partial^2 f}{\partial \xi_i \partial \xi_j}$ with $j \ne i$. A sufficient condition for (**C1′**) to be satisfied is that

$$\forall i \ne j, \quad \forall (x, \eta, \xi) \in [\sigma, \tau] \times Q \times C, \quad \frac{\partial^2 f(x, \eta, \xi)}{\partial \xi_i \partial \xi_j} \le 0.$$

C2′) $\forall (x, \eta) \in [\sigma, \tau] \times Q$, $\forall \alpha, \beta \in C$ with $\alpha \le \beta$,

$$\begin{aligned} &\left\langle \beta - \alpha; \frac{\partial f}{\partial \eta}(x, \eta, \alpha) + \frac{\partial f}{\partial \eta}(x, \eta, \beta) \right\rangle \\ &- \left\langle \beta - \alpha; \int_0^1 \Big[\frac{\partial f}{\partial \eta} + \frac{\partial^2 f}{\partial \eta \partial \xi} + \frac{\partial}{\partial \eta}\Big(\frac{\partial f}{\partial \xi}\Big) \cdot \xi \Big](x, \eta, \alpha + t(\beta - \alpha))\, dt \right\rangle \ \ge \ 0. \end{aligned}$$

Let us give the main result of this section.

Theorem 2.1 (Local rearrangement inequality). *Let $f \in \mathcal{C}^2([0,1] \times \mathbb{R}^N \times \mathbb{R}^N; \mathbb{R})$, $[\sigma, \tau]$, $Q \subset \mathbb{R}^N$ and $C \subset \mathbb{R}^N$ be closed and convex. We assume that C satisfies condition* (**C0**) *and that f satisfies conditions (***C1′***) and (***C2′***). Let $\bar\eta \in Q$ be such that*

$$\bar\eta + [0, (\tau - \sigma)]C \subset Q. \tag{2}$$

Then, for any $u \in W^{1,\infty}((0,1); \mathbb{R}^N)$, such that

$$u(\sigma) = \bar\eta \text{ and } \ u'(x) \in C \ \text{ for a.e. } \quad x \in [\sigma, \tau], \tag{3}$$

the following symmetrization inequality holds true

$$\int_\sigma^\tau f(x, u(x), u'(x))\, dx \ge \int_\sigma^\tau f(x, v(x), v'(x))\, dx, \tag{4}$$

where $v(x) = u(\sigma) + \int_\sigma^x (u')_*(t)\, dt$ $v'(x)$ and $(u')_*$ is the nondecreasing rearrangement of u' on $[\sigma, \tau]$.

Remarks. 1) $[0,(\tau-\sigma)]C$ denotes

$$[0,(\tau-\sigma)]C \;=\; \left\{\xi\in\mathbb{R}^N \;:\; \exists s\in[0,(\tau-\sigma)] \text{ and } \xi'\in C \text{ with } \xi=s\xi\right\}\,.$$

2) We can notice that Theorem 1.1 is an application of Theorem 2.1 with $\sigma=0$, $\tau=1$, $Q=\mathbb{R}^N$ and $C=\mathbb{R}^N$.

3) Theorem 1.1 and 2.1 still hold true for $u\in W^{1,p}((0,1);\mathbb{R}^N_+)$ under suitable growth assumptions on f.

4) Condition ($\mathbf{C1'}$) does not imply that f be convex or concave with respect to ξ. For this reason, the previous result is used below for studying nonconvex problems.

Proof of Theorem 2.1. We have to prove that, if $f:[0,1]\times\mathbb{R}^N\times\mathbb{R}^N\to\mathbb{R}$ satisfies the conditions of Theorem 2.1, then the symmetrization inequality holds true. The proof is rather long. It shall be divided into two parts, the first one being rather technical.

Since the map $v\to v_*$ is continuous in L^1, it is enough to do the proof for piecewise-affine maps such that u' is constant on intervals of the form $[\sigma+k(\tau-\sigma)/n,\sigma+(k+1)(\tau-\sigma)/n)$ where $0\le k\le n$.

Throughout the proof, we keep the following notations:

$$\theta=\frac{\tau-\sigma}{n},\text{ and }\tau_k=\sigma+\theta k\quad(\text{for }k\in\{0,\dots,n\}).$$

If u is some piecewise-affine map on intervals of the form $[\tau_k,\tau_{k+1})$, we denote by $u'(\tau_k)$ the value of $u'(x)$ for $x\in(\tau_k,\tau_{k+1})$.

The proof is divided in two steps. In the first step, we prove that, if u is piecewise affine on intervals of the form $[\tau_k,\tau_{k+1})$, and if v is such that v' is the rearrangements of u' on some fixed interval $[\tau_k,\tau_{k+2}]$ and such that $v(\tau_k)=u(\tau_k)$, then

$$\int_{\tau_k}^{\tau_{k+2}} f(x,u(x),u'(x))dx\ge\int_{\tau_k}^{\tau_{k+2}} f(x,v(x),v'(x))dx.$$

In the second step, given a piecewise-affine map on intervals of the form $[\tau_k,\tau_{k+1}]$, we construct by induction a sequence (u_p), where u'_{p+1} is obtained by a rearrangement of u'_p on some interval $[\tau_{k_p},\tau_{k_{p+2}}]$. We prove that this sequence converges, in a finite number of steps, to the rearrangement v of u on $[\sigma,\tau]$, and that inequality (4) holds true.

First step of the proof. We claim the following result: let u be some piecewise-affine function on intervals of the form $[\tau_k,\tau_{k+1})$. We assume

$u(\sigma) = \bar{\eta}$ and $u'(x) \in C$ for a.e. $x \in [\sigma, \tau]$. Let $k \leq n-2$ and denote by $v : [\tau_k, \tau_{k+2}] \to \mathbb{R}$ the map defined by the following: v' is the nondecreasing rearrangement of the restriction of u' to the interval $[\tau_k, \tau_{k+2}]$ and

$$v(x) = u_k(\tau_k) + \int_{\tau_k}^{x} v'(s)\, ds.$$

Then

$$\int_{\tau_k}^{\tau_{k+2}} f(x, u(x), u'(x))\, dx \geq \int_{\tau_k}^{\tau_{k+2}} f(x, v(x), v'(x))\, dx, \tag{5}$$

and moreover

$$v'(x) \in C \quad \text{for a.e.} \quad x \in [\tau_k, \tau_{k+2}]. \tag{6}$$

In order to prove our claim, we have to consider several cases:

Case 1. If $u'(\tau_k) \leq u'(\tau_{k+1})$, then u', which is piecewise constant, is nondecreasing on $[\tau_k, \tau_{k+2}]$. Thus there is nothing to prove since $u = v$.

Case 2. We now assume that $u'(\tau_k) \geq u'(\tau_{k+1})$.

To simplify the proof, we introduce some notations. We set $\eta = u(\tau_k)$, $\beta = u'(\tau_k) \in C$ and $\alpha = u'(\tau_{k+1}) \in C$. We have

$$u(x) = \begin{cases} \eta + \beta(x - \tau_k) & \text{if } x \in [\tau_k, \tau_{k+1}] \\ \eta + \beta\theta + \alpha(x - \tau_{k+1}) & \text{if } x \in [\tau_{k+1}, \tau_{k+2}] \end{cases}$$

since $\tau_{k+1} - \tau_k = \theta$. We also have

$$(u')_*(x) = v'(x) = \begin{cases} \alpha & \text{if } x \in [\tau_k, \tau_{k+1}) \\ \beta & \text{if } x \in [\tau_{k+1}, \tau_{k+2}). \end{cases}$$

Since we have set $v(x) = \eta + \int_0^x v'(s)\, ds$ for almost every $x \in [\tau_k, \tau_{k+2}]$, v satisfies

$$v(x) = \begin{cases} \eta + \alpha(x - \tau_k) & \text{if } x \in [\tau_k, \tau_{k+1}] \\ \eta + \theta\alpha + \beta(x - \tau_{k+1}) & \text{if } x \in [\tau_{k+1}, \tau_{k+2}]. \end{cases}$$

Note that $v'(x) \in C$ for almost every $x \in [\tau_k, \tau_{k+2}]$, so that (6) holds true. Let us define

$$I = \int_{\tau_k}^{\tau_{k+2}} f(x, u(x), u'(x)) dx - \int_{\tau_k}^{\tau_{k+2}} f(x, v(x), v'(x)) dx.$$

We have to prove that $I \geq 0$. We have

$$I = \int_{\tau_k}^{\tau_{k+1}} f(x, \eta + \beta(x - \tau_k), \beta)dx + \int_{\tau_{k+1}}^{\tau_{k+2}} f(x, \eta + \theta\beta + \alpha(x - \tau_{k+1}), \alpha)dx$$
$$- \int_{\tau_k}^{\tau_{k+1}} f(x, \eta + \alpha(x - \tau_k), \alpha)dx - \int_{\tau_{k+1}}^{\tau_{k+2}} f(x, \eta + \theta\alpha + \beta(x - \tau_{k+1}), \beta)dx.$$

Thus, after several natural changes of variables, we obtain

$$\begin{aligned} I \;=\; & \int_0^\theta f(x + \tau_k, \eta + \beta x, \beta) - f(x + \tau_k + \theta, \eta + \alpha\theta + \beta x, \beta)\, dx \\ & + \int_0^\theta f(x + \tau_k + \theta, \eta + \beta\theta + \alpha x, \alpha) - f(x, \eta + \alpha x, \alpha)\, dx. \end{aligned}$$

Thanks to Taylor's formula, I can be written as

$$\begin{aligned} I \;=\; & -\int_0^\theta \int_0^\theta \Big(\frac{\partial f}{\partial x} + \sum_{i=1}^N \alpha_i \frac{\partial f}{\partial \eta_i}\Big)(x + \tau_k + s, \eta + \alpha s + \beta x, \beta)\, ds\, dx \\ & + \int_0^\theta \int_0^\theta \Big(\frac{\partial f}{\partial x} + \sum_{i=1}^N \beta_i \frac{\partial f}{\partial \eta_i}\Big)(x + \tau_k + s, \eta + \beta s + \alpha x, \alpha)\, ds\, dx. \end{aligned}$$

After another change of variables, we finally have

$$\begin{aligned} I \;=\; & -\int_0^\theta \int_0^\theta \Big(\frac{\partial f}{\partial x} + \sum_{i=1}^N \alpha_i \frac{\partial f}{\partial \eta_i}\Big)(x + \tau_k + s, \eta + \alpha s + \beta x, \beta)\, ds\, dx \\ & + \int_0^\theta \int_0^\theta \Big(\frac{\partial f}{\partial x} + \sum_{i=1}^N \beta_i \frac{\partial f}{\partial \eta_i}\Big)(x + \tau_k + s, \eta + \beta x + \alpha s, \alpha)\, ds\, dx. \end{aligned}$$

For simplicity, we denote by J the integrand of the last equality:

$$I := \int_0^\theta \int_0^\theta J\, dx\, ds\,.$$

Our aim is to use conditions (**C1′**) and (**C2′**) at $(\tau_k + x + s, \eta + \beta x + \alpha s, \alpha + s(\beta - \alpha))$. For doing so, we have to prove that $\forall (x, s) \in [0, \theta]^2$, $(\eta + \beta x + \alpha s) \in Q$. For proving that claim, we first note that

$$\eta + \beta x + \alpha s = \eta + \frac{\beta x + \alpha s}{x + s}(x + s) \in \eta + [0, 2\theta]C,$$

because α and β belong to C, which is convex. On the other hand, since $u(\sigma) = \bar{\eta}$ and $u'(x) \in C$ for almost every $x \in [\sigma, \tau]$, we have $\eta = u(\tau_k) \in \bar{\eta} + [0, (\tau_k - \sigma)]C$, because C is closed and convex. Thus we have, since $\tau_{k+2} \leq \tau$, $(\eta + \beta x + \alpha s) \in \bar{\eta} + [0, (\tau_{k+2} - \sigma)]C \subset Q$ from assumption (2). So, omitting for simplicity the dependence of J with respect to $(\tau_k + x + s, \bar{\eta} + \beta x + \alpha s)$, we have

$$\begin{aligned} J &= \Big(\frac{\partial f}{\partial x} + \sum_i \beta_i \frac{\partial f}{\partial \eta_i}\Big)(\alpha) - \Big(\frac{\partial f}{\partial x} + \sum_i \alpha_i \frac{\partial f}{\partial \eta_i}\Big)(\beta) \\ &= \sum_{i=1}^N (\beta_i - \alpha_i)\Big(\frac{\partial f}{\partial \eta_i}(\beta) + \frac{\partial f}{\partial \eta_i}(\alpha)\Big) \\ &\quad + \Big(\frac{\partial f}{\partial x} + \sum_{i=1}^N \frac{\partial f}{\partial \eta_i}\alpha_i\Big)(\alpha) - \Big(\frac{\partial f}{\partial x} + \sum_{i=1}^N \frac{\partial f}{\partial \eta_i}\beta_i\Big)(\beta) \\ &= \sum_{i=1}^N (\beta_i - \alpha_i)\Big(\frac{\partial f}{\partial \eta_i}(\beta) + \frac{\partial f}{\partial \eta_i}(\alpha)\Big) \\ &\quad - \sum_{i=1}^N (\beta_i - \alpha_i) \int_0^1 \Big(\frac{\partial f}{\partial \eta_i} + \frac{\partial^2 f}{\partial x \partial \xi_i} + \sum_{k=1}^N \frac{\partial^2 f}{\partial \eta_k \partial \xi_i}\xi_k\Big)(\alpha + s(\beta - \alpha))\, ds \end{aligned}$$

which is nonnegative from assumption (**C2′**) applied at the point $(\tau_k + x + s, \eta + \beta x + \alpha s, \alpha + s(\beta - \alpha))$ which belongs to $[\sigma, \tau] \times Q \times C$. So $I \geq 0$ and inequality (5) is proved in this case.

Case 3. We now assume that we are neither in case 1 nor in case 2; i.e., there are some i_0 with $u'_{i_0}(\tau_k) < u'_{i_0}(\tau_{k+1})$ and some j_0 with $u'_{j_0}(\tau_k) > u'_{j_0}(\tau_{k+1})$. Up to a permutation on the indices, we can assume without loss of generality that there is some $1 < l < n$ such that $u'_i(\tau_k) > u'_i(\tau_{k+1})$ if $i \in \{1, \dots, l\}$ and $u'_i(\tau_k) \leq u'_i(\tau_{k+1})$ otherwise.

To simplify the notations, let us introduce the following conventions. If $s = (s_1, \dots, s_N)$ belongs to $\mathbb{R}^N$, we denote by $\widetilde{s}$ the vector of $\mathbb{R}^l$ defined by $\widetilde{s} = (s_1, \dots, s_l)$ and by $\widehat{s}$ the vector of $\mathbb{R}^{N-l}$ defined by $\widehat{s} = (s_{l+1}, \dots, s_N)$, so that $s = (\widetilde{s}, \widehat{s})$. Note that any component of $\widehat{u}$ is convex on $[\tau_k, \tau_{k+2}]$. Let $\phi_\epsilon : [\tau_k, \tau_{k+2}] \to \mathbb{R}^{N-l}$ be a regularization $\mathcal{C}^2$ of $\widehat{u}$, such that a) $\phi''_\epsilon \geq 0$, b) $\phi'_\epsilon(x) \in [\widehat{u}'(\tau_k), \widehat{u}'(\tau_{k+2})]$ for $x \in [\tau_k, \tau_{k+2})$, c) $\phi_\epsilon(\tau_k) = \widehat{u}(\tau_k)$, d) ϕ_ϵ converges strongly in $W^{1,1}((0,1))$ to $\widehat{u}$ when $\epsilon \to 0$.

Following the proof of case 2, we have to compute

$$I = \int_{\tau_k}^{\tau_{k+2}} f(x, u(x), u'(x))\, dx - \int_{\tau_k}^{\tau_{k+2}} f(x, v(x), v(x))\, dx,$$

where v' is equal to the nondecreasing rearrangement of the restriction of u' on $[\tau_k, \tau_{k+2}]$ and $v(x) = u(\tau_k) + \int_{\tau_k}^x v'(s)\, ds$. For computing I, we approximate I by

$$I_\epsilon = \int_{\tau_k}^{\tau_{k+2}} f(x, \widetilde{u}(x), \phi_\epsilon(x), \widetilde{u}'(x), \phi_\epsilon'(x)) dx - \int_{\tau_k}^{\tau_{k+2}} f(x, \widetilde{v}(x), \phi_\epsilon(x), \widetilde{v}'(x), \phi_\epsilon(x))\, dx.$$

Let us point out that $\widetilde{v}'$ (respectively $\widetilde{v}$) is not only the l-first component of v' (respectively of v), but also the nondecreasing rearrangement of the restriction of $\widetilde{u}'$ to the interval $[\tau_k, \tau_{k+2}]$ (respectively $\widetilde{x} = \widetilde{u}(\tau_k) + \int_{\tau_k}^x \widetilde{v}'(s)\, ds$). It is easily checked that $I_\epsilon \to I$ if $\epsilon \to 0$.

Let us set $\widetilde{\beta} = \widetilde{u}'(x)$ for $x \in [\tau_k, \tau_{k+1})$, $\widetilde{\alpha} = \widetilde{u}'(x)$ if $x \in [\tau_{k+1}, \tau_{k+2})$ and $\eta = \widetilde{u}(\tau_k)$. We apply the proof of case 2 to the map $g(x, \widetilde{\eta}, \widetilde{\xi}) := f(x, \widetilde{\eta}, \phi_\epsilon(x), \widetilde{\xi}, \phi_\epsilon'(x))$. Using Taylor's formula and after some changes of variables, we obtain, as in the second step,

$$\begin{aligned}
I_\epsilon = &-\int_0^\theta \int_0^\theta \Big(\frac{\partial f}{\partial x} + \sum_{i=1}^{l} \widetilde{\alpha}_i \frac{\partial f}{\partial \eta_i} + \sum_{i=l+1}^{N} \Big[\frac{\partial f}{\partial \eta_i} \phi_{\epsilon,i}'(\tau_k + x + s) + \frac{\partial f}{\partial \xi_i} \phi_{\epsilon,i}''(\tau_k + x + s) \Big] \Big) \\
&\times (\tau_k + x + s, \eta + \widetilde{\beta} x + \widetilde{\alpha} s, \phi_\epsilon(\tau_k + x + s), \widetilde{\beta}, \phi_\epsilon'(\tau_k + x + s))\, ds\, dx \\
&+ \int_0^\theta \int_0^\theta \Big(\frac{\partial f}{\partial x} + \sum_{i=1}^{l} \widetilde{\beta}_i \frac{\partial f}{\partial \eta_i} + \sum_{i=l+1}^{N} \Big[\frac{\partial f}{\partial \eta_i} \phi_{\epsilon,i}'(\tau_k + x + s) + \frac{\partial f}{\partial \xi_i} \phi_{\epsilon,i}''(\tau_k + x + s) \Big] \Big) \\
&\times (\tau_k + x + s, \eta + \widetilde{\beta} x + \widetilde{\alpha} s, \phi_\epsilon(\tau_k + x + s), \widetilde{\alpha}, \phi_\epsilon'(\tau_k + x + s))\, ds\, dx.
\end{aligned}$$

As in the first step, we set J_ϵ equal to the integrand of the previous equality, namely

$$I_\epsilon := \int_0^\theta \int_0^\theta J_\epsilon\, dx\, ds\,.$$

To apply conditions (**C1′**) and (**C2′**), we have to check that the arguments of J_ϵ belong to $[\sigma, \tau] \times Q \times C$. Namely, we have to check that $(\widetilde{\alpha}, \phi_\epsilon'(\tau_k + x + s)) \in C$ and $(\widetilde{\beta}, \phi_\epsilon'(\tau_k + x + s)) \in C$ and that $(\eta + \widetilde{\beta} x + \widetilde{\alpha} s, \phi_\epsilon(\tau_k + x + s)) \in Q$.

Let us first prove that $(\widetilde{\alpha}, \phi_\epsilon'(\tau_k + x + s))$ belongs to C. From property (**C0**) of C, $(\widetilde{\alpha}, \widetilde{u}'(\tau_k))$ belongs to C because $(\widetilde{\alpha}, \widehat{u}'(\tau_k)) = \min\{u'(\tau_k), u'(\tau_{k+1})\}$ (recall that $u'(\tau_k) = (\widetilde{\beta}, \widehat{u}'(\tau_k))$ and $u'(\tau_{k+1}) = (\widetilde{\alpha}, \widehat{u}'(\tau_{k+1}))$. Then $(\widetilde{\alpha}, \phi_\epsilon'(\tau_k + x + s)) \in [(\widetilde{\alpha}, \widehat{u}'(\tau_k)), (\widetilde{\alpha}, \widehat{u}'(\tau_{k+1}))] \subset C$. The same argument, with $\max\{u'(\tau_k), u'(\tau_{k+1})\}$ instead of $\min\{u'(\tau_k), u'(\tau_{k+1})\}$, proves that $(\widetilde{\beta}, \phi_\epsilon'(\tau_k + x + s))$

belongs to C. Note that

$$(\eta+\widetilde{\beta}x+\widetilde{\alpha}s,\phi_\epsilon(\tau_k+x+s)) = u(\tau_k)+(\tau_k+x+s)\Big(\frac{\widetilde{\beta}x+\widetilde{\alpha}s}{x+s},\frac{1}{x+s}\int_{\tau_k}^{x+s}\phi'_\epsilon(\tau)d\tau\Big),$$

because $u(\tau_k)=(\eta,\phi_\epsilon(\tau_k))$. Then

$$\Big(\frac{\widetilde{\beta}x+\widetilde{\alpha}s}{x+s},\frac{1}{x+s}\int_0^{x+s}\phi'_\epsilon(\tau)\,d\tau\Big)\in[\widetilde{\alpha},\widetilde{\beta}]\times[\widehat{u}'(\tau_k),\widehat{u}'(\tau_{k+1})];$$

thus the left-hand side belongs to C since $(\widetilde{\alpha},\widehat{u}'(\tau_k))$, $(\widetilde{\alpha},\widehat{u}'(\tau_{k+1}))$, $(\widetilde{\beta},\widehat{u}'(\tau_k))$ and $(\widetilde{\beta},\widehat{u}'(\tau_{k+1}))$ belong to C. Finally, let us recall that $u(\tau_k)\in\bar{\eta}+(\tau_k-\sigma)C$, so that (see the first step) $(\eta+\widetilde{\beta}x+\widetilde{\alpha}s,\phi_\epsilon(\tau_k+x+s))\in\bar{\eta}+(\tau_k+2\theta-\sigma)C$. Since $(\tau_k+2\theta-\sigma)\le\tau-\sigma$, from assumption (2) we obtain $(\eta+\widetilde{\beta}x+\widetilde{\alpha}s,\phi_\epsilon(\tau_k+x+s))\in Q$.

Omitting for simplicity the dependence of J with respect to (τ_k+x+s), $(\eta+\widetilde{\beta}x+\widetilde{\alpha}s)$, $\phi_\epsilon(\tau_k+x+s)$ and $\phi'_\epsilon(\tau_k+x+s)$, and keeping only the dependence of J with respect to $\widetilde{\alpha}$ and $\widetilde{\beta}$, we have

$$\begin{aligned}
J_\epsilon &= \Big(\frac{\partial f}{\partial x}+\sum_{i=1}^{l}\widetilde{\beta}_i\frac{\partial f}{\partial\eta_i}+\sum_{i=l+1}^{N}\Big[\frac{\partial f}{\partial\eta_i}\phi'_{\epsilon,i}+\frac{\partial f}{\partial\xi_i}\phi''_{\epsilon,i}\Big]\Big)(\widetilde{\alpha})\\
&\quad-\Big(\frac{\partial f}{\partial x}+\sum_{i=1}^{l}\widetilde{\alpha}_i\frac{\partial f}{\partial\eta_i}+\sum_{i=l+1}^{N}\Big[\frac{\partial f}{\partial\eta_i}\phi'_{\epsilon,i}+\frac{\partial f}{\partial\xi_i}\phi''_{\epsilon,i}\Big]\Big)(\widetilde{\beta})\\
&=\sum_{i=1}^{l}(\widetilde{\beta}_i-\widetilde{\alpha}_i)\Big(\frac{\partial f}{\partial\eta_i}(\widetilde{\beta})+\frac{\partial f}{\partial\eta_i}(\widetilde{\alpha})\Big)\\
&\quad-\sum_{i=1}^{l}(\widetilde{\beta}_i-\widetilde{\alpha}_i)\int_0^1\Big(\frac{\partial f}{\partial\eta_i}+\frac{\partial^2 f}{\partial x\partial\xi_i}+\sum_{j=1}^{l}\frac{\partial^2 f}{\partial\eta_j\partial\xi_i}\xi_j\\
&\quad+\sum_{j=l+1}^{N}\Big[\frac{\partial^2 f}{\partial\eta_j\partial\xi_i}\phi'_{\epsilon,j}+\frac{\partial^2 f}{\partial\xi_i\partial\xi_j}\phi''_{\epsilon,j}\Big]\Big)(\widetilde{\alpha}+s(\widetilde{\beta}-\widetilde{\alpha}))\,ds.
\end{aligned}$$

We know that $\phi'_\epsilon(x)$ belongs to the segment $[\widehat{u}'(\tau_k),\widehat{u}'(\tau_{k+1})]$, so that, for any $x\in[\tau_k,\tau_{k+1}]$, there is some $\rho_x\in\mathbb{R}$ such that $\phi''_\epsilon(x)=\rho_x\left(\widehat{u}'(\tau_{k+1})-\widehat{u}'(\tau_k)\right)$. Note that ρ_x is nonnegative since $\phi''_\epsilon(x)\ge 0$. Thus, from assumption $(\mathbf{C1'})$, we have

$$\sum_{i=1}^{l}\sum_{j=l+1}^{N}\frac{\partial^2 f}{\partial\xi_i\partial\xi_j}(\widetilde{\beta}_i-\widetilde{\alpha}_i)\phi''_{\epsilon,j}=\rho_x\left\langle\frac{\partial^2 f}{\partial\xi^2}(\max(u(\tau_k),u(\tau_{k+1}))\right.$$
$$\left.-u(\tau_{k+1})),u(\tau_{k+1})-\min((u(\tau_k),u(\tau_{k+1}))\right\rangle\le 0.$$

In addition, the following expression

$$\sum_{i=1}^{l}(\widetilde{\beta}_i-\widetilde{\alpha}_i)\Big(\frac{\partial f}{\partial\eta_i}(\widetilde{\beta})+\frac{\partial f}{\partial\eta_i}(\widetilde{\alpha})\Big)-\sum_{i=1}^{l}(\widetilde{\beta}_i-\widetilde{\alpha}_i)\int_0^1\Big(\frac{\partial f}{\partial\eta_i}+\frac{\partial^2 f}{\partial x\partial\xi_i}$$
$$+\sum_{j=1}^{l}\frac{\partial^2 f}{\partial\eta_j\partial\xi_i}\xi_j+\sum_{j=l+1}^{N}\frac{\partial^2 f}{\partial\eta_j\partial\xi_i}\phi'_{\epsilon,j}\Big)(\widetilde{\alpha}+s(\widetilde{\beta}-\widetilde{\alpha}))\,ds$$

is nonnegative from assumption (**C2′**) applied to $\alpha=(\widetilde{\alpha},\widehat{u}'(\tau_k))$ and $\beta=(\widetilde{\beta},\widehat{u}'(\tau_{k+1}))$. This proves that $I_\epsilon\ge 0$. Letting $\epsilon\to 0$ gives $I\ge 0$. Thus inequality (5) is proved in any case.

It remains to prove that (6) holds true in the third case. In this case,

$$v'(x)=\begin{cases}(\widetilde{u}'(\tau_{k+1}),\widehat{u}'(\tau_k)) & \text{if} \quad x\in[\tau_k,\tau_{k+1}]\\ (\widetilde{u}'(\tau_k),\widehat{u}'(\tau_{k+1})) & \text{if} \quad x\in[\tau_{k+1},\tau_{k+2}],\end{cases}$$

where $(\widetilde{u}'(\tau_{k+1}),\widehat{u}'(\tau_k))=\min(u(\tau_k),u(\tau_{k+1}))$ and $(\widetilde{u}'(\tau_k),\widehat{u}'(\tau_{k+1}))=\max(u(\tau_k),u(\tau_{k+1}))$ belong to C from property (**C0**). So $v'(x)\in C$ for almost every $x\in[\tau_k,\tau_{k+2}]$. This completes the proof of the first step.

Second step of the proof. Let us now consider the following algorithm:

- $u_0=u$.
- if u_p is already defined and not convex, then we define u_{p+1} in the following way. Let $k_p\le n-2$ be the smallest integer such that u_p is not convex on $[\tau_{k_p},\tau_{k_p+2}]$. Then we define u_{p+1} by setting

$$u_{p+1}(x)\ =\ \begin{cases}u_p(x) & \text{if} \quad x\notin[\tau_{k_p},\tau_{k_p+2}]\\ v_{p+1}(x) & \text{if} \quad x\in[\tau_{k_p},\tau_{k_p+2}]\end{cases}$$

where v'_{p+1} is the nondecreasing rearrangement of the restriction of u'_p to the interval $[\tau_{k_p},\tau_{k_p+2}]$ and

$$v_{p+1}(x)=u_p(\tau_{k_p})+\int_{\tau_{k_p}}^{x}v'_{p+1}(s)\,ds\ \text{ on } [\tau_{k_p},\tau_{k_p+2}].$$

Let us recall that, from the equimeasurability property of rearrangements, $v_{p+1}(\tau_{k_p+2}) = u(\tau_{k_p+2})$, so that u_{p+1} belongs to $W^{1,1}((\sigma,\tau))$. Moreover, from the construction of the first step, we can prove by induction that, for any p, we have $u'_p(x) \in C$ for almost every $x \in [\sigma,\tau]$. Indeed, this holds clearly true for $u_0 = u$, and, if the result holds true for some p, then $u'_{p+1}(x) \in C$ on $[\sigma,\tau]\backslash[\tau_{k_p},\tau_{k_p+2}]$, because $u_{p+1} = u_p$ on this interval, and $u'_{p+1}(x) \in C$ on $[\tau_{k_p},\tau_{k_p+2}]$ from the first step, property (6).

On the other hand, from inequality (5) of the first step, we deduce that

$$\begin{aligned}&\int_\sigma^\tau f(x,u_p(x),u'_p(x))\,dx - \int_\sigma^\tau f(x,u_{p+1}(x),u'_{p+1}(x))\,dx\\ &= \int_{\tau_{k_p}}^{\tau_{k_p+2}} f(x,u_p(x),u'_p(x))\,dx - \int_{\tau_{k_p}}^{\tau_{k_p+2}} f(x,u_{p+1}(x),u'_{p+1}(x))\,dx\end{aligned}$$

is nonnegative.

It is possible, but rather tedious, to prove that the previous algorithm converges in a finite number of steps to some u_q which is equal to v, where v is defined by $v' = (u')_*$ and $v(x) = u(\sigma) + \int_\sigma^x v'(s)\,ds$. Applying the previous inequalities to $p = q$, we have finally

$$\int_\sigma^\tau f(x,u(x),u'(x))\,dx \geq \int_\sigma^\tau f(x,v(x),v'(x))\,dx.$$

So the second part of the proof of Theorem 2.1 is achieved. □

2.3. Several remarks on conditions (C1) and (C2).

2.3.1. A sufficient condition for (C2). Let $f \in \mathcal{C}^2([0,1]\times\mathbb{R}^N\times\mathbb{R}^N;\mathbb{R})$ satisfy condition (**C2**). Let us set $\alpha = \xi$ and $\beta = \xi + he_i$, with $h > 0$ and (e_i) being the canonical basis of $\mathbb{R}^N$. Dividing inequality (**C2**) by h and letting $h \to 0+$ gives

$$\frac{\partial f}{\partial \eta_i}(x,\eta,\xi) - \frac{\partial^2 f}{\partial x\partial \xi_i}(x,\eta,\xi) - \frac{\partial^2 f}{\partial \eta\partial \xi_i}(x,\eta,\xi)\cdot\xi \;\geq\; 0.$$

So we have the following result:

Proposition 2.2. *If f satisfies condition* (**C2**)*, then f also satisfies the following condition:*

C2loc) $\forall (x,\eta,\xi) \in [0,1]\times\mathbb{R}^N\times\mathbb{R}^N$,

$$\frac{\partial f}{\partial \eta}(x,\eta,\xi) - \frac{\partial^2 f}{\partial x\partial \xi}(x,\eta,\xi) - \frac{\partial}{\partial \eta}\Big(\frac{\partial f}{\partial \xi}(x,\eta,\xi)\Big)\cdot\xi \;\geq\; 0.$$

Let us point out that condition (**C2loc**) is of local nature while condition (**C2**) is of global nature. So (**C2loc**) is much easier to check in practice.

Unfortunately, condition (**C2loc**) does not imply, in general, condition (**C2**). For instance, for $N = 1$, $f(\eta,\xi) = \eta(1-\xi^2)$ satisfies (**C2loc**) but not (**C2**). Moreover, the result of Theorem 1.1 does not hold true under condition (**C2loc**). For the same f, if $u(x) = x$ on $[0,1/2]$ and $u(x) = -2x+3/2$ on $[1/2,1]$, then symmetrization inequality (1) is not satisfied.

Let us give a hypothesis under which condition (**C2loc**) implies condition (**C2**):

Proposition 2.3. *If each component of* $\frac{\partial f}{\partial \eta}$ *is convex with respect to* ξ*, then* (**C2loc**) *implies condition* (**C2**).

Proof. Since $\frac{\partial f}{\partial \eta}$ is convex with respect to ξ, we have, for any $\alpha \le \beta$ (omitting the dependence with respect to (x,η)),

$$\int_0^1 \frac{\partial f}{\partial \eta}(\alpha+s(\beta-\alpha))ds \le \int_0^1 \big((1-s)\frac{\partial f}{\partial \eta}(\alpha)+s\frac{\partial f}{\partial \eta}(\beta)\big)ds \le \frac{1}{2}\big(\frac{\partial f}{\partial \eta}(\alpha)+\frac{\partial f}{\partial \eta}(\beta)\big).$$

Then we have

$$\begin{aligned}
&\left\langle \beta-\alpha; \frac{\partial f}{\partial \eta}(\alpha)+\frac{\partial f}{\partial \eta}(\beta)\right\rangle \\
&-\left\langle \beta-\alpha; \int_0^1 \big[\frac{\partial f}{\partial \eta}+\frac{\partial^2 f}{\partial x\partial \xi}+\frac{\partial^2 f}{\partial \eta\partial \xi}\cdot\xi\big](x,\eta,\alpha+t(\beta-\alpha))\,dt\right\rangle \\
&\ge \left\langle (\beta-\alpha); \int_0^1 \big[\frac{\partial f}{\partial \eta}-\frac{\partial^2 f}{\partial x\partial \xi}(x,\eta,\xi)-\frac{\partial^2 f}{\partial \eta\partial \xi}(x,\eta,\xi)\cdot\xi\big](\alpha+s(\beta-\alpha))\,ds\right\rangle
\end{aligned}$$

which is nonnegative since f satisfies condition (**C2loc**). So f satisfies condition (**C2**). $\square$

2.3.2. Conditions (C1) and (C2) are optimal. We now prove that conditions (**C1**) and (**C2**) cannot be improved in general.

Proposition 2.4. *Let* $f \in \mathcal{C}^2([0,1]\times\mathbb{R}^N\times\mathbb{R}^N;\mathbb{R})$*. Assume that* f *does not satisfy either condition* (**C1**) *or condition* (**C2**)*. Then there is some interval* $[\sigma,\tau]\subset[0,1]$ *and some* $u \in W^{1,\infty}((\sigma,\tau),\mathbb{R}^N)$ *such that*

$$\int_\sigma^\tau f(x,u(x),u'(x))\,dx \;<\; \int_\sigma^\tau f(x,v(x),v'(x))\,dx$$

where $v'(x)$ *is the nondecreasing rearrangement of* u' *on* $[\sigma,\tau]$ *and* $v(x) = u(\sigma)+\int_\sigma^x v'(t)\,dt$*.*

This proposition states that the Hardy-Littlewood inequalities obtained in Theorem 2.7 are optimal. It is only used there. Hence we omit the proof not to lengthen the paper (the proof can be found in [5]).

2.3.3. What are conditions (C1) and (C2) becoming when f is not smooth?. Later on, we need to work with maps f which are only $\mathcal{C}^1$. For this reason, we reformulate the assumptions of Theorem 2.1 in this framework.

Let $f : [0,1] \times \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ be a $\mathcal{C}^1$ map, let $[\sigma, \tau]$, C and Q be closed, convex subsets of $[0,1]$, $\mathbb{R}^N$ and $\mathbb{R}^N$. We assume that C satisfies condition (**C0**), but we replace condition (**C1**) by

$$(\mathbf{C1''}) \begin{cases} \text{There is some } C' \subset \mathbb{R}^N \text{ neighborhood of } C \text{ such that} \\ \forall (x,\eta) \in [\sigma,\tau] \times Q,\ \forall \xi_1, \xi_2 \in C', \\ f(x,\eta,\xi_1) + f(x,\eta,\xi_2) \geq f(x,\eta,\max(\xi_1,\xi_2)) + f(x,\eta,\min(\xi_1,\xi_2)) \end{cases}$$

and (**C2**) by

$$(\mathbf{C2''}) \begin{cases} \text{There is some } C' \subset \mathbb{R}^N \text{ neighborhood of } C \text{ such that} \\ \forall (x,\eta) \in [\sigma,\tau] \times Q,\ \forall \alpha, \beta \in C',\ \alpha \leq \beta \\ \left[\frac{\partial f}{\partial x} + \frac{\partial f}{\partial \eta} \cdot \beta\right](x,\eta,\alpha) - \left[\frac{\partial f}{\partial x} + \frac{\partial f}{\partial \eta} \cdot \alpha\right](x,\eta,\beta) \ \geq\ 0. \end{cases}$$

It is easy to check that, if f is smooth, then conditions (**C1″**) and (**C2″**) are equivalent to conditions (**C1**) and (**C2**). Indeed, conditions (**C1″**) and (**C2″**) are nothing but discretized versions of conditions (**C1**) and (**C2**).

Corollary 2.5. *Assume that C satisfies condition* (**C0**)*, that f satisfies conditions* (**C1″**) *and* (**C2″**)*. Let $\bar\eta \in Q$ be such that $\bar\eta + [0, (\tau - \sigma)]C \subset Q$. Then, for any $u \in W^{1,\infty}((0,1); \mathbb{R}^N)$, such that $u(\sigma) = \bar\eta$ and $u'(x) \in C$ for almost every $x \in [\sigma, \tau]$, the following symmetrization inequality holds true:*

$$\int_\sigma^\tau f(x, u(x), u'(x))\, dx \geq \int_\sigma^\tau f(x, v(x), v'(x))\, dx$$

where $v'(x)$ is the nondecreasing rearrangement of u' on $[\sigma, \tau]$ and $v(x) = u(\sigma) + \int_\sigma^x v'(t)\, dt$.

Proof. Let C' be a neighborhood of C on which (**C1″**) and (**C2″**) hold true. Fix $\bar\epsilon > 0$ such that $C + 2\bar\epsilon B \ \subset C'$, where B is the closed unit ball.

Let f_ϵ be a regularization of f by convolution:

$$f_\epsilon(x,\eta,\xi)=\int_{\mathbb{R}^N} f(x,\eta,\xi-\xi')\phi_\epsilon(\xi')\,d\xi'\ ,$$

with, as usual, $\psi_\epsilon(\xi')=\frac{1}{\epsilon^N}\psi(\frac{\xi'}{\epsilon})$ where ψ is smooth, nonnegative, with compact support and such that $\int_{\mathbb{R}^N}\psi(x)dx=1$ and where $\epsilon>0$ is such that $Supp(\psi_\epsilon)\subset B(0,\epsilon)$. Recall that f_ϵ is smooth and that f_ϵ converges uniformly to f on compacts. We now prove that f_ϵ satisfies conditions ($\mathbf{C1'}$) and ($\mathbf{C2'}$) of Theorem 2.1.

Let ξ_1 and ξ_2 belong to $C+\bar\epsilon B$. If $|\xi'|\le\bar\epsilon$, then we have

$$\begin{aligned}&f(x,\eta,\xi_1-\xi')+f(x,\eta,\xi_2-\xi')\\&\ge f(x,\eta,\max(\xi_1,\xi_2)-\xi')+f(x,\eta,\min(\xi_1,\xi_2)-\xi')\end{aligned}$$

because $\max(\xi_1-\xi',\xi_2-\xi')=\max(\xi_1,\xi_2)-\xi'$ and $\xi_i-\xi'\in C'$ for $i=1,2$. Since $\psi_\epsilon(\xi')\ge 0$ and vanishes for $|\xi'|\ge\bar\epsilon$, we deduce by convolution the following inequality on f_ϵ:

$$f_\epsilon(x,\eta,\xi_1)+f_\epsilon(x,\eta,\xi_2)\ge f_\epsilon(x,\eta,\max(\xi_1,\xi_2))+f_\epsilon(x,\eta,\min(\xi_1,\xi_2))\,.$$

Let $\xi\in C$ and $\{e_i\}$ be the canonical basis of $\mathbb{R}^N$. Fix $i\neq j$. Choosing $\xi_1=\xi+he_i$ and $\xi_2=\xi+he_j$ in the previous inequality (for $h\le\bar\epsilon$ so that ξ_1 and ξ_2 belong to $C+\bar\epsilon B$), dividing by h^2 and letting $h\to 0^+$ gives

$$\frac{\partial^2 f_\epsilon(x,\eta,\xi)}{\partial\xi_i\partial\xi_j}\le 0\,.$$

So f_ϵ satisfies a condition which obviously implies condition ($\mathbf{C1'}$) of Theorem 2.1. In the same way, we can prove that, for any $\alpha,\beta\in C$, with $\alpha\le\beta$,

$$\big[\frac{\partial f_\epsilon}{\partial x}+\frac{\partial f_\epsilon}{\partial\eta}\cdot\beta\big](x,\eta,\alpha)-\big[\frac{\partial f_\epsilon}{\partial x}+\frac{\partial f_\epsilon}{\partial\eta}\cdot\alpha\big](x,\eta,\beta)\ \ge\ 0.$$

Following the proof of Theorem 2.1 gives that

$$\begin{aligned}&\big[\frac{\partial f_\epsilon}{\partial x}+\frac{\partial f_\epsilon}{\partial\eta}\cdot\beta\big](x,\eta,\alpha)-\big[\frac{\partial f_\epsilon}{\partial x}+\frac{\partial f_\epsilon}{\partial\eta}\cdot\alpha\big](x,\eta,\beta)\\&=\left\langle\beta-\alpha;\frac{\partial f_\epsilon}{\partial\eta}(x,\eta,\alpha)+\frac{\partial f_\epsilon}{\partial\eta}(x,\eta,\beta)\right\rangle\\&-\left\langle\beta-\alpha;\int_0^1\big[\frac{\partial f_\epsilon}{\partial\eta}+\frac{\partial^2 f_\epsilon}{\partial x\partial\xi}+\frac{\partial^2 f_\epsilon}{\partial\eta\partial\xi}\cdot\xi\big](x,\eta,\alpha+t(\beta-\alpha))\,dt\right\rangle\ \ge\ 0\end{aligned}$$

so that condition (**C2′**) of Theorem 2.1 is satisfied by f_ϵ. From Theorem 2.1,

$$\int_\sigma^\tau f_\epsilon(x, u(x), u'(x))\, dx \geq \int_\sigma^\tau f_\epsilon(x, v(x), v'(x))\, dx$$

Letting $\epsilon \to 0^+$ gives the desired result. $\square$

2.4. Generalized Hardy-Littlewood inequalities. We are now interested in the Schwartz symmetrization of a map $u \in L^\infty(\Omega; \mathbb{R}^N)$, where Ω is an open subset of some $\mathbb{R}^P$ $(P \geq 1)$.

When $N = 2$, $u = (u_1, u_2)$, a well-known inequality, due to Hardy and Littlewood, states that

$$\int_\Omega u_1(x) u_2(x)\, dx \leq \int_{\tilde{\Omega}} \tilde{u_1}(x) \tilde{u_2}(x)\, dx$$

where $\tilde{u_i}$ for $i = 1, 2$ is the Schwartz symmetrization of u_i and $\tilde{\Omega}$ is the ball of $\mathbb{R}^P$ such that $|\tilde{\Omega}| = |\Omega|$. In [25], the second author gives the following extension of the Hardy-Littlewood inequality. Let $f : \mathbb{R}^+ \times (\mathbb{R}^+)^2 \to \mathbb{R}$ be a regular function. Then we have

$$\int_\Omega f(|x|, u_1(x), u_2(x))\, dx \geq \int_{\tilde{\Omega}} f(|x|, \tilde{u_1}(x), \tilde{u_2}(x))\, dx$$

if f satisfies the following inequalities:

i) $\frac{\partial^3 f}{\partial x \partial t \partial s}(x, t, s) \geq 0$, ii) $\frac{\partial^2 f}{\partial t \partial s}(x, t, s) \leq 0$, iii) $\frac{\partial^2 f}{\partial x \partial s}(x, t, s) \geq 0$,
iv) $\frac{\partial^2 f}{\partial x \partial t}(x, t, 0) \geq 0$, v) $\frac{\partial f}{\partial x}(x, t, s) \geq 0$.

In [25], the author conjectured that the generalized Hardy-Littlewood inequality still holds true if f satisfies only (ii), (v),

$$\frac{\partial^2 f}{\partial x \partial s}(x, t, s) \geq 0 \quad \text{and} \quad \frac{\partial^2 f}{\partial x \partial t}(x, t, s) \geq 0.$$

In the following theorem, we give a positive answer to this conjecture and extend the result for $u = (u_1, \ldots, u_N)$ for $N \geq 2$. Let us point out that the proof is completely different than that of [25].

Theorem 2.6. *Let Ω be an open, bounded subset of $\mathbb{R}^P$. Let $f : \mathbb{R}^N \to \mathbb{R}$ be a $\mathcal{C}^2$ map satisfying the following inequalities:*

$$\forall \xi \in \mathbb{R}^N, \quad \frac{\partial^2 f}{\partial \xi_i \partial \xi_j}(\xi) \leq 0 \ \text{ if } i \neq j.$$

Then, for any $u(\cdot) \in L^\infty(\Omega;\mathbb{R}^N)$, the following symmetrization inequality holds true:

$$\int_\Omega f(u(x))\,dx \geq \int_{\tilde{\Omega}} f(\tilde{u}(x))\,dx$$

where $\tilde{u}$ is the Schwartz symmetrization of u and where $\tilde{\Omega}$ is the ball of $\mathbb{R}^P$ such that $|\tilde{\Omega}| = |\Omega|$.

Remark. We obtain a Hardy-Littlewood inequality by setting $f(\xi_1,\xi_2) = -\xi_1\xi_2$.

Proof of Theorem 2.6. It is enough to do the proof for maps u of the form $u(x) = \sum_{i=1}^k \alpha_i \chi_{A_i}$, where $\alpha_i \in \mathbb{R}^N$ and χ_E is the characteristic function of a set E; i.e., $\chi_E(x) = 1$ if $x \in E$, $\chi_E(x) = 0$ otherwise. The A_i are measurable subsets of Ω such that $A_i \cap A_j = \emptyset$ if $i \neq j$ and $\cup_i A_i = \Omega$.

Let R be the radius of the ball $\tilde{\Omega}$ such that $|\tilde{\Omega}| = |\Omega|$. Let us divide the interval $[0,R]$ into k sub-intervals $[r_i, r_{i+1})$ (with $i = 1,\ldots,k+1$), where $r_1 = 0$ and $\forall i \leq k$, $|B(0,r_{i+1})\backslash B(0,r_i)| = |A_i|$. Then we set $v(s) = \alpha_i$ if $s \in [r_i, r_{i+1})$. We have

$$\int_\Omega f(u(x))\,dx = \sum_i f(\alpha_i)|A_i| = \int_{\tilde{\Omega}} f(v(|x|))\,dx$$

On the other hand,

$$\int_{\tilde{\Omega}} f(v(|x|))\,dx = \sigma \int_0^R f(v(s))\,ds$$

where $\sigma = |B(0,1)|$. From the assumption on f, $g(\xi) = f(-\xi)$ satisfies the conditions of Theorem 1.1, so that

$$\begin{aligned}\int_0^R f(v(s))\,ds = \int_0^R g(-v(s))\,ds &\geq \int_0^R g((-v)_*(s))\,ds \\ &\geq \int_0^R g(-v^*(s))\,ds \geq \int_0^R f(v^*(s))\,ds,\end{aligned}$$

where v^* is the nonincreasing unidimensional rearrangement of v. Note that $\tilde{u}(x) = v^*(|x|)$ for $x \in \tilde{\Omega}$, so that

$$\int_\Omega f(u(x))\,dx \;\geq\; \sigma \int_0^R f(v^*(s))\,ds \;=\; \int_{\tilde{\Omega}} f(\tilde{u}(x))\,dx. \quad \square$$

If $f = f(t,\xi)$ depends also on $|x|$, we have

Theorem 2.7. *Let Ω be an open bounded subset of $\mathbb{R}^P$. Let $f : \mathbb{R}^+ \times \mathbb{R}^N \to \mathbb{R}$ be a $\mathcal{C}^2$ map satisfying the following inequalities: $\forall t \geq 0, \forall \xi \in \mathbb{R}^N$,*

i) $\frac{\partial^2 f}{\partial \xi_i \partial \xi_j}(t, \xi) \leq 0$ if $i \neq j$, ii) $\frac{\partial^2 f}{\partial t \partial \xi_i}(t, \xi) \geq 0$ for $i \in \{1, \dots, P\}$, iii) $\frac{\partial f}{\partial t}(t, \xi) \geq 0$.

Then, for any $u(\cdot) \in L^\infty(\Omega; \mathbb{R}^N)$, the following symmetrization inequality holds true:

$$\int_\Omega f(|x|, u(x))\, dx \geq \int_{\tilde{\Omega}} f(|x|, \tilde{u}(x))\, dx$$

where $\tilde{u}$ is the Schwartz symmetrization of u and where $\tilde{\Omega}$ is the ball of $\mathbb{R}^P$ such that $|\tilde{\Omega}| = |\Omega|$.

Proof of Theorem 2.7. Let $u_0(x) = -|x|$ belong to $L^\infty(\Omega)$. From inequalities (i) and (ii) and from Theorem 2.6 applied to $g(t, \xi) = f(-t, \xi)$, we have

$$\int_\Omega f(-u_0(x), u(x))\, dx \geq \int_{\tilde{\Omega}} f(-\tilde{u_0}(x), \tilde{u}(x))\ dx.$$

Let us now prove that $-\tilde{u_0}(x) \geq |x|$ on $\tilde{\Omega}$. Indeed, if $\tilde{u_0}(x) \geq r$ for some $r \leq 0$, then $|B(0, |x|)| = \sigma |x|^P \leq |\{y \in \Omega : u_0(y) \geq r\}|$ from the very definition of Schwartz symmetrization. On another hand, $|\{y \in \Omega : u_0(y) \geq r\}| = |\{y \in \Omega : |y| \leq -r\}|$ from the definition of u_0. Since $|\{y \in \Omega : |y| \leq -r\}| \leq |B(0, -r)| = \sigma(-r)^P$ we have $|x| \leq -r$. Thus if $\tilde{u_0}(x) \geq r$, then $-|x| \geq r$. So we have $\tilde{u_0}(x) \leq -|x|$ on $\tilde{\Omega}$, i.e., $|x| \leq -\tilde{u_0}(x)$ a.e. $x \in \tilde{\Omega}$. Since f is nondecreasing with respect to t, we have

$$\int_{\tilde{\Omega}} f(-\tilde{u_0}(x), \tilde{u}(x))\, dx \geq \int_{\tilde{\Omega}} f(|x|, \tilde{u}(x))\, dx$$

which completes the proof. $\square$

3. Applications to the calculus of variations.

3.1. Problems with constraints. Let us consider an integrand $f : [0, 1] \times \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$, satisfying the usual growth assumptions: there are constants $p > 1$, $a \geq 0$, $b_i > 0$, $c_i > 0$, $d_i > 0$ such that

$$(\mathbf{G}) \qquad \begin{aligned} & -a - b_2|\eta|^p + c_2|\xi|^p \leq f(x, \eta, \xi) \leq a + b_1|\eta|^p + c_1|\xi|^p \\ & \Big|\frac{\partial f^{**}}{\partial \eta}(x, \eta, \xi)\Big| \leq d_1(1 + |\eta|^{p-1} + |\xi|^p) \\ & \Big|\frac{\partial f^{**}}{\partial \xi}(x, \eta, \xi)\Big| \leq d_2(1 + |\eta|^p + |\xi|^{p-1}) \end{aligned}$$

with $\lambda_1 c_2 - b_2 > 0$, where $\lambda_1 = \inf \left[\frac{\int_0^1 |v'|^p\,dx}{\int_0^1 |v|^p\,dx},\ v \in W_0^{1,p}((0,1)) \right]$ (cf. [3]).

Let us consider a nonempty subset $\mathcal{C}$ of $W^{1,p}((0,1),\mathbb{R}^N)$ which is the set of constraints of the problem. Our purpose is to find a minimum to the following problem:

$$(\mathcal{P}) \qquad \inf\Big\{ \int_0^1 f(x,u(x),u'(x))\,dx,\ \ u \in \mathcal{C},\ u(0)=\alpha,\ u(1)=\beta \Big\}.$$

We are going to prove that, under suitable conditions on f and $\mathcal{C}$, problem $(\mathcal{P})$ has at least one solution. Since the proof of this result relies heavily on symmetrization, we first give conditions on the subset $\mathcal{C}$ to be stable by symmetrization.

Definition 3.1. We say that a subset $\mathcal{C}$ of $W^{1,p}((0,1),\mathbb{R}^N)$ is stable by symmetrization if, for any $u \in \mathcal{C}$, if v is defined by $v'(x) = (u')_*(x)$ and $v(x) = u(0) + \int_0^x (u')_*(\tau)\,d\tau$, then $v \in \mathcal{C}$.

We have the following criteria of stability:

Proposition 3.2. *Let $\mathcal{C} \subset X = W^{1,p}((0,1);\mathbb{R}^N)$ be any intersection of constraints of the form*

i) $\{u \in X,\ \int_0^1 \phi(x,u(x))\,dx \le 0\ \}$,

ii) $\{u \in X,\ \sum_i \int_0^1 \psi_i(u_i'(x))\,dx \le 0\}$,

iii) $\{u \in X,\ \forall x \in [0,1],\ \phi(x,u(x)) \le 0\}$,

iv) $\{u \in X,\ u'(x) \le w(x)$ for a.e. $x \in [0,1]\}$,

v) $\{u \in X,\ \forall i \ne i_0,\ a_i(u_i'(x)) \le u_{i_0}'(x) \le b_i(u_i'(x))$ for a.e. $x \in [0,1]\}$,

vi) $\{u \in X,\ u'(x) \in K_0$ for a.e. $x \in [0,1]\}$,

where $\phi : [0,1]\times\mathbb{R}^N \to \mathbb{R}$ is nondecreasing with respect to u, $w \in L^1([0,1])$ is nondecreasing, the $\psi_i : \mathbb{R}^N \to \mathbb{R}$ (for $i = 1,\dots,N$) are continuous and satisfies the growth condition $\forall s \in \mathbb{R}$, $|\psi_i(s)| \le \lambda_i|s|^q + \mu_i$, where $q \in [1,p)$, $\lambda_i > 0$ and $\mu_i \ge 0$, $a_i : \mathbb{R} \to \mathbb{R}$ and $b_i : \mathbb{R} \to \mathbb{R}$ are continuous and nondecreasing, $i_0 \in \{1,\dots,N\}$ and K_0 satisfies condition **C0**.

Then $\mathcal{C}$ is stable by symmetrization.

Remarks. 1) As an important particular case, we have the constraints of the form $\mathcal{C} = \{u \in W^{1,p}((0,1);\mathbb{R}^N),\ \|u'\|_{L^q} = (\sum_i \|u_i'\|_q^q)^{1/q} = 1\}$ for some $q \in [1,p)$.

2) If $\mathcal{C}$ is any intersection of constraints of the form (i), … , (vi) where the exponent q for (ii) is uniform, then $\mathcal{C}$ is closed in $W^{1,q}([0,1])$.

Proof. It is enough to check that any constraint of type (i) to (vi) is stable by symmetrization. Let $u \in \mathcal{C}$, v be defined by $v' = u'_*$ and $v(x) = u(0) + \int_0^x u'_*(\tau)d\tau$.

If $\mathcal{C}$ is of the form (i): Since $v \le u$ and ϕ is nondecreasing, we have

$$\int_0^1 \phi(x, v(x))\, dx \le \int_0^1 \phi(x, u(x))\, dx \le 0.$$

So $v \in \mathcal{C}$.

If $\mathcal{C}$ is of the form (ii): From the equimeasurability property of rearrangement, we have

$$\sum_i \int_0^1 \psi_i(v_i'(x))\, dx = \sum_i \int_0^1 \psi_i(u_i'(x))\, dx \le 0.$$

So v belongs to $\mathcal{C}$.

If $\mathcal{C}$ is of the form (iii): Since $v \le u$ and ϕ is nondecreasing, we have $\phi(x, v(x)) \le \phi(x, u(x)) \le 0$. So $v \in \mathcal{C}$.

If $\mathcal{C}$ is of the form (iv): Since $u'(x) \le w(x)$ almost everywhere in $[0, 1]$, we have $v'(x) \le w(x)$ almost everywhere on [0,1]. So $v \in \mathcal{C}$.

If $\mathcal{C}$ is of the form (v): We follow the construction of the proof of Theorem 2.1. It is enough to do the proof for piecewise maps which are constant on sets of the form $[k/n, (k+1)/n)$. Moreover, we only do the proof for $N = 2$; the general case can be easily deduced from this one.

Without loss of generality, we assume that $n_0 = 1$. As in the proof of Theorem 2.1, we begin with considering what happens on the interval $[k/n, (k+2)/n)$. Let us set v_i' (for $i = 1, 2$) the nondecreasing rearrangement of u_i' on $[k/n, (k+2)/n)$. Then there are several cases:

1) If $v_i' = u_i'$ for $i = 1, 2$, then there is nothing to prove.

2) If $v_i' \ne u_i'$ for $i = 1, 2$, then

$$u_i'(k/n) = v_i'((k+1)/n) \;>\; u_i'((k+1)/n) = v_i'(k/n) \text{ for } i = 1, 2.$$

Moreover, since

$$a_2(u_2'(x)) \le u_1'(x) \le b_2(u_2'(x)) \text{ for almost every } \; x \in [k/n, (k+2)/n)\ ,$$

we have, on the one hand,

$$a_2(v_2'(k/n)) = a_2(u_2'((k+1)/n)) \le u_1'((k+1)/n) = v_1'(k/n),$$

and on the other hand,

$$v_1'(k/n) = u_1'((k+1)/n) \le a_2(v_2'((k+1)/n)) = b_2(v_2'(k/n)) \,.$$

The other inequality is verified in the same way.

3) If $u_1' = v_1'$ and $u_2' \neq v_2'$, then

$$a_2(v_2'(k/n)) = a_2(u_2'((k+1)/n)) \le a_2(u_2'(k/n)) \le u_1'(k/n) = v_1'(k/n)$$

because a_2 is nondecreasing, and

$$v_1'(k/n) = u_1'(k/n) \le u_1'((k+1)/n) \le b_2(u_2'((k+1)/n)) = b_2(v_2'(k/n)) \,.$$

The other inequality can be proved in the same way.

4) If $u_1' \neq v_1'$ and $u_2' = v_2'$, then

$$a_2(v_2'(k/n)) = a_2(u_2'(k/n)) \le a_2(u_2'((k+1)/n)) \le u_1'((k+1)/n) = v_1'(k/n)$$

and

$$v_1'(k/n) = u_1'((k+1)/n) \le u_1'(k/n) \le b_2(u_2'(k/n)) = b_2(v_2'(k/n)) \,.$$

The other inequality can be proved in the same way.

We complete the proof as in the second part of the proof of Theorem 2.1.

If $\mathcal{C}$ is of the form (vi): We do the same proof as for constraints of the form (v). $\square$

We have the following sufficient condition of existence of a minimum for problem $(\mathcal{P})$:

Theorem 3.3. *Assume that the integrand* $f \in \mathcal{C}^2([0,1] \times \mathbb{R}^N \times \mathbb{R}^N, \mathbb{R})$ *satisfies growth condition* (**G**) *and conditions* (**C1**) *and* (**C2**) *of Theorem* 1.1. *Also assume that the constraint* $\mathcal{C}$ *is nonempty, closed in* $W^{1,q}((0,1);\mathbb{R}^N)$ *(for some* $q \in [1,p)$*) and stable by symmetrization. Then problem* $(\mathcal{P})$ *has a solution. Moreover, this solution can be chosen convex.*

Remark. There are very few result concerning nonconvex problems with state constraint in the literature. See for instance [8] for constrained problems on u'.

The main step of the proof of Theorem 3.3 is the following Lemma (see Rockafellar [21]).

Lemma 3.4. *Assume that a sequence $\phi_n : [0,1] \to \mathbb{R}$ of convex functions converges pointwise to some (convex) function ϕ. Then $\phi_n'(x)$ converges to $\phi'(x)$ for almost every $x \in [0,1]$.*

Proof of Theorem 3.3. Let u_n be a minimizing sequence of problem $(\mathcal{P})$. Let us define v_n by setting: $v_n' = (u_n')_*$ and $v_n(x) = v_n(0) + \int_0^x v_n'(\tau)\, d\tau$.

Since f satisfies conditions (**C1**) and (**C2**), from Theorem 1.1 we have

$$\int_0^1 f(x, u_n(x), u_n'(x))\, dx \geq \int_0^1 f(x, v_n(x), v_n'(x))\, dx.$$

Moreover, since $\mathcal{C}$ is stable by symmetrization, the v_n belong to $\mathcal{C}$. From the equimeasurability property of the rearrangements, $v_n(0) = \alpha$ and $v_n(1) = \beta$. So v_n is a minimizing sequence.

Thanks to the growth assumption, it is clear that $\|v_n\|_{W^{1,p}}$ is bounded. Thus v_n is uniformly continuous and, from Ascoli Theorem a subsequence again denoted v_n converges uniformly to some v. Moreover, since the v_n are convex, Lemma 3.4 yields that v_n' converges for almost every x to $v'(x)$. So, from standard arguments, we have

$$\lim_n \int_0^1 f(x, v_n(x), v_n'(x)) dx = \int_0^1 f(x, v(x), v'(x))\, dx. \tag{7}$$

It remains to check that v belongs to $\mathcal{C}$. For doing so, we remark that, since each component of v_n' is nondecreasing and since $\|v_n'\|_p$ is bounded, the sequence v_n' converges to v' in $L^q([0,1])$ (see Lemma 4.1 given in the appendix). Thus the sequence v_n converges to v in $W^{1,q}([0,1])$. The set $\mathcal{C}$ being closed in $W^{1,q}([0,1])$, v belongs to $\mathcal{C}$. In conclusion, the sequence v_n being a minimizing sequence, equality (7) implies that v is a minimum. Moreover, from its construction, v is convex because so are the v_n. $\square$

Example. Let $f : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ be of the form

$$f(\eta, \xi) = g(\xi) + \sum_{i=1}^{N} \frac{\partial a}{\partial \eta_i}(\eta) \left(1 + \xi_i^2\right)^{\frac{1}{2}} .$$

We assume that

1) The map g is $\mathcal{C}^2$ and satisfies growth condition (**G**).
2) $\frac{\partial^2 g}{\partial \xi_i \partial \xi_j} \leq 0$ for $i \neq j$.

3) The map a is $\mathcal{C}^2$ with $\frac{\partial^2 a}{\partial \eta_i \partial \eta_j} \geq 0$.
Note that the above assumptions do not imply that f is convex. Using Proposition 2.3, it is easy to check that the map f satisfies the conditions of Theorem 3.3.

3.2. Regularity of the minimizers of a constrained problem. As pointed out in Theorem 3.3, Theorem 1.1 can be used for proving existence results for constrained problems. Theorem 1.1 can also be used for giving regularity results for problems for which existence is known. We do not intend to give a theory here. We only give an example of such a situation to illustrate our ideas. Let us consider the following problem:

$$(\mathcal{P}) \qquad \inf \Big\{ \int_0^1 f(x, u(x), u'(x))\, dx, \ \ u \in \mathcal{C},\ u(0) = \alpha,\ u(1) = \beta \Big\},$$

where $\mathcal{C} = \{u \in W^{1,p}((0,1), \mathbb{R}^N) : u'(x) \in K,\ \|u_i'\|_{L^1(0,1)} \leq 1\ \forall i\}$ with, for instance, $K = \{\xi \in \mathbb{R}^N : \xi_1 \in [a,b]\}$. We assume that f is of the form

$$f(x, \eta, \xi) = f(\eta, \xi) = \sum_i \xi_i^2 + h(\eta)$$

where h is $\mathcal{C}^1$. We assume that $\|\nabla h\|_\infty < +\infty$.

Note that, since f is convex and the constraint $\mathcal{C}$ weakly closed, problem $(\mathcal{P})$ has a solution.

Proposition 3.5. *Under the above assumptions, any solution of problem $(\mathcal{P})$ is $\mathcal{C}^{1,1}(0,1)$. Moreover, $\|u''\|_\infty \leq \|\nabla h\|_\infty$.*

Remarks. 1) The same problem, without constraint, has obviously a $\mathcal{C}^{1,1}$ solution as can be checked via the Euler-Lagrange equation. The constraints here prevent the use of the Euler equation.

2) In other problems, it can be interesting to prove that the components of the solution are semiconcave or semiconvex. This can be checked by the same method.

Proof. Let u be a solution of problem $(\mathcal{P})$. We start with some preliminaries. Let us set $M = \|\nabla h\|_\infty$ and

$$\begin{aligned} g(x, \eta, \xi) &= f(-\eta + Mx^2\mathbf{e}, -\xi + 2Mx\mathbf{e}) + M\langle \mathbf{e}, -\eta \rangle \\ &= \sum_{i=1}^N (-\xi_i + 2Mx)^2 + h(-\eta + Mx^2\mathbf{e}) + M\langle \mathbf{e}, -\eta \rangle, \end{aligned}$$

where $\mathbf{e} = (1, \ldots, 1) \in \mathbb{R}^N$. For proving Proposition 3.5, we apply Theorem 1.1 to the map g. So we have to verify the conditions of this theorem. The map g clearly satisfies conditions (**C1**). Let us now check that g also satisfies condition (**C2**) (equivalent here to condition (**C2loc**)). We have

$$\frac{\partial g}{\partial \eta} - \frac{\partial^2 g}{\partial x \partial \xi} - \frac{\partial}{\partial \eta}\left(\frac{\partial g}{\partial \xi}\right) \cdot \xi \quad = -\nabla h - M\mathbf{e} + 4M\mathbf{e}.$$

From the very definition of M, this last quantity is nonnegative. Therefore, g satisfies condition (**C2**).

For proving the proposition, it is enough to verify that the function $u(x) + Mx^2\mathbf{e}$ is convex and $u(x) - Mx^2\mathbf{e}$ is concave. We only prove that $u(x) - Mx^2\mathbf{e}$ is concave, the other case being similar.

For doing so, let us set $w(x) = -u(x) + Mx^2\mathbf{e}$. Then $w'(x) = -u'(x) + 2Mx\mathbf{e}$. Applying Theorem 1.1 to g and w gives

$$\int_0^1 g(x, s(x), s'(x))\, dx \le \int_0^1 g(x, w(x), w'(x))\, dx$$

where $s(x) = -\alpha + \int_0^x (w')_*(\tau)\, d\tau$. Making explicit the previous inequality gives

$$\begin{aligned} &\int_0^1 [f(v(x), v'(x)) + M \left\langle \mathbf{e}, v(x) - Mx^2\mathbf{e} \right\rangle]\, dx \\ \le &\int_0^1 [f(u(x), u'(x)) + M \left\langle \mathbf{e}, u(x) - Mx^2\mathbf{e} \right\rangle]\, dx \end{aligned}$$

thanks to the relation $-(w')_* = (-w')^*$, where

$$v(x) = \alpha + \int_0^x (-w')^*(\tau)\, d(\tau) + Mx^2\mathbf{e}\ .$$

Obviously, v satisfies $v(0) = \alpha$. Let us compute $v(1)$.

$$\begin{aligned} v(1) &= \alpha + \int_0^1 (-w')^*(\tau)\, d(\tau) + M\mathbf{e} = \alpha + \int_0^1 (-w')(\tau)\, d(\tau) + M\mathbf{e} \\ &= \alpha - (-\beta + M\mathbf{e} - (-\alpha)) + M\mathbf{e} = \beta. \end{aligned}$$

So v satisfies the boundary conditions.

Let us now prove that, if v belongs to $\mathcal{C}$, then $u = v$ and u is concave. We know that

$$\int_0^x (-w')^*(\tau)\, d(\tau) \;\geq\; \int_0^x (-w')(\tau)\, d(\tau)\,.$$

Therefore we have $v(x) \geq u(x)$ for any $x \in [0,1]$. If we prove that v belongs to $\mathcal{C}$, then we have

$$\int_0^1 f(v(x), v'(x))\, dx \;\geq\; \int_0^1 f(u(x), u'(x))\, dx$$

because u is a minimizer, and thus

$$\int_0^1 M \langle \mathbf{e}, v(x)\rangle \; dx \leq \int_0^1 M \langle \mathbf{e}, u(x)\rangle \; dx$$

which means, since $v \geq u$, that $u = v$. So

$$u(x) = v(x) = \alpha + \int_0^x (-w')^*(\tau)\, d(\tau) + Mx^2\mathbf{e}\,,$$

and so $u - Mx^2\mathbf{e}$ is concave.

It remains to prove that v satisfies the constraints. Let us first check that $\|v_i'\|_1 \leq \|u_i'\|_1$. For doing so, we consider, for $\epsilon > 0$, $\phi_\epsilon : [0,1] \times \mathbb{R} \to \mathbb{R}$ defined by $\phi_\epsilon(x,t) = \left(\epsilon + (-t + 2Mx)^2\right)^{\frac{1}{2}}$. It is easy to check that ϕ_ϵ satisfies condition (**C2**) (equivalent to (**C2loc**) here). Therefore Theorem 1.1 applied to ϕ_ϵ and to $w_i' = -(u_i' - 2Mx)$, gives

$$\int_0^1 \left(\epsilon + (v_i'(x))^2\right)^{\frac{1}{2}} \leq \int_0^1 \left(\epsilon + (u_i'(x))^2\right)^{\frac{1}{2}}.$$

Letting ϵ tend to 0^+ gives the desired result. Therefore, v' satisfies the constraint $\|v_i'\|_1 \leq 1$.

It remains to prove that $v'(x) \in K$ almost everywhere. Indeed, since $u_1'(x) \in [a,b]$, we have $a - 2Mx \leq u_1'(x) - 2Mx \leq b - 2Mx$. Thus $a - 2Mx \leq (u_1'(x) - 2Mx)^* \leq b - 2Mx$, which finally implies that $v_1(x) \in [a,b]$ almost everywhere.

In conclusion, we have proved that v belongs to $\mathcal{C}$, which completes the proof of the proposition. $\square$

3.3. Existence result of a convex solution under local conditions. We now consider a minimization problem without state constraint:

$$(\mathcal{P}) \qquad \inf\left\{ \int_0^1 f(x,u(x),u'(x))\,dx, \quad \begin{array}{l} u \in W^{1,p}((0,1)\mathbb{R}^N), \\ u(0)=a,\ u(1)=b \end{array} \right\}.$$

With problem $(\mathcal{P})$, we can associate the relaxed problem (see Ekeland-Temam [14]):

$$(\mathcal{PR}) \qquad \inf\left\{ \int_0^1 f^{**}(x,u(x),u'(x))\,dx, \quad \begin{array}{l} u \in W^{1,p}((0,1)\mathbb{R}^N), \\ u(0)=a,\ u(1)=b \end{array} \right\}$$

where f^{**} is the convex envelope of f with respect to ξ. Under suitable assumptions satisfied below, the value of the relaxed problem is known to be equal to the value of the original problem. Moreover, the relaxed problem has a solution. If $\bar{u}$ is a solution of the relaxed problem, then $\bar{u}$ is a solution of the initial problem if and only if $f(x,\bar{u}(x),\bar{u}'(x)) = f^{**}(x,\bar{u}(x),\bar{u}'(x))$ for almost every $x \in [0,1]$. In the sequel of the section, we assume that f is $\mathcal{C}^2$. Then it is known that f^{**} is $\mathcal{C}^{1,\alpha}$ (see Griewank-Rabier [16]). Here, we assume moreover that the derivatives $\frac{\partial^2 f^{**}}{\partial x \partial \xi}$ and $\frac{\partial^2 f^{**}}{\partial \eta \partial \xi}$ exist and are continuous with respect to their arguments.

In the sequel, we denote by $(\mathbf{C1}^{**})$ and $(\mathbf{C2}^{**})$ the following assumptions on f^{**}:

$(\mathbf{C1}^{**})$ $\forall (x,\eta) \in [0,1]\times\mathbb{R}^N$, $\forall \alpha,\beta \in \mathbb{R}^N$, we have

$$f^{**}(x,\eta,\alpha) + f^{**}(x,\eta,\beta) \geq f^{**}(x,\eta,\max(\alpha,\beta)) + f^{**}(x,\eta,\min(\alpha,\beta)).$$

$(\mathbf{C2}^{**})$ $\forall (x,\eta,\xi) \in [0,1]\times\mathbb{R}^N\times\mathbb{R}^N$,

$$\frac{\partial f^{**}}{\partial \eta}(x,\eta,\xi) - \frac{\partial^2 f^{**}}{\partial x \partial \xi}(x,\eta,\xi) - \frac{\partial}{\partial \eta}\Big(\frac{\partial f^{**}}{\partial \xi}(x,\eta,\xi)\Big).\xi \ \geq\ 0.$$

Let us now state the main result of this section.

Theorem 3.6. *Assume that the integrand f satisfies growth assumption* $(\mathbf{G})$*, is $\mathcal{C}^2$ and that f^{**} satisfies conditions* $(\mathbf{C1}^{**})$ *and* $(\mathbf{C2}^{**})$*. Then problem $(\mathcal{P})$ has a solution. Moreover, this solution can be chosen convex.*

Remark. If $N=1$, the above conditions are reduced to $(\mathbf{C2}^{**})$. For this dimension, the result is already known (see Aubert and Tahraoui [1] and Raymond [19]).

The new point in Theorem 3.6 is condition ($\mathbf{C1}^{**}$), which only appears for $N > 1$. This condition is not related to the method and cannot be omitted in the statement of Theorem 3.6. For instance, let us consider the following function f:

$$f(\eta_1,\eta_2,\xi_1,\xi_2) = \left(1-\frac{1}{4}\left(\xi_1-\xi_2\right)^2\right)^2 - e^{-\frac{1}{4}(\eta_1-\eta_2)^2} + \left(\xi_1+\xi_2\right)^2 + \frac{\alpha}{2}(\eta_1+\eta_2)$$

with $\alpha = 1 + 4\sup_{t\in\mathbb{R}} e^{-t^2} t$. With this choice of α, it is easy to check that condition ($\mathbf{C2}^{**}$) is fulfilled (and is even a strict inequality). However, problem ($\mathcal{P}$), for the boundary data $u(0) = u(1) = 0$, has no solution: indeed, if we set $w_1 = \frac{1}{2}(\eta_1+\eta_2)$ and $w_1 = \frac{1}{2}(\eta_1-\eta_2)$, then the new problem is actually the sum of two one-dimensional problems, one of them being nothing but a slight modification of the Bolza problem.

The key point in the proof of Theorem 3.6 is Proposition 3.7 below. Before giving the statement of this proposition, let us recall that, if $\bar{u}(\cdot)$ is a solution of relaxed problem ($\mathcal{PR}$), then the map $p(x) = \frac{\partial f^{**}}{\partial \xi}(x,\bar{u}(x),\bar{u}'(x))$ is continuous. We also have $\bar{u}'(x) \in \partial f^*(x,\bar{u}(x),p(x))$ for almost every $x \in [0,1]$.

In the statement of Proposition 3.7, we do not assume that conditions ($\mathbf{C1}^{**}$) and ($\mathbf{C2}^{**}$) hold true everywhere, but only on an open subset $\mathcal{O}$ of $[0,1]\times\mathbb{R}^N\times\mathbb{R}^N$. This localization is the key-point of Theorem 3.12 below.

Conditions ($\mathbf{C1}^{**}$) and ($\mathbf{C2}^{**}$) become on $\mathcal{O}$:

($\mathbf{C1}^{**}\mathbf{loc}$) $\forall(x,\eta,\alpha)\in\mathcal{O}$, $\forall(x,\eta,\beta)\in\mathcal{O}$, we have

$$f^{**}(x,\eta,\alpha) + f^{**}(x,\eta,\beta) \geq f^{**}(x,\eta,\max(\alpha,\beta)) + f^{**}(x,\eta,\min(\alpha,\beta)).$$

($\mathbf{C2}^{**}\mathbf{loc}$) There is some $\gamma > 0$ such that, $\forall(x,\eta,\xi)\in\mathcal{O}$,

$$\frac{\partial f^{**}}{\partial \eta}(x,\eta,\xi) - \frac{\partial^2 f^{**}}{\partial x\partial \xi}(x,\eta,\xi) - \frac{\partial^2 f^{**}}{\partial \eta\partial \xi}(x,\eta,\xi)\cdot\xi \;\geq\; \gamma\mathbf{e}$$

where $\mathbf{e} = (1,\dots,1)\in\mathbb{R}^N$.

We are now ready to state Proposition 3.7:

Proposition 3.7. *Assume that the integrand f is $\mathcal{C}^2$ and satisfies growth assumption ($\mathbf{G}$), that f^{**} satisfies conditions ($\mathbf{C1}^{**}\mathbf{loc}$) and ($\mathbf{C2}^{**}\mathbf{loc}$) on an open set $\mathcal{O}\subset[0,1]\times\mathbb{R}^N\times\mathbb{R}^N$. Let $\bar{u}$ be a solution of problem ($\mathcal{PR}$),* *and let us denote by* $p(\cdot)$ *the continuous map* $p(x) = \frac{\partial f^{**}}{\partial \xi}(x, \bar{u}(x), \bar{u}'(x))$. *Assume that there are* $x_0 \in (0,1)$ *and* $\epsilon_0 > 0$ *such that*

$$\forall x \in [x_0 - \epsilon_0, x_0 + \epsilon_0], \;\; \{x\} \times \{\bar{u}(x)\} \times \partial f^*(x, \bar{u}(x), p(x)) \;\subset\; \mathcal{O} \, .$$

Then $f(x, \bar{u}(x), \bar{u}'(x)) = f^{**}(x, \bar{u}(x), \bar{u}'(x))$ *a.e. on* $[x_0 - \epsilon_0, x_0 + \epsilon_0]$. *Moreover, the restriction of* $\bar{u}$ *to the interval* $[x_0 - \epsilon_0, x_0 + \epsilon_0]$ *is convex.*

We give the proof of Proposition 3.7 in the next section. Before this, we derive from this Proposition the proof of Theorem 3.6.

Proof of Theorem 3.6. Let us set, for any $\gamma > 0$,

$$f_\gamma(x, \eta, \xi) = f(x, \eta, \xi) + \gamma \langle \mathbf{e}, \eta \rangle \;\; .$$

Note that f_γ satisfies growth conditions $(\mathbf{G})$ and that

$$f_\gamma^{**}(x, \eta, \xi) = f^{**}(x, \eta, \xi) + \gamma \langle \mathbf{e}, \eta \rangle \;\; .$$

Let us denote by $(\mathcal{P}_\gamma)$ the (relaxed) problem

$$(\mathcal{PR}_\gamma) \qquad \inf \left\{ \int_0^1 f_\gamma^{**}(x, u(x), u'(x))\, dx, \quad \begin{array}{l} u \in W^{1,p}((0,1), \mathbb{R}^N), \\ u(0) = a, \;\; u(1) = b \end{array} \right\}.$$

Let u_γ be a solution of problem $(\mathcal{PR}_\gamma)$. Since f_γ and f_γ^{**} satisfy the conditions of Proposition 3.7 for $\mathcal{O} = [0,1] \times \mathbb{R}^N \times \mathbb{R}^N$, $x_0 = 1/2$ and $\epsilon_0 = 1/2$, Proposition 3.7 states that $f_\gamma^{**}(x, u_\gamma(x), u_\gamma'(x)) = f_\gamma(x, u_\gamma(x), u_\gamma'(x))$ almost everywhere on $[0,1]$ and that u_γ is convex. In particular, u_γ is a solution of the following problem:

$$(\mathcal{P}_\gamma) \qquad \inf \left\{ \int_0^1 f_\gamma(x, u(x), u'(x))\, dx, \quad \begin{array}{l} u \in W^{1,p}((0,1), \mathbb{R}^N), \\ u(0) = a, \;\; u(1) = b \end{array} \right\}.$$

Thanks to the growth assumption on f, the sequence u_γ is bounded in $W^{1,p}((0,1), \mathbb{R}^N)$. Therefore, the family u_γ is equicontinuous and a sequence $u_n = u_{\gamma_n}$ converges to some map $\bar{u}$ when $\gamma_n \to 0^+$. Since the u_n are convex, $\bar{u}$ is convex. Moreover, from Lemma 3.4, u_n' converges almost everywhere to $\bar{u}'$. Then from standard arguments in the calculus of variations,

$$\int_0^1 f_{\gamma_n}(x, u_n(x), u_n'(x))\, dx \to \int_0^1 f(x, \bar{u}(x), \bar{u}'(x))\, dx,$$

and it is easy to check that $\bar{u}$ is a minimum of problem $(\mathcal{P})$. $\square$

3.4. Proof of Proposition 3.7. Throughout this section, we consider $\bar{u}(\cdot)$ a solution of the relaxed problem $(\mathcal{PR})$. Recall that

$$p(x) = \frac{\partial f^{**}}{\partial \xi}(x, \bar{u}(x), \bar{u}'(x))$$

is a continuous map and that we have $\bar{u}'(x) \in \partial f^*(x, \bar{u}(x), p(x))$ a.e. $x \in [0,1]$. Let $\mathcal{O}$ be an open subset of $[0,1] \times \mathbb{R}^N \times \mathbb{R}^N$. Recall that f satisfies the following conditions on $\mathcal{O}$:

(**C1******loc**) $\forall (x,\eta,\alpha) \in \mathcal{O}$, $\forall (x,\eta,\beta) \in \mathcal{O}$, we have

$$f^{**}(x,\eta,\alpha) + f^{**}(x,\eta,\beta) \geq f^{**}(x,\eta,\max(\alpha,\beta)) + f^{**}(x,\eta,\min(\alpha,\beta)).$$

(**C2******loc**) There is some $\gamma > 0$ such that, $\forall (x,\eta,\xi) \in \mathcal{O}$,

$$\frac{\partial f^{**}}{\partial \eta}(x,\eta,\xi) - \frac{\partial^2 f^{**}}{\partial x \partial \xi}(x,\eta,\xi) - \frac{\partial^2 f^{**}}{\partial \eta \partial \xi}(x,\eta,\xi) \cdot \xi \ \geq \ \gamma \mathbf{e}$$

where $\mathbf{e} = (1, \ldots, 1) \in \mathbb{R}^N$.

The proof of Proposition 3.7 is divided into several lemmas:

Lemma 3.8. *If f satisfies growth assumption* (**G**) *and if f^{**} satisfies condition* (**C1******loc**), *then, for any $(x,\eta,p) \in [0,1] \times \mathbb{R}^N \times \mathbb{R}^N$ such that $\{x\} \times \{\eta\} \times \partial f^*(x,\eta,p) \subset \mathcal{O}$, the set $\partial f^*(x,\eta,p)$ satisfies condition* (**C0**) *of Theorem* 2.1.

Proof. Let ξ_1, ξ_2 belong to $C = \partial f^*(x,\eta,p)$. We have to prove that C satisfies condition (**C0**); i.e., $\max(\xi_1,\xi_2)$ and $\min(\xi_1,\xi_2)$ belong to C.

From the very definition of C and the convexity of f^{**}, we have (omitting the dependence of f^{**} with respect to (x,η)),

$$f^{**}(\xi_1) - \langle p, \xi_1 \rangle \leq f^{**}(\max(\xi_1,\xi_2)) - \langle p, \max(\xi_1,\xi_2) \rangle \tag{8}$$
$$f^{**}(\xi_2) - \langle p, \xi_2 \rangle \leq f^{**}(\min(\xi_1,\xi_2)) - \langle p, \min(\xi_1,\xi_2) \rangle \ . \tag{9}$$

Thus

$$\begin{aligned} f^{**}(\xi_1) + f^{**}(\xi_2) \ \leq \ & f^{**}(\max(\xi_1,\xi_2)) + f^{**}(\min(\xi_1,\xi_2)) \\ & - < p, \max(\xi_1,\xi_2) + \min(\xi_1,\xi_2) - \xi_1 - \xi_2 > \ . \end{aligned}$$

Since $\max(\xi_1,\xi_2) + \min(\xi_1,\xi_2) - \xi_1 - \xi_2 = 0$, we have on the one hand

$$f^{**}(\xi_1) + f^{**}(\xi_2) \leq f^{**}(\max(\xi_1,\xi_2)) + f^{**}(\min(\xi_1,\xi_2)) \,,$$

and, on the other hand, from assumption (**C1******loc**),

$$f^{**}(\xi_1) + f^{**}(\xi_2) \geq f^{**}(\max(\xi_1, \xi_2)) + f^{**}(\min(\xi_1, \xi_2))\,.$$

So the last two inequalities are in fact equalities, and so are inequalities (8) and (9). This means that $\max(\xi_1, \xi_2)$ and $\min(\xi_1, \xi_2)$ actually belong to $C = \partial f^*(x, \eta, p)$. $\square$

Lemma 3.9. *Assume that a convex compact set $C \subset \mathbb{R}^N$ satisfies condition* (**C0**). *Then, for any $\epsilon > 0$, the set $C + \epsilon B_\infty$ also satisfies condition* (**C0**), *where B_∞ is the closed unit ball for the infinite-norm.*

The proof is easy and left to the reader.

The crucial remark about condition (**C2******loc**) is that it implies (locally) condition (**C2**$''$) of Corollary 2.5. Let us point out that, whereas condition (**C2******loc**) is of local nature, condition (**C2**$''$) is of global nature.

Lemma 3.10. *Let f satisfy growth condition* (**G**) *and be such that f^{**} satisfies conditions* (**C2******loc**). *Let $(\bar{x}, \bar{\eta}, p) \in (0,1) \times \mathbb{R}^N \times \mathbb{R}^N$ be such that $\{\bar{x}\} \times \{\bar{\eta}\} \times \partial f^*(\bar{x}, \bar{\eta}, p) \subset \mathcal{O}$. Then there is some $\delta = \delta(\bar{x}, \bar{\eta}, p) > 0$ such that, if we set $Q_\delta = B(\bar{\eta}, \delta)$ and $C_\delta = \partial f^*(\bar{x}, \bar{\eta}, p) + \delta B_\infty$, (where $B(\bar{\eta}, \delta)$ is the ball centered at $\bar{\eta}$ with radius δ), then we have $\forall (x, \eta) \in [\bar{x} - \delta, \bar{x} + \delta] \times Q_\delta$, $\forall \alpha, \beta \in C_\delta$, with $\alpha \leq \beta$,*

$$\Big[\frac{\partial f^{**}}{\partial x} + \frac{\partial f^{**}}{\partial \eta}\beta\Big](\bar{x}, \bar{\eta}, \alpha) - \Big[\frac{\partial f^{**}}{\partial x} + \frac{\partial f^{**}}{\partial \eta}\alpha\Big](\bar{x}, \bar{\eta}, \beta) \geq \ (\gamma/2)\,\langle \mathbf{e}, \beta - \alpha\rangle.$$

Proof. Assume that, contrary to our claim, there are $\delta_n > 0$ converging to 0, for which the above condition does not hold true. Then there are $(x_n, \eta_n) \in [x - \delta_n, x + \delta_n] \times Q_{\delta_n}$ and $\alpha_n, \beta_n \in C_{\delta_n}$ with $\alpha_n \leq \beta_n$ such that

$$\begin{aligned}\Big[\frac{\partial f^{**}}{\partial x} + \frac{\partial f^{**}}{\partial \eta}\beta_n\Big](x_n, \eta_n, \alpha_n) - \Big[\frac{\partial f^{**}}{\partial x} + \frac{\partial f^{**}}{\partial \eta}\alpha_n\Big](x_n, \eta_n, \beta_n) \\ < (\gamma/2)\,\langle \mathbf{e}, \beta_n - \alpha_n\rangle\,.\end{aligned} \tag{10}$$

Then, $x_n \to \bar{x}$, $\eta_n \to \bar{\eta}$ and, up to a subsequence, $\alpha_n \to \alpha$ and $\beta_n \to \beta$ where α and β belong to $C_0 = \partial f^*(\bar{x}, \bar{\eta}, p)$ because $\partial f^*(\bar{x}, \bar{\eta}, p)$ is compact from growth assumption (**G**). We have to distinguish whether $\alpha = \beta$ or not.

First case. If $\beta \neq \alpha$, then, passing to the limit in the above inequality gives

$$\begin{aligned}&\Big[\frac{\partial f^{**}}{\partial x} + \frac{\partial f^{**}}{\partial \eta}.\beta\Big](\bar{x},\bar{\eta},\alpha) - \Big[\frac{\partial f^{**}}{\partial x} + \frac{\partial f^{**}}{\partial \eta}\cdot\alpha\Big](\bar{x},\bar{\eta},\beta)\\ &\qquad \leq\ (\gamma/2)\left\langle \mathbf{e}, \beta-\alpha\right\rangle,\end{aligned}$$

and so, from Taylor's formula,

$$\begin{aligned}&\left\langle \beta-\alpha; \frac{\partial f^{**}}{\partial \eta}(\bar{x},\bar{\eta},\alpha) + \frac{\partial f^{**}}{\partial \eta}(\bar{x},\bar{\eta},\beta)\right\rangle\\ &- \left\langle \beta-\alpha; \int_0^1 \left[\frac{\partial f^{**}}{\partial \eta} + \frac{\partial^2 f^{**}}{\partial x \partial \xi} + \frac{\partial^2 f^{**}}{\partial \eta \partial \xi}\cdot\xi\right](\bar{x},\bar{\eta},\alpha + t(\beta-\alpha))\,dt\right\rangle\\ &\qquad \leq\ (\gamma/2)\left\langle \mathbf{e}, \beta-\alpha\right\rangle .\end{aligned}$$

Since f^{**} is $\mathcal{C}^1$ and is affine on C_0, we have, thanks to Lemma 4.2 given in the appendix, $\forall \mu \in [0,1]$,

$$\frac{\partial f^{**}}{\partial \eta}(\bar{x},\bar{\eta},\mu\alpha + (1-\mu)\beta) = \mu\frac{\partial f^{**}}{\partial \eta}(\bar{x},\bar{\eta},\alpha) + (1-\mu)\frac{\partial f^{**}}{\partial \eta}(\bar{x},\bar{\eta},\beta).$$

Then it is easy to check that

$$\begin{aligned}&\left\langle \beta-\alpha, \frac{\partial f^{**}}{\partial \eta}(\bar{x},\bar{\eta},\alpha) + \frac{\partial f^{**}}{\partial \eta}(\bar{x},\bar{\eta},\beta)\right\rangle\\ &= 2\left\langle \beta-\alpha, \int_0^1 \frac{\partial f^{**}}{\partial \eta}(\bar{x},\bar{\eta},\alpha + t(\beta-\alpha))\,dt\ \right\rangle.\end{aligned}$$

Thus

$$\begin{aligned}(\gamma/2)\left\langle \mathbf{e}, \beta-\alpha\right\rangle\ \geq\ &\left\langle \beta-\alpha; \frac{\partial f^{**}}{\partial \eta}(\bar{x},\bar{\eta},\alpha) + \frac{\partial f^{**}}{\partial \eta}(\bar{x},\bar{\eta},\beta)\right\rangle\\ &- \left\langle \beta-\alpha; \int_0^1 \left[\frac{\partial f^{**}}{\partial \eta} + \frac{\partial^2 f^{**}}{\partial x \partial \xi} + \frac{\partial^2 f^{**}}{\partial \eta \partial \xi}\cdot\xi\right](\bar{x},\bar{\eta},\alpha + t(\beta-\alpha))\,dt\right\rangle\\ \geq\ &\left\langle \beta-\alpha, \int_0^1 \left[\frac{\partial f^{**}}{\partial \eta} - \frac{\partial^2 f^{**}}{\partial x \partial \xi} - \frac{\partial^2 f^{**}}{\partial \eta \partial \xi}\cdot\xi\right](\bar{x},\bar{\eta},\alpha + t(\beta-\alpha))\,dt\right\rangle\\ \geq\ &\gamma\left\langle \mathbf{e}, \beta-\alpha\right\rangle\end{aligned}$$

from assumption (**C2******loc**). Therefore, there is a contradiction since $\alpha \leq \beta$ and $\alpha \neq \beta$.

Second case. If $\alpha = \beta$, then $(\beta_n - \alpha_n)/|\beta_n - \alpha_n|$ converges, up to a subsequence, to some vector $v \geq 0$, with $v \neq 0$. Dividing inequality (10) by $|\beta_n - \alpha_n|$ and passing to the limit when $n \to +\infty$ gives on the one hand

$$\left\langle v; \Big[\frac{\partial f^{**}}{\partial \eta} - \frac{\partial^2 f^{**}}{\partial x \partial \xi} - \frac{\partial^2 f^{**}}{\partial \eta \partial \xi} \cdot \xi\Big](\bar{x}, \bar{\eta}, \alpha)\right\rangle \;\leq\; (\gamma/2)\,\langle \mathbf{e}, v\rangle\,.$$

On the other hand, we have, from assumption (**C2******loc**),

$$\left\langle v; \Big[\frac{\partial f^{**}}{\partial \eta} - \frac{\partial^2 f^{**}}{\partial x \partial \xi} - \frac{\partial^2 f^{**}}{\partial \eta \partial \xi} \cdot \xi\Big](\bar{x}, \bar{\eta}, \alpha)\right\rangle \;\geq\; \gamma\,\langle \mathbf{e}, v\rangle\,.$$

Since $v \geq 0$ and $|v| = 1$, there is a contradiction. $\square$

In the previous lemma, we have explained that conditions (**C2******loc**) and (**C2**$''$) are equivalent in a neighborhood of sets of the form $\{x\} \times \{\eta\} \times \partial f^*(x, \eta, p)$. We now prove that it is indeed possible to localize the solutions in a neighborhood of such sets.

Lemma 3.11. *Let f^{**} satisfy growth condition* (**G**)*. Let us fix $\bar{x} \in (0,1)$ and $\delta' > 0$ and let us set $C_{\delta'} = \partial f^*(\bar{x}, \bar{u}(\bar{x}), p(\bar{x})) + \delta' B_\infty$ and $Q_{\delta'} = B(\bar{u}(\bar{x}), \delta')$. Then there is some $\epsilon \in (0, \delta')$ such that*

i) $\forall x \in [\bar{x} - \epsilon, \bar{x} + \epsilon]\,, \qquad \partial f^*(x, \bar{u}(x), p(x)) \subset C_{\delta'}\,,$
ii) $\forall x \in [\bar{x} - \epsilon, \bar{x} + \epsilon]\,, \qquad \bar{u}(x) \in Q_{\delta'}\,,$
iii) $u(\bar{x} - \epsilon) + [0, 2\epsilon] C_{\delta'} \;\subset\; Q_{\delta'}\,.$

Remark. From property (i), we have $\bar{u}'(x) \in \partial f^*(x, \bar{u}(x), p(x)) \subset C_{\delta'}$ for almost every $x \in [\bar{x} - \epsilon, \bar{x} + \epsilon]$. Since $C_{\delta'}$ is bounded, $\bar{u}$ is Lipschitz continuous on $[\bar{x} - \epsilon, \bar{x} + \epsilon]$.

Proof. Since the set-valued map $(x, \eta, p) \rightsquigarrow \partial f^*(x, \eta, p)$ is upper semicontinuous (see Lemma 4.4 in the appendix), there is some $\epsilon \in (0, \delta')$ such that $\forall x \in [\bar{x} - \epsilon, \bar{x} + \epsilon]$, $\partial f^*(x, \bar{u}(x), p(x)) \subset C_{\delta'}$ We can choose $\epsilon > 0$ sufficiently small so that $\forall x \in [\bar{x} - \epsilon, \bar{x} + \epsilon]$, $\bar{u}(x) \in Q_{\delta'} = B(\bar{u}(\bar{x}), \delta')$ and $\bar{u}(\bar{x} - \epsilon) + [0, 2\epsilon] C_{\delta'} \;\subset\; Q_{\delta'}$. $\square$

We are now ready to prove Proposition 3.7.

Proof of Proposition 3.7. 1) Let $x_0 \in (0,1)$ and $\epsilon_0 > 0$ be such that $\forall x \in [x_0 - \epsilon_0, x_0 + \epsilon_0]$, $\{x\} \times \{\bar{u}(x)\} \times \partial f^*(x, \bar{u}(x), p(x)) \;\subset\; \mathcal{O}$. Let γ be as in condition (**C2******loc**). Let us fix $\bar{x} \in (x_0 - \epsilon_0, x_0 + \epsilon_0)$. Then we have

$$\{\bar{x}\} \times \{\bar{u}(\bar{x})\} \times \partial f^*(x, \bar{u}(\bar{x}), p(\bar{x})) \;\subset\; \mathcal{O}.$$

Let us now set $\bar\eta = \bar u(\bar x)$ and $p = p(\bar x)$. As before, for any $\delta > 0$, we set $Q_\delta = B(\bar\eta, \delta)$ and $C_\delta = \partial f^*(\bar x, \bar\eta, p) + \delta B_\infty$.

2) From Lemma 3.10 applied to $(\bar x, \bar\eta, p)$, there is some $\delta > 0$ such that $\forall (x, \eta) \in [\bar x - \delta, \bar x + \delta] \times Q_\delta$, $\forall \alpha, \beta \in C_\delta$, with $\alpha \le \beta$,

$$\Big[\frac{\partial f^{**}}{\partial x} + \frac{\partial f^{**}}{\partial \eta}\cdot\beta\Big](\bar x, \bar\eta, \alpha) - \Big[\frac{\partial f^{**}}{\partial x} + \frac{\partial f^{**}}{\partial \eta}\cdot\alpha\Big](\bar x, \bar\eta, \beta) \ge (\gamma/2)\left\langle \mathbf{e}, \beta - \alpha\right\rangle .$$

Choosing δ sufficiently small, we can also assume that $[\bar x - \delta, \bar x + \delta] \times Q_\delta \times C_\delta \subset \mathcal{O}$.

3) Let $\epsilon \in (0, \delta')$ be defined by Lemma 3.11 for $\delta' = \delta/3$. Then we have

$$\bar u'(x) \in C_{\delta/3} \qquad \text{almost everywhere on} \quad [\bar x - \epsilon, \bar x + \epsilon] . \tag{11}$$

Moreover, from assumption (**C1******loc**) and Lemmas 3.11 and 3.10, f^{**} satisfies conditions (**C1**$''$) and (**C2**$''$) of Corollary 2.5 on the neighborhood $C_{\delta'}$.

4) Let us define, for any $n > 0$ and $\sigma > 0$:

$$g_n(x, \eta, \xi) = f^{**}(x, \eta, \xi) + \frac{1}{n}\left(f(x, \eta, \xi) - M(\langle \xi, \mathbf{e}\rangle)^2\right) + \sigma |w_n(x) - \eta|^2$$

where

$$M = \sup_{\substack{x \in [\bar x - \epsilon, \bar x + \epsilon]\,, \\ \eta \in Q_\delta,\ \xi \in C_\delta\,, i \neq j}} \left| \frac{\partial^2 f(x, \eta, \xi)}{\partial \xi_i \partial \xi_j} \right|$$

and w_n is a $\mathcal{C}^1$ regularization of $\bar u$ on $[\bar x - \epsilon, \bar x + \epsilon]$ such that $w_n \to \bar u$ in $W^{1,p}((\bar x - \epsilon, \bar x + \epsilon))$ and $\|w_n'\|_\infty$ is uniformly bounded. This is possible since $\bar u$ is Lipschitz continuous on $[\bar x - \epsilon, \bar x + \epsilon]$ from inclusion (11).

We claim that there is some $\sigma > 0$ and some $n_0 > 0$ such that, for any $n \ge n_0$, g_n satisfies conditions (**C1**$''$) and (**C2**$''$) of Corollary 2.5, with $C' = C_{\delta'}$, $C = C_{\delta'/2}$, $Q = Q_{\delta'}$ and $[\sigma, \tau] = [\bar x - \epsilon, \bar x + \epsilon]$.

Proof of the claim. Indeed, we already know on the one hand, that f^{**} satisfies (**C1**$''$). On the other hand, for $i \neq j$,

$$\frac{\partial^2}{\partial \xi_i \partial \xi_j}\left(f(x, \eta, \xi) - M(\langle \xi, \mathbf{e}\rangle)^2\right) \le 0$$

from the very definition of M. Thus g_n satisfies condition (**C1**$''$) for any n.

If we set $\phi(x,\eta) = |w_n(x) - \eta|^2$, then

$$\Big|\Big[\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial\eta}\cdot\beta\Big](x,\eta,\alpha) - \Big[\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial\eta}\cdot\alpha\Big](x,\eta,\beta)\Big| \\ \leq 2|\beta-\alpha||w_n(x)-\eta| \leq K_1|\beta-\alpha|$$

for some constant K_1 because $\|w_n\|_\infty$ is uniformly bounded and $\eta \in Q_{\delta'}$ with $Q_{\delta'}$ bounded. If we set $\psi(x,\eta,\xi) = \big(f(x,\eta,\xi) - M(\langle\xi,\mathbf{e}\rangle)^2\big)$, then

$$\Big|\Big[\frac{\partial\psi}{\partial x} + \frac{\partial\psi}{\partial\eta}\cdot\beta\Big](x,\eta,\alpha) - \Big[\frac{\partial\psi}{\partial x} + \frac{\partial\psi}{\partial\eta}\cdot\alpha\Big](x,\eta,\beta)\Big| \\ \leq |\beta-\alpha|\Big(\Big\|\frac{\partial\psi}{\partial\eta}\Big\|_\infty + \Big\|\frac{\partial^2\psi}{\partial x\partial\xi}\Big\|_\infty + \Big\|\frac{\partial\psi}{\partial\eta\partial\xi}\cdot\xi\Big\|_\infty\Big)$$

where $\|\cdot\|_\infty$ denotes the supremum on $[\bar{x}-\epsilon,\bar{x}+\epsilon]\times Q_{\delta'}\times C_{\delta'}$. If we set

$$K_2 = \Big(\Big\|\frac{\partial\psi}{\partial\eta}\Big\|_\infty + \Big\|\frac{\partial^2\psi}{\partial x\partial\xi}\Big\|_\infty + \Big\|\frac{\partial\psi}{\partial\eta\partial\xi}\cdot\xi\Big\|_\infty\Big),$$

then, for any $(x,\eta) \in [\bar{x}-\epsilon,\bar{x}+\epsilon]\times Q_{\delta'}$, for any $\beta,\alpha \in C_\delta$, with $\alpha \leq \beta$, we have, thanks to step (3.4) of the proof,

$$\Big[\frac{\partial g_n}{\partial x} + \frac{\partial g_n}{\partial\eta}\cdot\beta\Big](x,\eta,\alpha) - \Big[\frac{\partial g_n}{\partial x} + \frac{\partial g_n}{\partial\eta}\cdot\alpha\Big](x,\eta,\beta) \\ \geq \big((\gamma/2) - \sigma K_1 - \tfrac{1}{n}K_2\big)\langle\beta-\alpha\,;\,\mathbf{e}\rangle\,.$$

This last quantity is nonnegative for $n \geq n_0$ (for some $n_0 > 0$) and $\sigma > 0$ sufficiently small.

5) From now on, σ is fixed as above and $n \geq n_0$. Let us now consider the constrained problems:

$$(\mathcal{P}_n) \qquad \min\left\{\int_{\bar{x}-\epsilon}^{\bar{x}+\epsilon} g_n(x,u(x),u'(x))\,dx, \quad \begin{array}{l} u \in W^{1,p}((\bar{x}-\epsilon,\bar{x}+\epsilon)), \\ u(\bar{x}-\epsilon) = \bar{u}(\bar{x}-\epsilon), \\ u(\bar{x}+\epsilon) = \bar{u}(\bar{x}+\epsilon), \\ u'(x) \in C_{\delta'/2} \text{ a.e.} \end{array}\right\}.$$

We have proved that the compact set $C_{\delta'/2}$ satisfies condition (**C0**) (see Lemmas 3.8 and 3.9) and that g_n satisfies conditions (**C1**$''$) and (**C2**$''$). Then, from Theorem 3.3, there is a minimum u_n of problem $(\mathcal{P}_n)$. Moreover, u_n is convex.

6) Since problems $(\mathcal{P}_n)$ are constrained problems, we have to introduce

$$\tilde{g}_n(x,\eta,\xi) \;=\; \begin{cases} g_n(x,\eta,\xi) & \text{if } \xi \in C_{\delta/2} \\ +\infty & \text{otherwise.} \end{cases}$$

Then problem $(\mathcal{P}_n)$ is equivalent to the unconstrained problem

$$(\mathcal{P}_n) \qquad \min\left\{ \int_{\bar{x}-\epsilon}^{\bar{x}+\epsilon} \tilde{g}_n(x,u(x),u'(x))\,dx, \quad \begin{array}{l} u \in W^{1,p}((\bar{x}-\epsilon,\bar{x}+\epsilon)), \\ u(\bar{x}-\epsilon) = \bar{u}(\bar{x}-\epsilon), \\ u(\bar{x}+\epsilon) = \bar{u}(\bar{x}+\epsilon) \end{array} \right\}.$$

Let $\tilde{g}_n^{**}$ be the convex envelope of $\tilde{g}_n$. The u_n being solutions of problem $(\mathcal{P}_n)$, the u_n are also solutions of the relaxed problem

$$(\mathcal{PR}_n) \qquad \min\left\{ \int_{\bar{x}-\epsilon}^{\bar{x}+\epsilon} \tilde{g}_n^{**}(x,u(x),u'(x))\,dx, \quad \begin{array}{l} u \in W^{1,p}((\bar{x}-\epsilon,\bar{x}+\epsilon)), \\ u(\bar{x}-\epsilon) = \bar{u}(\bar{x}-\epsilon), \\ u(\bar{x}+\epsilon) = \bar{u}(\bar{x}+\epsilon) \end{array} \right\},$$

and we have $\tilde{g}_n(x,u_n(x),u_n'(x)) = \tilde{g}_n^{**}(x,u_n(x),u_n'(x))$ for almost every $x \in [\bar{x}-\epsilon,\bar{x}+\epsilon]$.

7) Since the u_n are uniformly continuous (because $u_n' \in C_{\delta/2}$ and $C_{\delta/2}$ is bounded), the u_n converge weakly, up to a subsequence, in $W^{1,p}([0,1])$ to some $\tilde{u}$ a solution of the following problem:

$$(\mathcal{PR}_\infty) \qquad \min\left\{ \int_{\bar{x}-\epsilon}^{\bar{x}+\epsilon} \tilde{g}(x,u(x),u'(x))\,dx, \quad \begin{array}{l} u \in W^{1,p}((\bar{x}-\epsilon,\bar{x}+\epsilon)), \\ u(\bar{x}-\epsilon) = \bar{u}(\bar{x}-\epsilon), \\ u(\bar{x}+\epsilon) = \bar{u}(\bar{x}+\epsilon) \end{array} \right\}$$

where the convex map $\tilde{g}$ is defined by

$$\tilde{g}(x,\eta,\xi) \;=\; \begin{cases} f^{**}(x,\eta,\xi) + \sigma|\bar{u}(x)-\eta|^2 & \text{if } \xi \in C_{\delta/2} \\ +\infty & \text{otherwise.} \end{cases}$$

8) We claim that $\tilde{u} = \bar{u}$ on $[\bar{x}-\epsilon,\bar{x}+\epsilon]$.

Proof of the claim. Let us set $g(x,\eta,\xi) = f^{**}(x,\eta,\xi) + \sigma|\bar{u}(x)-\eta|^2$. On the one hand, $\bar{u}$ is clearly the *unique* minimum of the following problem:

$$(\mathcal{PR}_{loc}) \;\; \min\left\{ \int_{\bar{x}-\epsilon}^{\bar{x}+\epsilon} g(x,u(x),u'(x))\,dx \quad \begin{array}{l} u \in W^{1,p}((\bar{x}-\epsilon,\bar{x}+\epsilon)), \\ u(\bar{x}-\epsilon) = \bar{u}(\bar{x}-\epsilon), \\ u(\bar{x}+\epsilon) = \bar{u}(\bar{x}+\epsilon) \end{array} \right\}.$$

On the other hand, we have, thanks to (11), $\bar{u}'(x) \in C_{\delta'/3}$ for almost every $x \in [\bar{x}-\epsilon, \bar{x}+\epsilon]$, and thus $\tilde{g}(x, \bar{u}(x), \bar{u}'(x)) = g(x, \bar{u}(x), \bar{u}'(x))$ for almost every $x \in [\bar{x}-\epsilon, \bar{x}+\epsilon]$. Since

$$g(x, \eta, \xi) \leq \tilde{g}(x, \eta, \xi) \; , \tag{12}$$

$\bar{u}$ is also a solution of problem $(\mathcal{PR}_\infty)$. Therefore, the minimum of problem $(\mathcal{PR}_\infty)$ is equal to the minimum of problem $(\mathcal{PR}_{loc})$:

$$\min(\mathcal{PR}_\infty) \; = \; \min(\mathcal{PR}_{loc}). \tag{13}$$

Therefore, since, $\tilde{u}$ is a solution of $(\mathcal{PR}_\infty)$, we have

$$\begin{aligned} \min(\mathcal{PR}_\infty) &= \int_0^1 \tilde{g}(x, \tilde{u}(x), \tilde{u}'(x))\, dx \\ &\geq \int_0^1 g(x, \tilde{u}(x), \tilde{u}'(x))\, dx \;\; \text{thanks to (12)} \geq \min(\mathcal{PR}_{loc}). \end{aligned}$$

From equality (13), we finally deduce that $\tilde{u}$ is also a solution of $(\mathcal{PR}_{loc})$. But we have proved that problem $(\mathcal{PR}_{loc})$ has a unique solution $\bar{u}$. Thus $\tilde{u} = \bar{u}$.

In conclusion, we have proved that the u_n converge uniformly to $\bar{u}$. The u_n being convex, $\bar{u}$ is convex, and the derivatives u_n' converge to $\bar{u}'$ almost everywhere.

9) We claim that, for almost every $x \in [\bar{x}-\epsilon, \bar{x}+\epsilon]$, there is some $n_1 = n_1(x)$ such that, for any $n \geq n_1(x)$,

$$\tilde{g}_n^{**}(x, u_n(x), u_n'(x)) = g_n^{**}(x, u_n(x), u_n'(x)) \; .$$

Proof of the claim. Let x be such that the $u_n'(x)$ exist and converge to $\bar{u}'(x)$. Let $p_n \in \partial_\xi \tilde{g}_n^{**}(x, u_n(x), u_n'(x))$. Since the $\tilde{g}_n^{**}$ converge uniformly on compacts of $[0,1] \times \mathbb{R}^N \times C_{\delta'/2}$ to $\tilde{g}$, and since the u_n converge uniformly to $\bar{u}$, the p_n converge to some p satisfying on the one hand $p \in \partial_\xi \tilde{g}(x, \bar{u}(x), \bar{u}'(x))$. On another hand, since $\bar{u}'(x) \in C_{\delta'/3}$, there is a neighborhood of $\bar{u}'(x)$ on which $f^{**}(x, \bar{u}(x), \xi) = \tilde{g}(x, \bar{u}(x), \xi)$. Therefore, $p = p(x) = \frac{\partial f^{**}}{\partial \xi}(x, \bar{u}(x), \bar{u}'(x))$. From Lemma 4.3 given in the appendix, we have

$$\operatorname{Limsup}_n \; \partial g_n^*(x, u_n(x), p_n) \; \subset \; \partial f^*(x, \bar{u}(x), p(x)) \; \subset \; C_{\delta'/3}$$

(where Limsup denotes the set of cluster points of sequences of elements of $\partial g_n^*(x, u_n(x), p_n)$). Therefore, there is some $n_1(x) > 0$ such that $\forall n \geq n_1(x)$, $\partial g_n^*(x, u_n(x), p_n) \subset C_{\delta'/2}$.

Let us now fix $n \geq n_1(x)$. We obviously have

$$g_n^{**}(x, u_n(x), u_n'(x)) \leq \tilde{g}_n^{**}(x, u_n(x), u_n'(x)) \ .$$

It remains to prove the converse inequality. From the definition of the p_n and the Carathéodory theorem, there are $\lambda_i \in [0,1]$, $\xi_i \in \partial g_n^*(x, u_n(x), p_n)$ (for $i = 1, \ldots, N+1$) such that

$$\lambda_i \in [0,1], \quad u_n'(x) = \sum_{i=1}^{N+1} \lambda_i \xi_i$$

and

$$g_n^{**}(x, u_n(x), u_n'(x)) = \sum_{i=1}^{N+1} \lambda_i g_n(x, u_n(x), \xi_i) \ .$$

Since $\xi_i \in \partial g_n^*(x, u_n(x), p_n)$ for all i, we have $\xi_i \in C_{\delta'/2}$ because $n \geq n_1(x)$. Therefore, $g_n(x, u_n(x), \xi_i) = \tilde{g}_n(x, u_n(x), \xi_i)$ for any i, and

$$g_n^{**}(x, u_n(x), u_n'(x)) = \sum_{i=1}^{N+1} \lambda_i \tilde{g}_n(x, u_n(x), \xi_i) \geq \tilde{g}_n^{**}(x, u_n(x), u_n'(x))$$

by convexity. This proves our claim.

10) We finally prove that

$$f^{**}(x, \bar{u}(x), \bar{u}'(x)) = f(x, \bar{u}(x), \bar{u}'(x)) \quad \text{a.e. on } [\bar{x} - \epsilon, \bar{x} + \epsilon] \, . \tag{14}$$

Proof of (14). The u_n being solutions of problem $(\mathcal{P}_n)$, we have

$$\tilde{g}_n(x, u_n(x), u_n'(x)) = \tilde{g}_n^{**}(x, u_n(x), u_n'(x)) \tag{15}$$

for almost every $x \in [\bar{x} - \epsilon, \bar{x} + \epsilon]$, thanks to step 6.

Let x be such that the previous equality holds for x and such that the $u_n'(x)$ exist and converge to $\bar{u}'(x)$. If $n \geq n_1(x)$, we have from the previous step and (15) the following:

$$g_n^{**}(x, u_n(x), u_n'(x)) = \tilde{g}_n^{**}(x, u_n(x), u_n'(x)) = \tilde{g}_n(x, u_n(x), u_n'(x)) \, .$$

Therefore, for $n \geq n_1(x)$,

$$g_n^{**}(x, u_n(x), u_n'(x)) = \tilde{g}_n(x, u_n(x), u_n'(x)) = g_n(x, u_n(x), u_n'(x))$$

because $u_n'(x) \in C_{\delta'/2}$. Since $\xi \to -\langle \xi, \mathbf{e} \rangle^2$ is concave, Lemma 4.5 in the appendix implies that

$$\begin{aligned}&\left[f^{**}(x, u_n(x), u_n'(x)) + \tfrac{1}{n} f(x, u_n(x), u_n'(x))\right]^{**} \\ &\quad = f^{**}(x, u_n(x), u_n'(x)) + \tfrac{1}{n} f(x, u_n(x), u_n'(x))\,;\end{aligned}$$

i.e., after some elementary computations,

$$f^{**}(x, u_n(x), u_n'(x)) = f(x, u_n(x), u_n'(x)) \text{ almost everywhere on } [\bar{x} - \epsilon, \bar{x} + \epsilon].$$

Letting $n \to +\infty$ gives the desired equality:

$$f^{**}(x, \bar{u}(x), \bar{u}'(x)) = f(x, \bar{u}(x), \bar{u}'(x))\,.$$

3.5. Existence of a not-necessarily-convex solution. We investigate further the question of existence of a solution to problem $(\mathcal{P})$. We are now seeking for solutions which are not necessarily convex.

For that purpose, let us consider the following (very technical) assumptions (**H**).

(**H**) If $(u(\cdot), p(\cdot))$ is a solution to the following system:

$$(\mathbf{E}) \;\; p(x) = \frac{\partial f^{**}}{\partial \xi}(x, u(x), u'(x)) \text{ and } p'(x) = \frac{\partial f^{**}}{\partial \eta}(x, u(x), u'(x))\,,$$

then there is $F \subset [0,1]$ of zero measure such that, for any $x_0 \in [0,1] \backslash F$, we have either $f(x_0, u(x_0), u'(x_0)) = f^{**}(x_0, u(x_0), u'(x_0))$ or the following conditions: there is some open set $\mathcal{O} \subset [0,1] \times \mathbb{R}^N \times \mathbb{R}^N$ and an invertible $N \times N$ matrix $X = X(x_0, u(x_0), p(x_0))$ such that

(**h0**) $\{x_0\} \times \{u(x_0)\} \times \partial f^*(x_0, u(x_0), p(x_0)) \;\subset\; \mathcal{O}$.

(**h1**) $\forall (x, \eta, \alpha) \in \mathcal{O}$, $\forall (x, \eta, \beta) \in \mathcal{O}$, we have

$$f^{**}(x, \eta, \alpha) + f^{**}(x, \eta, \beta) \geq f^{**}(x, \eta, \max\nolimits_X(\alpha, \beta)) + f^{**}(x, \eta, \min\nolimits_X(\alpha, \beta))$$

where

$$\max\nolimits_X(\alpha, \beta) = X \max(X^{-1}\alpha, X^{-1}\beta), \;\; \min\nolimits_X(\alpha, \beta) = X \min(X^{-1}\alpha, X^{-1}\beta).$$

(**h2**) There is some $\gamma > 0$ such that, $\forall (x, \eta, \xi) \in \mathcal{O}$,

$$X^T \big[\frac{\partial f^{**}}{\partial \eta}(x, \eta, \xi) - \frac{\partial^2 f^{**}}{\partial x \partial \xi}(x, \eta, \xi) - \frac{\partial^2 f^{**}}{\partial \eta \partial \xi}(x, \eta, \xi).\xi\big] \geq \gamma \mathbf{e}$$

where $\mathbf{e} = (1, \ldots, 1) \in \mathbb{R}^N$ and X^T denotes the transpose matrix of X.

Remarks. 1) If $\bar{u}$ is a solution of the relaxed problem $(\mathcal{PR})$ and if we set $p(x) = \frac{\partial f^{**}}{\partial \xi}(x, u(x), u'(x))$, then $(\bar{u}, p)$ is a solution of system (**E**).

2) Assumptions (**H**) contain the qualitative description of the conditions of existence for a solution of problem $(\mathcal{P})$. We intend to investigate further these conditions in a later work.

We have the following theorem.

Theorem 3.12. *Assume that f satisfies growth assumption* (**G**) *and condition* (**H**). *Then problem $(\mathcal{P})$ has a solution. Moreover, any solution of the relaxed problem $(\mathcal{PR})$ is a solution of problem $(\mathcal{P})$.*

Proof. Let $\bar{u}$ be a solution of problem $(\mathcal{PR})$. We have to prove that

$$E = \{x \in (0,1) : f(x, \bar{u}(x), \bar{u}'(x)) > f^{**}(x, \bar{u}(x), \bar{u}'(x))\}$$

has a zero measure. As before, if we set $p(x) = \frac{\partial f^{**}}{\partial \xi}(x, \bar{u}(x), \bar{u}'(x))$, then $p(\cdot) : [0,1] \to \mathbb{R}^N$ is a continuous map, and we also have

$$\bar{u}'(x) \in \partial f^*(x, \bar{u}(x), p(x)) \text{ for almost every } x \in [0,1].$$

Let $F \subset E$ be as in assumptions (**H**) for $\bar{u}(\cdot)$ and $p(\cdot)$. We know that F has a zero measure. Therefore it remains to prove that the measure of $E \backslash F$ is zero. Let $x_0 \in E \backslash F$, and let us set $\eta_0 = \bar{u}(x_0)$, $p_0 = p(x_0)$. Let also $\mathcal{O}$ and X be as in assumption (**H**). We claim that the map $g : [0,1] \times \mathbb{R}^N \times \mathbb{R}^N$ defined by $g(x', \eta', \xi') = f(x', X\eta', X\xi')$ satisfies conditions (**C1******loc**) and (**C2******loc**) of the previous subsection on the open set $\mathcal{O}'$ defined by

$$\mathcal{O}' = \{(x', \eta', \xi') : (x', X\eta', X\xi') \in \mathcal{O}\}.$$

Indeed, let us first point out that $g^{**}(x', \eta', \xi') = f^{**}(x', X\eta', X\xi')$ since $\xi' \to X\xi'$ is an invertible linear map. If (x', η', α') and (x', η', β') belong to $\mathcal{O}'$, then $\max\{\alpha', \beta'\} = X^{-1}\max_X\{\alpha, \beta\}$ and $\min\{\alpha', \beta'\} = X^{-1}\min_X\{\alpha, \beta\}$, where $\alpha = X\alpha'$ and $\beta = X\beta'$. So assumption (**h1**) means that condition (**C1******loc**) is satisfied for g^{**}. Moreover we have

$$\frac{\partial g^{**}}{\partial \eta}(x',\eta',\xi') = X^T \frac{\partial f^{**}}{\partial \eta}(x', X\eta', X\xi')\,,$$

$$\frac{\partial^2 g^{**}}{\partial x \partial \xi}(x',\eta',\xi') = X^T \frac{\partial^2 f^{**}}{\partial x \partial \xi}(x', X\eta', X\xi')\,,$$

$$\frac{\partial^2 g^{**}}{\partial \eta \partial \xi}(x',\eta',\xi') = X^T \frac{\partial^2 f^{**}}{\partial \eta \partial \xi}(x', X\eta', X\xi')X\,,$$

so that

$$\frac{\partial^2 g^{**}}{\partial \eta \partial \xi}(x',\eta',\xi')\xi' = X^T \frac{\partial^2 f^{**}}{\partial \eta \partial \xi}(x', X\eta', X\xi')X\xi',$$

where $\eta_0 = \bar{u}(x_0)$ and $p_0 = p(x_0)$.

Thus assumption (**h2**) implies that condition (**C2******loc**) is satisfied for g^{**}. It is finally easy to check that $\partial g^{**}(x',\eta',\xi') = (X^{-1})^T \partial f^{**}(x', X\eta', X\xi')$ so that $\partial g^*(x_0, X^{-1}\eta_0, (X^{-1})^T p_0) = X\partial f^*(x_0,\eta_0,p_0)$. Therefore, from assumption (**h0**), we have $\{x_0\} \times \{X^{-1}\eta_0\} \times \partial g^*(x_0, X^{-1}\eta_0, (X^{-1})^T p_0) \subset \mathcal{O}'$. Then Lemma 4.4 in the appendix states that there is some $\epsilon_0 > 0$ such that, for any $x \in [x_0 - \epsilon_0, x_0 + \epsilon_0]$,

$$\{x\} \times \{X^{-1}\bar{u}(x)\} \times \partial g^*(x, X^{-1}\bar{u}(x), (X^{-1})^T p(x)) \;\subset\; \mathcal{O}'\,.$$

Let us set $\tilde{u}(x) = X^{-1}\bar{u}(x)$ and $\tilde{p}(x) = (X^{-1})^T p(x)$. It is easily checked that $\tilde{u}$ is a solution to the following problem of minimization:

$$\inf \left\{ \int_0^1 g^{**}(x, u(x), u'(x))\, dx, \quad \begin{array}{l} u \in W^{1,p}((0,1);\mathbb{R}^N), \\ u(0) = X^{-1}a,\ u(1) = X^{-1}b \end{array} \right\},$$

and that

$$\tilde{p}(x) = \frac{\partial g^{**}}{\partial \xi}(x, \tilde{u}(x), \tilde{u}'(x))\,.$$

So we are in position to apply Proposition 3.7, from which we deduce that

$$g(x, \tilde{u}(x), \tilde{u}'(x)) = g^{**}(x, \tilde{u}(x), \tilde{u}'(x)) \quad \text{a.e. on } [x_0 - \epsilon_0, x_0 + \epsilon_0]\,.$$

Therefore, for any $x_0 \in E$, there is some ϵ_0 such that

$$f(x, \bar{u}(x), \bar{u}'(x)) = f^{**}(x, \bar{u}(x), \bar{u}'(x)) \quad \text{a.e. on } [x_0 - \epsilon_0, x_0 + \epsilon_0]\,.$$

This clearly means that $E \backslash F$ has a zero measure. $\square$

3.6. An example. We complete this section with a (quite elementary) example. In this example, we show the role of the matrix X. Here $N = 2$, and we consider simplifying f of the form

$$f(\eta_1, \eta_2, \xi_1, \xi_2) = a(\eta_2)g(\xi_1) + \xi_2^2 + b(\eta_2) \ .$$

We assume that the following conditions are satisfied by a, g and b:

1) The functions a, g and b belong to $\mathcal{C}^2(\mathbb{R})$.
2) $\exists \bar{a} > 0$ with $\forall \eta_2 \in \mathbb{R}\,, a(\eta_2) \geq \bar{a}$.
3) $\forall \eta_2 \in \mathbb{R}\,,\ a'(\eta_2) > 0$ and $b'(\eta_2) > 0$.
4) $\exists \bar{\xi_1} > 0$ with $g(\xi_1) = g^{**}(\xi_1)$ on $(-\infty, \bar{\xi_1}]$ and $(g^{**})' > 0$ on $[\bar{\xi_1}, +\infty)$.
5) The map g (and therefore f) satisfies growth condition $(\mathbf{G})$.

Proposition 3.13. *Under the previous assumptions on a, g and b, f satisfies the conditions* $(\mathbf{H})$ *of Theorem 3.12. In particular, problem $(\mathcal{P})$ has a solution.*

Proof of Proposition 3.13. Let (u, p) be a solution of the Euler-Lagrange equation $(\mathbf{E})$:

$$(\mathbf{E})\ \ p(x) = \frac{\partial f^{**}}{\partial \xi}(x, u(x), u'(x)) \text{ and } p'(x) = \frac{\partial f^{**}}{\partial \eta}(x, u(x), u'(x)) \ .$$

Let $x_0 \in [0, 1]\backslash F$, where $F = \{x \in [0, 1] : u_2'(x) = 0\}$. We prove below that F has a zero measure. Let us assume that

$$f^{**}(x_0, u(x_0), u'(x_0)) < f(x_0, u(x_0), u'(x_0)) \ .$$

We have to check that conditions $(\mathbf{h1})$ and $(\mathbf{h2})$ are satisfied at x_0. With the following choice of X (that we justify below),

$$X = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \text{ or } X = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} ,$$

it is easy to check that condition $(\mathbf{h1})$ is fulfilled. Let us set $\eta_0 = u(x_0)$. In order to check that condition $(\mathbf{h2})$ holds, we compute

$$I(x_0, \eta_0, \xi) = (I_1(x_0, \eta_0, \xi), I_2(x_0, \eta_0, \xi)) = \Big[\frac{\partial f^{**}}{\partial \eta} - \frac{\partial^2 f^{**}}{\partial x \partial \xi} - \frac{\partial^2 f^{**}}{\partial \eta \partial \xi} \cdot \xi\Big](x_0, \eta_0, \xi)$$

for any $\xi \in \partial f^*(x_0, u(x_0), p(x_0))$. Note that

$$\partial f^*(x, \eta, p) = a(\eta_2)\partial g^*(\frac{p_1}{a(\eta_2)}) \times \{\frac{1}{2}p_2\} , \quad p = (p_1, p_2) ,$$

and therefore, $\forall \xi = (\xi_1, \xi_2) \in \partial f^*(x, \eta, p)$, we have $\xi_2 = p_2$.

The first component I_1 of I at (x_0, η_0, ξ) is

$$I_1 = \frac{\partial f^{**}}{\partial \eta_1} - \frac{\partial^2 f^{**}}{\partial x \partial \xi_1} - \frac{\partial^2 f^{**}}{\partial \eta_1 \partial \xi_1} \cdot \xi_1 - \frac{\partial^2 f^{**}}{\partial \eta_2 \partial \xi_1} \cdot \xi_2 = -a'(\eta_{0,2})(g^{**})'(\xi_1) \cdot \xi_2$$

where $\eta_0 = (u_1(x_0), u_2(x_0)) = (\eta_{0,1}, \eta_{0,2})$. The right-hand side of the equality is independent of ξ_1 (because g^{**} is affine on ∂g^*). Moreover, it does not vanish since $a'(\eta_2)$ and $(g^{**})'(\xi_1)$ are positive and $\xi_2 = u_2'(x_0) \neq 0$. Hence $I_1 > 0$ if $u_2'(x_0) > 0$ and $I_1 < 0$ if $u_2'(x_0) < 0$. Therefore, we choose $X = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ if $u_2'(x_0) > 0$ and $X = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ if $u_2'(x_0) < 0$.

The second component I_2 of I at (x_0, η_0, ξ) is

$$I_2 = \frac{\partial f^{**}}{\partial \eta_2} - \frac{\partial^2 f^{**}}{\partial x \partial \xi_2} - \frac{\partial^2 f^{**}}{\partial \eta_1 \partial \xi_2} \cdot \xi_1 - \frac{\partial^2 f^{**}}{\partial \eta_2 \partial \xi_2} \cdot \xi_2 = a'(\eta_{0,2})g^{**}(\xi_1) + b'(\eta_2)$$

which is positive on $\partial f^*(x_0, u(x_0), p(x_0))$ by hypothesis.

In conclusion, we have found a matrix X such that

$$\forall \xi \in \partial f^*(x_0, \eta_0, p), \quad X^T\big[\frac{\partial f^{**}}{\partial \eta} - \frac{\partial^2 f^{**}}{\partial x \partial \xi} - \frac{\partial^2 f^{**}}{\partial \eta \partial \xi} \cdot \xi\big](x_0, \eta_0, \xi) \ > \ 0.$$

Since the inequality is strict, it also holds in a neighborhood $\mathcal{O}$ of $\{x_0\} \times \{\eta_0\} \times \partial f^*(x_0, \eta_0, p)$. Thus condition (**h2**) is fulfilled.

To complete the proof, it only remains to prove that the set F has a zero measure. From the very definition of $p(\cdot)$, we have $p_2(x) = 2u_2'(x)$ and

$$p_2'(x) = a'(u_2(x))g^{**}(u_1'(x)) + b'(u_2(x)) ,$$

which is positive from the assumptions. Thus

$$2u_2''(x) = a'(u_2(x))g^{**}(u_1'(x)) + b'(u_2(x))$$

is positive for almost every x, which proves that F has zero measure from standard arguments. □

Remark. Of course, this example can be treated directly by using the Euler equation.

4. Appendix.

In this section, we collect several technical lemmas needed throughout the paper. We have put these results in the appendix for the sake of clarity. However, their proof is easy and left to the reader.

4.1. A convergence result.

Lemma 4.1. *Assume that a sequence* $u_n \in L^p([0,1],\mathbb{R})$ *(for some* $p>1$*) satisfies the following conditions:*

i) u_n *is nondecreasing on* $[0,1]$.

ii) *There is some* $C \geq 0$ *such that* $\|u_n\|_{L^p([0,1])} \leq C$.

iii) $u_n(x)$ *converges almost everywhere to some* $u(x)$.

Then, for any $q \in (1,p)$, u_n *converges to* u *in* $L^q([0,1])$.

Remarks. 1) The result does not hold for $q=p$. 2) A more general result (in $\Omega \subset \mathbb{R}^N$) can also be found in ([26]).

4.2. Some results in convex analysis.

Lemma 4.2. *Let* $f^{**} : [0,1]\times\mathbb{R}^N\times\mathbb{R}^N$ *be* $\mathcal{C}^1$. *Let* (x,η,p) *belong to* $[0,1]\times\mathbb{R}^N\times\mathbb{R}^N$ *and* α *and* β *belong to* $\partial f^*(x,\eta,p)$. *Then*

$$\forall \mu \in [0,1], \quad \frac{\partial f^{**}}{\partial \eta}(x,\eta,\mu\alpha+(1-\mu)\beta) = \mu\frac{\partial f^{**}}{\partial \eta}(x,\eta,\alpha)+(1-\mu)\frac{\partial f^{**}}{\partial \eta}(x,\eta,\beta).$$

Lemma 4.3. *Assume that a sequence of functions* ϕ_n *converges to some function* ϕ *uniformly on compacts; then, for any sequence* $p_n \in \mathbb{R}^N$ *converging to some* $p \in \mathbb{R}^N$, *we have*

$$\mathrm{Limsup}_n \ \ \partial\phi_n^*(p_n) \ \subset \ \partial\phi^*(p)$$

where Limsup denotes the set of cluster points of sequences of $\partial\phi_n^*(p_n)$.

Remark. In particular, if $\phi_n = \phi$, we have that the set-valued map $\partial\phi$ is upper semicontinuous.

Lemma 4.4. *Assume that* f *satisfies growth assumption (***G***). Then, for any continuous maps* $u:[0,1]\to\mathbb{R}^N$ *and* $p:[0,1]\to\mathbb{R}^N$, *the set-valued map*

$$x \rightsquigarrow \partial f^*(x,u(x),p(x))$$

is upper semicontinuous with convex compact images.

Lemma 4.5. *Let* $f:\mathbb{R}^N\to\mathbb{R}$ *and* $g:\mathbb{R}^N\to\mathbb{R}$ *be lower semicontinuous maps. If* g *is concave and if, for some* $\xi \in \mathbb{R}^N$, *we have* $(f+g)^{**}(\xi)=(f+g)(\xi)$, *then we have* $f^{**}(\xi)=f(\xi)$.

REFERENCES

[1] G. Aubert and R. Tahraoui, *Théorèmes d'existence pour des problèmes de calcul des variations*, J. Diff. Eq., 33 (1979), 1–15.

[2] G. Aubert and R. Tahraoui, *Théorèmes d'existence en optimisation non convexe*, Appl. Anal., 18 (1984), 75–100.

[3] G. Aubert and R. Tahraoui, *Young measures relaxation of functionals and existence results without weak lower semi continuity*, Advances in Differential Equations, 3 (1998).

[4] C. Bandle, "Isoperimetric Inequalities and Applications," London, Pitman, 1980.

[5] P. Cardaliaguet and R. Tahraoui, *Extended Hardy-Littlewood inequalities and applications to the calculus of variations*, Cahiers du Centre de Recherche V.J.C., 9806 (1998).

[6] A. Cellina, *On minima of a functional of the gradient: necessary conditions*, Nonlinear Anal., T.M.A., 20(4) (1993), 337–341.

[7] A. Cellina, *On minima of a functional of the gradient: sufficient conditions*, Nonlinear Anal.: T.M.A., 20(4) (1993), 343–347.

[8] A. Cellina and G. Colombo, *On a classical problem of the calculus of variations without convexity assumptions*, Ann. Inst. Henri Poincaré, 7(2) (1990), 97–106.

[9] L. Cesari, "Optimization Theory and Applications," (1983) Springer-Verlag, New York.

[10] K.M. Chong and N.M. Rice, "Equimeasurable Rearrangements of Functions," Queen's papers in pure and applied Mathematics No 28 (1971). Queen's University, Kingston, Ontario, Canada.

[11] J.A. Crowe, P.C. Rosenbloom, and J.A. Zweibel, *Rearrangements of functions*, J. Funct. Anal., 66 (1986), 432–438.

[12] B. Dacorogna and P. Marcellini, *Existence of minimizers for non quasiconvex integrals*, 131(4) (1995) 359–399.

[13] I. Ekeland, *Discontinuités de champs hamiltoniens et existence de solutions optimales en calcul des variations*, Publications Math. de l'I.H.E.S., 47 (1977), 5–32.

[14] I. Ekeland and R. Temam, Convex Analysis and Variational Problems, North Holland, Amsterdam, 1976.

[15] G. Friesecke, *A necessary and sufficient condition for nonattainment and formation of microstructure almost everywhere in scalar variational problems*, Proc. Royal Soc. Edinburgh., 124A (1994), 437–479.

[16] A. Griewank and P.J. Rabier, *On the smoothness of convex envelopes*, Trans. Amer. Soc., 322 (1990), 691–709.

[17] J. Mossino, Inégalités isopérimétriques et applications en physique, Travaux en cours, Hermann, Paris, 1984.

[18] C. Olech, *Integrals of set-valued functions and linear optimal control problems*, Colloque sur la théorie mathématique du contrôle optimal, C.B.R.M. Vander Louvain, (1970), 109–125.

[19] J.-P. Raymond, *Théorème d'existence pour des problèmes variationnels non convexes*, Proc. Royal Soc. Edinburgh, 107A (1987), 43–64.

[20] J.-P. Raymond, *Conditions nécessaires et suffisantes d'existence de solutions en calcul des variations*, Ann. Inst. Henri Poincaré, 4(2) (1987), 169–202.

[21] R.T. Rockafellar, Convex Analysis, Princeton University Press, Princeton, 1970.

[22] R. Tahraoui, *Théorèmes d'existence en calcul des variations et applications à l'elasticité non linéaire*, Proc. Royal Soc. Edinburgh, 109 A (1988), 51–78.

[23] R. Tahraoui, *Sur une classe de fonctionnelles non convexes et applications*, S.I.A.M. J. Math. Anal., 21 (1988), 37–52.

[24] R. Tahraoui, *Régularité de la solution d'un problème variationnel*, Ann. Inst. Henri Poincaré, 9(1) (1996), 51–99.

[25] R. Tahraoui, *Symmetrization inequalities*, Nonlinear Anal. T.M.A., 27(8) (1996), 933–955.

[26] R. Tahraoui, *Unpublished work* (1997).