Introduction to differential games

(First version[1])

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Contents

Chapter 1

Introduction

Differential game theory investigates conflict problems in systems which are driven by differential equations. This topic lies at the intersection of game theory (several players are involved) and of controlled systems (the differential equations are controlled by the players).

The scope of these notes is to review some basic results in the area: we will start with some insight of Isaacs' Theory, then will introduce the notion of strategies. The main part of these notes will be devoted to the existence of a value for two-player zerosum differential games and its characterization as the unique viscosity solution of some partial differential equation. Then we will analyze the notion of Nash equilibrium payoff with delay (or memory) strategies. The last part will be devoted to differential games with incomplete information.

This chapter is mainly dedicated with the more basic aspects of Isaacs' theory. We start with a simple game, through which we introduce a first notion of strategy. Then we discuss a verification Theorem, which is typical of Isaacs' approach.

Before this, let us fix some notations used throughout the text:

Some notations :

- $\langle x, y\rangle$ is the usual scalar product between two vectors x and y of a finite dimensional space and $|\cdot|$ the associated euclidean norm.
- $B_r(x)$ is the closed ball of center x and of radius r
- If E is a set, then $\mathbf{1}_E$ is the indicator function of E (equal to 1 if E and to 0 outside of E).
- $\mathcal{S}_N$ is the set of $N \times N$ symmetric matrices.
- If $\mathcal{O}$ is an open subset of $\mathbb{R}^N$ and $\mathbf{V} : \mathcal{O} \to \mathbb{R}$, then $D\mathbf{V}(x)$ and $D^2\mathbf{V}(x)$ respectively denote the gradient and the Hessian matrix of $\mathbf{V}$ at x whenever they exist.
- In the same way, if $\mathcal{O}$ is an open subset of $\mathbb{R} \times \mathbb{R}^N$ and $\mathbf{V} = \mathbf{V}(t, x) : \mathcal{O} \to \mathbb{R}$ depends on time and space, we denote by $\partial_t \mathbf{V}(t, x)$ the time derivative of $\mathbf{V}$ and by $D\mathbf{V}(t, x)$ and $D^2\mathbf{V}(t, x)$ the spatial gradient and spatial Hessian matrix of $\mathbf{V}$ at a point (t, x).

1.1 A simple game as an appetizer

Let us start the description of differential games with a very popular exemple: the game "Lion and Man". The story goes as follows: a lion and a man are enclosed in a closed circular arena. The lion is famished and—most surprizingly—the man is not so keen to be used as breakfast. Several questions arise then:

1. How long (at least) can the man escape the lion?
2. If he manages to escape all the time, what is the minimal distance he can put between him and the lion?

These questions are formulated from the man's viewpoint. For the lion they become:

1. How long (at most) does it take for the lion to catch the man?

2. If he does not succeed in catching the man, how close to him can he go?

These questions are typical of game theory: the lion try to minimize some quantity (capture time or distance to the man) while the man aims at maximizing such quantities.

What is unusual for game theory here is that we are dealing with a continuous time problem. Dynamics can be expressed in terms of instantaneous speeds for the lion and the man, which choose it at each time to realize their goal. This kind of problem is typical from control theory.

Let us try to formalize a little bit these ideas. We assume that the lion can run with a maximal speed denoted by $L > 0$ and that the man can run at a maximal speed $M > 0$. Let $y(t) \in \mathbb{R}^2$ be the position of the man at time t and $z(t) \in \mathbb{R}^2$ be the position of the lion at time t. Then the man chooses at each time its velocity $y'(t)$ in the set of possible velocities $U = \{u \in \mathbb{R}^2\ ,\ |u| \leq M\}$ while the lion chooses its velocity $z'(t)$ in the set $V = \{v \in \mathbb{R}^2\ ,\ |v| \leq L\}$. The sets U and V are called control sets of the man and lion respectively.

The dynamics of the problem is:

$$y'(t) = u(t) \text{ where } u(t) \in U, \text{ and } z'(t) = v(t) \text{ where } v(t) \in V\ .$$

Since man and lion have to stay in the arena, we also have to add to these equations the state constraints

$$|y(t)| \leq R \text{ and } |z(t)| \leq R \qquad \forall t \geq 0\ .$$

Let us now try to formulate, in quite a naive way, some ideas to solve this game. This will oblige us to think about a key notion in differential game theory: the notion of *strategies*.

First case : $\underline{L > M}$ This is the worse situation for the man. Obviously if the lion runs directly in the man's direction, capture will occur in a finite time. Mathematically this means that lion chooses at each time the speed

$$\bar{v}(y(t), z(t)) = L\frac{y(t) - z(t)}{|y(t) - z(t)|}\ .$$

So the lion will solve at each time the differential equation

$$z'(t) = \bar{v}(y(t), z(t))\ . \tag{1.1}$$

Since the man always stay in the arena ($|y(t)| \leq R$ for all t), it is an easy exercise to show that the solution $z(t)$ of the above equation also remains in the arena. The map $\bar{v}$ is a strategy, i.e., a decision rule which allows to the lion to decide at each time what control to play. One can also check that this is a good strategy, in the sense that it garanties the capture in controlled time whatever the man does: assume that the man chooses at each time the control $u(t)$ in U, then

$$\begin{aligned}
\tfrac{d}{dt}|y(t) - z(t)|^2 = &\ 2\langle y(t) - z(t), y'(t) - z'(t)\rangle \\
= &\ 2\langle y(t) - z(t), y'(t)\rangle - 2\langle y(t) - z(t), \bar{v}(y(t), z(t))\rangle \\
\leq &\ 2M|y(t) - z(t)| - 2L|y(t) - z(t)|
\end{aligned}$$

since the man runs with a maximal speed of M (i.e., $|u(t)| \leq M$) and $z'(t)$ is given by (1.1). So

$$\frac{d}{dt}|y(t) - z(t)|^2 \leq -2(L - M)|y(t) - z(t)|$$

which implies that

$$\frac{d}{dt}|y(t) - z(t)| \leq -(L - M)$$

Therefore capture holds before $|y(0) - z(0)|/(L - M)$, even if the man knows that the lion plays the strategy $\bar{v}$.

Note that there is no reason why this strategy would be "optimal" for the lion, since it does not take into account the fact that that the man is constrained to stay in the arena.

Second case : $\underline{L < M}$ This case is much more auspicious for the man, although it is not completely obvious that he can avoid the capture, since he is constrained to remain in the arena. We are going to show that this is actually the case and, moreover, that he can stay at some fixed distance from the lion.

Let us assume, to fix the ideas and simplify the computation that the radius of the arena is $R = 1$ and that the man starts from the boundary of the arena (Exercice : show that the man can indeed ensure to be in this situation after a while by enlarging the initial distance between him and the lion). We now work in polar coordonates. Let (ρ_M, θ_M) and (ρ_L, θ_L) the polar coordonates of the man and the lion. The constraints on the speed become

$$(\rho'_M)^2 + \rho_M^2(\theta'_M)^2 \leq M^2 \text{ and } (\rho'_L)^2 + \rho_L^2(\theta'_L)^2 \leq L^2 \ .$$

In order to avoid cumbersome technicalities, we just define the man's strategy in a neighbourhood of the "bad set" $\rho_M = \rho_L$ and $\theta_M = \theta_L$ (mod. 2π). The man is going to stay of the boundary of the arena: $\rho_M(t) = 1$ for all t. So $\rho'_M = 0(t)$ and one can choose $|\theta'_M(t)| = M$. The man also chooses his radial speed in feedback form, i.e.,

$$\theta'_M(t) = \overline{s}_M(\theta_M(t), \theta_L(t)) \tag{1.2}$$

where $\overline{s}_M$ in the following way:

$$\overline{s}_M(\theta_M, \theta_L) = \begin{cases} M & \text{if } \theta_M - \theta_L \geq 0 \text{ and } |\theta_M - \theta_L| < \frac{\pi}{2} \text{ mod. } 2\pi \\ -M & \text{if } \theta_M - \theta_L < 0 \text{ and } |\theta_M - \theta_L| < \frac{\pi}{2} \text{ mod. } 2\pi \end{cases}$$

This is a "strategy", meaning that it depends on the current position of θ_M and θ_L. Let $\epsilon \in (0, 1)$ such that

$$|\theta_M(0) - \theta_L(0)| + 1 - \rho_L(0) > \epsilon$$

and such that

$$L\left(1 + \frac{1}{(1-\epsilon)^2}\right)^{1/2} < M \ .$$

This is possible since the positions of lion and man are distinct at the initial time and $L < M$. We claim that, whatever speed $(\rho'_L(t), \theta'_L(t))$ the lion plays, one always has

$$\rho(t) := |\theta_M(t) - \theta_L(t)| + 1 - \rho_L(t) \geq \epsilon/2 \ ,$$

which just means that the distance between lion and man is bounded from below by a positive constant. Note that, since the strategy $\overline{s}_M$ is discontinuous, there could be an issue in defining the solution of equation (1.2). This is however not the case because, if $\theta_M(t) = \theta_L(t)$ (mod. 2π) for some t and $\rho_L(t)$ is sufficiently close to 1, then $|\theta'_L(t)| < M = |\theta'_M(t)|$, so that θ_L is well-defined and $\theta_M \neq \theta_L$ on some small interval $(t, t+h)$.

Let us now argue by contradiction and suppose that there is a first time $t_1 > 0$ such that $\rho(t_1) = \epsilon/2$. Then there is a nonempty interval (t_0, t_1) on which $\rho(t) \in (\epsilon/2, \epsilon)$ for $t \in (t_0, t_1)$. Let us show that ρ is nondecreasing on this interval, which clearly yields to a contradiction. Note that, on this interval, we have $1 - \rho_M(t) \leq \epsilon$, which implies that $\rho_M(t) \geq 1 - \epsilon > 0$.

We have, for almost all $t \in (t_0, t_1)$, using the Cauchy-Schwarz inequality and the fact that $\theta_M(t) \neq \theta_L(t)$ for an coutable number of t,

$$\begin{aligned} \frac{d}{dt}\rho(t) &= \operatorname{sgn}(\theta_M(t) - \theta_L(t))(\theta'_M(t) - \theta'_L(t)) - \rho'_L(t) \\ &\geq -\left((\rho'_L(t))^2 + (\rho_L(t)\theta'_L(t))^2\right)^{1/2}\left(1 + \frac{1}{(\rho_L(t))^2}\right)^{1/2} + M \\ &\geq -L\left(1 + \frac{1}{(1-\epsilon)^2}\right)^{1/2} + M \ \geq \ 0 \end{aligned}$$

thanks to the choice of ϵ.

Third case : $\underline{L = M}$ This problem has been a mathematical challenge for some time. The solution is due to Berkovitz (see the historical comments in Hajek's monograph [124]).

The result is the following: the lion can get as close as he wants to the man, but the man can always avoid the capture. In other words, capture never occurs, but the minimal distance between lion and man becomes arbitrarily small.

Let us show that the lion can get as close as he wants to the man. For this, we will assume to simplify the computations that at time 0 the lion is at the center of the arena. His strategy will be the following: he is going to move as fast as possible in the man's direction while always remaining on the same radius as him, that is, by ensuring that $z(t)$ remains on the segment $[0, y(t)]$. In polar coordinates this means that $\theta_L(t) = \theta_M(t)$ for all t. So $\overline{\theta}'_L = \theta'_M$. Since $\rho_L(t) \leq \rho_M(t)$, the lion maximises $\overline{\rho}'_L$ under the constraint $\overline{\theta}'_L = \overline{\theta}'_M$. We get

$$\overline{\rho}'_L = \left[M^2 - \bar{\rho}_L^2(\overline{\theta}'_M)^2\right]^{\frac{1}{2}} .$$

The pair $(\overline{\theta_L}', \overline{\rho_L}')$ is again a "strategy", i.e., a decision rule for the lion. Let us underline that it now depends on his opponent's speed.

Let us check that the distance between the lion and the man goes to 0. Let us again argue by contradiction and suppose that it is bounded from below by a positive constant $\epsilon > 0$. Then

$$\rho_M(t) \geq \rho_L(t) + \epsilon \qquad \forall t \geq 0 .$$

Note that

$$(\bar{\rho}_L(t) + \epsilon)|\bar{\theta}'_L(t)| \leq \rho_M(t)|\theta'_M(t)| \leq M$$

so that

$$\overline{\rho}'_L(t) = \left[M^2 - \rho_L^2(\overline{\theta}'_M)^2\right]^{\frac{1}{2}} \geq \left[M^2 - \frac{\rho_L^2 M^2}{(\epsilon + \bar{\rho}_L)^2}\right]^{\frac{1}{2}} \geq \frac{M\epsilon}{1+\epsilon}$$

since $0 \leq \bar{\rho}_M(t) \leq 1$. But the above inequality implies that $\bar{\rho}$ is increasing with a speed not smaller than $M\epsilon/(1+\epsilon)$, which contradicts the fact that $\bar{\rho}_M(t) \leq 1$. □

Let us finally show that the man can avoid the capture. In order to make our life simpler we are going to assume that the lion plays the strategy described above, the general case being much more involved.

Without loss of generality (why ?) we can assume that the man's initial position is in the interior of the arena ($\rho_M(0) \in (0,1)$). Then he is going to spiral up to the boundary. More precisely he plays

$$\theta'_M = M^{\frac{1}{2}} \left(\frac{2-\rho_M}{\rho_M}\right)^{\frac{1}{2}} \text{ and } \rho'_M = \left(M - (\rho_M \theta'_M)^2\right)^{\frac{1}{2}}$$

This choice is possible because

$$M - (\rho_M \theta'_M)^2 = M(1 - 2\rho_M + \rho_M^2) = M(1-\rho_M)^2$$

which gives $\rho'_M = M^{\frac{1}{2}}(1-\rho_M)$ and so

$$\forall t \geq 0,\ \rho_M(t) = 1 - (1-\rho_M(0))e^{-\sqrt{M}t} .$$

Let us now check that the capture does not occur in finite time: we have

$$\begin{aligned}
\tfrac{d}{dt}|y(t) - z(t)| = \tfrac{d}{dt}(\rho_M(t) - \rho_L(t)) &= \rho'_M(t) - \rho'_L(t) \\
&= \left(M - (\rho_M\theta'_M)^2\right)^{\frac{1}{2}} - \left(M - (\rho_L\theta'_M)^2\right)^{\frac{1}{2}} \\
&= -(\theta'_M)^2(\rho_M^2 - \rho_L^2)/[\rho'_M(t) + \rho'_L(t)] \\
&\geq -2(\theta'_M)^2(\rho_M - \rho_L)/[2\rho'_M(t)] \\
&\geq -(\theta'_M)^2 e^{\sqrt{M}t}(\rho_M - \rho_L)/(1-\rho_M(0))
\end{aligned}$$

Gronwall's Lemma then implies that $\rho_M(t) - \rho_L(t) > 0$ for all $t \geq 0$. □

The above approach is very naive: indeed we did not define rigourously the problem, nor what kind of strategies the player are allowed to play. We have argued as if the order in which the players play did not influence the solution, etc... One of the aims of differential game theory is to make rigourous the analysis of such games. However, in order to better understand the difficulties, it is convenient to start with a more computational aspect of the theory: Isaacs approach.

1.2 A verification theorem for pursuit-evasion games

In this section we start the analysis of two-player zero-sum differential games by presenting some ideas due to Isaacs and his followers. Although most of these ideas rely on regularity assumptions (on the value function) that do not hold in general, they shed a precious light on the problem by revealing—in an idealized and simplified framework—many phenomena that will be encountered in a more technical setting throughout the other chapters. Another interesting aspect of these techniques is that they allow to solve explicitely several games, in contrast with the subsequent theories which are mainly concerned with theoretical issues.

We restrict ourselves to present the very basic aspects of this theory, a deeper analysis exceeding largely the scope of these notes. The interested reader can find significant developments in the monographs by Isaacs himself [133], Blaquière, Gérard and Leitman [35], Baçsar and Olsder [15], Bernhard [40], Lewin [151] and Melikyan [162].

1.2.1 A general framework

A pursuit-evasion differential game is given by a dynamics and a target: the dynamics is an (ordinary) differential equation controlled by two controllers:

$$X_t' = f(X_t, u_t, v_t)$$

where u_t belongs for any t to some control set U and is chosen at each time by the first Player and v_t belongs to some control set V and is chosen by the second Player. The state of the system X_t lies in $\mathbb{R}^N$ and we will always assume that $f : \mathbb{R}^N \times U \times V \to \mathbb{R}^N$ is smooth and bounded, so that the solution of the above differential equation will be defined on $[0, +\infty)$.

The target C is a subset of $\mathbb{R}^N$. In the pursuit-evasion game we investigate a game in which the first Player tries to maintain the state of the system as long as possible outside of the target C while the second Player aims at reaching C as soon as possible.

We now have to explain the nature of information the Players are allowed to use in order to choose their control all along the play. This is our first attempt to rigourously formalize the fondamental notion of strategies.

Definition 1.1 (Feedback strategies) *A feedback strategy for the first Player (resp. for the second Player) is a map* $\bar{u} : \mathbb{R}^+ \times \mathbb{R}^N \to U$ *(resp.* $\bar{v} : \mathbb{R}^+ \times \mathbb{R}^N \to V$*).*

This means that each Player chooses at each time t its control as a function of t and of the current position of the system. As will be extensively explained later on, other definitions of strategies are possible (and in fact more appropriate to prove existence of the value).

The main issue here is that, given two arbitrary strategies $\bar{u} : \mathbb{R}^+ \times \mathbb{R}^N \to U$ and $\bar{v} : \mathbb{R}^+ \times \mathbb{R}^N \to V$ and an initial position $x_0 \in \mathbb{R}^N$, the system

$$\left\{ \begin{array}{l} X_t' = f(X_t, \bar{u}(t, X_t), \bar{v}(t, X_t)) \\ x(0) = x_0 \end{array} \right. \tag{1.3}$$

does not necessarily have a solution. Hence we have to restrict the choice of strategies for the Players.

Definition 1.2 *A pair* $(\bar{U}, \bar{V})$ *of sets of strategies is* **admissible** *if:*

1. *all the Lebesgue measurable maps* $u : [0, +\infty[\to U$ *(resp.* $v : [0, +\infty[\to V$*) belong to* $\bar{U}$ *(resp.* $\bar{V}$*).*
2. *for any pair* $(\bar{u}, \bar{v})$ *of* $(\bar{U}, \bar{V})$ *and any initial position* x_0*, equation (1.3) has a unique solution.*
3. *(concatenation property) if* $\bar{u}_1$ *and* $\bar{u}_2$ *belong to* $\bar{U}$*, then for any time* $\tau > 0$ *the strategy* u_3 *defined by*

   $$\bar{u}_3(t, x) = \bar{u}_1(t, x) \text{ if } t \in [0, \tau] \text{ and } \bar{u}_3(t, x) = \bar{u}_2(t, x) \text{ otherwise}$$

   also belongs to $\bar{U}$ *(and symmetrically for* $\bar{V}$*).*
4. *(shift property) if* $\bar{u}$ *belongs to* $\bar{U}$ *and* $\tau > 0$*, then* $\bar{u}_1(t, x) = \bar{u}(t + \tau, x)$ *belongs to* $\bar{U}$ *(and symmetrically for* $\bar{V}$*).*

Notation: From now on we fix a given admissble pair $\bar{U} \times \bar{V}$ of set of strategies. For any $(\bar{u}, \bar{v}) \in \bar{U} \times \bar{V}$ we denote by $X_t^{x_0, \bar{u}, \bar{v}}$ the unique solution of (1.3).

Let us explain what means reaching the target: for given a trajectory $X : [0, +\infty) \to \mathbb{R}^N$ let

$$\theta_C(X) := \inf\{t \geq 0 \mid X_t \in C\} .$$

We set $\theta_C(X) := +\infty$ if $X_t \notin C$ for all $t \geq 0$. For $x_0 \notin C$, $\bar{u} \in \bar{U}$ and $\bar{v} \in \bar{V}$ we define

$$\mathcal{J}(x_0, \bar{u}, \bar{v}) := \theta_C(X^{x_0, \bar{u}, \bar{v}}) .$$

Definition 1.3 *For a fixed admissible pair $(\bar{U}, \bar{V})$ of sets of strategies the lower value function is given by*

$$\mathbf{V}^-(x_0) := \sup_{\bar{u} \in \bar{U}} \inf_{\bar{v} \in \bar{V}} \mathcal{J}(x_0, \bar{u}, \bar{v})$$

while the upper value function is

$$\mathbf{V}^+(x_0) := \inf_{\bar{v} \in \bar{V}} \sup_{\bar{u} \in \bar{U}} \mathcal{J}(x_0, \bar{u}, \bar{v})$$

We say that the game has a value if $\mathbf{V}^+ = \mathbf{V}^-$. In this case we say that the map $\mathbf{V} := \mathbf{V}^+ = \mathbf{V}^-$ is the value of the game. We say that a strategy $\bar{u}^* \in \bar{U}$ (resp. $\bar{v}^* \in \bar{V}$) is optimal for the first Player (resp. the second Player) if

$$\mathbf{V}^-(x_0) := \inf_{\bar{v} \in \bar{V}} \mathcal{J}(x_0, \bar{u}^*, \bar{v}) \qquad (\text{resp. } \mathbf{V}^+(x_0) := \sup\nolimits_{\bar{u} \in \bar{U}} \mathcal{J}(x_0, \bar{u}, \bar{v}^*))$$

Let us note that upper and lower value functions *a priori* depend on the admissible sets of strategies $(\bar{U}, \bar{V})$. In fact, with this definition of value function, the existence of a value is completely open. This is the reason why we will be forced to give up this approach to obtain a more general result. However we shall now see that it is nevertheless useful to understand some basic facts on the problem.

1.2.2 A verification Theorem

The following verification Theorem—largely due to Isaacs—allows to show that a given function is indeed the value function of the game. For this, it will be enough to check that this (supposedly smooth) function is the solution of a Partial Differential Equation (P.D.E.) called Hamilton-Jacobi-Isaacs equation (in short Isaacs'equation).

Let us associate with the dynamics of the game

$$X'(t) = f(X_t, u_t, v_t)$$

two functions called *Hamiltonians* of the system:

$$H^-(x,p) := \sup_{u \in U} \inf_{v \in V} \langle p, f(x,u,v) \rangle \text{ and } H^+(x,p) := \inf_{v \in V} \sup_{u \in U} \langle p, f(x,u,v) \rangle .$$

Obviously

$$H^-(x,p) \leq H^+(x,p) \qquad \forall (x,p) \in \mathbb{R}^N \times \mathbb{R}^N .$$

We work here under *Isaacs'condition*, which suppose the equality between these two quantities:

$$H^-(x,p) = H^+(x,p) \qquad \forall (x,p) \in \mathbb{R}^N \times \mathbb{R}^N . \tag{1.4}$$

In this case we set

$$H(x,p) = H^-(x,p) = H^+(x,p) .$$

Let us recall that the argument of an optimization problem is the set of optimizers (if any) of this problem. Let us denote by $\tilde{u}(x,p)$ and $\tilde{v}(x,p)$ elements of $\arg\max_{u \in U} \min_{v \in V} \langle p, f(x,u,v) \rangle$ and of $\arg\min_{v \in V} \max_{u \in U} \langle p, f(x,u,v) \rangle$:

$$\tilde{u}(x,p) \in \arg\max\nolimits_{u \in U} \min_{v \in V} \langle p, f(x,u,v) \rangle \quad \text{and} \quad \tilde{v}(x,p) \in \arg\min\nolimits_{v \in V} \max_{u \in U} \langle p, f(x,u,v) \rangle \qquad \forall (x,p) .$$

Theorem 1.4 (Verification Theorem) *Let us assume that the target is closed and that Isaacs'condition (1.4) holds. Suppose that there is a nonnegative map* $\mathbf{V} : \mathbb{R}^N \to \mathbb{R}$*, continuous on* $\mathbb{R}^N$ *and of class* $\mathcal{C}^1$ *over* $\mathbb{R}^N \backslash C$*, with* $\mathbf{V}(x) = 0$ *on* C *and satisfying the Isaacs'equation:*

$$H(x, DV(x)) + 1 = 0 \qquad \forall x \in \mathbb{R}^N \backslash C \, . \tag{1.5}$$

Let us furthermore assume that the maps $x \to \bar{u}^*(x) := \tilde{u}(x, DV(x))$ *and* $x \to \bar{v}^*(x) := \tilde{v}(x, DV(x))$ *belong to* $\bar{U}$ *and* $\bar{V}$ *respectively.*

Then the game has a value and $\mathbf{V}$ *is the value of the game. Moreover the strategies* $\bar{u}^*(x)$ *and* $\bar{v}^*(x)$ *are optimal, in the sense that*

$$\mathbf{V}(x) = \mathcal{J}(x, \bar{u}^*, \bar{v}^*) = \sup_{\bar{u} \in \bar{U}} \mathcal{J}(x, \bar{u}, \bar{v}^*) = \inf_{\bar{v} \in \bar{V}} \mathcal{J}(x, \bar{u}^*, \bar{v})$$

for all $x \in \mathbb{R}^N \backslash C$*.*

This result is quite striking since it reduces the resolution of the game to that of a P.D.E. and furthermore provides in the same time the optimal feedbacks of the players. Unfortunately it is of limited interest because the value function is very seldom smooth enough for this result to apply. In fact Isaacs' theory is mainly concerned with the singularities of the value function, i.e., the set of points where the value function fails to be either continuous, or differentiable.

The reverse result also holds true: if the game has a value and if this value has the regularity described in the Theorem, then it satisfies Isaacs' equation (1.5). This statement shall be proved, under much more general conditions, in a subsequent chapter.

Proof: Let us fix $x_0 \in \mathbb{R}^N$. We first show that

$$\sup_{\bar{u}} \mathcal{J}(x_0, \bar{u}, \bar{v}^*) \leq V(x_0) \, .$$

For this, let $\bar{u} \in \bar{U}$ and let us set $X_t = X^{x_0, \bar{u}, \bar{v}^*}$ and $\tau := \mathcal{J}(x_0, \bar{u}, \bar{v}^*)$. Then, for any $t \in [0, \tau)$, we have

$$\begin{array}{rcl} \frac{d}{dt}\mathbf{V}(X_t) & = & \langle D\mathbf{V}(X_t), f(X_t, \bar{u}(t, X_t), \bar{v}^*(X_t)) \rangle \\ & \leq & \max_{u \in U} \langle D\mathbf{V}(X_t), f(X_t, u, \bar{v}^*(X_t)) \rangle \\ & = & H(X_t, D\mathbf{V}(X_t)) \\ & = & -1 \end{array}$$

Let us integrate the above inequality between 0 and $t \leq \tau$. We get

$$\mathbf{V}(X_t) - \mathbf{V}(x_0) \leq -t \, .$$

Since $\mathbf{V}(X_t)$ is nonnegative and since this inequality holds for any $t \leq \tau$, τ has to be finite. For $t = \tau$, we have $\mathbf{V}(x(\tau)) = 0$ since X_τ belongs to C. So by continuity of $\mathbf{V}$ on $\mathbb{R}^N$,

$$\mathbf{V}(x_0) \geq \tau = \mathcal{J}(x_0, \bar{u}, \bar{v}^*) \, .$$

One can show in the same way that $\mathbf{V}(x_0) \leq \mathcal{J}(x_0, \bar{u}^*, \bar{v})$ for any $\bar{v} \in \bar{V}$. Hence

$$\mathcal{J}(x_0, \bar{u}, \bar{v}^*) \leq \mathbf{V}(x_0) \leq \mathcal{J}(x_0, \bar{u}^*, \bar{v})$$

which proves that the game has a value, that this value is $\mathbf{V}$ and that the strategies $\bar{u}^*(x)$ and $\bar{v}^*(x)$ are optimal. □

Corollary 1.5 *Under the notations and assumptions of Theorem 1.4, one has*

$$\mathbf{V}(X_t^{x_0, \bar{u}^*, \bar{v}^*}) = \mathbf{V}(x_0) - t \qquad \forall t \in [0, \mathbf{V}(x_0)], \ \forall x_0 \in \mathbb{R}^N \backslash C \, .$$

Proof: Indeed

$$\begin{array}{rcl} \frac{d}{dt}\mathbf{V}(X_t) & = & \langle D\mathbf{V}(X_t), f(X_t, \bar{u}^*(X_t), \bar{v}^*(X_t)) \rangle \\ & = & H(X_t, D\mathbf{V}(X_t)) \ = \ -1 \end{array}$$

□

1.2.3 The Hamiltonian system

Let us now give some hints about the explicit computation of the value function $\mathbf{V}$. The key idea is that it is possible to compute $\mathbf{V}$ along the characteristics associated with Isaacs' equation (1.5). This system of ordinary differential equations is a Hamiltonian system.

Throughout this section we assume that $\mathbf{V}$ is as in Theorem 1.4 and of class $\mathcal{C}^2$ in $\mathbb{R}^N\backslash C$. We also assume that the maps $\tilde{u}(x,p)$ and $\tilde{v}(x,p)$ are uniquely defined. Finally we suppose that the Hamiltonian $H(x,p) := H^-(x,p) = H^+(x,p)$ is of class $\mathcal{C}^2$ on $\mathbb{R}^N \times \mathbb{R}^N_*$ (which implies in general some regularity on f).

Theorem 1.6 *Let $x_0 \in \mathbb{R}^N\backslash C$ be an initial position, $\bar{u}^*$ and $\bar{v}^*$ be the optimal strategies given in Theorem (1.4) and let us set $X_t = X^{x_0,\bar{u}^*,\bar{v}^*}$. Then the pair (X,P), where $P_t := D\mathbf{V}(X_t)$, is a solution of the Hamiltonian system*

$$\begin{cases} X'_t = \frac{\partial H}{\partial p}(X_t,P_t) \\ P'_t = -\frac{\partial H}{\partial x}(X_t,P_t) \end{cases} \tag{1.6}$$

on $[0,\mathbf{V}(x_0))$.

Remarks 1.7

1. The variable P is often called the adjoint variable of X.
2. In control theory (i.e., when f only depends on u or on v), the existence of such an adjoint is an optimality condition for a given trajectory X. This statement is known as the Pontryagin maximum principle.

Proof of Theorem 1.6: We first use the following Lemma, proved below:

Lemma 1.8

$$\frac{\partial H}{\partial p}(x,p) = f(x,\tilde{u}(x,p),\tilde{v}(x,p))$$

In particular $X'_t = \frac{\partial H}{\partial p}(X_t,P_t)$. Let us now notice that, by definition of P,

$$P'_t = D^2\mathbf{V}(X_t)X'_t = D^2\mathbf{V}(X_t)f(X_t,\bar{u}^*(X_t),\bar{v}^*(X_t))) = D^2\mathbf{V}(X_t)\frac{\partial H}{\partial p}(X_t,P_t).$$

Differentiating Isaacs' equation (1.5), we get

$$\frac{\partial H}{\partial x}(x,D\mathbf{V}(x)) + D^2\mathbf{V}(x)\frac{\partial H}{\partial p}(x,D\mathbf{V}(x)) = 0 \qquad \forall x \in \mathbb{R}^N\backslash C\ ,$$

from which we deduce the equation satisfied by P_t. □

Proof of Lemma 1.8: Let us recall that the maximum and the minimum $\tilde{u}(x,p)$ et $\tilde{v}(x,p)$ are supposed to be unique. Since f is continuous and U and V are compact, this easily implies that the maps $(x,p) \to \tilde{u}(x,p)$ and $(x,p) \to \tilde{v}(x,p)$ are continuous on $\mathbb{R}^N \times \mathbb{R}^N_*$. Let $q \in \mathbb{R}^N$ and $h > 0$. We have (omitting the x variable for simplicity)

$$H(p+hq) - H(p) = \langle f(\tilde{u}(p+hq),\tilde{v}(p+hq)),p+hq\rangle - \langle f(\tilde{u}(p),\tilde{v}(p)),p\rangle$$

But

$$\langle f(\tilde{u}(p+hq),\tilde{v}(p+hq)),p+hq\rangle \ \leq\ \langle f(\tilde{u}(p+hq),\tilde{v}(p)),p+hq\rangle$$

while

$$\langle f(\tilde{u}(p),\tilde{v}(p)),p\rangle \ \geq\ \langle f(\tilde{u}(p+hq),\tilde{v}(p)),p\rangle\ ,$$

so that

$$H(p+hq) - H(p) \ \leq\ h\langle f(\tilde{u}(p+hq),\tilde{v}(p)),q\rangle\ .$$

Dividing by $h > 0$ and letting $h \to 0^+$, we obtain

$$\langle DH(p),q\rangle \ \geq\ \langle f(\tilde{u}(p),\tilde{v}(p)),q\rangle\ .$$

The same inequality for $-q$ leads to the opposite inequality. □

Let us now assume that the target is the closure of an open set with a smooth boundary: more precisely we suppose that $C = \{x \in \mathbb{R}^N,\ \phi(x) \leq 0\}$ where $\phi : \mathbb{R}^N \to \mathbb{R}$ is a $\mathcal{C}^2$ function such that $D\phi(x) \neq 0$ whenever $\phi(x) = 0$.

Proposition 1.9 *If we assume furthermore that* $\mathbf{V}$ *is of class* $\mathcal{C}^1$ *on* $\overline{\mathbb{R}^N\backslash C}$*, then*

$$\forall x \in \partial C,\ H(x, D\phi(x)) < 0 \text{ et } D\mathbf{V}(x) = -\frac{D\phi(x)}{H(x, D\phi(x))}$$

Remarks 1.10

1. The pair (X, P) in Theorem 1.6 admits for an initial condition for X: $X_0 = x_0$ and a terminal condition for P: $P_T = -\frac{D\phi(X_T)}{H(x, D\phi(X_T))}$ where $T = \mathbf{V}(x_0)$.
2. We shall see below that the condition $H(x, D\phi(x)) \leq 0$ in ∂C is necessary in order to ensure the value function to be continuous in a neighborhood of ∂C.

Proof of Proposition 1.9: Let $y \in \partial C$. Then $\mathbf{V}(y) = 0$. Since $\mathbf{V} = 0$ on ∂C, $\mathbf{V}$ has a local maximum at y on $\partial C = \{x \in \mathbb{R}^N\ ;\ \phi(x) = 0\}$, so that, by the Euler-Lagrange condition, there is some $\lambda \in \mathbb{R}$ such that $D\mathbf{V}(y) = \lambda D\phi(y)$. Now note that, since $D\phi(y) \neq 0$,

$$\phi\left(y + hD\phi(y)\right) = \phi(y) + h|D\phi(y)|^2 + o(h) = h|D\phi(y)|^2 + o(h)\ >\ 0$$

for any $h > 0$ sufficiently small. So the point $y + hD\phi(y)$ belongs to $\mathbb{R}^N\backslash C = \{x \in \mathbb{R}^N,\ \phi(x) > 0\}$ for $h > 0$ sufficiently small. Since $\mathbf{V}$ is positive on $\mathbb{R}^N\backslash C$, this implies that

$$0\ <\ \mathbf{V}\left(y + hD\phi(y)\right) = \mathbf{V}(y) + h\langle D\phi(y), D\mathbf{V}(y)\rangle + o(y) = h\lambda|D\phi(y)|^2 + o(h)\ .$$

Dividing by $h > 0$ and letting $h \to 0^+$ we get $\lambda \geq 0$.

Recall that $H(x, D\mathbf{V}(x)) = -1$, so that $H(x, \lambda D\phi(x)) = -1$. Since H is $1-$positively homogeneous with respect to p, one gets $\lambda = -1/H(x, D\phi(x))$. □

The idea in order to compute $\mathbf{V}$ at some point x_0 could be the following: one looks at a solution to (1.6) with limit condition $X_0 = x_0$ and $P_T = -\frac{D\phi(X_T)}{H(x, D\phi(X_T))}$ where $T = \theta_C(X)$. Then $T = \theta_C(X)$ is a good candidate for $\mathbf{V}(x_0)$.

Unfortunately such a solution does not always exist, or there might be several solutions. Moreover, even if it exists and is unique, it does not necessarily give the right answer. This is due to the fact that the value function is not $\mathcal{C}^1$—nor even continuous—in general.

In pratice one looks at all the solutions of the backward system

$$\begin{cases} X_t' = -\frac{\partial H}{\partial p}(X_t, P_t) \\ P_t' = \frac{\partial H}{\partial x}(X_t, P_t) \\ X_0 = \xi,\ P_0 = -\frac{D\phi(\xi)}{H(\xi, D\phi(\xi))} \end{cases}$$

for $\xi \in \partial C$ and try to deduce from this the function $\mathbf{V}$.

1.2.4 Usable part of the boundary and discontinuities

Usable part of the boundary

We have seen above (Proposition 1.9) that a necessary condition for the value to be of class $\mathcal{C}^1$ on $\overline{\mathbb{R}^N\backslash C}$ is that

$$\forall x \in \partial C,\ H(x, D\phi(x)) < 0$$

where $C := \{x \in \mathbb{R}^N \mid \phi(x) \leq 0\}$. This leads us to call *usable part of the boundary* ∂C the set of points $x \in \partial C$ such that $H(x, D\phi(x)) < 0$. We denote this set by UP.

Proposition 1.11 *Let us assume that the game has a value* $\mathbf{V}$ *and that Isaacs' condition (1.4) holds. If* $x \in UP$ *then*

$$\lim_{y \to x,\ y \notin C} \mathbf{V}(y)\ =\ 0$$

On the contrary, if $H(x, D\phi(x)) > 0$*, then*

$$\exists \tau > 0,\ \exists r > 0, \text{ such that, if } |y - x| \leq r \text{ and } y \notin C, \text{ then } \mathbf{V}(y)\ \geq\ \tau$$

Remarks 1.12

1. The Proposition states that the second Player (which plays with v) can ensure an almost immediate capture in a neighborhood of the points of UP.
2. On the contrary, in a neighborhood of a point $x \in \partial C$ such that $H(x, D\phi(x)) > 0$ holds, the first Player can postpone the capture at least for a positive time.
3. What happens at points $x \in \partial C$ such that $H(x, D\phi(x)) = 0$ is much more intricate. The set of such points—improperly called Boundary of the Usable Part (BUP)—plays a central role in the computation of the boundary of the domain of the value function.

Proof of Proposition 1.11: If $x \in UP$ then, from the definition of H,

$$H(x, D\phi(x)) = \inf_{v\in V} \sup_{u\in U} \langle f(x,u,v), D\phi(x)\rangle \ < \ 0$$

and thus there are $\theta > 0$, $v_0 \in V$ such that, for any $u \in U$,

$$\langle f(x,u,v_0), D\phi(x))\rangle \ \leq \ -2\theta.$$

By continuity, there is some $r' > 0$ with

$$\forall y \in B(x,r'),\ \forall u \in U,\ \langle f(y,u,v_0), D\phi(y)\rangle \ \leq \ -\theta.$$

Let M be an upper bound of $|f|$ and of $|D\phi|$ on $B(x,r') \times U \times V$ and $B(x,r')$ respectively. Let $r \in (0, (r'\theta)/(M^2+\theta))$ (note that $r < r'$). Then $\phi(y) \leq rM$ in $B(x,r)$ because $\phi(x) = 0$ and ϕ is $M-$Lipschitz continuous.

Let us now fix $y \in B(x,r)\backslash C$ and $\bar{u} \in \bar{U}$ a strategy for the first Player. Let us set $X_t = X_t^{y,\bar{u},\bar{v}}$, where $\bar{v} := v_0$ (constant strategy). Then $X_t \in B(x,r')$ for $t \in [0,(r'-r)/M]$ since $|x'(t)| \leq M$. On this interval one has

$$\frac{d}{dt}\phi(X_t) = \langle D\phi(X_t), f(X_t, \bar{u}(t,X_t), v_0)\rangle \ \leq \ -\theta$$

So

$$\phi(X_t) \leq \phi(x) - \theta t \leq Mr - \theta t\ .$$

For $t = Mr/\theta$ (such a t belongs to $[0,(r'-r)/M]$ by the choice of r), this implies that $\phi(X_t) \leq 0$. Hence $\mathcal{J}(y, \bar{u}, \bar{v}_0) \leq Mr/\theta$. So, for any $y \in B(x,r)$

$$\mathbf{V}(y) \leq \sup_{\bar{u}\in\bar{U}} \mathcal{J}(y,\bar{u},\bar{v}_0) \leq Mr/\theta\ ,$$

which shows that $\lim_{y\to x,\ y\notin C} \mathbf{V}(y) = 0$ since r is arbitrary.

If $H(x, D\phi(x)) > 0$, one can do similar estimates with a constant strategy $\bar{u} := u_0$ for Player 1, where $u_0 \in U$ is such that

$$\inf_{v\in V} \langle f(x,u_0,v), D\phi(x)\rangle \ > \ 0\ .$$

□

Discontinuities

We now investigate what happens at points of discontinuity of the value function.

Proposition 1.13 *Let us still assume that the game has a value* $\mathbf{V}$ *and that Isaacs' condition (1.4) holds. Let* $x \notin C$ *and* $r > 0$ *such that* $B(x,r) \subset \mathbb{R}^N\backslash C$. *Let us assume that there are two maps* $\mathbf{V}_1$ *and* $\mathbf{V}_2$ *of class* $\mathcal{C}^1$ *in* $B(x,r)$ *and a map* $\psi : \mathbb{R}^N \to \mathbb{R}$ *such that*

a) $\forall y \in B(x,r),$ if $\psi(y) = 0,$ then $D\psi(y) \neq 0.$
b) $\forall y \in B(x,r),$ if $\psi(y) > 0,$ then $\mathbf{V}(y) = \mathbf{V}_1(y).$
c) $\forall y \in B(x,r),$ if $\psi(y) < 0,$ then $\mathbf{V}(y) = \mathbf{V}_2(y).$
d) $\forall y \in B(x,r),$ if $\psi(y) = 0,$ then $\mathbf{V}_1(y) > \mathbf{V}_2(y).$

Then

$$H(x, D\psi(x)) = 0.$$

Remarks 1.14

1. The set $\{x \mid \psi(x)=0\}$ is called a *semi-permeable surface* or *barrier* for the game. One can show that each player can prevent the other one from crossing the barrier in one direction.

2. Equation $H(x, D\psi(x)) = 0$ is also called Isaacs' equation. It is a geometric equation, in the sense that one is interested not in the solution ψ itself, but in the set $\{\psi(x)=0\}$. Since H is $1-$positively homogeneous, if ψ is a solution, then so is $\theta\circ\psi$ for any smooth nondecreasing map $\theta:\mathbb{R}\to\mathbb{R}$.

3. An important particular application of the Proposition concerns the domain of the value function (i.e., the set $dom(\mathbf{V}) := \{x \mid \mathbf{V}(x) < +\infty\}$). If the boundary of the domain is smooth, then it satisfies Isaacs' equation.

4. If ψ satisfies Isaacs' equation, then the set $\{x\in\mathbb{R}^N \ ; \ \psi(x)=0\}$ is invariant for the solution of (1.6) in the following sense: the solution (X,P) of (1.6) with initial conditions $X_0 = x$ and $P_0 = D\psi(x)$ satisfies $\psi(X_t) = 0$ pour tout $t\geq 0$ small.

Proof of Proposition 1.13: Up to reducing r if necessary, we can assume that there is $\eta>0$ such that

$$\inf_{y\in B(x,r)} \mathbf{V}_1(y) \geq \sup_{y\in B(x,r)} \mathbf{V}_2(y)+\eta \tag{1.7}$$

Let M be an upper bound of $|f|$ and $|D\psi|$ on $B(x,r)$. Let us argue by contradiction by assuming for instance that $H(x,D\psi(x))>0$. Then there is $u_0\in U$ and $\theta>0$ such that

$$\langle f(x,u_0,v), D\psi(x)\rangle \geq 2\theta \qquad \forall v\in V\ .$$

By continuity we can find $r'\in(0,r)$ such that

$$\langle f(y,u_0,v), D\psi(y)\rangle \geq \theta \qquad \forall y\in B(x,r'),\ \forall v\in V\ .$$

As in the proof of Proposition 1.11, we can then show that there is a ball $B(x,r'')$ (for some $0<r''<r'$) such that, if $y\in B(x,r'')$ with $\psi(y)<0$, then there is a time $t^*\leq Mr''/\theta$ such that $\psi(X_{t^*}^{y,\bar{u}_0,\bar{v}})>0$, where $\bar{u}_0 := u_0$ and $\bar{v}\in\bar{V}$ is any strategy of Player 2.

Let now $\bar{v}^*$ be an optimal strategy for Player 2 when starting from a reference point $y_0\in B(x,r'')$ with $\psi(y_0)<0$. Let $X_t = X_t^{y_0,u_0,\bar{v}^*}$. Then there is a time $t^*\in(0,Mr''/\theta)$ such that $\psi(X_{t^*})>0$ and $y^* := X_{t_*}\in B(x,r)$. Let now u_1 be an $\epsilon-$optimal response to the strategy $\bar{v}^*(\cdot+t^*,\cdot)$. By definition of the value one has

$$\mathcal{J}(y^*,u_1,\bar{v}^*(\cdot+t^*,\cdot)) \geq \mathbf{V}(y^*)-\epsilon = \mathbf{V}_1(y^*)-\epsilon\ .$$

Let us now define the new control u by $u(t)=u_0$ on $[0,t^*)$ and $u(t)=u_1(t)$ on $[t^*,+\infty)$. Then

$$\mathbf{V}(y_0)=\mathbf{V}_2(y_0)\geq \mathcal{J}(y_0,u,\bar{v}^*) = \mathcal{J}(y^*,u_1,\bar{v}^*(\cdot+t^*,\cdot))-t^* \geq \mathbf{V}_1(y^*)-\epsilon-Mr''/\theta$$

which yields to a contradiction with (1.7) if we choose ϵ and r'' small enough. □

1.2.5 Some classical differential games

We complete this section by introducing other classes of two-player zero-sum differential that are classicaly studied in the literature.

Bolza problem

Bolza problem is a problem with finite horizon: the game ends up at some fixed terminal time denoted here T. Let us fix an admissible pair $(\bar{U},\bar{V})$ of feedback strategies. Given an initial condition $(t_0,x_0)\in[0,T]\times\mathbb{R}^N$a strategy $\bar{u}\in\bar{U}$ for the first Player and a strategy $\bar{v}\in\bar{V}$ for the second player, the payoff is given by

$$\mathcal{J}(t_0,x_0,\bar{u},\bar{v}) := \int_{t_0}^T \ell(s,X_s,\bar{u}(s,X_s),\bar{v}(s,X_s))ds + g(x(T))$$

where X is the unique solution to the following differential equation

$$\begin{cases} X_t' = f(X_t, \bar{u}(t, X_t), \bar{v}(t, X_t)) \\ x(t_0) = x_0 \end{cases} \tag{1.8}$$

We assume that $\ell : [0, T] \times \mathbb{R}^N \times U \times V \to \mathbb{R}$ and $g : \mathbb{R}^N \to \mathbb{R}$ are smooth and bounded. The first Player minimizes the payoff while the second Player maximises it. This leads to the definition of the value functions:
Upper value function:

$$\mathbf{V}^+(t_0, x_0) := \inf_{\bar{u}\in\bar{U}} \sup_{\bar{v}\in\bar{V}} \mathcal{J}(t_0, x_0, \bar{u}, \bar{v})$$

Lower value function:

$$\mathbf{V}^-(t_0, x_0) := \sup_{\bar{v}\in\bar{V}} \inf_{\bar{u}\in\bar{U}} \mathcal{J}(t_0, x_0, \bar{u}, \bar{v})$$

Isaacs' condition takes the form:

$$H(t, x, p) := \inf_{u\in U} \sup_{v\in V} \{\ell(t, x, u, v) + \langle p, f(x, u, v)\rangle\} = \sup_{v\in V} \inf_{u\in U} \{\ell(t, x, u, v) + \langle p, f(x, u, v)\rangle\}$$

for any $(t, x, p) \in [0, T] \times \mathbb{R}^N \times \mathbb{R}^N$. We assume that there exists $\tilde{u}(t, x, p)$ and $\tilde{v}(t, x, p)$ such that

$$H(t, x, p) := \sup_{v\in V} \{\ell(t, x, \tilde{u}(t, x, p), v) + \langle p, f(x, \tilde{u}(t, x, p), v)\rangle\} = \inf_{u\in U} \{\ell(t, x, u, \tilde{v}(t, x, p)) + \langle p, f(x, u, \tilde{v}(t, x, p))\rangle\} \ .$$

Exercice 1.1 (Verification Theorem) *Show that if* $\mathbf{V}$ *is a* $\mathcal{C}^1$ *function on* $(0, T) \times \mathbb{R}^N$*, continuous on* $[0, T] \times \mathbb{R}^N$*, which satisfies the terminal condition* $\mathbf{V}(T, x) = g(x)$ *and the Hamilton-Jacobi-Isaacs equation:*

$$\partial_t \mathbf{V}(t, x) + H(t, x, D\mathbf{V}(t, x)) = 0 \qquad \forall x \in (0, T) \times \mathbb{R}^N,$$

and if, furthermore the maps $\bar{u}^*(t, x) := \tilde{u}(t, x, D\mathbf{V}(t, x))$ *and* $\bar{v}^*(t, x) := \tilde{v}(t, x, D\mathbf{V}(t, x))$ *belong to* $\bar{U}$ *and* $\bar{V}$ *respectively, the game has a value which is* $\mathbf{V}$*.*

Show that in this case the strategies $\bar{u}^*(t, x)$ *and* $\bar{v}^*(t, x)$ *are optimal for the Players.*

Mayer problem is a particular case of Bolza problem in which $\ell = 0$.

Exercice 1.2 *Let us suppose that the assumption of the previous exercise hold and that* $\ell = 0$*. Show that the value function* $\mathbf{V}$ *is constant along trajectories when the players play their optimal strategies.*

Infinite horizon problem

In this problem we are again dealing with an integral cost, but the horizon is now infinite. In order for the payoff to be well-defined we need to introduce a discount factor $\lambda > 0$ which indicates that a payoff today is more interesting than a payoff tomorrow.

Let us fix an admissible pair $(\bar{U}, \bar{V})$ of feedback strategies. Given an initial condition $x_0 \in \mathbb{R}^N$, a strategy $\bar{u} \in \bar{U}$ for the first Player and a strategy $\bar{v} \in \bar{V}$ for the second player, the outcome of the game is given by

$$\mathcal{J}(x_0, \bar{u}, \bar{v}) := \int_0^{+\infty} e^{-\lambda t} \ell(X_t, \bar{u}(t, X_t), \bar{v}(t, X_t))dt$$

where X is the unique solution to (1.8) with initial condition $(0, x_0)$. We assume that $\ell : \mathbb{R}^N \times U \times V \to \mathbb{R}$ is smooth and bounded. Here again the first player minimizes while the second player maximizes.
Upper value function:

$$\mathbf{V}^+(x_0) := \inf_{\bar{u}\in\bar{U}} \sup_{\bar{v}\in\bar{V}} \mathcal{J}(x_0, \bar{u}, \bar{v})$$

Lower value function:

$$\mathbf{V}^-(x_0) := \sup_{\bar{v}\in\bar{V}} \inf_{\bar{u}\in\bar{U}} \mathcal{J}(x_0, \bar{u}, \bar{v})$$

In this game Isaacs' condition takes the form

$$H(x, p) = \inf_{u\in U} \sup_{v\in V} \{L(x, u, v) + \langle p, f(x, u, v)\rangle\} \sup_{v\in V} \inf_{u\in U} \{L(x, u, v) + \langle p, f(x, u, v)\rangle\}$$

for any $(x, p) \in \mathbb{R}^N \times \mathbb{R}^N$. Note that the upper and lower value functions are bounded in $\mathbb{R}^N$.

Exercice 1.3 State and prove a verification Theorem for this game.

Indications :

i) The Hamilton-Jacobi-Isaacs' equation for this game is

$$-\lambda \mathbf{V}(x) + H(x, D\mathbf{V}(x)) = 0$$

ii) There is no terminal condition for the value function. To overcome this difficulty, it is natural to assume that the candidate value function is bounded.

Chapter 2

Strategies

In this chapter we introduce the different notions of strategies used in this notes. Since in the sequel we shall mainly work in the framework of nonanticipative strategies and of nonanticipative strategies with delay, the reader can first restrict his lecture to the first section, which is devoted to these notions. Some more refined notions of strategies are used when working outside of Isaacs' condition, or for differential games with incomplete information.

In order to fix the ideas, we mainly work in the framework of a two-player differential game and for bounded controls. The extension to other frameworks is straightforward in general.

We aim at formalizing the fact that the players play in continuous time and observe eachother continuously. This is not an easy task and, actually, no completely satisfactory definition has been found up to now.

2.1 Nonanticipative and delay strategies

Let U and V be metric spaces and $-\infty < t_0 < t_1 \leq +\infty$. We denote by $\mathcal{U}(t_0, t_1)$ the set of bounded, Lebesgue measurable maps $u : [t_0, t_1] \to U$. We set $\mathcal{U}(t_0) := \mathcal{U}(t_0, +\infty)$ (or, if the game has a fixed horizon $T, \mathcal{U}(t_0) := \mathcal{U}(t_0, T)$). Elements of $\mathcal{U}(t_0)$ are called the *open loop controls* played by Player 1. Symetrically let us denote by $\mathcal{V}(t_0, t_1)$ the set of bounded Lebesgue measurable maps $v : [t_0, t_1] \to V$. We will systematically call Player 1 the player playing with the control u and Player 2 the player playing with the control v. If $u_1, u_2 \in \mathcal{U}(t_0)$ and $t_1 \geq t_0$, we write $u_1 \equiv u_2$ on $[t_0, t_1]$ whenever u_1 and u_2 coincide almost everywhere on $[t_0, t_1]$.

A strategy for Player 1 is a map α from $\mathcal{V}(t_0)$ to $\mathcal{U}(t_0)$. This means that Player 1 answers to each control $v \in \mathcal{V}(t_0)$ of Player 2 by a control $u = \alpha(v) \in \mathcal{U}(t_0)$. However since we wish to formalize the fact that no player can guess in advance the future behaviour of the other player, we have to require that such a map α is nonanticipative.

Definition 2.1 (Nonanticipative strategy) *A map $\alpha : \mathcal{V}(t_0) \to \mathcal{U}(t_0)$ is nonanticipative if, for any time $t_1 > t_0$ and any controls $v_1, v_2 \in \mathcal{V}(t_0)$ such that $v_1 = v_2$ almost everywhere in $[t_0, t_1]$ we have $\alpha(v_1) = \alpha(v_2)$ almost everywhere in $[t_0, t_1]$.*

We denote by $\mathcal{A}(t_0)$ the set of Player 1's nonanticipative strategies $\alpha : \mathcal{V}(t_0) \to \mathcal{U}(t_0)$. In a symmetric way we denote by $\mathcal{B}(t_0)$ the set of Player 2's nonanticipative strategies, which are the nonanticipative maps $\beta : \mathcal{U}(t_0) \to \mathcal{V}(t_0)$.

In order to put the game under normal form, one should be able to say that, for any pair of nonanticipative strategies $(\alpha, \beta) \in \mathcal{A}(t_0) \times \mathcal{B}(t_0)$ there is a unique pair of controls $(u, v) \in \mathcal{U}(t_0) \times \mathcal{V}(t_0)$ such that

$$\alpha(v) = u \text{ and } \beta(u) = v \ .$$

The pair (u, v) would be the natural answer of the players to the strategies (α, β). Unfortunately this is not possible, as shows exercice 2.2. For this reason we are lead to introduce a more restrictive notion of strategy, the nonanticipative strategies with delay.

Definition 2.2 (Delay strategies) *A nonanticipative strategy with delay (in short delay strategy) for Player 1 is a map $\alpha : \mathcal{V}(t_0) \to \mathcal{U}(t_0)$ for which there is a* delay $\tau > 0$ *such that, for any two controls $v_1, v_2 \in \mathcal{V}(t_0)$ and for any $t \geq t_0$, if $v_1 \equiv v_2$ on $[t_0, t]$, then $\alpha(v_1) \equiv \alpha(v_2)$ on $[t_0, t+\tau]$.*

In particular, the restriction of $\alpha(v)$ to the interval $[t_0, t_0+\tau]$ is independent of v because any two controls $v_1, v_2 \in \mathcal{V}(t_0)$ coincide almost everywhere on $[t_0, t_0]$.

We also note that delay strategies are nonanticipative strategies, but the converse is false in general. For instance, if $\sigma : V \to U$ is Borel measurable, then the map

$$\alpha(v)(t) = \sigma(v(t)) \qquad \forall t \in [t_0, +\infty),\ \forall v \in \mathcal{V}(t_0)$$

is a nonanticipative strategy but not a delay strategy, unless σ is constant.
We denote by $\mathcal{A}_d(t_0)$ (resp. $\mathcal{B}_d(t_0)$) the set of delay strategies for Player 1 (resp. Player 2).

The key property of delay strategies is given in the following Lemma:

Lemma 2.3 *Let $\alpha \in \mathcal{A}(t_0)$ and $\beta \in \mathcal{B}(t_0)$. Assume that either α or β is a delay strategy. Then there is a unique pair of controls $(u, v) \in \mathcal{U}(t_0) \times \mathcal{V}(t_0)$ such that*

$$\alpha(v) = u \text{ and } \beta(u) = v \qquad \text{almost everywhere in } [t_0, +\infty)\ .$$

Proof : Let us assume to fix the ideas that α is a delay strategy. Let τ be the associated delay.

We first claim that, for any integer $k \geq 1$, there is a unique pair of Lebesgue measurable maps $(u_k, v_k) : [t_0, t_0 + k\tau] \to U \times V$ such that $\alpha(v_k) = u_k$ and $\beta(u_k) = v_k$ on $[t_0, t_0 + k\tau]$. We prove the claim by induction.

For $k = 1$, let us pick any control $v \in \mathcal{V}(t_0)$ and set $u_1 = \alpha(v)$ and $v_1 = \beta(u_1)$. Since α is a delay strategy, we know that the restriction of $\alpha(v)$ to $[t_0, t_0 + \tau]$ is independent of v, so that $\alpha(v_1) = \alpha(v) = u_1$ almost everywhere on $[t_0, t_0 + \tau]$. So the property holds for $k = 1$.

Let us now assume that the result holds for some $k \geq 1$: there is a unique pair $(u_k, v_k) \in \mathcal{U}(t_0)\times\mathcal{V}(t_0)$ such that $\alpha(v_k) = u_k$ and $\beta(u_k) = v_k$ a.e. on $[t_0, t_0 + k\tau]$. We extend u_k and v_k in arbitrary controls on $[t_0, +\infty)$. Let us now set $u_{k+1} = \alpha(v_k)$ and $v_{k+1} = \beta(u_{k+1})$ on $[t_0, +\infty)$. Then from our construction $u_{k+1} = u_k$ a.e. on $[t_0, t_0 + k\tau]$. Since β is a nonanticipative strategy, this implies that $v_k = \beta(u_k) = \beta(u_{k+1}) = v_{k+1}$ a.e. on $[t_0, t_0 + k\tau]$. But α is a delay strategy and therefore $u_{k+1} = \alpha(v_k) = \alpha(v_{k+1})$ a.e. on $[t_0, t_0 + (k + 1)\tau]$. This completes the proof of the claim by induction.

By construction the pair (u_k, v_k) is unique. Hence, if $l \leq k$, we have

$$(u_k, v_k) = (u_l, v_l) \text{ a.e. on } [t_0, t_0 + l\tau]\ .$$

So, if we set

$$(u, v) = (u_k, v_k) \qquad \text{on } [t_0, t_0 + k\tau]$$

the pair (u, v) satisfies the desired property. □

2.2 Random strategies

As before we fix (U, d_U) and (V, d_V) two compact metric spaces. Let us endow $\mathcal{U}(t_0)$ with the topology of the L^1_{loc} convergence: we say that a sequence (u_n) of controls in $\mathcal{U}(t_0)$ converges to some control $u \in \mathcal{U}(t_0)$ if

$$\lim_{n\to+\infty} \int_0^T d_U(u_n(t), u(t))dt = 0 \qquad \forall T \geq 0\ .$$

In the same way we endow $\mathcal{V}(t_0)$ with the topology of the L^1_{loc} convergence.

We now introduce the notions of random strategies. Let us fix a nonempty set $\mathcal{S}$ of probability spaces. It is the set where the players are going to find random variables in order to randomize their strategies. In practice $\mathcal{S}$ has to be "large": for instance

$$\mathcal{S} = \{([0, 1]^n, B([0, 1]^n), \mathcal{L}^n),\ \text{for some } n \in \mathbb{N}^*\}\ ,$$

where $B([0, 1]^n)$ is the class of Borel sets and $\mathcal{L}^n$ is the Lebesgue measure on $\mathbb{R}^n$.

Definition 2.4 (Random delay strategies) *A random delay strategy (in short, a "random strategy") for Player 1 is a pair* $((\Omega_\alpha, \mathcal{F}_\alpha, \mathbb{P}_\alpha), \alpha)$*, where* $(\Omega_\alpha, \mathcal{F}_\alpha, \mathbb{P}_\alpha)$ *belongs to the set of probability spaces* $\mathcal{S}$ *and* $\alpha : \Omega_\alpha \times \mathcal{V}(t_0) \mapsto \mathcal{U}(t_0)$ *satisfies*

(i) α *is a measurable map from* $\Omega_\alpha \times \mathcal{V}(t_0)$ *to* $\mathcal{U}(t_0)$*, with* Ω_α *endowed with the* $\sigma-$*field* $\mathcal{F}_\alpha$ *and* $\mathcal{U}(t_0)$ *and* $\mathcal{V}(t_0)$ *with the Borel* $\sigma-$*field associated with the* L^1_{loc} *topology,*

(ii) there is a delay $\tau > 0$ *such that, for any* $v_1, v_2 \in \mathcal{V}(t_0)$*, any* $t \geq t_0$*, and any* $\omega \in \Omega_\alpha$*,*

$$v_1 \equiv v_2 \text{ on } [t_0, t] \quad \Rightarrow \quad \alpha(\omega, v_1) \equiv \alpha(\omega, v_2) \text{ on } [t_0, t + \tau] \ .$$

We denote by $\mathcal{A}_r(t_0)$ the set of random delay strategies for Player 1. By abuse of notations, an element of $\mathcal{A}_r(t_0)$ is simply noted α—instead of $((\Omega_\alpha, \mathcal{F}_\alpha, \mathbb{P}_\alpha), \alpha)$—, the underlying probability space being always denoted by $(\Omega_\alpha, \mathcal{F}_\alpha, \mathbb{P}_\alpha)$.

Random delay strategies for Player 2 are defined symmetrically: a random strategy is a map $\beta : \Omega_\beta \times \mathcal{U}(t_0) \mapsto \mathcal{V}(t_0)$, where $(\Omega_\beta, \mathcal{F}_\beta, \mathbb{P}_\beta)$ belongs to $\mathcal{S}$, which satisfies the conditions:

(i) β is measurable from $\Omega_\beta \times \mathcal{U}(t_0)$ to $\mathcal{V}(t_0)$,

(ii) there is a delay $\tau > 0$ such that, for any $u_1, u_2 \in \mathcal{U}(t_0)$, any $t \geq t_0$ and any $\omega \in \Omega_\beta$,

$$u_1 \equiv u_2 \text{ on } [t_0, t] \quad \Rightarrow \quad \beta(\omega, u_1) \equiv \beta(\omega, u_2) \text{ on } [t_0, t + \tau] \ .$$

The set of random delay strategies for Player 2 is denoted by $\mathcal{B}_r(t_0)$. Elements of $\mathcal{B}_r(t_0)$ are denoted simply by β, and the underlying probability space by $(\Omega_\beta, \mathcal{F}_\beta, \mathbb{P}_\beta)$.

Lemma 2.5 *For any pair* $(\alpha, \beta) \in \mathcal{A}_r(t_0) \times \mathcal{B}_r(t_0)$ *and any* $\omega := (\omega_1, \omega_2) \in \Omega_\alpha \times \Omega_\beta$*, there is a unique pair* $(u_\omega, v_\omega) \in \mathcal{U}(t_0) \times \mathcal{V}(t_0)$*, such that*

$$\alpha(\omega_1, v_\omega) = u_\omega \text{ and } \beta(\omega_2, u_\omega) = v_\omega \ . \tag{2.1}$$

Furthermore the map $\omega \to (u_\omega, v_\omega)$ *is measurable from* $\Omega_\alpha \times \Omega_\beta$ *endowed with* $\mathcal{F}_\alpha \otimes \mathcal{F}_\beta$ *into* $\mathcal{U}(t_0) \times \mathcal{V}(t_0)$ *endowed with the Borel* $\sigma-$*field associated with the* L^1_{loc} *topology.*

Proof : Let us prove simultaneously the existence, uniqueness and measurability of $\omega \to (u_\omega, v_\omega)$. Without loss of generality we assume that the strategies α and β have the same associated delay $\tau > 0$. As for Lemma 2.3, we argue by induction to prove that the map $\omega \to (u_\omega, v_\omega)_{|_{[t_0, t+n\tau]}}$ from $\Omega_\alpha \times \Omega_\beta$ into $L^1([t_0, t+n\tau], U \times V)$ is well-defined and measurable.

Let us start with $n = 1$. Let us fix $\hat{u}$ and $\hat{v}$ in $\mathcal{U}(t_0)$ and $\mathcal{V}(t_0)$. Since $\alpha(\omega_1, \cdot)$ and $\beta(\omega_2, \cdot)$ have a delay τ, the restrictions of $\alpha(\omega_1, \hat{v})$ and $\beta(\omega_2, \hat{u})$ to $[t_0, t_0 + \tau]$ do not depend on $\hat{u}$ and $\hat{v}$. Hence we set $(u_\omega, v_\omega) := (\alpha(\omega, \hat{v}), \beta(\omega, \hat{u}))$ in $[t_0, t_0 + \tau]$, which uniquely defines (u_ω, v_ω) on this interval. In order to prove the measurability of the map $\omega \to (u_\omega, v_\omega)$, it is enough to show that, for any Borel subsets B_1 and B_2 of $\mathcal{U}(t_0, t_0 + \tau)$ and $\mathcal{V}(t_0, t_0 + \tau)$, the set

$$\Omega := \{\omega \in \Omega_\alpha \times \Omega_\beta \mid (u_\omega, v_\omega)_{|_{[t_0, t_0+\tau]}} \in B_1 \times B_2\}$$

is measurable. This is indeed the case because

$$\Omega = \{\omega_1 \in \Omega_\alpha \mid \alpha(\omega_1, \hat{v})_{|_{[t_0, t_0+\tau]}} \in B_1\} \times \{\omega_2 \in \Omega_\beta \mid \beta(\omega_2, \hat{u})_{|_{[t_0, t_0+\tau]}} \in B_2\} \ ,$$

which is measurable since α and β are measurable. So the result holds true for $n = 1$.

Let us now assume that $\omega \to (u_\omega, v_\omega)_{|_{[t_0, t_0+n\tau]}}$ from $\Omega_\alpha \times \Omega_\beta$ into $L^1([t_0, t_0 + n\tau])$ is well-defined and measurable, and let us show that this still holds true for $n+1$. Let us fix again $\hat{u}$ and $\hat{v}$ in $\mathcal{U}(t_0)$ and $\mathcal{V}(t_0)$. For any $(u, v) \in \mathcal{U}(t_0, t_0 + n\tau) \times \mathcal{V}(t_0, t_0 + n\tau)$, we denote by $\tilde{u}$ and $\tilde{v}$ the maps equal to u and v on $[t_0, t_0 + n\tau]$ and to $\hat{u}$ and $\hat{v}$ on $[t_n, T]$. Note that $(u, v) \mapsto (\tilde{u}, \tilde{v})$ is continuous from L^1 to L^1. We extend (u_ω, v_ω) to $[t_0, t_0 + (n+1)\tau]$ by setting $(u_\omega, v_\omega) := (\alpha(\omega_1, \widetilde{v_\omega}), \beta(\omega_2, \widetilde{u_\omega}))$ on $[t_0, t_0 + (n+1)\tau]$. Since α and β have a delay τ, this extension is unique and does not depend on the choice of $\hat{u}$ and $\hat{v}$. In order to prove the measurability of the map $\omega \to (u_\omega, v_\omega)_{|_{[t_0, t_0+(n+1)\tau]}}$, it is again enough to show that, for any Borel subsets B_1 and B_2 of $\mathcal{U}(t_0, t_0 + (n+1)\tau)$ and $\mathcal{V}(t_0, t_0 + (n+1)\tau)$, the set

$$\Omega := \{\omega \in \Omega_\alpha \times \Omega_\beta \mid (u_\omega, v_\omega)_{|_{[t_0, t_0+(n+1)\tau)}} \in B_1 \times B_2\}$$

is measurable. This is indeed the case because Ω is the preimage of the set $B_1 \times B_2$ by the map $\omega \to (\alpha(\omega_1, \widetilde{v_\omega}), \beta(\omega_2, \widetilde{u_\omega}))$ which is measurable as the composition of the mesurable maps $\omega \mapsto (u_\omega, v_\omega)_{|_{[t_0, t_0+n\tau]}}$, the map $(u, v) \mapsto (\tilde{u}, \tilde{v})$ and the maps α and β. $\square$

2.3 Exercises

Exercice 2.1 Let $f : [0,+\infty) \times \mathbb{R}^N \times U \times V \to \mathbb{R}^N$ be continuous in all its variables and uniformly Lipschitz continuous with respect to the space variable x. Let us fix an initial condition x_0 and let $\bar{u} : [0,+\infty) \times \mathbb{R}^N \times V \to U$ be a Borel measurable map which is also uniformly Lipschitz continuous with respect to the space variable x. For any control $v \in \mathcal{V}(t_0)$ let X^v be the unique solution to the diffential equation

$$\left\{ \begin{array}{ll} X_t' = f(t, X_t, \bar{u}(t, X_t, v_t), v_t) & t \geq 0 \\ X_{t_0} = x_0 \end{array} \right.$$

Then we set $\alpha : \mathcal{V}(t_0) \to \mathcal{U}(t_0)$ by

$$\alpha(v)(t) = \bar{u}(t, X_t^v, v_t) \qquad \forall v \in \mathcal{V}(t_0) .$$

Show that α is nonanticipative.

Exercice 2.2 Assume that $U = V = [-1, 1]$.

1. Let (α, β) be the pair of nonanticipative strategies defined by:

$$\alpha(v)(t) = v(t) \text{ and } \beta(u)(t) = u(t) \qquad \text{for a.e. } t \geq t_0,\ \forall (u, v) \in \mathcal{U}(t_0) \times \mathcal{V}(t_0) .$$

   Show that there is infinitely many pairs of controls $(u, v) \in \mathcal{U}(t_0) \times \mathcal{V}(t_0)$ such that

$$\alpha(v) = u \text{ and } \beta(u) = v \qquad \text{a.e. in } [t_0, +\infty) .$$

2. If now we define α and β by

$$\alpha(v)(t) = -v(t) \text{ and } \beta(u)(t) = \operatorname{sgn}(u(t)) \qquad \text{for a.e. } t \geq t_0,\ \forall (u, v) \in \mathcal{U}(t_0) \times \mathcal{V}(t_0) , \tag{2.2}$$

   (where $\operatorname{sgn}(s) = 1$ if $s \geq 0$ and -1 otherwise), show that there is no pair $(u, v) \in \mathcal{U}(t_0) \times \mathcal{V}(t_0)$ for which (2.2) holds.

Exercice 2.3 Let $\beta \in \mathcal{B}(t_0)$, $t_1 > t_0$ and $u_0 \in \mathcal{U}(t_0)$ be fixed. Show that the map $\beta_1 : \mathcal{U}(t_0) \to \mathcal{V}(t_0)$ defined by

$$\beta_1(u)(t) = \beta_0(\tilde{u})(t) \text{ where } \tilde{u} = \left\{ \begin{array}{ll} u_0 & \text{on } [t_0, t_1) \\ u & \text{on } [t_1, +\infty) \end{array} \right. \qquad \forall t \in [t_1, +\infty),\ \forall u \in \mathcal{U}(t_1) ,$$

is a nonanticipative strategy on $[t_1, +\infty)$.

2.4 Comments

Nonanticipative strategies were introduced by Varaiya [208], Roxin [184], Elliott-Kalton [92, 93]. They were extensively used in the viscosity solution approach of differential games and, in particular, in the seminal work by Evans-Souganidis [96]. Throughout these notes we prefer to work with the notion of delay strategies, which allows to put the game in the so-called normal form.

Chapter 3

Zerosum differential games: viscosity solution approach

This chapter is devoted to the analysis of two-player, zero-sum differential games. The main issue is to prove the existence of a value for such games and to characterize it as the unique solution of some partial differential equation, the Hamilton-Jacobi-Isaacs equation.

We start with a finite horizon problem, called Bolza problem, for which we explain the proof of the existence of a value in a rather simple framework. This leads us to introduce the notion of viscosity solution for Hamilton-Jacobi equations, notion that we discuss in some details in the second section. Then we give some "explicit solution" of the Bolza problem, and complete the chapter by the analysis of an infinite horizon problem.

3.1 Bolza problem

We start with the analysis of Bolza problem, which is a game with a finite horizon (the game ends at some fixed time T), where the payoffs of the players consists in a running payoff and a terminal one.

3.1.1 Definition of the value functions

Throughout the chapter, we denote by $T > 0$ the finite horizon of the game, i.e., the time at which the game ends.

Dynamics: For a fixed initial position $(t_0, x_0) \in [0, T] \times \mathbb{R}^N$ we consider the differential equation

$$\left\{ \begin{array}{ll} X'_t = f(t, X_t, u_t, v_t) & t \in [t_0, T] \\ X_{t_0} = x_0 \end{array} \right. \tag{3.1}$$

Throughout this section, we assume that

$$\left\{ \begin{array}{rl} (i) & U \text{ and } V \text{ are compact metric spaces,} \\ (ii) & \text{the map } f : [0, T] \times \mathbb{R}^N \times U \times V \text{ is bounded and continuous in all its variables} \\ (iii) & f \text{ is uniformly Lipschitz continuous with respect to the space variable:} \\ & |f(t, x, u, v) - f(t, y, u, v)| \leq \text{Lip}(f)|x - y| \\ & \qquad\qquad \forall (t, x, y, u, v) \in [0, T] \times \mathbb{R}^N \times \mathbb{R}^N \times U \times V \end{array} \right. \tag{3.2}$$

The Lebesgue measurable maps $u : [t_0, T] \to U$ and $v : [t_0, T] \to V$ are the controls played by the first and the second Player respectively. We denote by $\mathcal{U}(t_0)$ the set of Lebesgue measurable controls $u : [t_0, +\infty) \to U$ of the first Player and by $\mathcal{V}(t_0)$ the set of Lebesgue measurable controls $v : [t_0, +\infty) \to V$ of the second Player. For any pair $(u, v) \in \mathcal{U}(t_0) \times \mathcal{V}(t_0)$, equation (3.1) has a unique solution, denoted $X^{t_0, x_0, u, v}$.

Payoffs: The payoff of the players depends on a running payoff $\ell : [0, T] \times \mathbb{R}^N \times U \times V \to \mathbb{R}$ and on a terminal payoff $g : \mathbb{R}^N \to \mathbb{R}$. Namely, if the players play the controls $(u, v) \in \mathcal{U}(t_0) \times \mathcal{V}(t_0)$, then the cost the first Player is trying to minimize (it is a payoff for the second Player who is maximizing) is given by

$$\mathcal{J}(t_0,x_0,u,v)=\int_{t_0}^T \ell(s,X_s^{t_0,x_0,u,v},u_s,v_s)ds+g(X_T^{t_0,x_0,u,v})\ .$$

Throughout this section we assume that

$$\text{(3.3)}\qquad \begin{cases} (i) & g:\mathbb{R}^N\to\mathbb{R} \text{ is bounded and Lipschitz continuous,}\\ (ii) & \ell:[0,T]\times\mathbb{R}^N\times U\times V\to\mathbb{R} \text{ is continuous, bounded}\\ & \text{and Lipschitz continuous with respect to the } x \text{ variable.}\end{cases}$$

We denote by $\mathrm{Lip}(\ell)$ and $\mathrm{Lip}(g)$ the Lipschitz constants.

Strategies: In Chapter 2 we have defined a delay strategy for the first Player as a map $\alpha:\mathcal{V}(t_0)\to\mathcal{U}(t_0)$ for which there is a delay τ such that for any two controls $v_1,v_2\in\mathcal{V}(t_0)$ and for any $t\in[t_0,T]$, if $v_1\equiv v_2$ in $[t_0,t]$, then $\alpha(v_1)\equiv\alpha(v_2)$ in $[t_0,(t+\tau)\wedge T]$. The set of strategies for the first Player are denoted by $\mathcal{A}_d(t_0)$. Delay strategies for the second Player are defined in a symmetric way and the set of those strategies is denoted by $\mathcal{B}_d(t_0)$. Following Lemma 2.3, we shall systematically use the fact that if $(\alpha,\beta)\in\mathcal{A}_d(t_0)\times\mathcal{B}_d(t_0)$ is a pair of strategies, then there is a unique pair of controls $(u,v)\in\mathcal{U}(t_0)\times\mathcal{V}(t_0)$ such that

$$\text{(3.4)}\qquad \alpha(v)=u \text{ and } \beta(u)=v \qquad \text{a.e. in } [t_0,T]\ .$$

In particular we always use the notation (α_s,β_s) for (u_s,v_s) and $X_t^{t_0,x_0,\alpha,\beta}$ for $X_t^{t_0,x_0,u,v}$. The payoff associated to the two strategies $(\alpha,\beta)\in\mathcal{A}_d(t_0)\times\mathcal{B}_d(t_0)$ is given by

$$\mathcal{J}(t_0,x_0,\alpha,\beta)=\int_{t_0}^T \ell(s,X_s^{t_0,x_0,\alpha,\beta},\alpha_s,\beta_s)ds+g(X_T^{t_0,x_0,\alpha,\beta})\ .$$

Definition 3.1 (Value functions) *The* **upper value function** *is given by*

$$\text{(3.5)}\qquad \mathbf{V}^+(t_0,x_0):=\inf_{\alpha\in\mathcal{A}_d(t_0)}\sup_{\beta\in\mathcal{B}_d(t_0)}\mathcal{J}(t_0,x_0,\alpha,\beta)$$

while the **lower value function** *is*

$$\text{(3.6)}\qquad \mathbf{V}^-(t_0,x_0):=\sup_{\beta\in\mathcal{B}_d(t_0)}\inf_{\alpha\in\mathcal{A}_d(t_0)}\mathcal{J}(t_0,x_0,\alpha,\beta)\ .$$

Remark 3.2 Obviously, the following inequality always holds:

$$\mathbf{V}^-(t_0,x_0)\le\mathbf{V}^+(t_0,x_0)\qquad\forall(t_0,x_0)\in[0,T]\times\mathbb{R}^N\ .$$

So the key point is to prove the reverse one and to characterize the value $\mathbf{V}^+=\mathbf{V}^-$.

Lemma 3.3 (Equivalent definition of the value functions) *We have*

$$\mathbf{V}^+(t_0,x_0):=\inf_{\alpha\in\mathcal{A}_d(t_0)}\sup_{v\in\mathcal{V}(t_0)}\mathcal{J}(t_0,x_0,\alpha(v),v)$$

and

$$\mathbf{V}^-(t_0,x_0):=\sup_{\beta\in\mathcal{B}_d(t_0)}\inf_{u\in\mathcal{U}(t_0)}\mathcal{J}(t_0,x_0,u,\beta(u))$$

Proof : Let us check for instance the first equality. Obviously

$$\mathbf{V}^+(t_0,x_0)\ge\inf_{\alpha\in\mathcal{A}_d(t_0)}\sup_{v\in\mathcal{V}(t_0)}\mathcal{J}(t_0,x_0,\alpha(v),v)$$

because $\mathcal{V}(t_0)\subset\mathcal{B}_d(t_0)$. We also know that for any $(\alpha,\beta)\in\mathcal{A}_d(t_0)\times\mathcal{B}_d(t_0)$, there is a unique pair of controls $(u,v)\in\mathcal{U}(t_0)\times\mathcal{V}(t_0)$ for which (3.4) holds. Then

$$\mathcal{J}(t_0,x_0,\alpha,\beta)=\mathcal{J}(t_0,x_0,\alpha(v),v)\le\sup_{v'\in\mathcal{V}(t_0)}\mathcal{J}(t_0,x_0,\alpha(v'),v')\ .$$

Hence

$$\sup_{\beta\in\mathcal{B}_d(t_0)}\mathcal{J}(t_0,x_0,\alpha,\beta)\le\sup_{v'\in\mathcal{V}(t_0)}\mathcal{J}(t_0,x_0,\alpha(v'),v')\ .$$

Taking the infimum over $\alpha\in\mathcal{A}_d(t_0)$ completes the proof. □

3.1.2 Dynamic programming property and regularity of the value functions

The aim of this section is to prove some fundamental estimates for the value functions. For this we concentrate on the upper value. We can do so without loss of generality because

$$(-\mathbf{V}^-)(t_0,x_0) = \inf_{\beta\in\mathcal{B}_d(t_0)} \sup_{\alpha\in\mathcal{A}_d(t_0)} \left(-\mathcal{J}(t_0,x_0,\alpha,\beta)\right) ,$$

which means that the map $(-\mathbf{V}^-)$ can be seen as an upper value for the game with running payoff $-\ell$, terminal payoff $-g$, and where the first Player is maximizing while the second Player is minimizing. Hence any result for $\mathbf{V}^+$ directly translates into a result for $(-\mathbf{V}^-)$, and so for $\mathbf{V}^-$.

Dynamic programming property

The main result of this section is the following Theorem:

Theorem 3.4 (Dynamic programming) *Let* $(t_0,x_0)\in[0,T)\times\mathbb{R}^N$ *and* $h\in(0,T-t_0)$*. Then*

$$\text{(3.7)}\quad \mathbf{V}^+(t_0,x_0) = \inf_{\alpha\in\mathcal{A}_d(t_0)} \sup_{v\in\mathcal{V}(t_0)} \left\{ \int_{t_0}^{t_0+h} \ell(s,X_s^{t_0,x_0,\alpha(v),v},\alpha(v)_s,v_s)ds + \mathbf{V}^+(t_0+h,X_{t_0+h}^{t_0,x_0,\alpha(v),v}) \right\} .$$

Before proving the result we need some preliminary remarks. The first one is a easy statement which is used throughout these notes:

Lemma 3.5 *Let A and B be some sets and let $f,g : A\times B\to\mathbb{R}$ be two maps. Let us assume that there is a constant $k\geq 0$ such that*

$$\sup_{a\in A,\ b\in B} |f(a,b)-g(a,b)| \leq k .$$

Then

$$\text{(3.8)}\qquad \left|\inf_{a\in A}\sup_{b\in B} f(a,b) - \inf_{a\in A}\sup_{b\in B} g(a,b)\right| \leq k ,$$

as soon as one of the two $\inf\sup$ *is finite.*

Proof : Indeed, if for instance $\inf_{a\in A}\sup_{b\in B} f(a,b)$ is finite, then, since

$$f(a,b)\leq g(a,b)+k \qquad \forall a\in A,\ b\in B ,$$

we get, by taking the $\sup_b$ and then the $\inf_a$ in the above inequality,

$$\inf_{a\in A}\sup_{b\in B} f(a,b) \leq \inf_{a\in A}\sup_{b\in B} g(a,b)+k .$$

The reverse statement

$$\inf_{a\in A}\sup_{b\in B} f(a,b) \geq \inf_{a\in A}\sup_{b\in B} g(a,b)-k$$

can be proved in the same way, so that $\inf_{a\in A}\sup_{b\in B} g(a,b)$ is finite and (3.8) holds. □

The next remark deals with the space regularity of the value functions. We shall see later that the value functions are also Lipschitz continuous in time.

Lemma 3.6 *The map $\mathbf{V}^+$ is Lipschitz continuous with respect to the x variable uniformly in the time variable: namely there is some $C>0$ such that, for any $t\in[0,T]$ and any $x,y\in\mathbb{R}^N$, we have*

$$\text{(3.9)}\qquad |V^+(t,x)-V^+(t,y)| \leq C|x-y| .$$

Proof of Lemma 3.6 : Let us fix $(t_0,x_0,y_0)\in[0,T]\times\mathbb{R}^N\times\mathbb{R}^N$ and $(u,v)\in\mathcal{U}(t_0)\times\mathcal{V}(t_0)$. We set $X_t^1 = X_t^{t_0,x_0,u,v}$ and $X_t^2 = X_t^{t_0,y_0,u,v}$. Since f is globally Lipschitz continuous, Gronwall's Lemma implies that

$$|X_t^1-X_t^2| \leq |x_0-y_0|e^{\mathrm{Lip}(f)(t-t_0)} \qquad \forall t\in[t_0,T] .$$

Using the Lipschitz continuity of ℓ and g we get

$$
\begin{array}{rcl}
|\mathcal{J}(t_0,x_0,u,v)-\mathcal{J}(t_0,y_0,u,v)| & \leq & \int_{t_0}^T |\ell(s,X^1_s,u_s,v_s)-\ell(s,X^2_s,u_s,v_s)|ds+|g(X^1_T)-g(X^2_T)| \\
& \leq & \mathrm{Lip}(\ell)\int_{t_0}^T |X^1_s-X^2_s|ds+\mathrm{Lip}(g)|X^1_T-X^2_T| \\
& \leq & C|x_0-y_0|
\end{array} \tag{3.10}
$$

where C only depends on Lip(f), Lip(ℓ), Lip(g) and T. The above inequality holds for any pair of controls $(u,v)\in\mathcal{U}(t_0)\times\mathcal{V}(t_0)$, hence for any pair of strategies:

$$
|\mathcal{J}(t_0,x_0,\alpha,\beta)-\mathcal{J}(t_0,y_0,\alpha,\beta)| \;\leq\; C|x_0-y_0| \qquad \forall \alpha\in\mathcal{A}_d(t_0),\ \beta\in\mathcal{B}_d(t_0)\ .
$$

We complete the proof of (3.9) thanks to Lemma 3.5. □

Our next step is the following Lemma, which states that nearly optimal strategies for a given initial position (t_0,x_0) remain nearly optimal for points of the form (t_0,y_0) with y_0 sufficiently close to x_0.

Lemma 3.7 *For any $\epsilon>0$ there is some $\eta>0$ with the following property: for any $(t_0,x_0)\in[0,T]\times\mathbb{R}^N$, any $\epsilon-$optimal strategy $\bar\alpha\in\mathcal{A}_d(t_0)$ for $\mathbf{V}^+(t_0,x_0)$, i.e., such that*

$$
\sup_{v\in\mathcal{V}(t_0)} \mathcal{J}(t_0,x_0,\bar\alpha(v),v)\leq \mathbf{V}^+(t_0,x_0)+\epsilon\ ,
$$

remains $(2\epsilon)-$optimal for $\mathbf{V}^+(t_0,y_0)$ for any $y_0\in B(x_0,\eta)$:

$$
\sup_{v\in\mathcal{V}(t_0)} \mathcal{J}(t_0,y_0,\bar\alpha(v),v)\leq \mathbf{V}^+(t_0,y_0)+2\epsilon \qquad \forall y_0\in B(x_0,\eta)\ .
$$

Proof of Lemma 3.7: Let C be a Lipschitz constant of $\mathbf{V}^+$ with respect to x. According to inequality (3.10) established in the proof of Lemma 3.6, there is also a constant C' such that, for any $y_0\in\mathbb{R}^N$, for any pair of controls $(u,v)\in\mathcal{U}(t_0)\times\mathcal{V}(t_0)$,

$$
|\mathcal{J}(t_0,x_0,u,v)-\mathcal{J}(t_0,y_0,u,v)|\leq C'|x_0-y_0|\ .
$$

Hence

$$
\begin{array}{rcl}
\sup\limits_{v\in\mathcal{V}(t_0)} \mathcal{J}(t_0,y_0,\bar\alpha(v),v) & \leq & \sup\limits_{v\in\mathcal{V}(t_0)} \mathcal{J}(t_0,x_0,\bar\alpha(v),v)+C'|x_0-y_0| \\
& \leq & \mathbf{V}^+(t_0,x_0)+\epsilon+C'|x_0-y_0| \\
& \leq & \mathbf{V}^+(t_0,y_0)+\epsilon+(C'+C)|x_0-y_0|
\end{array}
$$

This proves that $\bar\alpha$ is $(2\epsilon)-$optimal for $\mathbf{V}^+(t_0,y_0)$ as soon as $|y_0-x_0|\leq\eta:=\epsilon/(C+C')$. □

Proof of Theorem 3.4 : Let us set

$$
W(t_0,t_0+h,x_0)=\inf_{\alpha\in\mathcal{A}_d(t_0)}\ \sup_{v\in\mathcal{V}(t_0)}\left\{\int_{t_0}^{t_0+h}\ell(s,X_s^{t_0,x_0,\alpha(v),v},\alpha(v)_s,v_s)ds+\mathbf{V}^+(t_0+h,X_{t_0+h}^{t_0,x_0,\alpha(v),v})\right\}
$$

We first show inequality $\mathbf{V}^+\leq W$.
Let us fix some $\epsilon>0$ and let α^0 be $\epsilon-$optimal for $W(t_0,t_0+h,x_0)$:

$$
\sup_{v\in\mathcal{V}(t_0)}\left\{\int_{t_0}^{t_0+h}\ell(s,X_s^{t_0,x_0,\alpha^0(v),v},\alpha^0_s,v_s)ds+\mathbf{V}^+(t_0+h,X_{t_0+h}^{t_0,x_0,\alpha^0(v),v})\right\}\leq W(t_0,t_0+h,x_0)+\epsilon\ . \tag{3.11}
$$

For any $x\in\mathbb{R}^N$, let α^x be $\epsilon-$optimal for the game $\mathbf{V}^+(t_0+h,x)$. From Lemma 3.7, there is some $\eta>0$ such that α^x is $(2\epsilon)-$optimal for $\mathbf{V}^+(t_0+h,y)$ for any $y\in B(x,\eta)$. Since $\mathbb{R}^N$ is locally compact, we can find a countable family $(x_i)_{i\in\mathbb{N}^*}$ such that the family of balls $(B(x_i,\eta/2))_{i\in\mathbb{N}^*}$ is a locally finite covering of $\mathbb{R}^N$: any point x belongs to some ball $B(x_i,\eta_{x_i})$ and to at most to a finite number of such balls. Let us set $\mathcal{O}_1=B(x_1,\eta/2)$ and $\mathcal{O}_i=B(x_i,\eta/2)\backslash\bigcup_{j<i}B(x_j,\eta/2)$ for $i\geq 2$. Let us also fix $\tau\in(0,\eta/(2\|f\|_\infty))$.

We are now ready to define a new strategy by setting, for any $t\in[t_0,T]$ and $v\in\mathcal{V}(t_0)$,

$$
\alpha(v)_t=\left\{\begin{array}{ll}
\alpha^0(v)_t & \text{if } t\in[t_0,t_0+h) \\
\alpha^{x_i}\left(v_{|_{[t_0+h,T]}}\right)_t & \text{if } t\in[t_0+h,T] \text{ and } X_{t_0+h-\tau}^{t_0,x_0,\alpha^0(v),v}\in\mathcal{O}_i
\end{array}\right.
$$

where $v_{|_{[t_0+h,T]}}$ is the restriction of v to $[t_0+h,T]$.

We claim that α is a delay strategy. Since f is bounded, we have

$$|X_{t_0+h}^{t_0,x_0,\alpha^0(v),v}| \leq |x_0| + \|f\|_\infty T \qquad \forall v\in\mathcal{V}(t_0)\ .$$

The covering $(B(x_i,\eta_{x_i}))_{i\in\mathbb{N}^*}$ being locally finite, there is some $k\in\mathbb{N}^*$ such that

$$B(0,|x_0|+\|f\|_\infty T)\subset \bigcup_{i\in\{1,\dots,k\}} B(x_i,\eta/2) = \bigcup_{i\in\{1,\dots,k\}} \mathcal{O}_i\ .$$

In particular the definition of α only involves a finite number of strategies: namely the α^{x_i} (for $i\in\{1,\dots,k\}$) and α^0. Up to reducing τ if necessary, we can assume that τ is a delay common to α^0 and to all the α^{x_i} for $i\in\{1,\dots,k\}$. We claim that τ is also a delay for α. Indeed let $v_1,v_2\in\mathcal{V}(t_0)$ be such that $v_1=v_2$ a.e. in $[t_0,t]$ for some $t\in[t_0,T]$. If $t\leq t_0+h-\tau$, then $\alpha(v_1)\equiv\alpha^0(v_1)\equiv\alpha^0(v_2)\equiv\alpha(v_2)$ on $[t_0,t+\tau]$. If $t\geq t_0+h-\tau$ then $X_{t_0+h-\tau}^{t_0,x_0,\alpha^0(v_1),v_1} = X_{t_0+h-\tau}^{t_0,x_0,\alpha^0(v_2),v_2}$ belongs to some $\mathcal{O}_i$, so that $\alpha(v_1)\equiv\alpha^0(v_1)\equiv\alpha^0(v_2)\equiv\alpha(v_2)$ on $[t\vee(t_0+h-\tau),t_0+h]$ and $\alpha(v_1)\equiv\alpha^{x_i}((v_1)_{|_{[t_0+h,T]}})\equiv\alpha^{x_i}((v_2)_{|_{[t_0+h,T]}})\equiv\alpha(v_2)$ on $[t\vee(t_0+h),T\wedge(t+\tau)]$, because α^{x_i} is has a delay τ. This proves that α is a delay strategy.

Next we claim that

$$\mathcal{J}(t_0,x_0,\alpha(v),v)\leq W(t_0,t_0+h,x_0)+3\epsilon \qquad \forall v\in\mathcal{V}(t_0)\ .$$

Let us fix $v\in\mathcal{V}(t_0)$ and set $X_s = X_s^{t_0,x_0,\alpha(v),v}$. We note that

$$X_s = \begin{cases} X_s^{t_0,x_0,\alpha^0(v),v} & \text{if } s\in[t_0,t_0+h] \\ X_s^{t_0+h,X_{t_0+h},\alpha^{x_i}(v^h),v^h} & \text{if } s\in[t_0+h,T] \text{ and } X_{t_0+h-\tau}\in\mathcal{O}_i \end{cases}$$

where $v^h = v_{|_{[t_0+h,T]}}$. Hence

$$\begin{aligned}\mathcal{J}(t_0,x_0,\alpha(v),v) \ &= \int_{t_0}^{t_0+h} \ell(s,X_s^{t_0,x_0,\alpha^0(v),v},\alpha^0(v)_s,v_s)ds \\ &+\sum_{i=1}^k \mathbf{1}_{\mathcal{O}_i}(X_{t_0+h-\tau})\left\{\int_{t_0+h}^T \ell(s,X_s^{t_0+h,X_{t_0+h},\alpha^{x_i}(v^h),v^h},\alpha^{x_i}(v^h)_s,v^h_s)ds + g(X_T^{t_0+h,X_{t_0+h},\alpha^{x_i}(v^h),v^h})\right\}\end{aligned}$$

If $X_{t_0+h-\tau}$ belongs to $\mathcal{O}_i$, then, by the definition of $\mathcal{O}_i$ and the choice of τ, X_{t_0+h} belongs to $B(x^i,\eta)$. Therefore α^{x_i} is $(2\epsilon)-$optimal for $\mathbf{V}^+(t_0+h,X_{t_0+h})$, which means that

$$\int_{t_0+h}^T \ell(s,X_s^{t_0+h,X_{t_0+h},\alpha^{x_i}(v^h),v^h},\alpha^{x_i}(v^h)_s,v^h_s)ds + g(X_T^{t_0+h,X_{t_0+h},\alpha^{x_i}(v^h),v^h}) \leq \mathbf{V}^+(t_0+h,X_{t_0+h})+2\epsilon\ .$$

Hence

$$\mathcal{J}(t_0,x_0,\alpha(v),v) \ \leq\ \int_{t_0}^{t_0+h} \ell(s,X_s^{t_0,x_0,\alpha^0(v),v},\alpha^0(v)_s,v_s)ds + \mathbf{V}^+(t_0+h,X_{t_0+h}^{t_0,x_0,\alpha^0,v})+2\epsilon$$

We now use the $\epsilon-$optimality of α^0 in $W(t_0,t_0+h,x_0)$ (i.e., (3.11)) to get

$$\mathcal{J}(t_0,x_0,\alpha(v),v)\ \leq\ W(t_0,t_0+h,x_0)+3\epsilon\ .$$

This inequality holds for any $v\in\mathcal{V}(t_0)$, so that

$$\mathbf{V}^+(t_0,x_0)\leq \sup_{v\in\mathcal{V}(t_0)} \mathcal{J}(t_0,x_0,\alpha(v),v)\ \leq\ W(t_0,t_0+h,x_0)+3\epsilon\ .$$

Since ϵ is arbitrary, inequality $\mathbf{V}^+\leq W$ is proved.

Let us now show the reverse inequality: $W\leq\mathbf{V}^+$. Let $\epsilon>0$ be fixed and let $\bar\alpha\in\mathcal{A}_d(t_0)$ be $\epsilon-$optimal for $\mathbf{V}^+(t_0,x_0)$. Let us fix some $\bar v\in\mathcal{V}(t_0)$ and define a new delay strategy $\alpha^{\bar v}\in\mathcal{A}_d(t_0+h)$ by setting:

$$\alpha^{\bar v}(v)_t = \bar\alpha(\tilde v)_t \text{ where } \tilde v_s = \begin{cases}\bar v_s & \text{if } s\in[t_0,t_0+h] \\ v_s & \text{otherwise}\end{cases} \qquad \forall t\in[t_0+h,T],\ v\in\mathcal{V}(t_0+h)\ .$$

Then, if we set $X_t = X_t^{t_0,x_0,\bar{\alpha}(\bar{v}),\bar{v}}$, we have

$$\mathbf{V}^+(t_0+h, X_{t_0+h}) \leq \sup_{v\in\mathcal{V}(t_0+h)} \mathcal{J}(t_0+h, X_{t_0+h}, \alpha^{\bar{v}}(v), v) \ . \tag{3.12}$$

Let $\mathcal{V}(t_0, t_0+h, \bar{v})$ be the set of controls $v \in \mathcal{V}(t_0)$ such that $v = \bar{v}$ on $[t_0, t_0+h]$. Then, by definition of $\alpha^{\bar{v}}$, we have

$$\mathcal{J}(t_0, x_0, \bar{\alpha}(v), v) = \int_{t_0}^{t_0+h} \ell(s, X_s, \bar{\alpha}(\bar{v})_s, \bar{v}_s)ds + \mathcal{J}(t_0+h, X_{t_0+h}, \alpha^{\bar{v}}(v_{|[t_0+h,T]}), v_{|[t_0+h,T]}) \qquad \forall v \in \mathcal{V}(t_0, t_0+h, \bar{v}) \ .$$

Hence

$$\sup_{v\in\mathcal{V}(t_0,t_0+h,\bar{v})} \mathcal{J}(t_0, x_0, \bar{\alpha}(v), v) = \int_{t_0}^{t_0+h} \ell(s, X_s, \bar{\alpha}(\bar{v})_s, \bar{v}_s)ds + \sup_{v\in\mathcal{V}(t_0+h)} \mathcal{J}(t_0+h, X_{t_0+h}, \alpha^{\bar{v}}(v), v) \ .$$

Combining (3.12) with the above inequality then leads to

$$\begin{aligned} \int_{t_0}^{t_0+h} \ell(s, X_s, \bar{\alpha}(\bar{v})_s, \bar{v}_s)ds + \mathbf{V}^+(t_0+h, X_{t_0+h}) &\leq \sup_{v\in\mathcal{V}(t_0,t_0+h,\bar{v})} \mathcal{J}(t_0, x_0, \bar{\alpha}(v), v) \\ &\leq \sup_{v\in\mathcal{V}(t_0)} \mathcal{J}(t_0, x_0, \bar{\alpha}(v), v) \leq \mathbf{V}^+(t_0, x_0) + \epsilon \end{aligned}$$

since $\bar{\alpha}$ is $\epsilon-$optimal for $\mathbf{V}^+(t_0, x_0)$. Taking the supremum over $\bar{v}$ and using the definition of W then implies that $W(t_0, t_0+h, x_0) \leq \mathbf{V}^+(t_0, x_0) + \epsilon$. This gives the desired result since ϵ is arbitrary. □

Regularity of the value functions

From the dynamic programming property we can deduce a space-time regularity for the value functions:

Corollary 3.8 *The maps $\mathbf{V}^+$ and $\mathbf{V}^-$ are bounded and Lipschitz continuous in all variables.*

Proof : As explained at the begining of the section, we only need to do the proof for $\mathbf{V}^+$. Since ℓ and g are bounded, so is $\mathbf{V}^+$. Since, from Lemma 3.6, $\mathbf{V}^+$ is Lipschitz continuous in space uniformly with respect to the time, it is enough to show that $\mathbf{V}^+$ is Lipschitz continuous in time uniformly with respect to the space variable. Recall that ℓ is bounded by some constant M and that according to Lemma 3.6, $\mathbf{V}^+$ is $C'-$Lipschitz continuous in the space variable for some C'.

Let $x_0 \in \mathbb{R}^N$ be fixed and $0 \leq t_0 < t_1 \leq T$. From the dynamic programming property applied to $h = t_1 - t_0$, we have

$$\mathbf{V}^+(t_0, x_0) = \inf_{\alpha\in\mathcal{A}_d(t_0)} \sup_{v\in\mathcal{V}(t_0)} \left\{ \int_{t_0}^{t_1} \ell(s, X_s^{t_0,x_0,\alpha(v),v}, \alpha_s, v_s)ds + \mathbf{V}^+(t_1, X_{t_1}^{t_0,x_0,\alpha(v),v}) \right\} \ .$$

We note that, for any $\alpha \in \mathcal{A}_d(t_0)$ and $v \in \mathcal{V}(t_0)$ we have

$$\left| \int_{t_0}^{t_1} \ell(s, X_s^{t_0,x_0,\alpha(v),v}, \alpha_s, v_s)ds \right| \leq M(t_1 - t_0)$$

while, from the growth condition on f we also have

$$|X_{t_1}^{t_0,x_0,\alpha(v),v} - x_0| \leq \|f\|_\infty (t_1 - t_0) \ .$$

Since $\mathbf{V}^+$ is $C'-$Lipschitz continuous in the space variable, we get

$$|\mathbf{V}^+(t_1, X_{t_1}^{t_0,x_0,\alpha(v),v}) - \mathbf{V}^+(t_1, x_0)| \leq C'\|f\|_\infty (t_1 - t_0) \ .$$

Hence

$$\begin{aligned} &\mathbf{V}^+(t_1, x_0) - (M + C'\|f\|_\infty)(t_1 - t_0) \\ &\quad \leq \int_{t_0}^{t_1} \ell(s, X_s^{t_0,x_0,\alpha(v),v}, \alpha_s, v_s)ds + \mathbf{V}^+(t_1, X_{t_1}^{t_0,x_0,\alpha(v),v}) \leq \mathbf{V}^+(t_1, x_0) + (M + C'\|f\|_\infty)(t_1 - t_0) \ . \end{aligned}$$

Taking the supremum over $v \in \mathcal{V}(t_0)$ and the infimum over $\alpha \in \mathcal{A}_d(t_0)$ in the previous inequalities it then implies, thanks to the dynamic programming property, that

$$|\mathbf{V}^+(t_0, x_0) - \mathbf{V}^+(t_1, x_0)| \leq (M + C'\|f\|_\infty)(t_1 - t_0) \ .$$

□

3.1.3 Isaacs' equation and viscosity solutions

Heuristic derivation of Isaacs' equation

Let us first show, in a purely heuristic way, that dynamic programming property is deeply related with a partial differential equation (in short PDE) called Hamilton-Jacobi-Isaacs' equation (in short Isaacs' equation). We still work with $\mathbf{V}^+$. Dynamic programming (3.7) can be rewritten as

$$\inf_{\alpha\in\mathcal{A}_d(t_0)}\sup_{v\in\mathcal{V}(t_0)}\left\{\frac{1}{h}\int_{t_0}^{t_0+h}\ell(s,X_s^{\alpha,v},\alpha(v)_s,v_s)ds+\frac{\mathbf{V}^+(t_0+h,X_{t_0+h}^{\alpha,v})-\mathbf{V}^+(t_0,x_0)}{h}\right\}=0 \tag{3.13}$$

where we have used the notation $X_t^{\alpha,v}=X_t^{t_0,x_0,\alpha,v}$. Letting h tend to 0^+, $\frac{X_{t_0+h}^{\alpha,v}-X_{t_0}}{h}$ behaves as $f(t_0,x_0,\alpha(v)_{t_0},v_{t_0})$. Hence $\frac{\mathbf{V}^+(t_0+h,X_{t_0+h}^{\alpha,v})-\mathbf{V}^+(t_0,x_0)}{h}$ is close to

$$\partial_t\mathbf{V}^+(t_0,x_0)+\langle D\mathbf{V}^+,f(t_0,x_0,\alpha(v)_{t_0},v_{t_0})\rangle\ .$$

Finally $\frac{1}{h}\int_{t_0}^{t_0+h}\ell(s,X_s^{\alpha,v},\alpha(v)_s,v_s)ds$ behaves as $\ell(t_0,x_0,\alpha(v)_{t_0},v_{t_0})$. Since α is a delay strategy, $\alpha(v)_{t_0}$ does not depend on v. Therefore equality (3.13) becomes

$$\inf_{u\in U}\sup_{v\in V}\left\{\ell(t_0,x_0,u,v)+\partial_t\mathbf{V}^+(t_0,x_0)+\langle D\mathbf{V}^+,f(t_0,x_0,u,v)\rangle\right\}=0$$

If we set

$$H^+(t,x,p)=\inf_{u\in U}\sup_{v\in V}\{\langle p,f(t,x,u,v)\rangle+\ell(t,x,u,v)\}\qquad\text{for }(t,x,p)\in[0,T]\times\mathbb{R}^N\times\mathbb{R}^N\ ,$$

the map $\mathbf{V}^+$ should satisfy the Hamilton-Jacobi-Isaacs' equation

$$\begin{cases}\partial_tW(t,x)+H^+(t,x,DW(t,x))=0 & \text{in }\ (0,T)\times\mathbb{R}^N\\ W(T,x)=g(x) & \text{in }\ \mathbb{R}^N\end{cases} \tag{3.14}$$

Applying the similar arguments for $\mathbf{V}^-$ we obtain that $\mathbf{V}^-$ should satisfy the symmetric equation

$$\begin{cases}\partial_tW(t,x)+H^-(t,x,DW(t,x))=0 & \text{in }\ (0,T)\times\mathbb{R}^N\\ W(T,x)=g(x) & \text{in }\ \mathbb{R}^N\end{cases} \tag{3.15}$$

where

$$H^-(x,p)=\sup_{v\in V}\inf_{u\in U}\{\langle p,f(t,x,u,v)\rangle+\ell(t,x,u,v)\}\ .$$

Now if Isaacs' condition holds, i.e., if $H^+=H^-$, then $\mathbf{V}^+$ and $\mathbf{V}^-$ satisfy the same equation and one can hope that this implies the equality $\mathbf{V}^+=\mathbf{V}^-$. This is indeed the case, but we have to be careful with the sense we give to equations (3.14) and (3.15).

Let us recall that, since $\mathbf{V}^+$ is Lipschitz continuous, Rademacher's Theorem states that $\mathbf{V}^+$ is differentiable almost everywhere. In fact one can show (see Exercise 3.4) that $\mathbf{V}^+$ indeed satisfies equation (3.14) at each point of differentiability. Unfortunately this is not enough to characterize the value functions. For instance, we show in Exercise 3.2 that one can find infinitely many Lipschitz continuous functions satisfying almost everywhere an equation of the form (3.14).

The idea of "viscosity solutions", introduced by Crandall-Lions [82], is that one should look closely even at points where the function is not differentiable.

A first glimpse at viscosity solutions

We now explain the proper meaning to equations of the form:

$$\partial_t\mathbf{V}(t,x)+H(t,x,D\mathbf{V}(t,x))=0\qquad\text{in }\ (0,T)\times\mathbb{R}^N \tag{3.16}$$

where $H:[0,T]\times\mathbb{R}^N\times\mathbb{R}^N\to\mathbb{R}$ is continuous.

Definition 3.9 (Viscosity solution)

- *A map* $\mathbf{V} : [0,T] \times \mathbb{R}^N \to \mathbb{R}$ *is a* **viscosity supersolution** *of (3.16) if* $\mathbf{V}$ *is lower semi-continuous (l.s.c.) in* $(0,T) \times \mathbb{R}^N$ *and if, for any test function* $\phi \in \mathcal{C}^1([0,T] \times \mathbb{R}^N)$ *such that* $\mathbf{V} - \phi$ *has a local minimum at some point* $(t,x) \in (0,T) \times \mathbb{R}^N$*, one has*
$$\partial_t \phi(t,x) + H(t,x,D\phi(t,x)) \leq 0 \ .$$
- *A map* $\mathbf{V} : [0,T] \times \mathbb{R}^N \to \mathbb{R}$ *is a* **viscosity subsolution** *of (3.16) if* $\mathbf{V}$ *is upper semi-continuous (u.s.c.) and if, for any test function* $\phi \in \mathcal{C}^1([0,T] \times \mathbb{R}^N)$ *such that* $\mathbf{V} - \phi$ *has a local maximum at some point* $(t,x) \in (0,T) \times \mathbb{R}^N$*, one has*
$$\partial_t \phi(t,x) + H(t,x,D\phi(t,x)) \geq 0 \ .$$
- *A* **viscosity solution** *to (3.16) if a map* $\mathbf{V}$ *which is a viscosity sub- and supersolution to (3.16).*

Remarks 3.10 1. Note that, with this definition, a solution is a continuous map.

2. One can easy check that, if $\mathbf{V} \in \mathcal{C}^1([0,T] \times \mathbb{R}^N)$, then $\mathbf{V}$ is a supersolution (resp. subsolution) of (3.16) if and only if, for any $(t,x) \in (0,T) \times \mathbb{R}^N$,
$$\partial_t \mathbf{V}(t,x) + H(t,x,D\mathbf{V}(t,x)) \leq 0 \qquad (\text{resp. } \geq 0) \ .$$

Lemma 3.11 *If* $\mathbf{V}$ *is a subsolution (respectively supersolution) of equation (3.16), then* $-\mathbf{V}$ *is a supersolution (resp. subsolution) of*
$$\partial_t V(t,x) + \tilde{H}(t,x,DV(t,x)) = 0 \qquad \text{in } \ (0,T) \times \mathbb{R}^N$$
where
$$\tilde{H}(t,x,p) = -H(t,x,-p) \qquad \forall (t,x,p) \in [0,T] \times \mathbb{R}^N \times \mathbb{R}^N \ .$$

Proof : We do the proof for instance in the case of subsolutions. Let $\mathbf{V}$ be a subsolution of (3.16). Then $(-\mathbf{V})$ is lower semi-continuous because $\mathbf{V}$ is upper semi-continuous. Let $\phi \in \mathcal{C}^1((0,T) \times \mathbb{R}^N)$ be a test function such that $(-\mathbf{V}) - \phi$ has a local minimum at some point $(t,x) \in (0,T) \times \mathbb{R}^N$. Then $\mathbf{V} + \phi = \mathbf{V} - (-\phi)$ has a local maximum at (t,x), so that, by definition of viscosity subsolutions, one has
$$-\partial_t \phi(t,x) + H(t,x,-D\phi(t,x)) \geq 0 \ .$$
Hence
$$\partial_t \phi(t,x) + \tilde{H}(t,x,D\phi(t,x)) = \partial_t \phi(t,x) - H(t,x,-D\phi(t,x)) \leq 0 \ .$$
□

The main point in considering viscosity solution is the following comparison principle, which implies that equation (3.16), supplemented with a terminal condition, has at most one solution. For this we need to assume that H satisfies the following conditions :

(3.17) $$|H(t_1,x_1,p) - H(t_2,x_2,p)| \leq C(1+|p|)|(t_1,x_1) - (t_2,x_2)|$$

and

(3.18) $$|H(t,x,p_1) - H(t,x,p_2)| \leq C|p_1 - p_2|$$

for some constant C.

Theorem 3.12 (Comparison principle) *Under assumption (3.17) and (3.18), let* $\mathbf{V}_1$ *be a subsolution of (3.16) which is u.s.c. on* $[0,T] \times \mathbb{R}^N$ *and* $\mathbf{V}_2$ *be a supersolution of (3.16) which is l.s.c. on* $[0,T] \times \mathbb{R}^N$*. Let us assume that* $\mathbf{V}_1(T,x) \leq \mathbf{V}_2(T,x)$ *for any* $x \in \mathbb{R}^N$*. Then*
$$\mathbf{V}_1(t,x) \leq \mathbf{V}_2(t,x) \qquad \forall (t,x) \in [0,T] \times \mathbb{R}^N \ .$$

The proof of Theorem 3.12 is a little intricate and postponed to section 3.2.7. From this result one easily deduces:

Corollary 3.13 *Let* $g : \mathbb{R}^N \to \mathbb{R}$ *be continuous. Then equation (3.16) has at most one continuous viscosity solution which satisfies the terminal condition* $\mathbf{V}(T,x) = g(x)$ *for any* $x \in \mathbb{R}^N$*.*

Proof of Corollary 3.13: Let $\mathbf{V}_1$ and $\mathbf{V}_2$ be two bounded and Lipschitz continuous viscosity solution of (3.16) such that $\mathbf{V}_1(T,x) = \mathbf{V}_2(T,x) = g(x)$ for any $x \in \mathbb{R}^N$. Since, in particular, $\mathbf{V}_1$ is a subsolution and $\mathbf{V}_2$ a supersolution and $\mathbf{V}_1(T,\cdot) = \mathbf{V}_2(T,\cdot)$, we have by comparison $\mathbf{V}_1 \leq \mathbf{V}_2$ in $[0,T] \times \mathbb{R}^N$. Reversing the roles of $\mathbf{V}_1$ and $\mathbf{V}_2$, one gets the opposite inequality, whence the equality. □

3.1.4 Existence and characterization of the value

We have seen that one can associate with our game two Hamilton-Jacobi equations:

$$\left\{\begin{array}{ll}\partial_t \mathbf{V}(t,x)+H^+(t,x,D\mathbf{V}(t,x))=0 & \text{in } (0,T)\times\mathbb{R}^N\\ \mathbf{V}(T,x)=g(x) & \text{in } \mathbb{R}^N\end{array}\right. \tag{3.19}$$

where H^+ is defined by

$$H^+(t,x,p)=\inf_{u\in U}\sup_{v\in V}\{\langle p,f(t,x,u,v)\rangle+\ell(t,x,u,v)\}\ , \tag{3.20}$$

and

$$\left\{\begin{array}{ll}\partial_t \mathbf{V}(t,x)+H^-(t,x,D\mathbf{V}(t,x))=0 & \text{in } (0,T)\times\mathbb{R}^N\\ \mathbf{V}(T,x)=g(x) & \text{in } \mathbb{R}^N\end{array}\right. \tag{3.21}$$

where H^- is defined by

$$H^-(t,x,p)=\sup_{v\in V}\inf_{u\in U}\{\langle p,f(t,x,u,v)\rangle+\ell(t,x,u,v)\}\ . \tag{3.22}$$

Theorem 3.14 *Under conditions (3.2) and (3.3) on f, ℓ and g, and if Isaacs' assumption holds:*

$$H^+(t,x,p)=H^-(t,x,p)\qquad \forall (t,x,p)\in[0,T]\times\mathbb{R}^N\times\mathbb{R}^N\ , \tag{3.23}$$

the game has a value:

$$\mathbf{V}^+(t,x)=\mathbf{V}^-(t,x)\qquad \forall (t,x)\in[0,T]\times\mathbb{R}^N\ .$$

Moreover $\mathbf{V}^+=\mathbf{V}^-$ is the unique viscosity solution of Isaacs' equation (3.19)=(3.21).

The key point of the proof of Theorem 3.14 is the following (half-)characterization of the value functions:

Lemma 3.15 *The upper value function $\mathbf{V}^+$ is a subsolution of equation (3.19) where H^+ is defined by (3.20) while the lower value function $\mathbf{V}^-$ is a viscosity supersolution to (3.21).*

Remark 3.16 The map $\mathbf{V}^+$ is actually a viscosity *solution* of equation (3.19) $\mathbf{V}^-$ is a viscosity *solution* to (3.21): see Exercice 3.5.

Proof of the Theorem 3.14: According to Corollary 3.8, $\mathbf{V}^+$ and $\mathbf{V}^-$ are both Lipschitz continuous and bounded. Since $H^-=H^+$, $\mathbf{V}^-$ is a supersolution of (3.19) while $\mathbf{V}^+$ is a subsolution of that same equation. Under assumptions (3.2) and (3.3) on f, ℓ and g, the Hamiltonian $H^+=H^-$ satisfies (3.17) and (3.18). Since $\mathbf{V}^+(T,\cdot)=\mathbf{V}^-(T,\cdot)=g$, the comparison principle then implies that $\mathbf{V}^+\leq\mathbf{V}^-$. Since the reverse inequality always holds, one gets the equality. □

Proof of Lemma 3.15 : As before, it is enough prove the result for $\mathbf{V}^+$: indeed, if we do so, then $(-\mathbf{V}^-)$—being the upper value function of the game with running payoff $-\ell$ and terminal payoff $-g$ and where the first Player maximizes—is a subsolution of

$$\left\{\begin{array}{ll}\partial_t \mathbf{V}(t,x)+\tilde{H}^+(t,x,D\mathbf{V}(t,x))=0 & \text{in } (0,T)\times\mathbb{R}^N\\ \mathbf{V}(T,x)=-g(x) & \text{in } \mathbb{R}^N\end{array}\right.$$

where

$$\tilde{H}^+(t,x,p)=\inf_{v\in V}\sup_{u\in U}\{-\langle p,f(t,x,u,v)\rangle-\ell(t,x,u,v)\}\ ,$$

Then Lemma 3.11 states that $\mathbf{V}^-$ is a supersolution of (3.21).

Let us now show that $\mathbf{V}^+$ is a subsolution of (3.19). Since $\mathbf{V}^+$ is Lipschitz continuous and satisfies $\mathbf{V}^+(T,x)=g(x)$, we only have to show that, if ϕ is a $\mathcal{C}^1$ test function such that $\mathbf{V}^+-\phi$ has a local maximum at $(t_0,x_0)\in(0,T)\times\mathbb{R}^N$, then $\partial_t\phi(t_0,x_0)+H^+(t_0,x_0,D\phi(t_0,x_0))\geq 0$.

Since $\mathbf{V}^+-\phi$ has a local maximum at (t_0,x_0), there is some $r>0$ such that

$$\mathbf{V}^+(t,x)\leq\phi(t,x)+\mathbf{V}^+(t_0,x_0)-\phi(t_0,x_0)\qquad\forall (t,x)\in B((t_0,x_0),r)\ .$$

From the dynamic programming property, we have

$$\mathbf{V}^+(t_0,x_0) = \inf_{\alpha\in\mathcal{A}_d(t_0)} \sup_{v\in\mathcal{V}(t_0)} \left\{ \int_{t_0}^{t_0+h} \ell(s, X_s^{t_0,x_0,\alpha(v),v}, \alpha_s, v_s)ds + \mathbf{V}^+(t_0+h, X_{t_0+h}^{t_0,x_0,\alpha(v),v}) \right\} .$$

Let us set $h_0 = r/(\|f\|_\infty + 1)$. Then, for any $h \in (0,h_0)$ and any $(\alpha, v) \in \mathcal{A}_d(t_0)\times\mathcal{V}(t_0)$, we have $(t_0+h, X_{t_0+h}^{t_0,x_0,\alpha(v),v}) \in B((t_0,x_0),r)$, so that

$$0 \le \inf_{\alpha\in\mathcal{A}_d(t_0)} \sup_{v\in\mathcal{V}(t_0)} \left\{ \int_{t_0}^{t_0+h} \ell(s, X_s^{t_0,x_0,\alpha(v),v}, \alpha_s, v_s)ds + \phi(t_0+h, X_{t_0+h}^{t_0,x_0,\alpha(v),v}) - \phi(t_0,x_0) \right\} . \tag{3.24}$$

Let us now fix any (time independent) control $u \in U$. From (3.24) for any $\epsilon > 0$ and any $h > 0$ small, there is some (time dependant) control $v_h \in \mathcal{V}(t_0)$ such that

$$-\epsilon h \le \int_{t_0}^{t_0+h} \ell(s, X_s^{t_0,x_0,u,v_h}, u, v_{h,s})ds + \phi(t_0+h, X_{t_0+h}^{t_0,x_0,u,v_h}) - \phi(t_0,x_0) . \tag{3.25}$$

Since ℓ is Lipschitz continuous and f is bounded, we have

$$\begin{aligned} &\left| \int_{t_0}^{t_0+h} \ell(s, X_s^{t_0,x_0,u,v_h}, u, v_{h,s})ds - \int_{t_0}^{t_0+h} \ell(t_0,x_0,u,v_{h,s})ds \right| \\ &\quad \le \int_{t_0}^{t_0+h} \left|\ell(s, X_s^{t_0,x_0,u,v_h}, u, v_{h,s})ds - \ell(t_0,x_0,u,v_{h,s})\right| \, ds \\ &\quad \le \mathrm{Lip}(\ell) \int_{t_0}^{t_0+h} (|s-t_0| + |X_s^{t_0,x_0,u,v_h} - x_0|) \, ds \\ &\quad \le \mathrm{Lip}(\ell) \int_{t_0}^{t_0+h} (1+\|f\|_\infty)(s-t_0) \, ds \;\le\; o(h) . \end{aligned}$$

Since ϕ is of class $\mathcal{C}^1$, we have

$$\phi(t_0+h, X_{t_0+h}^{t_0,x_0,u,v_h}) - \phi(t_0,x_0) = \int_{t_0}^{t_0+h} \partial_t\phi(s, X_s^{t_0,x_0,u,v_h}) + \langle D\phi(s, X_s^{t_0,x_0,u,v_h}), f(s, X_s^{t_0,x_0,u,v_h}, u, v_{h,s})\rangle ds$$

where, by uniform continuity of $\partial_t\phi$, $D\phi$ and f with respect to the (t,x) variables,

$$\left| \int_{t_0}^{t_0+h} \partial_t\phi(s, X_s^{t_0,x_0,u,v_h})ds - h\partial_t\phi(t_0,x_0) \right| \le o(h)$$

and

$$\left| \int_{t_0}^{t_0+h} \langle D\phi(s, X_s^{t_0,x_0,u,v_h}), f(s, X_s^{t_0,x_0,u,v_h}, u, v_{h,s})\rangle ds - \int_{t_0}^{t_0+h} \langle D\phi(t_0,x_0), f(t_0,x_0,u,v_{h,s})\rangle ds \right| \le o(h) .$$

Plugging the above estimates into (3.25) gives

$$-\epsilon h - o(h) \le h\partial_t\phi(t_0,x_0) + \int_{t_0}^{t_0+h} \ell(t_0,x_0,u,v_{h,s}) + \langle D\phi(t_0,x_0), f(t_0,x_0,u,v_{h,s})\rangle \; ds .$$

Since

$$\begin{aligned} &\int_{t_0}^{t_0+h} \ell(t_0,x_0,u,v_{h,s}) + \langle D\phi(t_0,x_0), f(t_0,x_0,u,v_{h,s})\rangle \; ds \\ &\quad \le \int_{t_0}^{t_0+h} \max_{v\in V} \{\ell(t_0,x_0,u,v) + \langle D\phi(t_0,x_0), f(t_0,x_0,u,v)\rangle\} \, ds \\ &\quad = h \max_{v\in V} \{\ell(t_0,x_0,u,v) + \langle D\phi(t_0,x_0), f(t_0,x_0,u,v)\rangle\} \; , \end{aligned}$$

we get

$$-\epsilon h - o(h) \le h \left\{ \partial_t\phi(t_0,x_0) + \max_{v\in V} \{\ell(t_0,x_0,u,v) + \langle D\phi(t_0,x_0), f(t_0,x_0,u,v)\rangle\} \right\} .$$

Dividing the above expression by h, letting $h\to 0^+$ and then $\epsilon \to 0^+$ gives:

$$0 \;\le\; \partial_t\phi(t_0,x_0) + \max_{v\in V} \{\ell(t_0,x_0,u,v) + \langle D\phi(t_0,x_0), f(t_0,x_0,u,v)\rangle\} .$$

Taking the infimum with respect to $u \in U$ then completes the proof. □

3.2 A closer look at viscosity solutions

Since Hamilton-Jacobi equations play a crucial role in the analysis of two-player zero-sum differential games, it is now time to develop a deeper analysis of the notion of viscosity solutions for these equations.

3.2.1 Definition of viscosity solutions: stationary problems

In order to introduce the notion of viscosity solutions in some generality it is convenient (and probably more convincing) to work with second order Hamilton-Jacobi equations.

Let us denote by $\mathcal{S}_N$ be the set of $N \times N$ symmetric matrices. For $X, Y \in \mathcal{S}_N$, we write $X \leq Y$ when $Y - X$ is a positive semidefinite matrix.

Let $\mathcal{O}$ be an open subset of $\mathbb{R}^N$. We consider the Hamilton-Jacobi equation

$$H(x, \mathbf{V}(x), D\mathbf{V}(x), D^2\mathbf{V}(x)) = 0 \qquad \text{in } \mathcal{O} \,, \tag{3.26}$$

where the real-valued Hamiltonian $H = H(x, r, p, X)$ is defined on $\mathcal{O} \times \mathbb{R} \times \mathbb{R}^N \times \mathcal{S}_N$ and continuous. Our standing assumption, in order to give a meaning to the notion of viscosity solution, is that H is elliptic, meaning that it is nondecreasing with respect to the Hessian matrix:

$$H(x, r, p, X) \leq H(x, r, p, Y) \qquad \text{whenever } X \leq Y \,. \tag{3.27}$$

For instance, this is the case for the Laplace equation

$$\Delta\mathbf{V}(x) = 0, \qquad \text{where } \Delta\mathbf{V}(x) = \sum_{i=1}^{N} \frac{\partial^2 \mathbf{V}}{\partial x_i^2}(x) \,.$$

Definition 3.17 (Viscosity solution)

- *A map* $\mathbf{V} : \mathcal{O} \to \mathbb{R}$ *is a* **viscosity supersolution** *of (3.26) if* $\mathbf{V}$ *is lower semi-continuous (l.s.c.) and if, for any test function* $\phi \in \mathcal{C}^2(\mathcal{O})$ *such that* $\mathbf{V} - \phi$ *has a local minimum at some point* $x \in \mathcal{O}$*, one has*

$$H(x, \mathbf{V}(x), D\phi(x), D^2\phi(x)) \leq 0 \,.$$

- *A map* $\mathbf{V} : \mathcal{O} \to \mathbb{R}$ *is a* **viscosity subsolution** *of (3.26) if* $\mathbf{V}$ *is upper semi-continuous (u.s.c.) and if, for any test function* $\phi \in \mathcal{C}^2(\mathcal{O})$ *such that* $\mathbf{V} - \phi$ *has a local maximum at some point* $x \in \mathcal{O}$*, one has*

$$H(x, \mathbf{V}(x), D\phi(t, x), D^2\phi(x)) \geq 0 \,.$$

- *A* **viscosity solution** *to (3.26) if a map* $\mathbf{V}$ *which is a viscosity sub- and supersolution to (3.26).*

Remark 3.18 1. With this definition, a solution is a continuous map.

2. **On the sign convention:** Note carefully that, being a viscosity solution of equation (3.26) is not equivalent to being a viscosity solution of

$$-H(x, \mathbf{V}(x), D\mathbf{V}(x), D^2\mathbf{V}(x)) = 0 \qquad \text{in } \mathcal{O} \,,$$

We have fixed here a sign convention which is adapted to the framework of differential games. Most often the opposite sign convention is used in the literature (with, of course, the opposite definition of ellipticity).

Proposition 3.19 *Let* $\mathbf{V} \in \mathcal{C}^2(\mathcal{O})$*. Then* $\mathbf{V}$ *is a supersolution (resp. subsolution) of (3.26) if and only if* $\mathbf{V}$ *is a classical supersolution (resp. subsolution) of (3.26), i.e., for any* $x \in \mathcal{O}$*,*

$$H(x, \mathbf{V}(x), D\mathbf{V}(x), D^2\mathbf{V}(x)) \leq 0 \qquad (\text{resp. } \geq 0).$$

Proof : Exercice. □

Proposition 3.20 *Let* $\mathbf{V}$ *be a subsolution of (3.26). Then* $-\mathbf{V}$ *is a supersolution of*

$$\tilde{H}(x, W(x), DW(x), D^2W(x)) = 0 \quad x \in \mathcal{O}$$

where

$$\tilde{H}(x, r, p, X) = -H(x, -r, -p, -X) \quad \forall (x, r, p, X) \in \mathcal{O} \times \mathbb{R} \times \mathbb{R}^N \times \mathcal{S}_N \,.$$

Remark 3.21 Note that $\tilde{H}$ also satisfies the ellipticity condition (3.27).

Proof : See the proof of Lemma 3.11. □

Here is a very useful equivalent definition of viscosity solutions:

Lemma 3.22 *If the Hamiltonian H is continuous, one can replace "local maximum" (resp. "local minimum") by "strict local maximum" (resp. "strict local minimum") in the definition of subsolution (resp. supersolution).*

Proof : Let us assume for instance that $\mathbf{V}$ is u.s.c. and that, for any test function $\phi \in \mathcal{C}^2$ such that $u - \phi$ has a strict local maximum at some point $x_0 \in \mathcal{O}$, we have $H(x_0, \mathbf{V}(x_0), D\phi(x_0), D^2\phi(x_0)) \geq 0$. In order to show that $\mathbf{V}$ is a subsolution, let us assume that $u - \phi$ has a local maximum at some point $x_0 \in \mathcal{O}$. Let us set $\phi_1(x) = \phi(x) + |x - x_0|^4$. Then $\mathbf{V} - \phi_1$ has a strict local maximum at x_0, and so $H(x_0, \mathbf{V}(x_0), D\phi_1(x_0), D^2\phi_1(x_0)) \geq 0$ by assumption. But $D\phi_1(x_0) = D\phi(x_0)$ and $D^2\phi_1(x_0) = D^2\phi(x_0)$. So $H(x_0, u(x_0), D\phi(x_0), D^2\phi(x_0)) \geq 0$. □

3.2.2 Definition of viscosity solution: evolution problems

The definition of viscosity solution introduced above can be directly applied to Hamilton-Jacobi equations of evolution type:

$$\partial_t \mathbf{V}(t,x) + H(t, x, \mathbf{V}(x), D\mathbf{V}(x), D^2\mathbf{V}(x)) = 0 \qquad \text{in } \mathcal{O} , \tag{3.28}$$

where $\mathcal{O}$ is an open subset of $\mathbb{R} \times \mathbb{R}^N$ and the real-valued Hamiltonian $H = H(t, x, r, p, X)$ is defined on $\mathcal{O} \times \mathbb{R} \times \mathbb{R}^N \times \mathcal{S}_N$. We again assume that H is elliptic:

$$H(t, x, r, p, X) \leq H(t, x, r, p, Y) \qquad \text{whenever } X \leq Y .$$

A typical example is the (backward) heat equation:

$$\partial_t \mathbf{V}(t,x) + \Delta \mathbf{V}(x) = 0, \qquad \text{where } \Delta \mathbf{V}(x) = \sum_{i=1}^N \frac{\partial^2 \mathbf{V}}{\partial x_i^2}(x) .$$

For sake of completeness let us explicitely explain what is a viscosity solution in this framework:

Definition 3.23 (Viscosity solution)

- *A map $\mathbf{V} : \mathcal{O} \to \mathbb{R}$ is a* **viscosity supersolution** *of (3.28) if $\mathbf{V}$ is lower semi-continuous and if, for any test function $\phi \in \mathcal{C}^2(\mathcal{O})$ such that $\mathbf{V} - \phi$ has a local minimum at some point $(t,x) \in \mathcal{O}$, one has*

$$\partial_t \phi(t,x) + H(x, \mathbf{V}(x), D\phi(x), D^2\phi(x)) \leq 0 .$$

- *A map $\mathbf{V} : \mathcal{O} \to \mathbb{R}$ is a* **viscosity subsolution** *of (3.28) if $\mathbf{V}$ is upper semi-continuous and if, for any test function $\phi \in \mathcal{C}^2(\mathcal{O})$ such that $\mathbf{V} - \phi$ has a local maximum at some point $(t,x) \in \mathcal{O}$, one has*

$$\partial_t \phi(t,x) + H(t, x, \mathbf{V}(x), D\phi(t,x), D^2\phi(x)) \geq 0 .$$

- *A* **viscosity solution** *to (3.28) if a map $\mathbf{V}$ which is a viscosity sub- and a supersolution to (3.28).*

Remark 3.24 According to Lemma 3.22, if H is continuous, then one can replace the assumption that $\mathbf{V} - \phi$ has a local maximum by the assumption that $\mathbf{V} - \phi$ has a *strict* local maximum in the definition of subsolution (and symmetrically for supersolution).

When, as it is often the case, the domain $\mathcal{O}$ is of the form $(0,T) \times \Omega$, where Ω is some open subset of $\mathbb{R}^N$, then viscosity solutions in $(0,T) \times \Omega$ are solutions "up to $t = 0$":

Lemma 3.25 *If H is continuous in $[0,T) \times \Omega$ and if W is a subsolution (resp. supersolution) of (3.28) on $(0,T) \times \Omega$, then W is still a subsolution (resp. supersolution) at $t = 0$, i.e., if a $\mathcal{C}^1$ test function ϕ is such that $W - \phi$ has a local maximum (resp. minimum) on $[0,T) \times \Omega$ at some point $(0,x)$, then*

$$\partial_t \phi(0,x) + H(0, x, D\phi(0,x), D^2\phi(0,x)) \geq 0 \qquad (\text{resp. } \leq 0) .$$

Proof of Lemma 3.25: Thanks to Proposition 3.20 it is enough to do the proof for subsolutions. Let W be a subsolution of (3.28) and let us assume that $W-\phi$ has a strict local maximum on $[0,T]\times\Omega$ at $(0,x)$ for some $\mathcal{C}^1$ test-function ϕ and some $x\in\mathbb{R}^N$. Then there is some $r>0$ such that $B_r(x)\subset\Omega$ and

$$(3.29)\qquad (W-\phi)(s,y)\ <\ (W-\phi)(0,x)\qquad \forall (s,y)\in([0,T]\times\mathbb{R}^N)\backslash\{(0,x)\},\ |(s,y)-(0,x)|\leq r\ .$$

Let us denote by $D=B_r((0,x))\cap((0,+\infty)\times\mathbb{R}^N)$ the half ball and by $S=(\partial B_r((0,x)))\cap((0,+\infty)\times\mathbb{R}^N)$ the half sphere. Let us fix $\sigma>0$ and consider the map $\Psi_\sigma(s,y)=W(s,y)-\phi(s,y)-\sigma/s$. Let (s_σ,y_σ) be a maximum point on D of Ψ_σ. We claim that (s_σ,y_σ) converges to $(0,x)$ as $\sigma\to 0^+$. Indeed let $(\bar{s},\bar{y})$ be a cluster point of (s_σ,y_σ) as $\sigma\to0^+$. Since, for any $(s,y)\in D$ we have

$$\Psi_\sigma(s,y)=W(s,y)-\phi(s,y)-\sigma/s\leq\Psi_\sigma(s_\sigma,y_\sigma)\leq W(s_\sigma,y_\sigma)-\phi(s_\sigma,y_\sigma)\ ,$$

letting $\sigma\to0^+$ gives

$$W(s,y)-\phi(s,y)\leq W(\bar{s},\bar{y})-\phi(\bar{s},\bar{y})\ ,$$

so that $(\bar{s},\bar{y})$ is a maximum point of $W-\phi$ on D. Then (3.29) implies that $(\bar{s},\bar{y})=(0,x)$.

Since, for σ sufficiently small, (s_σ,y_σ) is a local maximum of Ψ_σ and since W is a subsolution, we have

$$\phi_t(s_\sigma,y_\sigma)-\frac{\sigma}{s_\sigma^2}+H(s_\sigma,y_\sigma,D\phi(s_\sigma,y_\sigma),D^2\phi(s_\sigma,y_\sigma))\geq0\ .$$

Therefore

$$\phi_t(s_\sigma,y_\sigma)+H(s_\sigma,y_\sigma,D\phi(s_\sigma,y_\sigma),D^2\phi(s_\sigma,y_\sigma))\geq0\ .$$

Letting $\sigma\to0^+$ gives the result. □

3.2.3 Stability

We now describe one of the most striking properties of viscosity solutions, which is their robustness with respect to passage to the limit. We work in the framework of stationary equations of the form (3.26). Let $\mathcal{O}$ be an open subset of $\mathbb{R}^N$ and let $H_n,H:\mathcal{O}\times\mathbb{R}\times\mathbb{R}^N\times\mathcal{S}_N\to\mathbb{R}$ be continuous Hamiltonians.

Theorem 3.26 (Stability) *Let us assume that* $(\mathbf{V}_n)$ *is a sequence of continuous subsolutions of equation*

$$(3.30)\qquad H_n(x,W(x),DW(x),D^2W(x))=0\quad x\in\mathcal{O}$$

which locally uniformly converges to a map $\mathbf{V}:\mathcal{O}\to\mathbb{R}$ *and that* (H_n) *locally uniformly converges to some Hamiltonian* $H:\mathcal{O}\times\mathbb{R}\times\mathbb{R}^N\times\mathcal{S}_N\to\mathbb{R}$. *Then* $\mathbf{V}$ *is still a subsolution of (3.26).*

Remarks 3.27

1. Thanks to Proposition 3.20 a symmetric result also holds for supersolutions and for solutions. Note that this result is somewhat surprising since it states that one can pass to the limit in a second order equation with only uniform convergence.
2. A straightforward consequence of the Theorem is the following: let us consider a sequence of Bolza problems, with dynamics f_n, running payoffs ℓ_n and terminal payoffs g_n such that conditions (3.2) and (3.3) hold uniformly with respect to n. Assume that (f_n), (ℓ_n) and (g_n) locally uniformly converge to some f, ℓ and g. Then, under Isaacs' condition on the (f_n), (ℓ_n) and (g_n), the value function associated to (f_n), (ℓ_n) and (g_n) locally uniformly converges to the value function for f, ℓ and g.

The proof of Theorem 3.26 uses the following Lemma:

Lemma 3.28 *If a continuous map* $f:\mathcal{O}\to\mathbb{R}$ *has a strict local maximum at some point* x_0 *and if a sequence of continuous functions* (f_n) *locally uniformly converges to* f, *then there is a sequence* (x_n) *of local maxima of* f_n *which converges to* x_0.

Proof : Since the continuous function f has a strict local maximum at x_0, there is some $r>0$ such that $B_r(x_0)\subset\mathcal{O}$ and $f(x)<f(x_0)$ for $x\in B_r(x_0)\backslash\{x_0\}$. In particular

$$f(x_0)=\max_{B_r(x_0)}f\ >\ \max_{\partial B_r(x_0)}f\ .$$

Since (f_n) uniformly converges to f in $B_r(x_0)$, there is some n_0 such that

$$f_n(x_0) > \max_{\partial B_r(x_0)} f_n \qquad \forall n \geq n_0 \ . \tag{3.31}$$

Let x_n be a maximum point of f_n on $B_r(x_0)$. Then by (3.31) x_n belongs to the interior of $B_r(x_0)$, i.e., x_n is a local maximum of f_n. Let us now show that (x_n) converges to x_0. Let y be a cluster point of the sequence (x_n). Since, for any $z \in B_r(x_0)$, $f_n(x_n) \geq f_n(z)$ and since f_n uniformly converges to f, we have $f(y) \geq f(z)$. So y is a maximum point of f in $B_r(x_0)$, which implies that $y = x_0$ since x_0 is the unique maximum point on $B_r(x_0)$. The bounded sequence (x_n) has a unique cluster point, x_0, therefore it converges to x_0. □

Proof of Theorem 3.26 : Let $\phi \in \mathcal{C}^2$ be such that $\mathbf{V} - \phi$ has a strict local maximum at some point $x_0 \in \mathcal{O}$. Since $\mathbf{V}_n - \phi$ locally uniformly converges to $\mathbf{V} - \phi$, there is a sequence (x_n) of local mixima of $\mathbf{V}_n - \phi$ which converges to x_0. Since $\mathbf{V}_n$ is a subsolution of (3.30) one has $H_n(x_n, \mathbf{V}_n(x_n), D\phi(x_n), D^2\phi(x_n)) \geq 0$. The sequence (H_n) converging locally uniformly to H, we get, by letting $n \to +\infty$, $H(x_0, \mathbf{V}(x_0), D\phi(x_0), D^2\phi(x_0)) \geq 0$. □

The assumption of uniform convergence in Theorem 3.26 is actually unnecessarily strong. A uniform bound on the solution is enough provided that one uses Barles-Perthame notion of half-relaxed limit: let $(\mathbf{V}_n)$ be a uniformly bounded sequence of maps in $\mathcal{O}$. The upper half relaxed limit $\mathbf{V}^*$ and lower half relaxed limit $\mathbf{V}_*$ of the sequence $(\mathbf{V}_n)$ are defined by

$$\mathbf{V}^*(x) = \limsup_{x_n \to x,\ n \to +\infty} \mathbf{V}_n(x_n) \qquad \text{and} \qquad \mathbf{V}_*(x) = \liminf_{x_n \to x,\ n \to +\infty} \mathbf{V}_n(x_n) \qquad \forall x \in \mathcal{O} \ .$$

One easily checks that $\mathbf{V}^*$ is u.s.c. while $\mathbf{V}_*$ is l.s.c.. The stability Theorem 3.26 can be generalized to this kind of convergence.

Theorem 3.29 (Stability by half-relaxed limit) *Let $(\mathbf{V}_n)$ be a locally uniformly bounded sequence of subsolutions of (3.30). Assume that (H_n) locally uniformly converges to H. Then $\mathbf{V}^*$ is also a subsolution of (3.26).*

The proof is almost the same as for Theorem 3.26, provided that one replaces Lemma 3.28 by the following result, the proof of which is left as an exercise.

Lemma 3.30 *Let $f : \mathcal{O} \to \mathbb{R}$ be a u.s.c. map which has a strict local maximum at some point x_0 and let (f_n) be a sequence of u.s.c. maps such that $f(x_0) = \limsup_{z_n \to x_0,\ n \to +\infty} f_n(z_n)$. Then there is a subsequence $n_k \to +\infty$ and a sequence (x_k), such that x_k is a local maximum of f_{n_k}, the sequence (x_k) converges to x_0 and the sequence $(f_{n_k}(x_k))$ converges to $f(x_0)$.*

3.2.4 Some basic properties of viscosity solution

Proposition 3.31 *Let $\mathbf{V}_1$ and $\mathbf{V}_2$ be two subsolutions of (3.26). Then $\max\{\mathbf{V}_1, \mathbf{V}_2\}$ is still a subsolution of (3.26).*

Remark 3.32 In a symmetric way, the minimum of two supersolutions is still a supersolution.

Proof : Let $\mathbf{V} = \max\{\mathbf{V}_1, \mathbf{V}_2\}$. Then $\mathbf{V}$ is u.s.c., as the maximum of two u.s.c. maps. Let $\phi \in \mathcal{C}^1(\mathcal{O})$ be a test function such that $\mathbf{V} - \phi$ has a local maximum at a point x. Let us assume to fix the ideas that $\mathbf{V}(x) = \mathbf{V}_1(x)$. Then $\mathbf{V}_1 - \phi$ has also a local maximum at x because, for any y in a neighbourhood of x,

$$\mathbf{V}_1(y) - \phi(y) \leq \mathbf{V}(y) - \phi(y) \leq \mathbf{V}(x) - \phi(x) = \mathbf{V}_1(x) - \phi(x) \ .$$

Since $\mathbf{V}_1$ is a subsolution and $\mathbf{V}_1(x) = \mathbf{V}(x)$, we get: $H(x, \mathbf{V}(x), D\phi(t,x), D^2\phi(x)) \geq 0$, which is the desired inequality. □

Proposition 3.31 can be generalized to the supremum of an arbitrary number of subsolutions:

Proposition 3.33 *Let $(\mathbf{V}_\alpha)_{\alpha \in A}$ be a family of subsolutions of equation (3.26). Let us assume that the $\mathbf{V}_\alpha$ are locally uniformly bounded from above and let $\mathbf{V}$ be the upper semicontinuous envelope of $\sup_{\alpha \in A} \mathbf{V}_\alpha$. Then $\mathbf{V}$ is still a subsolution of (3.26).*

Proof : Let $\phi \in \mathcal{C}^1(\mathcal{O})$ be a test function such that $\mathbf{V} - \phi$ has a local maximum at a point x. By definition of the upper semicontinuous envelope, there are $\alpha_n \in A$ and $x_n \to x$ such that $\lim \mathbf{V}_{\alpha_n}(x_n) = \mathbf{V}(x)$. Let $\mathbf{V}^*$ be the half-relaxed upper limit of the $\mathbf{V}_{\alpha_n}$. Then, since $\mathbf{V}_{\alpha_n} \leq \mathbf{V}$ and $\mathbf{V}$ is u.s.c., one has $\mathbf{V}^* \leq \mathbf{V}$. In particular, $\mathbf{V}^*(x) = \mathbf{V}(x)$. From Theorem 3.29, $\mathbf{V}^*$ is still a subsolution of (3.26). Note that $\mathbf{V}^* - \phi$ has a local maximum at x: indeed, for any y in a neighbourhood of x, we have

$$\mathbf{V}^*(y) - \phi(y) \leq \mathbf{V}(y) - \phi(y) \leq \mathbf{V}(x) - \phi(x) = \mathbf{V}^*(x) - \phi(x) .$$

Since $\mathbf{V}^*$ is a subsolution and $\mathbf{V}^*(x) = \mathbf{V}(x)$, we get: $H(x, \mathbf{V}(x), D\phi(t,x), D^2\phi(x)) \geq 0$, which is the desired inequality. □

3.2.5 Comparison principle for first order stationnary equations in bounded domains

As we have already seen in the analysis of the Bolza problem, comparison principle is one of the basic tools for proving the existence of a value for zero-sum differential games. Unfortunately there is no "universal" comparison principle and one has to adapt it to the equation at hand. This is the reason why we shall analyse three different types of problems: stationary equation in bounded domains, stationary equations in $\mathbb{R}^N$, evolution equations in $\mathbb{R}^N$. Of course there are many other kinds of equations, but we shall not need them in this chapter: we refer to the monographs on viscosity solution quoted in section 3.6 for complements. Moreover, we restrict the analysis to first order Hamilton-Jacobi equations: second order ones require deeper arguments, which are not needed here.

Let us now start with the comparison principle for first order stationary Hamilton-Jacobi equations of the form:

$$H(x, W(x), DW(x)) = 0 \qquad \text{in } \mathcal{O} \tag{3.32}$$

where $\mathcal{O}$ is an open bounded subset of $\mathbb{R}^N$ and where $H : \mathcal{O} \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ satisfies the following assumption: There are constants $\gamma > 0$ and $C \geq 0$ such that

$$H(x, s_1, p) - H(x, s_2, p) \ \leq\ -\gamma(s_1 - s_2) \quad \text{if} \quad s_1 \geq s_2 \tag{3.33}$$

and

$$|H(x, s, p) - H(y, s, p)| \ \leq\ C(1 + |p|)|y - x| \tag{3.34}$$

Theorem 3.34 *Let $\mathbf{V}_1$ and $\mathbf{V}_2$ be a sub- and a supersolution of (3.32), with V_1 u.s.c in $\overline{\mathcal{O}}$, V_2 l.s.c. in $\overline{\mathcal{O}}$ and $V_1 \leq V_2$ in $\partial\mathcal{O}$. Then $V_1 \leq V_2$ in $\mathcal{O}$.*

Let us first note that, if $\mathbf{V}_1$ and $\mathbf{V}_2$ are smooth (say of class $\mathcal{C}^1$ in $\mathcal{O}$), then the proof of the Theorem is straightforward. Indeed, since $\mathbf{V}_1 - \mathbf{V}_2$ is u.s.c. in the compact set $\overline{\mathcal{O}}$, it has a maximum point x_0. In order to show that $\sup_{\mathcal{O}}(\mathbf{V}_1 - \mathbf{V}_2) \leq 0$, we argue by contradiction by assuming the $(\mathbf{V}_1 - \mathbf{V}_2)(x_0) > 0$. Then $x_0 \notin \partial\mathcal{O}$ because $\mathbf{V}_1 \leq \mathbf{V}_2$ on $\mathcal{O}$. Hence $x_0 \in \mathcal{O}$ which implies, by the necessary conditions of optimality, that $D\mathbf{V}_1(x_0) = D\mathbf{V}_2(x_0)$. Since $\mathbf{V}_1$ and $\mathbf{V}_2$ are respectively sub- and supersolutions of (3.32), we have, by setting $p = D\mathbf{V}_1(x_0) = D\mathbf{V}_2(x_0)$,

$$H(x_0, \mathbf{V}_1(x_0), p) \geq 0 \qquad \text{and} \qquad H(x_0, \mathbf{V}_2(x_0), p) \leq 0 ,$$

so that, using assumption (3.33),

$$0 \leq H(x_0, \mathbf{V}_1(x_0), p) - H(x_0, \mathbf{V}_2(x_0), p) \leq -\gamma(\mathbf{V}_1(x_0) - \mathbf{V}_2(x_0)) \ <\ 0 .$$

This is impossible. So all the difficulty in the proof of Theorem 3.34 lies in the nonsmoothness of the functions.

Proof of Theorem 3.34 : As in the formal proof we argue by contradiction by assuming that

$$M := \sup_{x \in \mathcal{O}} (\mathbf{V}_1 - \mathbf{V}_2)(x) > 0 .$$

In order to overcome the issue of the nonsmoothness of the functions $\mathbf{V}_1$ and $\mathbf{V}_2$ we introduce the doubling variable technique. This technique appears in almost all proofs of comparison principles. It goes back to Kruzkov in his work on the conservation laws [140].

For any $\epsilon > 0$ let

$$W_\epsilon(x,y) = \mathbf{V}_1(x) - \mathbf{V}_2(y) - \frac{1}{2\epsilon}|x-y|^2 \qquad (x,y) \in \mathcal{O} \times \mathcal{O} \, .$$

Note that W_ϵ is u.s.c. in $\overline{\mathcal{O} \times \mathcal{O}}$ and in particular it has a maximum point (x_ϵ, y_ϵ). Let us set

$$M_\epsilon = \max_{(x,y)\in\overline{\mathcal{O}\times\mathcal{O}}} W_\epsilon(x,y) = W_\epsilon(x_\epsilon, y_\epsilon) \, .$$

In the following Lemma we collect some estimates on (x_ϵ, y_ϵ):

Lemma 3.35 *(i)* $\lim_{\epsilon\to 0^+} M_\epsilon = M$,

(ii) $\lim_{\epsilon\to 0^+} \frac{1}{\epsilon}|x_\epsilon - y_\epsilon|^2 = 0$

(iii) For $\epsilon > 0$ *sufficiently small,* $(x_\epsilon, y_\epsilon) \in \mathcal{O} \times \mathcal{O}$.

Postponing the proof of Lemma 3.35, let us complete the proof of Theorem 3.34. Let us fix $\epsilon > 0$ sufficiently small so that (iii) holds and $M_\epsilon > M/2$. Let us note that

$$\mathbf{V}_1(x_\epsilon) \geq \mathbf{V}_2(y_\epsilon) \geq W_\epsilon(x_\epsilon, y_\epsilon) \geq M/2 > 0 \, . \tag{3.35}$$

Since W_ϵ has a maximum at $(x_\epsilon, y_\epsilon) \in \mathcal{O} \times \mathcal{O}$, the map $x \to \mathbf{V}_1(x) - [\mathbf{V}_2(y_\epsilon) + \frac{1}{2\epsilon}|x - y_\epsilon|^2]$ has a maximum point at x_ϵ with $x_\epsilon \in \mathcal{O}$. Using the test function $\phi(x) = \mathbf{V}_2(y_\epsilon) + \frac{1}{2\epsilon}|x - y_\epsilon|^2$ and the fact that $\mathbf{V}_1$ is a subsolution of (3.32), we get

$$H\left(x_\epsilon, \mathbf{V}_1(x_\epsilon), D\phi(x_\epsilon)\right) = H\left(x_\epsilon, \mathbf{V}_1(x_\epsilon), \frac{x_\epsilon - y_\epsilon}{\epsilon}\right) \geq 0 \, . \tag{3.36}$$

In the same way, the map $y \to \mathbf{V}_2(y) - [\mathbf{V}_1(x_\epsilon) - \frac{1}{2\epsilon}|y - x_\epsilon|^2]$ has a minimum at y_ϵ, with $y_\epsilon \in \mathcal{O}$, and therefore, since $\mathbf{V}_2$ is a supersolution,

$$H\left(y_\epsilon, \mathbf{V}_2(y_\epsilon), \frac{x_\epsilon - y_\epsilon}{\epsilon}\right) \leq 0 \, . \tag{3.37}$$

Computing the difference between (3.36) and (3.37) and using assumption (3.33) (since $\mathbf{V}_1(x_\epsilon) \geq \mathbf{V}_2(y_\epsilon)$ by (3.35)) and then assumption (3.34) we obtain

$$\begin{aligned}
0 &\leq H\left(x_\epsilon, \mathbf{V}_1(x_\epsilon), \frac{x_\epsilon - y_\epsilon}{\epsilon}\right) - H\left(y_\epsilon, \mathbf{V}_2(y_\epsilon), \frac{x_\epsilon - y_\epsilon}{\epsilon}\right) \\
&\leq -\gamma(\mathbf{V}_1(x_\epsilon) - \mathbf{V}_2(y_\epsilon)) + H\left(x_\epsilon, \mathbf{V}_2(y_\epsilon), \frac{x_\epsilon - y_\epsilon}{\epsilon}\right) - H\left(y_\epsilon, \mathbf{V}_2(y_\epsilon), \frac{x_\epsilon - y_\epsilon}{\epsilon}\right) \\
&\leq -\gamma M_\epsilon + C\left(1 + \frac{|x_\epsilon - y_\epsilon|}{\epsilon}\right)|x_\epsilon - y_\epsilon|
\end{aligned}$$

As $\epsilon \to 0^+$ we get $0 \leq -\gamma M$ thanks to the Lemma. This is impossible since $M > 0$. □

Proof of Lemma 3.35 : From the definition of W_ϵ, M_ϵ is nondecreasing with ϵ and $M_\epsilon \geq M$. Since $\mathcal{O}$ is bounded and $\mathbf{V}_1$ is u.s.c. in $\overline{\mathcal{O}}$ while $\mathbf{V}_2$ is l.s.c. in this set, there is a constant K such that $u \leq K$ and $v \geq -K$ in $\overline{\mathcal{O}}$. So

$$M \leq M_\epsilon = \mathbf{V}_1(x_\epsilon) - \mathbf{V}_2(y_\epsilon) - \frac{1}{2\epsilon}|x_\epsilon - y_\epsilon|^2 \leq 2K - \frac{1}{2\epsilon}|x_\epsilon - y_\epsilon|^2 \, .$$

This proves that $\frac{1}{\epsilon}|x_\epsilon - y_\epsilon|^2$ is bounded, and therefore that $x_\epsilon - y_\epsilon \to 0$ as $\epsilon \to 0^+$. Let us now argue by contradiction and assume that (i) or (ii) or (iii) does not hold. Since the sequences (x_ϵ) and (y_ϵ) are bounded, one can then find a sequence $\epsilon_n \to 0^+$ and $x \in \overline{\mathcal{O}}$ such that (i) or (ii) or (iii) does not hold along this sequence and such that $x_{\epsilon_n} \to x$ and $y_{\epsilon_n} \to x$. We have

$$M \leq \lim M_{\epsilon_n} \leq \liminf[\mathbf{V}_1(x_{\epsilon_n}) - \mathbf{V}_2(y_{\epsilon_n})] \leq \limsup[\mathbf{V}_1(x_{\epsilon_n}) - \mathbf{V}_2(y_{\epsilon_n})] \leq \mathbf{V}_1(x) - \mathbf{V}_2(x) \leq M \tag{3.38}$$

since $\mathbf{V}_1$ is u.s.c. and $\mathbf{V}_2$ is l.s.c.. Hence M_{ϵ_n} converges to M and x is a maximum point of $\mathbf{V}_1 - \mathbf{V}_2$. In particular, $x \in \mathcal{O}$ because $\mathbf{V}_1 \leq \mathbf{V}_2$ on $\partial\mathcal{O}$ while $(\mathbf{V}_1 - \mathbf{V}_2)(x) = M > 0$. So $(x_{\epsilon_n}, y_{\epsilon_n}) \in \mathcal{O} \times \mathcal{O}$ for n sufficiently large. Moreover (3.38) also implies that $\mathbf{V}_1(x_{\epsilon_n}) - \mathbf{V}_2(y_{\epsilon_n})$ converges to $\mathbf{V}_1(x) - \mathbf{V}_2(x) = M$, so that $\frac{1}{\epsilon_n}|x_{\epsilon_n} - y_{\epsilon_n}|^2 = \mathbf{V}_1(x_{\epsilon_n}) - \mathbf{V}_2(y_{\epsilon_n}) - M_{\epsilon_n}$ converges to 0. Therefore we have found a contradiction and (i), (ii) and (iii) holds. □

3.2.6 Comparison principle for stationary equations in unbounded domains

Next we investigate the comparison principle for first order Hamilton-Jacobi equations of the form:

$$H(x, W(x), DW(x)) = 0 \qquad \text{in } \mathbb{R}^N \tag{3.39}$$

where $H : \mathcal{O} \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ satisfies assumptions (3.33), (3.34) and

$$|H(x, r, p) - H(x, r, q)| \leq C|p - q| \tag{3.40}$$

for some constant $C > 0$.

Theorem 3.36 *Let $\mathbf{V}_1$ be a bounded subsolution of (3.39) and $\mathbf{V}_2$ a bounded supersolution of (3.39). Then $V_1 \leq V_2$ dans $\mathbb{R}^N$.*

Proof : The main difficulty now comes from the unboundness of $\mathbb{R}^N$. To overcome this problem, we are going to introduce a penalization term at infinity. As before we argue by contradiction by assuming that

$$M := \sup_{x \in \mathbb{R}^N} (\mathbf{V}_1(x) - \mathbf{V}_2(x)) \; > \; 0 \, .$$

For $\alpha > 0$ let us set

$$M_\alpha = \sup_{x \in \mathbb{R}^N} (\mathbf{V}_1(x) - \mathbf{V}_2(x) - \alpha |x|^2) \, .$$

Note that $M_\alpha \to M$ as $\alpha \to 0^+$ (see Exercice 3.1). Let $\epsilon > 0$ and

$$W_{\alpha,\epsilon}(x, y) := \mathbf{V}_1(x) - \mathbf{V}_2(y) - \frac{|x - y|^2}{\epsilon} - \frac{\alpha}{2}(|x|^2 + |y|^2) \, .$$

Since $\mathbf{V}_1$ and $-\mathbf{V}_2$ are u.s.c. and bounded, $W_{\alpha,\epsilon}$ is u.s.c. and coercive, i.e., $\lim_{|(x,y)| \to +\infty} W_{\alpha,\epsilon}(x, y) = -\infty$. So $W_{\alpha,\epsilon}$ has a maximum point $(x_{\alpha,\epsilon}, y_{\alpha,\epsilon})$ and we set

$$M_{\alpha,\epsilon} := \max_{(x,y) \in \mathbb{R}^{2N}} W_{\alpha,\epsilon}(x, y) = W_{\alpha,\epsilon}(x_{\alpha,\epsilon}, y_{\alpha,\epsilon}) \, .$$

Let us collect some estimates on $(x_{\alpha,\epsilon}, y_{\alpha,\epsilon})$:

Lemma 3.37 *(i)* $\lim_{\epsilon \to 0^+} M_{\alpha,\epsilon} = M_\alpha$,

(ii) for any $\alpha > 0$, $\lim_{\epsilon \to 0^+} \frac{1}{\epsilon}|x_{\alpha,\epsilon} - y_{\alpha,\epsilon}|^2 = 0$

(iii) There is a constant $C > 0$ such that $\alpha(|x_{\alpha,\epsilon}| + |y_{\alpha,\epsilon}|) \leq C\sqrt{\alpha}$

We now complete the proof of Theorem 3.36. Since $W_{\alpha,\epsilon}$ has a maximum at $(x_{\alpha,\epsilon}, y_{\alpha,\epsilon})$, the map $x \to W_{\alpha,\epsilon}(x, y_{\alpha,\epsilon})$ has a maximum point at $x_{\alpha,\epsilon}$. Hence, for any $x \in \mathbb{R}^N$ we have

$$\mathbf{V}_1(x) \leq \mathbf{V}_1(x_{\alpha,\epsilon}) + \frac{1}{\epsilon}\left[|x - y_{\alpha,\epsilon}|^2 - |x_{\alpha,\epsilon} - y_{\alpha,\epsilon}|^2\right] + \frac{\alpha}{2}\left[|x|^2 - |x_{\alpha,\epsilon}|^2\right]$$

Hence, if we denote by $\phi(x)$ the right-hand side of the above inequality, we have that ϕ is smooth and $\mathbf{V}_1 - \phi$ has a maximum at $x_{\alpha,\epsilon}$ because $\phi(x_{\alpha,\epsilon}) = \mathbf{V}_1(x_{\alpha,\epsilon})$. Since $\mathbf{V}_1$ is a subsolution of (3.39) this implies that

$$H\left(x_{\alpha,\epsilon}, \mathbf{V}_1(x_{\alpha,\epsilon}), D\phi(x_{\alpha,\epsilon})\right) = H\left(x_{\alpha,\epsilon}, \mathbf{V}_1(x_{\alpha,\epsilon}), \frac{x_{\alpha,\epsilon} - y_{\alpha,\epsilon}}{\epsilon} + \alpha x_{\alpha,\epsilon}\right) \geq 0 \, . \tag{3.41}$$

In the same way, the map $y \to W_{\alpha,\epsilon}(x_{\alpha,\epsilon}, y)$ has a maximum at $y_{\alpha,\epsilon}$, and therefore, since $\mathbf{V}_2$ is a supersolution,

$$H\left(y_{\alpha,\epsilon}, \mathbf{V}_2(y_{\alpha,\epsilon}), \frac{x_{\alpha,\epsilon} - y_{\alpha,\epsilon}}{\epsilon} - \alpha y_{\alpha,\epsilon}\right) \leq 0 \, . \tag{3.42}$$

Computing the difference between (3.41) and (3.42) and using first assumption (3.40), and then assumption (3.33) with $\mathbf{V}_1(x_\epsilon) \geq \mathbf{V}_2(y_\epsilon)$ and finally assumption (3.34), we obtain

$$\begin{aligned}
0 \leq\ & H\left(x_{\alpha,\epsilon}, \mathbf{V}_1(x_{\alpha,\epsilon}), \frac{x_{\alpha,\epsilon} - y_{\alpha,\epsilon}}{\epsilon} + \alpha x_{\alpha,\epsilon}\right) - H\left(y_{\alpha,\epsilon}, \mathbf{V}_2(y_{\alpha,\epsilon}), \frac{x_{\alpha,\epsilon} - y_{\alpha,\epsilon}}{\epsilon} - \alpha y_{\alpha,\epsilon}\right) \\
\leq\ & H\left(x_{\alpha,\epsilon}, \mathbf{V}_1(x_{\alpha,\epsilon}), \frac{x_{\alpha,\epsilon} - y_{\alpha,\epsilon}}{\epsilon}\right) - H\left(y_{\alpha,\epsilon}, \mathbf{V}_2(y_{\alpha,\epsilon}), \frac{x_{\alpha,\epsilon} - y_{\alpha,\epsilon}}{\epsilon}\right) + C\alpha(|x_{\alpha,\epsilon}| + |y_{\alpha,\epsilon}|) \\
\leq\ & -\gamma(\mathbf{V}_1(x_{\alpha,\epsilon}) - \mathbf{V}_2(y_{\alpha,\epsilon})) + H\left(x_{\alpha,\epsilon}, \mathbf{V}_2(y_{\alpha,\epsilon}), \frac{x_{\alpha,\epsilon} - y_{\alpha,\epsilon}}{\epsilon}\right) - H\left(y_{\alpha,\epsilon}, \mathbf{V}_2(y_{\alpha,\epsilon}), \frac{x_{\alpha,\epsilon} - y_{\alpha,\epsilon}}{\epsilon}\right) \\
& +C\alpha(|x_{\alpha,\epsilon}| + |y_{\alpha,\epsilon}|) \\
\leq\ & -\gamma M_{\alpha,\epsilon} - C\left(1 + \frac{|x_{\alpha,\epsilon} - y_{\alpha,\epsilon}|}{\epsilon}\right)|x_{\alpha,\epsilon} - y_{\alpha,\epsilon}| + C\alpha(|x_{\alpha,\epsilon}| + |y_{\alpha,\epsilon}|)
\end{aligned}$$

When we let $\epsilon \to 0^+$ and then $\alpha \to 0^+$ we get $0 \leq -\gamma M$ thanks to the Lemma. This is impossible. □

Proof of Lemma 3.37 : We already know that $M_{\alpha,\epsilon} \geq M_\alpha$. Let K be a bound for $\mathbf{V}_1$ and $\mathbf{V}_2$. Then, since $M_\alpha \to M > 0$, for $\alpha > 0$ small enough we have

$$\begin{aligned}
0 \leq M_\alpha \leq M_{\alpha,\epsilon} &= \mathbf{V}_1(x_{\alpha,\epsilon}) - \mathbf{V}_2(y_{\alpha,\epsilon}) - \frac{1}{2\epsilon}|x_{\alpha,\epsilon} - y_{\alpha,\epsilon}|^2 - \frac{\alpha}{2}(|x_{\alpha,\epsilon}|^2 + |y_{\alpha,\epsilon}|^2) \\
&\leq 2K - \frac{1}{2\epsilon}|x_{\alpha,\epsilon} - y_{\alpha,\epsilon}|^2 - \frac{\alpha}{2}(|x_{\alpha,\epsilon}|^2 + |y_{\alpha,\epsilon}|^2) .
\end{aligned}$$

This proves that (iii) holds and that $\frac{1}{\epsilon}|x_{\alpha,\epsilon} - y_{\alpha,\epsilon}|^2$ is bounded independently of α. So $x_{\alpha,\epsilon} - y_{\alpha,\epsilon}$ tends to 0 as $\epsilon \to 0^+$. Let us fix α and note that $(x_{\alpha,\epsilon})$ and $(y_{\alpha,\epsilon})$ are bounded thanks to (iii). Let x_α be a cluster point of $x_{\alpha,\epsilon}$ and $y_{\alpha,\epsilon}$ as $\epsilon \to 0$ and let $\epsilon_n \to 0^+$ be such that $x_{\alpha,\epsilon_n} \to x_\alpha$ and $y_{\alpha,\epsilon_n} \to y_\alpha$.

We have

$$\begin{aligned}
M_\alpha \leq \lim M_{\alpha,\epsilon_n} &\leq \liminf[\mathbf{V}_1(x_{\alpha,\epsilon_n}) - \mathbf{V}_2(y_{\alpha,\epsilon_n}) - \frac{\alpha}{2}(|x_{\alpha,\epsilon_n}|^2 + |y_{\alpha,\epsilon_n}|^2)] \\
&\leq \limsup[\mathbf{V}_1(x_{\alpha,\epsilon_n}) - \mathbf{V}_2(y_{\alpha,\epsilon_n}) - \frac{\alpha}{2}(|x_{\alpha,\epsilon_n}|^2 + |y_{\alpha,\epsilon_n}|^2)] \\
&\leq \mathbf{V}_1(x_\alpha) - \mathbf{V}_2(x_\alpha) - \alpha|x_\alpha|^2 \leq M_\alpha .
\end{aligned}$$

In particular M_{α,ϵ_n} converges to M_α as $\epsilon \to 0$. Moreover these inequalities also show that

$$\lim \mathbf{V}_1(x_{\alpha,\epsilon_n}) - \mathbf{V}_2(y_{\alpha,\epsilon_n}) - \frac{\alpha}{2}(|x_{\alpha,\epsilon_n}|^2 + |y_{\alpha,\epsilon_n}|^2) = \mathbf{V}_1(x_\alpha) - \mathbf{V}_2(x_\alpha) - \alpha|x_\alpha|^2 = M_\alpha ,$$

so that

$$\lim \frac{1}{\epsilon}|x_{\alpha,\epsilon_n} - y_{\alpha,\epsilon_n}|^2 = \lim \mathbf{V}_1(x_{\alpha,\epsilon_n}) - \mathbf{V}_2(y_{\alpha,\epsilon_n}) - \frac{\alpha}{2}(|x_{\alpha,\epsilon_n}|^2 + |y_{\alpha,\epsilon_n}|^2) - M_{\alpha,\epsilon_n} = 0 .$$

Since this holds true for any subsequence $\epsilon_n \to 0$ such that the bounded sequences (x_{ϵ_n}) and (y_{ϵ_n}) converge, a compactness argument allows to conclude that (i) and (ii) hold. □

3.2.7 Comparison principle for evolution equations in unbounded domains

Finally we turn to first order Hamilton-Jacobi evolution equations of the form:

(3.43) $$\partial_t W(t,x) + H(t,x,DW(t,x)) = 0 \qquad \text{in } [0,T) \times \mathbb{R}^N$$

where $T > 0$ is a fixed horizon and $H : [0,T] \times \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ satisfies the following conditions :

(3.44) $$|H(t_1,x_1,p) - H(t_2,x_2,p)| \leq C(1+|p|)|(t_1,x_1) - (t_2,x_2)|$$

and

(3.45) $$|H(t,x,p_1) - H(t,x,p_2)| \leq C|p_1 - p_2|$$

for some constant C.

Theorem 3.38 (Comparison principle) *Let $\mathbf{V}_1$ be a subsolution of (3.43) which is u.s.c. on $[0,T] \times \mathbb{R}^N$ and $\mathbf{V}_2$ be a supersolution of (3.43) which is l.s.c. on $[0,T] \times \mathbb{R}^N$. Let us assume that $\mathbf{V}_1(T,x) \leq \mathbf{V}_2(T,x)$ for any $x \in \mathbb{R}^N$. Then*

$$\mathbf{V}_1(t,x) \leq \mathbf{V}_2(t,x) \qquad \forall (t,x) \in [0,T] \times \mathbb{R}^N .$$

Remark 3.39 Note that there is no growth assumption on $\mathbf{V}_1$ nor on $\mathbf{V}_2$. The key point is that, thanks to condition (3.45) one can actually restrict $\mathbf{V}_1$ and $\mathbf{V}_2$ to bounded domains and still preserve the property of being sub- and supersolutions. This is the aim of the next Lemma.

Lemma 3.40 *Assume that H is continuous and satisfies (3.45). If W is u.s.c. on $[0,T]\times\mathbb{R}^N$ and a subsolution of (3.43) on $(0,T)\times\mathbb{R}^N$ (resp. is l.s.c. on $[0,T]\times\mathbb{R}^N$ and a supersolution of (3.43) on $(0,T)\times\mathbb{R}^N$), then, for any $(t_0,x_0)\in[0,T)\times\mathbb{R}^N$, W is still a subsolution (resp. supersolution) in the cone*

$$C_{t_0,x_0} = \{(t,x)\in[t_0,T]\times\mathbb{R}^N\ ,\ |x-x_0|\le C(t-t_0)\}\ , \tag{3.46}$$

i.e., if a $\mathcal{C}^1$ test function ϕ is such that $W-\phi$ has a local maximum (resp. minimum) on C_{t_0,x_0} at some point (t,x) with $t<T$, then

$$\partial_t\phi(t,x) + H(t,x,D\phi(t,x)) \ge 0 \qquad \text{(resp. } \le 0)\ .$$

Postponing the proof of Lemma 3.40, let us start the proof of Theorem 3.38. Note that it is enough to show that, for any $\sigma>0$, we have

$$\mathbf{V}_1(t,x)-\mathbf{V}_2(t,x)-\sigma(T-t)\ \le\ 0 \qquad \forall(t,x)\in[0,T]\times\mathbb{R}^N\ .$$

Let us argue by contradiction and assume that this does not hold. Then there is some $\sigma>0$ and (t_0,x_0) such that

$$M := \sup_{(t,x)\in C_{t_0,x_0}} \mathbf{V}_1(t,x)-\mathbf{V}_2(t,x)-\sigma(T-t)\ >\ 0\ ,$$

where C_{t_0,x_0} is defined by (3.46). From now on we fix such a σ and (t_0,x_0). We now use the doubling variable technique: for $\epsilon>0$ we set

$$\Phi_\epsilon((t,x),(s,y)) = \mathbf{V}_1(t,x)-\mathbf{V}_2(s,y)-\frac{1}{2\epsilon}|(s,y)-(t,x)|^2-\sigma(T-s) \qquad \forall(t,x),(s,y)\in C_{t_0,x_0}$$

and consider the problem

$$M_\epsilon := \sup_{(t,x),(s,y)\in C_{t_0,x_0}} \Phi_\epsilon((t,x),(s,y))\ .$$

Note that $M_\epsilon\ge M$. Since $\mathbf{V}_1$ and $-\mathbf{V}_2$ are u.s.c. in C_{t_0,x_0}, so is the map Φ_ϵ. Since the set C_{t_0,x_0} is compact, the above problem has a maximum point $((t_\epsilon,x_\epsilon),(s_\epsilon,y_\epsilon))$. Next we collect some estimates on $(t_\epsilon,x_\epsilon),(s_\epsilon,y_\epsilon)$.

Lemma 3.41 *(i)* $\lim_{\epsilon\to0^+} M_\epsilon = M$,

(ii) $\lim_{\epsilon\to0^+}\frac{1}{\epsilon}|(t_\epsilon,x_\epsilon)-(s_\epsilon,y_\epsilon)|^2=0$

(iii) for $\epsilon>0$ small enough, $t_\epsilon<T$ and $s_\epsilon<T$.

We are now ready to complete the proof of Theorem 3.38. Since the map $(t,x)\to\Phi_\epsilon((t,x),(s_\epsilon,y_\epsilon))$ has a maximum at the point (t_ϵ,x_ϵ) on C_{t_0,x_0}, we have, for any $(t,x)\in C_{t_0,x_0}$,

$$\mathbf{V}_1(t,x)\le\mathbf{V}_1(t_\epsilon,x_\epsilon)+\frac{1}{2\epsilon}\left[|(s_\epsilon,y_\epsilon)-(t,x)|^2-|(s_\epsilon,y_\epsilon)-(t_\epsilon,x_\epsilon)|^2\right]\ .$$

Let us denote by $\phi(t,x)$ the right-hand side of the above inequality. Then ϕ is a smooth function which coincides with $\mathbf{V}_1$ at (t_ϵ,x_ϵ). Therefore $\mathbf{V}_1-\phi$ has a maximum at the point (t_ϵ,x_ϵ) on C_{t_0,x_0}. Since $\mathbf{V}_1$ is a subsolution of (3.43) and $t_\epsilon<T$, Lemma 3.40 implies that

$$\partial_t\phi(t_\epsilon,x_\epsilon)+H(t_\epsilon,x_\epsilon,D\phi(t_\epsilon,x_\epsilon))\ge0\ ,$$

i.e.,

$$\frac{t_\epsilon-s_\epsilon}{\epsilon}+H\left(t_\epsilon,x_\epsilon,\frac{x_\epsilon-y_\epsilon}{\epsilon}\right)\ge0\ . \tag{3.47}$$

In a symmetric way, since the map $(s,y)\to\Phi_\epsilon((t_\epsilon,x_\epsilon),(s,y))$ has a maximum at (s_ϵ,y_ϵ) on C_{t_0,x_0}, one has, for any $(s,y)\in C_{t_0,x_0}$,

$$\mathbf{V}_2(s,y)\ge\mathbf{V}_2(s_\epsilon,y_\epsilon)-\frac{1}{2\epsilon}\left[|(s,y)-(t_\epsilon,x_\epsilon)|^2-|(s_\epsilon,y_\epsilon)-(t_\epsilon,x_\epsilon)|^2\right]+\sigma(s-s_\epsilon)$$

and, since $\mathbf{V}_2$ is a supersolution of (3.43) we obtain, again thanks to Lemma 3.40,

$$\frac{t_\epsilon - s_\epsilon}{\epsilon} + \sigma + H\left(s_\epsilon, y_\epsilon, \frac{x_\epsilon - y_\epsilon}{\epsilon}\right) \leq 0 \ . \tag{3.48}$$

Computing the difference between (3.47) and (3.48) gives

$$-\sigma + H\left(t_\epsilon, x_\epsilon, \frac{x_\epsilon - y_\epsilon}{\epsilon}\right) - H\left(s_\epsilon, y_\epsilon, \frac{x_\epsilon - y_\epsilon}{\epsilon}\right) \geq 0 \ .$$

We now use assumption (3.44) on H:

$$-\sigma + C\left[1 + \frac{|x_\epsilon - y_\epsilon|}{\epsilon}\right] |x_\epsilon - y_\epsilon| \geq 0 \ .$$

Letting finally $\epsilon \to 0^+$ and using Lemma 3.41 we get a contradiction since σ is positive. □

Proof of Lemma 3.41: We already know that $M_\epsilon \geq M$. Let K be an upper bound for $\mathbf{V}_1 - \mathbf{V}_2$ on C_{t_0,x_0}. Then

$$\begin{array}{rcl} 0 \ \leq \ M \ \leq \ M_\epsilon & = & \mathbf{V}_1(t_\epsilon, x_\epsilon) - \mathbf{V}_2(s_\epsilon, y_\epsilon) - \frac{1}{2\epsilon}|(t_\epsilon, x_\epsilon) - (s_\epsilon, y_\epsilon)|^2 - \sigma(T - s_\epsilon) \\ & \leq & K - \frac{1}{2\epsilon}|(t_\epsilon, x_\epsilon) - (s_\epsilon, y_\epsilon)|^2 \ . \end{array}$$

This proves that $\frac{1}{\epsilon}|(t_\epsilon, x_\epsilon) - (s_\epsilon, y_\epsilon)|^2$ is bounded and therefore that the difference $(t_\epsilon, x_\epsilon) - (s_\epsilon, y_\epsilon)$ tends to 0 as $\epsilon \to 0^+$. Let (t,x) be a cluster point of the bounded sequences (t_ϵ, x_ϵ) and (s_ϵ, y_ϵ) as $\epsilon \to 0$ and $\epsilon_n \to 0^+$ with $(t_{\epsilon_n}, x_{\epsilon_n}) \to (t,x)$ and $(t_{\epsilon_n}, x_{\epsilon_n}) \to (t,x)$. We have

$$\begin{array}{rcl} M \leq \lim M_{\epsilon_n} & \leq & \liminf[\mathbf{V}_1(t_{\epsilon_n}, x_{\epsilon_n}) - \mathbf{V}_2(s_{\epsilon_n}, y_{\epsilon_n}) - \sigma(T - s_{\epsilon_n})] \\ & \leq & \limsup[\mathbf{V}_1(t_{\epsilon_n}, x_{\epsilon_n}) - \mathbf{V}_2(s_{\epsilon_n}, y_{\epsilon_n}) - \sigma(T - s_{\epsilon_n})] \\ & \leq & \mathbf{V}_1(t,x) - \mathbf{V}_2(t,x) - \sigma(T - t) \leq M \ . \end{array}$$

Hence $M_{\epsilon_n} \to M$ holds and

$$\lim_{\epsilon \to 0^+} \mathbf{V}_1(t_{\epsilon_n}, x_{\epsilon_n}) - \mathbf{V}_2(s_{\epsilon_n}, y_{\epsilon_n}) - \sigma(T - s_{\epsilon_n}) = \mathbf{V}_1(t,x) - \mathbf{V}_2(t,x) - \sigma(T-t) = M \ ,$$

so that

$$\lim_{\epsilon \to 0^+} \frac{1}{2\epsilon}|(t_{\epsilon_n}, x_{\epsilon_n}) - (s_{\epsilon_n}, y_{\epsilon_n})|^2 = \lim_{\epsilon \to 0^+} \mathbf{V}_1(t_{\epsilon_n}, x_{\epsilon_n}) - \mathbf{V}_2(s_{\epsilon_n}, y_{\epsilon_n}) - \sigma(T - s_{\epsilon_n}) - M_{\epsilon_n} \ = \ 0 \ .$$

Let us finally point out that $t < T$. Indeed, if we had $t = T$, then

$$M \leq \mathbf{V}_1(T,x) - \mathbf{V}_2(T,x) \leq 0 \ ,$$

which is impossible since $M > 0$. So $t_{\epsilon_n} < T$ and $s_n < T$ for any n large enough. We can then complete the proof by a compactness argument. □

Proof of Lemma 3.40 : Thanks to Proposition 3.20 it is enough to do the proof for subsolutions. Let ϕ be such that $W - \phi$ has a strict local maximum on C_{t_0,x_0} at some point (t,x). For $\sigma > 0$ let us consider a maximum point (s_σ, y_σ) of the map

$$(s,y) \to \Phi_\sigma(s,y) := W(s,y) - \phi(s,y) + \frac{\sigma}{2}\ln\left(C^2(t-t_0)^2 - |x - x_0|^2\right)$$

on C_{t_0,x_0}. Note that this maximum point exists because C_{t_0,x_0} is compact and $\Phi_\sigma(s,y) \to -\infty$ as (s,y) converges to some point at the boundary of C_{t_0,x_0}.

Standard arguments then show that (s_σ, y_σ) converges to (t,x). In particular we have $s_\sigma < T$ for σ small enough because $t < T$. Since W is a subsolution of (3.43), we have

$$\partial_t\phi(t,x) - \frac{\sigma C^2(s_\sigma - t_0)}{A_\sigma} + H\left(s_\sigma, y_\sigma, D\phi(s_\sigma, y_\sigma) + \frac{\sigma(y_\sigma - x_0)}{A_\sigma}\right) \ \geq \ 0 \ ,$$

where we have set $A_\sigma = C^2(t_\sigma - t_0)^2 - |x_\sigma - x_0|^2$. Using condition (3.45) on H we get

$$\partial_t\phi(t,x) - \frac{\sigma C}{A_\sigma}\left[C(s_\sigma - t_0) - |y_\sigma - x_0|\right] + H\left(s_\sigma, y_\sigma, D\phi(s_\sigma, y_\sigma)\right) \ \geq \ 0 \ .$$

Since $A_\sigma > 0$ and $C(s_\sigma - t_0) - |y_\sigma - x_0| > 0$, this implies that

$$\partial_t\phi(t,x) + H\left(s_\sigma, y_\sigma, D\phi(s_\sigma, y_\sigma)\right) \ \geq \ 0 \ ,$$

and we obtain the desired inequality by letting $\sigma \to 0^+$. □

3.3 Further properties of the value function of Bolza problem

3.3.1 Explicit solutions

In this section we aim at giving a representation formula for the value of our game when g is convex and when ℓ and f are independent of (t,x). In this case the Hamiltonian $H = H(p)$ only depends on the gradient variable.

Let us recall that the value function $\mathbf{V}$ is the unique viscosity solution of the Hamilton-Jacobi-Isaacs' equation

$$\left\{ \begin{array}{ll} \partial_t \mathbf{V}(t,x) + H(D\mathbf{V}(t,x)) = 0 & \text{in } (0,T)\times\mathbb{R}^N \\ W(T,x) = g(x) & \text{in } \mathbb{R}^N \end{array} \right. \tag{3.49}$$

In particular, if the terminal condition g is linear: $g(x) = \langle a, x\rangle$ where $a \in \mathbb{R}^N$, then there is an obvious solution to (3.49): $\mathbf{V}(t,x) = (T-t)H(a) + \langle a, x\rangle$. Hopf-Lax formula provides a surprising generalization of that remark.

Proposition 3.42 (Hopf-Lax representation formula) *If g is convex and super-linear, i.e.,*

$$\lim_{\|x\|\to+\infty} \frac{g(x)}{\|x\|} = +\infty ,$$

and $H : \mathbb{R}^N \to \mathbb{R}$ satisfies condition (3.45), then the solution to (3.49) is given by the formula

$$\mathbf{V}(t,x) = \left(g^*(q) - (T-t)H(q)\right)^*(x)$$

where ϕ^* is the Fenchel conjugate of the function ϕ:

$$\phi^*(x) = \sup_{q\in\mathbb{R}^N} \langle q, x\rangle - \phi(q) \qquad \forall x \in \mathbb{R}^N .$$

Remark 3.43 In particular the solution $\mathbf{V}$ is convex, which is not obvious *a priori.*

Proof of the Proposition: Let us set $\mathbf{V}(t,x) = \left(g^*(q) - (T-t)H(q)\right)^*(x)$ and let us check that $\mathbf{V}$ is a continuous solution of Isaacs' equation (3.49) and that $\mathbf{V}(T,x) = g(x)$ for all $x \in \mathbb{R}^N$. The continuity of $\mathbf{V}$ is just a consequence of the super-linearity of g and is left to the reader. Since g is convex and l.s.c. we have

$$\mathbf{V}(T,x) = g^{**}(x) = g(x) ,$$

so that $\mathbf{V}$ satisfies the terminal condition.

Let us now show that $\mathbf{V}$ is a subsolution (this is the easy part). For any $q \in \mathbb{R}^N$, let us set

$$\mathbf{V}_q(t,x) = \langle q, x\rangle - g^*(q) + (T-t)H(q) \qquad \forall (t,x) \in [0,T]\times\mathbb{R}^N .$$

One easily checks that, for any $q \in \mathbb{R}^N$, $\mathbf{V}_q$ is a $\mathcal{C}^1$ solution of Isaacs' equation (3.49). Since $\mathbf{V} = \sup_q \mathbf{V}_q$ by definition and since the supremum of subsolutions is still a subsolution (from Proposition 3.33), $\mathbf{V}$ is also a subsolution.

The fact that $\mathbf{V}$ is also a supersolution is a small miracle. Indeed there is no reason in general that the supremum of solutions is a solution (and this is in fact generally false). Here again convexity plays a key role.

Let ϕ be a $\mathcal{C}^1$ test function such that $\mathbf{V} - \phi$ has a local minimum at some point $(t_0, x_0) \in (0,T)\times\mathbb{R}^N$. We have to show that $\partial_t\phi(_0, x_0) + H(\phi_x(t_0,x_0)) \leq 0$. Let us fix $\tau \geq 0$ and $v \in \mathbb{R}^N$. For $h > 0$, we set $(t_h, x_h) = (t_0, x_0) + h(\tau, v)$. By definition of $\mathbf{V}$ there is some $q_h \in \mathbb{R}^N$ such that $\mathbf{V}(t_h,x_h) = \mathbf{V}_{q_h}(t_h,x_h)$. Moreover, since g is super-linear, the family q_h is bounded as h tends to 0^+. So there is a subsequence $h_n \to 0^+$ such that q_{h_n} converges to some $q \in \mathbb{R}^N$. From standard arguments we have $\mathbf{V}(t_0,x_0) = \mathbf{V}_q(t_0,x_0)$. Let us now set to simplify the notations: $q_{h_n} = q_n$, etc...

Since $\mathbf{V} - \phi$ has a local minimum at (t_0,x_0), we have, for n large enough,

$$\mathbf{V}(t_n,x_n) - \phi(t_n,x_n) \geq \mathbf{V}(t_0,x_0) - \phi(t_0,x_0) .$$

By definition of $\mathbf{V}$, we have $\mathbf{V}(t_0, x_0) \geq \mathbf{V}_{q_n}(t_0, x_0)$ and by definition of q_n, we also have $\mathbf{V}(t_n, x_n) = \mathbf{V}_{q_n}(t_n, x_n)$. Hence

$$\mathbf{V}_{q_n}(t_n, x_n) - \mathbf{V}_{q_n}(t_0, x_0) \geq \phi(t_n, x_n) - \phi(t_0, x_0) ,$$

which can be rewritten as

$$h_n(\langle q_n, v\rangle - \tau H(q_n)) \geq \phi(t_0 + h_n\tau, x_0 + h_n v) - \phi(t_0, x_0)$$

Dividing this inequality by $h_n > 0$ and letting n tend to $+\infty$ gives

$$\langle q, v\rangle - \tau H(q) \geq \partial_t\phi(t_0, x_0)\tau + \langle \phi_x(t_0, x_0), v\rangle .$$

Let Q be the set of vectors $q \in \mathbb{R}^N$ such that $\mathbf{V}(t_0, x_0) = \mathbf{V}_q(t_0, x_0)$. Let us recall that Q is compact. We have just proved that

$$\min_{\tau\in[0,1],\ v\in B(0,1)} \max_{q\in Q} \left(\langle q, v\rangle - \tau H(q) - (\partial_t\phi\tau + \langle \phi_x, v\rangle)\right) \ \geq\ 0 \text{ at } (t_0, x_0) . \tag{3.50}$$

Let us assume for a while that one can exchange the min and the max in the above expression. Then the resulting inequality states that there is some $q \in \mathbb{R}^N$ such that, for any $\tau \in [0, 1]$ and $v \in B(0, 1)$,

$$\langle q, v\rangle - \tau H(q) \geq \partial_t\phi(t_0, x_0)\tau + \langle \phi_x(t_0, x_0), v\rangle .$$

Therefore

$$\partial_t\phi(t_0, x_0) \leq -H(q) \text{ and } \phi_x(t_0, x_0) = q .$$

Hence $\partial_t\phi + H(\phi_x) \leq 0$ and the proof is complete. The main problem is that the exchange of the min and the max is not allowed, first because Q is not convex and, second, because H is not concave.

Here is the miracle. Let $q_0 \in Q$. Since q_0 is optimal, we have $\mathbf{V}_q(t_0, x_0) \leq \mathbf{V}_{q_0}(t_0, x_0)$ for any $q \in \mathbb{R}^N$, i.e.,

$$\langle q, x_0\rangle - g^*(q) + (T - t_0)H(q) \leq \langle q_0, x_0\rangle - g^*(q_0) + (T - t_0)H(q_0) .$$

Hence

$$H(q) \leq \frac{1}{T - t_0}\left(\langle q_0 - q, x_0\rangle + g^*(q) - g^*(q_0) + (T - t_0)H(q_0)\right)$$

Let $r(q)$ denote the right-hand side of the above inequality. The map r is convex. Moreover the above inequality holds for any $q \in Q$ (by inverting the roles of q_0 and q). So

$$\forall q \in Q,\ r(q) = H(q) .$$

Rewriting inequality (3.50) by replacing H by r we get:

$$\min_{\tau\in[0,1],\ v\in B(0,1)} \max_{q\in Q} \left(\langle q, v\rangle - \tau r(q) - (\partial_t\phi\tau + \langle \phi_x, v\rangle)\right) \ \geq\ 0 .$$

Let $co(Q)$ be the convex envelope of Q. Since Q is compact, so is $co(Q)$. Since $Q \subset co(Q)$ the previous inequality implies that

$$\min_{\tau\in[0,1],\ v\in B(0,1)} \max_{q\in co(Q)} \left(\langle q, v\rangle - \tau r(q) - (\partial_t\phi\tau + \langle \phi_x, v\rangle)\right) \ \geq\ 0 .$$

Now the map r is convex, so that the above expression is concave with respect to q and convex with respect to (τ, v). From the min-max we get:

$$\max_{q\in co(Q)} \min_{\tau\in[0,1],\ v\in B(0,1)} \left(\langle q, v\rangle - \tau r(q) - (\partial_t\phi\tau + \langle \phi_x, v\rangle)\right) \ \geq\ 0 .$$

So there is some $q \in co(Q)$ such that, for any $\tau \in [0, 1]$ and any $v \in B(0, 1)$,

$$\langle q, v\rangle - \tau r(q) \geq \partial_t\phi(t_0, x_0)\tau + \langle \phi_x(t_0, x_0), v\rangle .$$

This inequality implies that

$$\partial_t\phi(t_0, x_0) \leq -r(q) \text{ and } \phi_x(t_0, x_0) = q .$$

But we already know that $-H(q) \geq -r(q)$. Hence

$$\partial_t\phi + H(\phi_x) \leq -H(q) + H(q) = 0 .$$

This proves that $\mathbf{V}$ is a supersolution and completes the proof by uniqueness of the solution. □

3.3.2 Long time average

3.4 The infinite horizon problem

In this section we study a two-player zerosum differential game in which the payoff is given by a discounted integral.

Dynamics: For a fixed initial position $x_0 \in \mathbb{R}^N$ we consider the differential equation

$$\left\{ \begin{array}{ll} X_t' = f(X_t, u_t, v_t) & t \in [0, +\infty) \\ X_0 = x_0 \end{array} \right. \tag{3.51}$$

Throughout the section, we assume that

$$\left\{ \begin{array}{rl} (i) & U \text{ and } V \text{ are compact metric spaces,} \\ (ii) & \text{the map } f : \mathbb{R}^N \times U \times V \text{ is bounded and continuous in all its variables} \\ (iii) & f \text{ is uniformly Lipschitz continuous with respect to the space variable:} \\ & |f(x,u,v) - f(y,u,v)| \le \mathrm{Lip}(f)|x-y| \qquad \forall (x,y,u,v) \in \mathbb{R}^N \times \mathbb{R}^N \times U \times V \end{array} \right. \tag{3.52}$$

The controls of Player 1 and Player 2 are now Lebesgue measurable maps $u : [0, +\infty) \to U$ and $v : [0, +\infty) \to V$. The set of such controls are simply denoted by $\mathcal{U}$ and $\mathcal{V}$, since the starting time is always $t_0 = 0$. For any pair $(u, v) \in \mathcal{U} \times \mathcal{V}$, equation (3.1) has a unique solution, denoted $X^{x_0,u,v}$.

Payoffs: The payoff of the players depends on a discount rate $\lambda > 0$ and on a running payoff $\ell : \mathbb{R}^N \times U \times V \to \mathbb{R}$. Namely, if the players play the controls $(u, v) \in \mathcal{U} \times \mathcal{V}$, then the cost the first Player is trying to minimize is given by

$$\mathcal{J}(x_0, u, v) = \int_0^{+\infty} e^{-\lambda s} \ell(X_s^{x_0,u,v}, u_s, v_s) ds \ .$$

Throughout this section we assume that

$$\ell \text{ is bounded and Lipschitz continuous.} \tag{3.53}$$

We denote by $\mathrm{Lip}(\ell)$ the Lipschitz constant of ℓ.

Strategies: A delay strategy for the first Player is a now map $\alpha : \mathcal{V} \to \mathcal{U}$ for which there is a delay $\tau > 0$ such that for any two controls $v_1, v_2 \in \mathcal{V}$ and for any $t \ge 0$, if $v_1 = v_2$ a.e. in $[0, t]$, then $\alpha(v_1) = \alpha(v_2)$ a.e. in $[0, t + \tau]$. The set of delay strategies for the first Player are denoted by $\mathcal{A}_d$, while the set of delay strategies for the second Player is denoted by $\mathcal{B}_d$. As before, we systematically use the fact that if $(\alpha, \beta) \in \mathcal{A}_d \times \mathcal{B}_d$ is a pair of strategies, then there is a unique pair of controls $(u, v) \in \mathcal{U} \times \mathcal{V}$ such that

$$\alpha(v) = u \text{ and } \beta(u) = v \qquad \text{a.e. in } [0, +\infty) \ . \tag{3.54}$$

In particular we always use the notation (α_s, β_s) for (u_s, v_s) and $X_t^{x_0,\alpha,\beta}$ for $X_t^{x_0,u,v}$, where (u_s, v_s) is defined by (3.54). The payoff associated to the two strategies $(\alpha, \beta) \in \mathcal{A}_d \times \mathcal{B}_d$ is given by

$$\mathcal{J}(x_0, \alpha, \beta) = \int_0^{+\infty} e^{-\lambda s} \ell(X_s^{x_0,\alpha,\beta}, \alpha_s, \beta_s) ds \ .$$

Definition 3.44 (Value functions) *The* **upper value function** *is given by*

$$\mathbf{V}^+(x_0) := \inf_{\alpha \in \mathcal{A}_d} \sup_{\beta \in \mathcal{B}_d} \mathcal{J}(x_0, \alpha, \beta) \tag{3.55}$$

while the **lower value function** *is*

$$\mathbf{V}^-(x_0) := \sup_{\beta \in \mathcal{B}_d} \inf_{\alpha \in \mathcal{A}_d} \mathcal{J}(x_0, \alpha, \beta) \ . \tag{3.56}$$

Remarks 3.45 1. Note that now the value function only depends on the space variable. Obviously, the following inequality always holds:

$$\mathbf{V}^-(x_0) \le \mathbf{V}^+(x_0) \qquad \forall x_0 \in \mathbb{R}^N \ .$$

So the key point is to prove the reverse one and to characterize the value $\mathbf{V}^+ = \mathbf{V}^-$.

2. As in Lemma 3.3, we have
$$\mathbf{V}^+(x_0) := \inf_{\alpha\in\mathcal{A}_d}\sup_{v\in\mathcal{V}} \mathcal{J}(x_0,\alpha(v),v)$$
and
$$\mathbf{V}^-(x_0) := \sup_{\beta\in\mathcal{B}_d}\inf_{u\in\mathcal{U}} \mathcal{J}(x_0,u,\beta(u))$$

3.4.1 Regularity of the value functions

Lemma 3.46 *The value functions $\mathbf{V}^+$ and $\mathbf{V}^-$ are bounded.*

Proof : Indeed, since ℓ is bounded,
$$|\mathcal{J}(x_0,u,v)| \le \int_0^{+\infty} e^{-\lambda s}|\ell(X_s^{x_0,\alpha,\beta},\alpha_s,\beta_s)|\,ds \le \int_0^{+\infty} e^{-\lambda s}\|\ell\|_\infty ds = \frac{\|\ell\|_\infty}{\lambda}\ ,$$
so that $\mathbf{V}^+$ and $\mathbf{V}^-$ are also bounded by $\|\ell\|_\infty/\lambda$. □

Lemma 3.47 *The value functions $\mathbf{V}^+$ and $\mathbf{V}^-$ are Hölder continuous in $\mathbb{R}^N$.*

Proof : Let us denote by L the Lipschitz constant for ℓ and f (with respect to the x variable) and let M be a bound for ℓ. Without loss of generality we can assume that $L > \lambda$. Let $x_0, x_1 \in \mathbb{R}^N$, $\alpha \in \mathcal{A}_d$ and $v \in \mathcal{V}$. We have, thanks to Gronwall Lemma
$$|X_t^{x_0,\alpha,v} - X^{x_1,\alpha,v}| \le |x_0 - x_1|e^{Lt} \qquad \forall t \ge 0\ .$$
Let us fix $T > 0$ to be chosen later. We now compare $\mathcal{J}(x_0,\alpha,v)$ and $\mathcal{J}(x_1,\alpha,v)$. Since ℓ is $L-$Lipschitz continuous with respect to the x variable and bounded by M we have
$$\begin{array}{rcl} |\mathcal{J}(x_0,\alpha,v) - \mathcal{J}(x_0,\alpha,v)| & \le & \int_0^{+\infty} e^{-\lambda s}|\ell(X_s^{x_0,\alpha,v},\alpha,v) - \ell(X_s^{x_1,\alpha,v},\alpha,v)|ds \\ & \le & L\int_0^T e^{-\lambda s}e^{-\lambda s}|X_s^{x_0,\alpha,v} - X_s^{x_1,\alpha,v}|ds + \int_T^{+\infty} e^{-\lambda s}2Mds \\ & \le & L\int_0^T e^{-\lambda s}|x_0 - x_1|e^{Ls}ds + 2Me^{-\lambda T}/M \\ & \le & |x_0 - x_1|\frac{L}{L-\lambda}(e^{(L-\lambda)T} - 1) + 2Me^{-\lambda T} \end{array}$$
We now optimize the above expression with respect to T. The choice $T = (1/L)\ln(2M/(L|x_0 - x_1|))$ is the best one, provided that $|x_0 - x_1|$ is not too large. With this choice we get
$$|\mathcal{J}(x_0,\alpha,v) - \mathcal{J}(x_1,\alpha,v)| \le C|x_0 - x_1|^{\lambda/L}\ ,$$
for some constant C independent of x_0, x_1 and of α and v. Since the above inequality holds true for any α and v, we get, thanks to Lemma 3.5,
$$|\mathbf{V}^+(x_0) - \mathbf{V}^+(x_1)| \le C|x_0 - x_1|^{\lambda/L}\ .$$
□

In the proof of the dynamic programming property we shall need the fact that a nearly optimal strategy at a point remains nearly optimal in a neighbourhood.

Lemma 3.48 *For any $\epsilon > 0$ there is some $\eta > 0$ with the following property: for any $x_0 \in \mathbb{R}^N$, any $\epsilon-$optimal strategy $\bar\alpha \in \mathcal{A}_d$ for $\mathbf{V}^+(x_0)$, i.e., such that*
$$\sup_{v\in\mathcal{V}} \mathcal{J}(x_0,\bar\alpha(v),v) \le \mathbf{V}^+(x_0) + \epsilon\ ,$$
remains $(2\epsilon)-$optimal for $\mathbf{V}^+(y_0)$ for any $y_0 \in B(x_0,\eta)$:
$$\sup_{v\in\mathcal{V}} \mathcal{J}(y_0,\bar\alpha(v),v) \le \mathbf{V}^+(y_0) + 2\epsilon \qquad \forall y_0 \in B(x_0,\eta)\ .$$

Proof : Let C be a Lipschitz constant of $\mathbf{V}^+$ with respect to x. As in the proof of Lemma 3.47, there is also a constant C' such that, for any $y_0 \in \mathbb{R}^N$, for any pair of controls $(u,v) \in \mathcal{U} \times \mathcal{V}$,

$$|\mathcal{J}(t_0, x_0, u, v) - \mathcal{J}(t_0, y_0, u, v)| \le C'|x_0 - y_0|^{\lambda/L} ,$$

where L is a Lipschitz constant for f and ℓ. Hence

$$\begin{array}{rcl} \sup\limits_{v \in \mathcal{V}(t_0)} \mathcal{J}(y_0, \bar{\alpha}(v), v) & \le & \sup\limits_{v \in \mathcal{V}(t_0)} \mathcal{J}(x_0, \bar{\alpha}(v), v) + C'|x_0 - y_0|^{\lambda/L} \\ & \le & \mathbf{V}^+(x_0) + \epsilon + C'|x_0 - y_0|^{\lambda/L} \\ & \le & \mathbf{V}^+(y_0) + \epsilon + (C' + C)|x_0 - y_0|^{\lambda/L} \end{array}$$

This proves that $\bar{\alpha}$ is $(2\epsilon)-$optimal for $\mathbf{V}^+(t_0, y_0)$ as soon as $|y_0 - x_0| \le \eta := (\epsilon/(C + C'))^{L/\lambda}$. □

3.4.2 Dynamic programming property

Theorem 3.49 (Dynamic programming property) *Let $x_0 \in \mathbb{R}^N$ and $h > 0$. Then*

$$\mathbf{V}^+(x_0) = \inf_{\alpha \in \mathcal{A}_d} \sup_{v \in \mathcal{V}} \left\{ \int_0^h e^{-\lambda s} \ell(X_s^{x_0, \alpha(v), v}, \alpha(v)_s, v_s) ds + e^{-\lambda h} \mathbf{V}^+(X_h^{x_0, \alpha(v), v}) \right\} .$$

Proof of Theorem 3.49 : Let us set

$$W(h, x_0) = \inf_{\alpha \in \mathcal{A}_d} \sup_{v \in \mathcal{V}} \left\{ \int_0^h \ell(X_s^{x_0, \alpha(v), v}, \alpha(v)_s, v_s) ds + e^{-\lambda h} \mathbf{V}^+(X_h^{x_0, \alpha(v), v}) \right\}$$

We first show inequality $\mathbf{V}^+ \le W$.
Let us fix some $\epsilon > 0$ and let α^0 be $\epsilon-$optimal for $W(h, x_0)$:

$$\sup_{v \in \mathcal{V}} \left\{ \int_0^h \ell(X_s^{x_0, \alpha^0(v), v}, \alpha_s^0, v_s) ds + e^{-\lambda h} \mathbf{V}^+(X_h^{x_0, \alpha^0(v), v}) \right\} \le W(h, x_0) + \epsilon . \quad (3.57)$$

For any $x \in \mathbb{R}^N$, let α^x be $\epsilon-$optimal for the game $\mathbf{V}^+(x)$. From Lemma 3.48, there is some $\eta > 0$ such that α^x is $(2\epsilon)-$optimal for $\mathbf{V}^+(y)$ for any $y \in B(x, \eta)$. Since $\mathbb{R}^N$ is locally compact, we can find a countable family $(x_i)_{i \in \mathbb{N}^*}$ such that the family of balls $(B(x_i, \eta/2))_{i \in \mathbb{N}^*}$ is a locally finite covering of $\mathbb{R}^N$: any point x belongs to some ball $B(x_i, \eta/2)$ and to at most a finite number of such balls. Let us set $\mathcal{O}_1 = B(x_1, \eta/2)$ and $\mathcal{O}_i = B(x_i, \eta/2) \setminus \bigcup_{j<i} B(x_j, \eta/2)$ for $i \ge 2$.

Let $\tau \in (0, \eta/(2\|f\|_\infty) \wedge h)$. We are now ready to define a new strategy by setting, for any $t \in [0, +\infty)$ and $v \in \mathcal{V}$,

$$\alpha(v)_t = \begin{cases} \alpha^0(v)_t & \text{if } t \in [0, h) \\ \alpha^{x_i}\left(v^h\right)_{t-h} & \text{if } t \in [h, +\infty) \text{ and } X_{h-\tau}^{x_0, \alpha^0(v), v} \in \mathcal{O}_i \end{cases}$$

where $v_t^h = v_{t-h}$ for $t \ge h$. As in the proof of Lemma 3.4 one can show that α is a delay strategy because $X_{h-\tau}^{x_0, \alpha^0(v), v}$ remains in a bounded set, and therefore in $\bigcup_{i=1}^k \mathcal{O}_i$ for some integer k. Next we claim that

$$\mathcal{J}(x_0, \alpha(v), v) \le W(h, x_0) + 3\epsilon \qquad \forall v \in \mathcal{V} .$$

Let us fix $v \in \mathcal{V}$ and set $X_s = X_s^{x_0, \alpha(v), v}$. We note that

$$X_s = \begin{cases} X_s^{x_0, \alpha^0(v), v} & \text{if } s \in [0, h] \\ X_{s-h}^{X_h, \alpha^{x_i}(v^h), v^h} & \text{if } s \in [h, +\infty) \text{ and } X_{h-\tau} \in \mathcal{O}_i \end{cases}$$

where $v_t^h = v_{t-h}$ for $t \ge h$. Then

$$\begin{array}{rcl} \mathcal{J}(x_0, \alpha(v), v) & = & \displaystyle\int_0^h e^{-\lambda s} \ell(X_s^{x_0, \alpha^0(v), v}, \alpha^0(v)_s, v_s) ds \\ & & \displaystyle + \sum_{i=1}^k \mathbf{1}_{\mathcal{O}_i}(X_{h-\tau}) \left\{ \int_h^{+\infty} e^{-\lambda s} \ell(X_{s-h}^{X_h, \alpha^{x_i}(v^h), v^h}, \alpha^{x_i}(v^h)_{s-h}, v_{s-h}^h) ds \right\} \end{array}$$

where

$$\int_h^{+\infty} e^{-\lambda s}\ell(X_{s-h}^{X_h,\alpha^{x_i}(v^h),v^h},\alpha^{x_i}(v^h)_{s-h},v^h_{s-h})\;ds = e^{-\lambda h}\int_0^{+\infty} e^{-\lambda s}\ell(X_s^{X_h,\alpha^{x_i}(v^h),v^h},\alpha^{x_i}(v^h)_s,v^h_s)\;ds\;.$$

Let us assume that $X_{h-\tau}$ belongs to $\mathcal{O}_i$. Then, from the definition of $\mathcal{O}_i$ and of τ, X_h belongs to the ball $B(x_i,\eta)$. Since α^{x_i} is $(2\epsilon)-$optimal for $\mathbf{V}^+(\cdot)$ on $\mathcal{O}_i$, we have therefore

$$\int_0^{+\infty} e^{-\lambda s}\ell(X_s^{X_h,\alpha^{x_i}(v^h),v^h},\alpha^{x_i}(v^h)_s,v^h_s)\;ds \leq \mathbf{V}^+(X_h)+2\epsilon\;.$$

Hence

$$\mathcal{J}(x_0,\alpha(v),v)\;\leq\;\int_0^h e^{-\lambda s}\ell(X_s^{x_0,\alpha^0(v),v},\alpha^0(v)_s,v_s)ds+e^{-\lambda h}\left(\mathbf{V}^+(X_h^{x_0,\alpha^0,v})+2\epsilon\right)$$

We now use the $\epsilon-$optimality of α^0 in $W(t_0,t_0+h,x_0)$ (i.e., (3.57)) to get

$$\mathcal{J}(x_0,\alpha(v),v)\;\leq\;W(h,x_0)+3\epsilon\;.$$

This inequality holds for any $v\in\mathcal{V}(t_0)$, so that

$$\mathbf{V}^+(x_0)\leq\sup_{v\in\mathcal{V}}\mathcal{J}(x_0,\alpha(v),v)\;\leq\;W(h,x_0)+3\epsilon\;.$$

Since ϵ is arbitrary, inequality $\mathbf{V}^+\leq W$ is proved.

Let us now show the reverse inequality: $W\leq\mathbf{V}^+$.
Let $\epsilon>0$ be fixed and let $\bar\alpha\in\mathcal{A}_d$ be $\epsilon-$optimal for $\mathbf{V}^+(x_0)$. Let us fix some $\bar v\in\mathcal{V}$ and define a new delay strategy $\alpha^{\bar v}\in\mathcal{A}_d$ by setting:

$$\alpha^{\bar v}(v)_t=\bar\alpha(\tilde v)_{t+h}\text{ where }\tilde v_s=\left\{\begin{array}{ll}\bar v_s & \text{if } s\in[0,h]\\ v_s & \text{otherwise}\end{array}\right.\qquad\forall t\geq 0,\;v\in\mathcal{V}\;.$$

Then, if we set $X_t=X_t^{x_0,\bar\alpha(\bar v),\bar v}$, we have

$$\mathbf{V}^+(X_h)\leq\sup_{v\in\mathcal{V}}\mathcal{J}(X_h,\alpha^{\bar v}(v),v)\;. \tag{3.58}$$

Let $\mathcal{V}(h,\bar v)$ be the set of controls $v\in\mathcal{V}$ such that $v=\bar v$ on $[0,h]$. Then, by definition of $\alpha^{\bar v}$, we have

$$\mathcal{J}(x_0,\bar\alpha(v),v)=\int_0^h\ell(X_s,\bar\alpha(\bar v)_s,\bar v_s)ds+e^{-\lambda h}\mathcal{J}(X_h,\alpha^{\bar v}(v_{\cdot-h}),v_{\cdot-h})\qquad\forall v\in\mathcal{V}(t_0,t_0+h,\bar v)\;.$$

Hence

$$\sup_{v\in\mathcal{V}(h,\bar v)}\mathcal{J}(x_0,\bar\alpha(v),v)=\int_0^h\ell(X_s,\bar\alpha(\bar v)_s,\bar v_s)ds+\sup_{v\in\mathcal{V}}\mathcal{J}(X_{t_0+h},\alpha^{\bar v}(v),v)\;.$$

Combining (3.58) with the above inequality then leads to

$$\begin{array}{rcl}\displaystyle\int_0^h\ell(X_s,\bar\alpha(\bar v)_s,\bar v_s)ds+\mathbf{V}^+(X_h) & \leq & \displaystyle\sup_{v\in\mathcal{V}(h,\bar v)}\mathcal{J}(x_0,\bar\alpha(v),v)\\ & \leq & \displaystyle\sup_{v\in\mathcal{V}}\mathcal{J}(x_0,\bar\alpha(v),v)\leq\mathbf{V}^+(x_0)+\epsilon\end{array}$$

since $\bar\alpha$ is $\epsilon-$optimal for $\mathbf{V}^+(x_0)$. Taking the supremum over $\bar v$ and using the definition of W then implies that $W(h,x_0)\leq\mathbf{V}^+(x_0)+\epsilon$. This gives the desired result since ϵ is arbitrary. □

3.4.3 Existence and characterization of the value

Together with our value functions let us associate two Hamilton-Jacobi equations:

$$-\lambda \mathbf{V}(t,x) + H^+(x, D\mathbf{V}(t,x)) = 0 \text{ in } \mathbb{R}^N \tag{3.59}$$

where H^+ is defined by

$$H^+(t,x,p) = \inf_{u\in U}\sup_{v\in V}\left\{\langle p, f(x,u,v)\rangle + \ell(x,u,v)\right\}\ , \tag{3.60}$$

and

$$-\lambda \mathbf{V}(t,x) + H^-(x, D\mathbf{V}(t,x)) = 0 \text{ in } \mathbb{R}^N \tag{3.61}$$

where H^- is defined by

$$H^-(x,p) = \sup_{v\in V}\inf_{u\in U}\left\{\langle p, f(x,u,v)\rangle + \ell(x,u,v)\right\}\ . \tag{3.62}$$

Theorem 3.50 *Under conditions (3.52) and (3.53) on f and ℓ, and if Isaacs' assumption holds:*

$$H^+(x,p) = H^-(x,p) \qquad \forall (x,p)\in\mathbb{R}^N\times\mathbb{R}^N\ , \tag{3.63}$$

then the game has a value:

$$\mathbf{V}^+(x) = \mathbf{V}^-(x) \qquad \forall x\in\mathbb{R}^N\ .$$

Moreover $\mathbf{V}^+ = \mathbf{V}^-$ is the unique viscosity solution of Isaacs' equation (3.59)=(3.61).

The proof of Theorem 3.50 relies on a (half-)characterization of the value functions and on the comparison principle (Theorem 3.36).

Lemma 3.51 *The upper value function $\mathbf{V}^+$ is a subsolution of equation (3.59) where H^+ is defined by (3.60) while the lower value function $\mathbf{V}^-$ is a viscosity supersolution to (3.61).*

Proof of Theorem 3.50: According to Lemma 3.46 and Lemma 3.47, $\mathbf{V}^+$ and $\mathbf{V}^-$ are both bounded and Hölder continuous. Since $H^- = H^+$, $\mathbf{V}^-$ is a supersolution of (3.59) while $\mathbf{V}^+$ is a subsolution of that equation. Under assumptions (3.52) and (3.53) on f and ℓ, the Hamiltonian $H^+ = H^-$ satisfies (3.33), (3.34) and (3.40). The comparison principle (Theorem 3.36) then implies that $\mathbf{V}^+ \le \mathbf{V}^-$. Since the reverse inequality always holds, one gets the equality and the characterization of the value. □

Proof of Lemma 3.51 : As usual it is enough to prove the result for $\mathbf{V}^+$. We have to show that, if ϕ is a $\mathcal{C}^1$ test function such that $\mathbf{V}^+ - \phi$ has a local maximum at $x_0\in\mathbb{R}^N$, then $-\lambda\mathbf{V}^+(x_0) + H^+(x_0, D\phi(x_0)) \ge 0$.

Since $\mathbf{V}^+ - \phi$ has a local maximum at x_0, there is some $r>0$ such that

$$\mathbf{V}^+(x) \le \phi(x) + \mathbf{V}^+(x_0) - \phi(x_0) \qquad \forall x\in B(x_0,r)\ .$$

From the dynamic programming property, we have

$$\mathbf{V}^+(x_0) = \inf_{\alpha\in\mathcal{A}_d}\sup_{v\in\mathcal{V}}\left\{\int_0^h e^{-\lambda s}\ell(X_s^{x_0,\alpha(v),v}, \alpha(v)_s, v_s)ds + e^{-\lambda h}\mathbf{V}^+(X_h^{x_0,\alpha(v),v})\right\}\ .$$

Let us set $h_0 = r/(\|f\|_\infty + 1)$. Then, for any $h\in(0,h_0)$ and any $(\alpha,v)\in\mathcal{A}_d\times\mathcal{V}$, we have $X_h^{x_0,\alpha(v),v}\in B(x_0,r)$, so that

$$0 \le \inf_{\alpha\in\mathcal{A}_d}\sup_{v\in\mathcal{V}}\left\{\int_0^h e^{-\lambda s}\ell(X_s^{x_0,\alpha(v),v}, \alpha(v)_s, v_s)\ ds + e^{-\lambda h}\phi(X_h^{x_0,\alpha(v),v}) - \phi(x_0)\right\}\ . \tag{3.64}$$

Let us now fix any (time independent) control $u\in U$. From (3.64) for any $\epsilon>0$ and any $h>0$ small, there is some (time dependant) control $v_h\in\mathcal{V}$ such that

$$-\epsilon h \le \int_0^h e^{-\lambda s}\ell(X_s^{x_0,\alpha(v),v}, \alpha(v)_s, v_s)\ ds + e^{-\lambda h}\phi(X_h^{x_0,\alpha(v),v}) - \phi(x_0)\ . \tag{3.65}$$

Since ℓ is Lipschitz continuous and f is bounded, we have (as in the proof of Lemma 3.15)

$$\left|\int_0^h e^{-\lambda s}\ell(X_s^{x_0,\alpha(v),v}, \alpha(v)_s, v_s)\ ds - \int_0^h e^{-\lambda s}\ell(x_0, \alpha(v)_s, v_s)\ ds\right| \ \le\ o(h)\ .$$

To estimate the last two terms in (3.65) we note that

$$e^{-\lambda h}\phi(X_h^{x_0,\alpha(v),v}) - \phi(x_0) = e^{-\lambda h}(\phi(X_h^{x_0,\alpha(v),v}) - \phi(x_0)) + (e^{-\lambda h}-1)\phi(x_0)$$

where

$$(e^{-\lambda h}-1)\phi(x_0) = -\lambda\phi(x_0)h + o(h)\ .$$

Since ϕ is of class $\mathcal{C}^1$, we have

$$\phi(X_h^{x_0,\alpha(v),v}) - \phi(x_0) = \int_0^h \langle D\phi(X_s^{x_0,u,v_h}), f(X_s^{x_0,u,v_h}, u, v_{h,s})\rangle ds$$

where, by uniform continuity of $D\phi$ and f,

$$\left|\int_0^h \langle D\phi(X_s^{x_0,u,v_h}), f(X_s^{x_0,u,v_h}, u, v_{h,s})\rangle ds - \int_0^h \langle D\phi(x_0), f(x_0,u,v_{h,s})\rangle ds\right| \leq o(h)\ .$$

Plugging the above estimates into (3.65) gives

$$-\epsilon h - o(h) \leq -\lambda\phi(x_0)h + \int_0^h \ell(x_0,u,v_{h,s}) + \langle D\phi(x_0), f(x_0,u,v_{h,s})\rangle\ ds\ .$$

Since

$$\begin{aligned}
&\int_0^h \ell(x_0,u,v_{h,s}) + \langle D\phi(x_0), f(x_0,u,v_{h,s})\rangle\ ds\\
&\quad\leq \int_0^h \max_{v\in V}\{\ell(x_0,u,v) + \langle D\phi(x_0), f(x_0,u,v)\rangle\}\, ds\\
&\quad= h\max_{v\in V}\{\ell(x_0,u,v) + \langle D\phi(x_0), f(x_0,u,v)\rangle\}\ ,
\end{aligned}$$

we get

$$-\epsilon h - o(h) \leq h\left\{-\lambda\phi(x_0)h + \max_{v\in V}\{\ell(x_0,u,v) + \langle D\phi(x_0), f(x_0,u,v)\rangle\}\right\}\ .$$

Dividing the above expression by h, letting $h \to 0^+$ and then $\epsilon \to 0^+$ gives:

$$0 \ \leq\ -\lambda\phi(x_0) + \max_{v\in V}\{\ell(x_0,u,v) + \langle D\phi(x_0), f(x_0,u,v)\rangle\}\ .$$

Taking the infimum with respect to $u \in U$ then completes the proof. □

3.5 Exercices

Exercice 3.1 Let $\mathcal{O}$ be an open subset of $\mathbb{R}^N$ and $f_n : \mathcal{O} \to \mathbb{R}$ which converges pointwise to some map $f : \mathcal{O} \to \mathbb{R}$ with $f_n \leq f$ for all n.

1. Show that
$$\lim_{n\to+\infty}\sup_{x\in\mathcal{O}} f_n(x) = \sup_{x\in\mathcal{O}} f(x)\ .$$

2. Show that the equality does not hold in general if one removes the assumption $f_n \leq f$.

Exercice 3.2 Show that the map $\mathbf{V} : [0,T]\times\mathbb{R} \to \mathbb{R}$

$$\mathbf{V}(t,x) = \begin{cases} 0 & \text{si } |x| \geq T-t \\ (T-t)-|x| & \text{sinon}\end{cases}$$

is Lipschitz continuous and satisfies the Hamilton-Jacobi equation $\partial_t\mathbf{V}(t,x) + |D\mathbf{V}(t,x)| = 0$ at any point of differentiability of $\mathbf{V}$. Deduce from this that there are infinitely many solutions to that equation with terminal condition $\mathbf{V}(T,x) = 0$.

What is the viscosity solution of this equation with terminal condition $V(T,\cdot) = 0$?

Exercice 3.3 Let $\mathcal{O}$ be an open subset of $\mathbb{R}^N$ and let us consider a partition $\mathcal{O} = \mathcal{O}_1 \cup \mathcal{O}_2 \cup \Gamma$ of $\mathcal{O}$ (i.e., $\mathcal{O}_1$, $\mathcal{O}_2$ and Γ are disjoint). We assume that $\mathcal{O}_1$ and $\mathcal{O}_2$ are open, that Γ is a closed, smooth, connected surface. For $x \in \Gamma$ we denote by $n(x)$ the unit vector normal to Γ at x pointing towards $\mathcal{O}_2$, so that $n(x)$ is an outward normal to $\mathcal{O}_1$ at x and an inward normal to $\mathcal{O}_2$ at x. Let $v_1 : \overline{\mathcal{O}_1} \to \mathbb{R}$ and $v_2 : \overline{\mathcal{O}_2} \to \mathbb{R}$ be of class $\mathcal{C}^1$ in their respective domain and be solutions of some Hamilton-Jacobi equation $H(x, u(x), Du(x)) = 0$ in $\mathcal{O}_1$ and $\mathcal{O}_2$ respectively. We assume that $v_1 = v_2$ on Γ and define the continuous map $v : \mathcal{O} \to \mathbb{R}$ by $v = v_1$ in $\mathcal{O}_1$, $v = v_2$ in $\mathcal{O}_2$ and $v = v_1 = v_2$ on Γ. Finally we suppose that $Dv_1(x) \neq Dv_2(x)$ for $x \in \Gamma$.

1. Show that there is a continuous map $\lambda : \Gamma \to \mathbb{R}_*$ such that $Dv_2(x) - Dv_1(x) = \lambda(x)n(x)$ for any $x \in \Gamma$. To fix the ideas, we assume that $\lambda(x) > 0$ for all $x \in \Gamma$.

2. Show that v is a viscosity subsolution of $H(x, u, Du) = 0$ in $\mathcal{O}$ (Hint: prove that there is no test function ϕ such that $v - \phi$ has a local maximum at $x \in \Gamma$).

3. Let us assume further that $H(x, v_1(x), Dv_1(x) + sn(x)) \leq 0$ for all $s \in [0, \lambda(x)]$ and all $x \in \Gamma$. Show that v is a solution of $H(x, u, Du) = 0$ in $\mathcal{O}$.

Exercice 3.4 Let $\mathbf{V}$ be a viscosity subsolution of equation

$$\partial_t \mathbf{V}(t,x) + H(t, x, D\mathbf{V}(t,x)) = 0 \qquad \text{in } (0,T) \times \mathbb{R}^N .$$

Let (t_0, x_0) be a point of differentiability of $\mathbf{V}$. The aim of this exercise is to show that

$$\partial_t \mathbf{V}(t_0, x_0) + H(t_0, x_0, D\mathbf{V}(t_0, x_0)) \geq 0 .$$

Let

$$\sigma(r) = \max_{|(t,x)-(t_0,x_0)| \leq r} \frac{\mathbf{V}(t,x) - \mathbf{V}(t_0,x_0) - \partial_t \mathbf{V}(t_0,x_0)(t-t_0) - \langle D\mathbf{V}(t_0,x_0), x - x_0 \rangle}{|(t,x) - (t_0,x_0)|}$$

and

$$\rho(r) = \int_0^r \sigma(\tau) d\tau .$$

1. Show that the map $\phi(t,x) = \mathbf{V}(t_0,x_0) - \partial_t \mathbf{V}(t_0,x_0)(t-t_0) - \langle D\mathbf{V}(t_0,x_0), x - x_0 \rangle + \rho(|(t,x) - (t_0,x_0)|)$ is of class $\mathcal{C}^1$ with $\phi(t_0,x_0) = \mathbf{V}(t_0,x_0)$, $\partial_t \phi(t_0,x_0) = \partial_t \mathbf{V}(t_0,x_0)$ and $D\phi(t_0,x_0) = D\mathbf{V}(t_0,x_0)$.

2. Show that $\mathbf{V} \leq \phi$.

3. Conclude.

Exercice 3.5 Let $\mathbf{V}^+$ be the upper value function of the Bolza problem defined by (3.6). Under assumptions (3.2) and (3.3) on the dynamics and payoffs of the game, but without assuming Isaacs' condition, show that $\mathbf{V}^+$ is a viscosity solution of Isaacs' equation (3.19).

Exercice 3.6 (Value in nonanticipative strategies) We consider Bolza problem under the assumptions of Theorem 3.14 and denote by $\mathbf{V}$ the value for the game played in delay strategies. Let us recall that, for any nonanticipative strategy $\alpha \in \mathcal{A}(t_0)$ and any delay strategy $\beta \in \mathcal{B}_d(t_0)$ there is a unique pair of controls $(u,v) \in \mathcal{U}(t_0) \times \mathcal{V}(t_0)$ such that

$$\alpha(v) = u \qquad \text{and } \beta(u) = v \qquad \text{a.e. on } [t_0, T],$$

and that the symmetric result holds for any $\alpha \in \mathcal{A}_d(t_0)$ and $\beta \in \mathcal{B}(t_0)$.

Show that

$$\mathbf{V}(t_0,x_0) = \inf_{\alpha \in \mathcal{A}(t_0)} \sup_{v \in \mathcal{V}(t_0)} \mathcal{J}(t_0, x_0, \alpha(v), v) = \sup_{\beta \in \mathcal{B}(t_0)} \inf_{u \in \mathcal{U}(t_0)} \mathcal{J}(t_0, x_0, u, \beta(u))$$

for any $(t_0, x_0) \in [0,T] \times \mathbb{R}^N$.

3.6 Comments

This Chapter is largely inspired by the seminal work Evans-Souganidis [96] which is the first paper in which viscosity solution techniques are used to prove the existence of a value function. There is by now a large litterature on that approach and we refer the reader to the monographs by Bardi-Capuzzo Dolcetta [27] and by Elliott [94] for further references. The unique difference with our presentation is that these papers consider differential games in the framework of nonanticipative strategies, while we have chosen to work here with delay strategies.

Viscosity solutions of Hamilton-Jacobi equations where introduced by Crandall and Lions [82] in the early 80's for first order Hamilton-Jacobi equations and later generalized to second order equations: see the monographs by Bardi-Capuzzo Dolcetta [27], Barles [31], Fleming-Soner [104], Lions [152].

The first existence result of a value for differential games goes back to the 60's and the early 70's, with works by Fleming [101], Varaiya [208], Varaiya-Lin [209], Roxin [184], Friedman [108, 109, 110], Elliot-Kalton [92, 93], Berkovitz [38]. Another very interesting approach was developed by Krasovskii-Subbotin [139], who introduced the notion of positional strategies, and proved the existence of a value for qualitative, and then quantitative, games.

deal

Chapter 4

Nash equilibrium payoffs for nonzero-sum differential games

In this chapter, we consider a differential game played by I Players (where $I \geq 2$) and with dynamics

$$X'_t = f(t, X_t, u^1_t, \ldots, u^I_t)\ .$$

In order to simplify the notations, we restrict the analysis to games with a terminal payoff. The extension to games with integral and terminal payoff is straightforward. In this game, Player i plays with the control u^i which takes its values in some compact set U^i. He aims at maximizing his terminal payoff $g^i(X_T)$. Our aim is to define the notion of Nash equilibrium payoffs for such a game and characterize it. Moreover, if Isaacs' condition holds, we also prove the existence of such payoffs.

4.1 Definition of Nash equilibrium payoffs

We assume the following conditions on the data:

$$(4.1) \qquad \left\{ \begin{array}{rl} i) & \text{The sets } U^i \ (i = 1, \ldots, I) \text{ are compact subsets of some finite dimentional spaces} \\ ii) & f : [0,T] \times \mathbb{R}^N \times U^1 \times \ldots \times U^I \to \mathbb{R}^N \text{ is continuous and bounded,} \\ & \text{and globally Lipschitz continuous with respect to } x \\ iii) & \text{The maps } g_i : \mathbb{R}^N \to \mathbb{R} \text{ are Lipschitz continuous and bounded for } i = 1 \ldots, I. \end{array} \right.$$

Let us set

$$U^{-i} = U^1 \times \ldots \times U^{-i-1} \times U^{-i+1} \times \ldots \times U^I\ .$$

As usual, for $i \in \{1, \ldots, I\}$, we denote by $\mathcal{U}^i(t_0)$ the set of measurable controls $u^i : [t_0, T] \to U^i$. We set

$$\mathcal{U}^{-i}(t_0) = \mathcal{U}^1(t_0) \times \ldots \times \mathcal{U}^{i-1}(t_0) \times \mathcal{U}^{i+1}(t_0) \times \ldots \times \mathcal{U}^I(t_0)\ .$$

We denote by (u^{-i}) a generic element of $\mathcal{U}^{-i}(t_0)$. For any $I-$tuple of controls $(u^i) = (u^1, \ldots, u^I) \in \mathcal{U}^1(t_0) \times \ldots \times \mathcal{U}^I(t_0)$, we denote by $X^{t_0,x_0,(u^i)}$ the unique solution to

$$\left\{ \begin{array}{l} X'_t = f(t, X_t, u^1(t), \ldots, u^I(t)) \\ X_{t_0} = x_0 \end{array} \right.$$

For $i \in \{1, \ldots, I\}$ a map $\alpha : \mathcal{U}^{-i}(t_0) \to \mathcal{U}^i(t_0)$ is a **delay strategy** for Player i if there is a delay $\tau > 0$ such that, for any $t \in (t_0, T]$, for any $(I-1)-$tuple of controls $(u^{-i}) \in \mathcal{U}^{-i}(t_0)$ and $(v^{-i}) \in \mathcal{U}^{-i}(t_0)$ which coincide almost everywhere on a subinterval $[t_0, t]$, the images $\alpha((u^{-i}))$ and $\alpha((v^{-i}))$ coincide almost everywhere on $[t_0, (t + \tau) \wedge T]$. As usual we denote by $\mathcal{A}^i_d(t_0)$ the set of delay strategies of Player i.

Following (a slight extension of) Lemma 2.3 we shall systematically use the fact that if $(\alpha^1, \ldots, \alpha^I) \in \mathcal{A}^1_d(t_0) \times \ldots \times \mathcal{A}^I_d(t_0)$, then there is a unique $I-$tuple of controls $(u^i) \in \mathcal{U}^1(t_0) \times \ldots \times \mathcal{U}^I(t_0)$ such that

$$(4.2) \qquad \alpha^i((u^{-i})) = u^i \qquad \text{a.e. in } [t_0, T],\ \forall i \in \{1, \ldots, I\}\ .$$

In particular we always use the notation $X^{t_0,x_0,(\alpha^i)}_t$ for $X^{t_0,x_0,(u^i)}_t$. For any $I-$tuple $(\alpha^i) \in \mathcal{A}^1_d(t_0) \times \ldots \times \mathcal{A}^I_d(t_0)$ of delay strategies, we set

$$\mathcal{J}_i(t_0, x_0, (\alpha^i)) = g^i(X^{t_0,x_0,(\alpha^i)}_T)\ .$$

Definition 4.1 (Equilibrium payoffs) *Let $(t_0, x_0) \in [0, T) \times \mathbb{R}^N$ be fixed. A $I-$tuple $(e_1, \ldots, e_I) \in \mathbb{R}^I$ is a Nash equilibrium payoff at the point (t_0, x_0) if, for any $\epsilon > 0$, there is a $I-$tuple $(\bar{\alpha}^1, \ldots, \bar{\alpha}^i) \in \mathcal{A}_d^i(t_0) \times \ldots \times \mathcal{A}_d^I(t_0)$ of delay strategies such that, for $i \in \{1, \ldots, I\}$,*

$$|e_i - \mathcal{J}_i(t_0, x_0, (\bar{\alpha}^i))| \leq \epsilon \tag{4.3}$$

and

$$\mathcal{J}_i(t_0, x_0, \bar{\alpha}-i, \bar{\alpha}^i) \geq \mathcal{J}_1(t_0, x_0, \bar{\alpha}^{-i}, \alpha^i) - \epsilon \qquad \forall \alpha^i \in \mathcal{A}_d^i(t_0) \; . \tag{4.4}$$

Remark 4.2 One can replace conditions (4.4) by

$$\mathcal{J}_i(t_0, x_0, \bar{\alpha}^{-i}, \bar{\alpha}^i) \geq \mathcal{J}_1(t_0, x_0, \bar{\alpha}^{-i}, u^i) - \epsilon \qquad \forall u^i \in \mathcal{U}^i(t_0)$$

Let $\mathcal{E}(t_0, x_0)$ be the set of Nash equilibrium payoffs of the game at the point (t_0, x_0). Our aim is to prove that the set $\mathcal{E}(t_0, x_0)$ is non empty and to characterize it.

For that purpose, let us introduce the upper value functions of the zero-sum differential games associated to g_i:

$$\mathbf{V}_i^+(t_0, x_0) = \inf_{(\alpha^{-i}) \in \mathcal{A}_d^{-i}(t_0)} \sup_{u^i \in \mathcal{U}^i(t_0)} \mathcal{J}_i(t_0, x_0, (\alpha^{-i}), u^i) \; .$$

Let us recall that, under assumptions (4.1), the value function $\mathbf{V}_i^+$ is Lipschitz continuous and bounded on $[0, T] \times \mathbb{R}^N$. Moreover, under Isaacs' condition

$$\inf_{(u^{-i}) \in U^{-i}} \sup_{u^i \in U^i} \langle f(x, (u^{-i}), u^i), p \rangle = \sup_{u^i \in U^i} \inf_{(u^{-i}) \in U^{-i}} \langle f(x, (u^{-i}), u^i), p \rangle \qquad \forall (x, p) \in \mathbb{R}^N \times \mathbb{R}^N, \tag{4.5}$$

we have the following equalities, which mean that $\mathbf{V}_i^+$ is indeed the value of some games:

$$\mathbf{V}_i^+(t_0, x_0) = \sup_{u^i \in \mathcal{U}^i(t_0)} \inf_{(\alpha^{-i}) \in \mathcal{A}_d^{-i}(t_0)} \mathcal{J}_i(t_0, x_0, (\alpha^{-i}), u^i) \; .$$

4.2 Characterization of Nash equilibrium payoffs

In order to characterize the Nash equilibrium payoffs, let us introduce the notion of reachable and consistent payoff:

Definition 4.3 *Let $(t_0, x_0) \in [0, T) \times \mathbb{R}^N$ be fixed.*

We say that a $I-$tuple $(e_i) \in \mathbb{R}^I$ is a reachable and consistent payoff at (t_0, x_0) if, for any $\epsilon > 0$, there is some $I-$tuple of controls $(u^i) \in \mathcal{U}^1(t_0) \times \ldots \times \mathcal{U}^I(t_0)$ such that:

1. *(Reachable) $\forall i \in \{1, \ldots, I\}$, $|e_i - g_i(X_T^{t_0, x_0, (u^i)})| \leq \epsilon$,*
2. *(Consistent) $\forall i \in \{1, \ldots, I\}$, $\forall t \in [t_0, T[$, $e_i \geq \mathbf{V}_i^+(t, X_t^{t_0, x_0, (u^i)}) - \epsilon$.*

Let us denote by $\mathcal{R}(t_0, x_0)$ the set of reachable and consistent payoffs at (t_0, x_0).

Theorem 4.4 *Let us assume that f and g satisfy assumption (4.1). Then, for any $(t_0, x_0) \in [0, T] \times \mathbb{R}^N$, a $I-$tuple $(e_i) \in \mathbb{R}^I$ is a Nash equilibrium payoff at (t_0, x_0) if and only if it is a reachable and consistent payoff at (t_0, x_0):*

$$\mathcal{E}(t_0, x_0) = \mathcal{R}(t_0, x_0) \qquad \forall (t_0, x_0) \in [0, T) \times \mathbb{R}^N \; .$$

Proof of Theorem 4.4: Let us start with the proof of the inclusion $\mathcal{E}(t_0, x_0) \subset \mathcal{R}(t_0, x_0)$. Let (e_i) belong to $\mathcal{E}(t_0, x_0)$. For any fixed $\epsilon > 0$, there is some $I-$tuple $(\bar{\alpha}^I)$ of delay strategies such that

$$|e_i - \mathcal{J}_i(t_0, x_0, (\bar{\alpha}^i))| \leq \epsilon \tag{4.6}$$

and

$$\mathcal{J}_1(t_0, x_0, (\bar{\alpha}^{-i}), \bar{\alpha}^i) \geq \mathcal{J}_1(t_0, x_0, u, (\bar{\alpha}^{-i}), u^i) - \frac{\epsilon}{2} \qquad \forall u^i \in \mathcal{U}^i(t_0) \; . \tag{4.7}$$

Let $(\bar{u}^i)$ be the unique $I-$tuple of controls such that $\bar{\alpha}^i((\bar{u}^{-i})) = \bar{u}^i$ and let us set

$$X_t = X_t^{t_0, x_0, (\bar{u}^i)} \qquad \text{and} \qquad e_i' = \mathcal{J}_i(t_0, x_0, (\bar{\alpha}^i)) \;\text{ for } i = 1, \ldots, I \; .$$

Note that (4.6) means that the $I-$tuple (e_i) is reachable. We claim that it is consistent:

(4.8) $$e_i' \geq \mathbf{V}_i^+(t, X_t) - \epsilon \qquad \forall t \in [t_0, T]\ .$$

Let $t_1 \in [t_0, T]$ and $i \in \{1, \ldots, I\}$. Let us set $x_1 = X_{t_1}$ and let us define, for $j \neq i$, the strategy $\alpha^j \in \mathcal{A}_d^j(t_1)$ by $\alpha^j(u)_t = \bar{\alpha}^j((\hat{u})^{-j})_t$ for $t \in [t_1, T]$, where $(\hat{u}^{-j}) = (\bar{u}^{-j})$ on $[t_0, t_1]$ and $(\hat{u}^{-j}) = u$ on $[t_1, T]$. Let $u^i \in \mathcal{U}^i(t_1)$ be an $\epsilon/2$-optimal control for $\mathbf{V}_i^+(t_1, x_1)$ against (α^{-i}):

(4.9) $$\mathcal{J}_i(t_1, x_1, (\alpha^{-i}), u^i) \ \geq \sup_{v^i \in \mathcal{U}^i(t_1)} \mathcal{J}_i(t_1, x_1, (\alpha^{-i}), v^i) - \frac{\epsilon}{2} \ \geq\ \mathbf{V}_i^+(t_1, x_1) - \frac{\epsilon}{2}\ .$$

Let us finally define the control $\tilde{u}^i \in \mathcal{U}^i(t_0)$ by $\tilde{u}_t^i = \bar{u}_t^i$ if $t \in [t_0, t_1)$ and $\tilde{u}_t^i = u_t^i$ if $t \in [t_1, T]$. Since the strategies $(\bar{\alpha}^{-i})$ are nonanticipative, we have $X_{t_1}^{t_0, x_0, (\bar{\alpha}^{-i}), \tilde{u}^i} = x_1$ and, by (4.7) and (4.9), we have

$$e_i' = \mathcal{J}_i(t_0, x_0, (\bar{\alpha}^i)) \geq \mathcal{J}_i(t_0, x_0, (\bar{\alpha}^{-i}), \tilde{u}^i) - \frac{\epsilon}{2} = \mathcal{J}_1(t_1, x_1, (\alpha^{-i}), u^i) - \frac{\epsilon}{2} \geq \mathbf{V}_1^+(t_1, x_1) - \epsilon\ .$$

So (4.8) holds.

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Let us now prove inclusion : $\mathcal{R}(t_0, x_0) \subset \mathcal{E}(t_0, x_0)$. Let $(e_i) \in \mathcal{R}(t_0, x_0)$. For any $\epsilon > 0$, there is some $I-$tuple of controls $(\bar{u}^i) \in \mathcal{U}^1(t_0) \times \ldots \mathcal{U}^I(t_0)$ such that, for $i = 1, \ldots, I$,

(4.10) $$|\mathcal{J}_i(t_0, x_0, (\bar{u}^i)) - e_i| \leq \epsilon \qquad \text{and} \qquad \mathbf{V}_i^+(t, X_t^{t_0, x_0, (\bar{u}^i)}) \leq e_i' + \epsilon/2 \qquad \forall t \in [t_0, T]$$

where $e_i' = \mathcal{J}_i(t_0, x_0, (\bar{u}^i))$. In order to show that (e_i) is a Nash equilibrium payoff we are going to build delay strategies $(\bar{\alpha}^i)$ such that

(4.11) $$\bar{\alpha}^i((\bar{u}^{-i})) = \bar{u}^i$$

and such that

(4.12) $$\mathcal{J}_i(t_0, x_0, (\bar{\alpha}^{-i}), u^i) \leq e_i' + \epsilon \qquad \forall u^i \in \mathcal{U}^i(t_0)\ .$$

This clearly implies that (e_i) belongs to $\mathcal{R}(t_0, x_0)$. Let us set $X_t = X_t^{t_0, x_0, (\bar{u}^i)}$. The heuristic idea is that the Players agree to play the control $(\bar{u}^i)$. They punish the first Player which deviates (say Player j, deviating at time τ) by playing its worse strategy (α^{-j}) in the zero-sum game $\mathbf{V}^+(\tau, X_\tau)$. The only issue is to build delay strategies doing this. For this we will have to discretise in space and time.

Let us fix $i \in \{1, \ldots, I\}$, let n be a large integer to be defined later and set $t_k = t_0 + (T - t_0)k/n$ for $k \in \{0, \ldots, n\}$. For $k \in \{0, \ldots, n\}$, $x \in \mathbb{R}^N$ and $j \neq i$, let $(\alpha^{-j,k}) \in \mathcal{A}_d^{-j}(t_0)$ be an $(\epsilon/4)-$ optimal strategy for the game $\mathbf{V}_j^+(t_k, X_{t_k})$:

$$\sup_{u^j \in \mathcal{U}^j(t_k)} \mathcal{J}_j(t_k, X_{t_k}, (\alpha^{-j,k}), u^j) \leq \mathbf{V}_j^+(t_k, X_{t_k}) + \frac{\epsilon}{4}\ .$$

Then, using assumptions (4.1) on f, g_1 and g_2 and the Lipschitz continuity of $\mathbf{V}_2^+$, one can find some $\eta > 0$ such that

(4.13) $$\sup_{u^j \in \mathcal{U}^j(t_k)} \mathcal{J}_j(t_k, y, (\alpha^{-j,k}), u^j) \leq \mathbf{V}_j^+(t_k, X_{t_k}) + \frac{\epsilon}{2} \qquad \forall y \in B(X_{t_k}, \eta)$$

(see Lemma 3.7 for the proof of a similar statement). We denote by $\alpha^{i,j,k}$ the $i-$th component of $(\alpha^{-j,k})$. We are now ready to define the delay strategy $\bar{\alpha}^i$. Let $(u^{-i}) \in \mathcal{U}^{-i}(t_0)$. If $(u^{-i}) = (\bar{u}^{-i})$ a.e. on $[t_0, t_{n-1}]$, then we set $\bar{\alpha}^i((u^{-i})) = \bar{u}^i$ (in particular, (4.11) holds). Otherwise, let

$$\bar{k} = \sup\left\{k \in \{1, \ldots, n-1\}\ ,\ (u^{-i}) = (\bar{u}^{-i}) \text{ a.e. on } [t_0, t_{k-1}]\right\}\ ,$$

($t_{\bar{k}}$ is the first time the deviation is detected)

$$\bar{j} = \min\left\{j \neq i\ ,\ u^j \not\equiv \bar{u}^j \text{ a.e. on } [t_{\bar{k}-1}, t_{\bar{k}})\right\}$$

(Player $\bar{j}$ is the deviator with the smallest index) and set

$$\bar{\alpha}^i((u^{-i}))_t = \begin{cases} \bar{u}_t^i & \text{if } t \in [t_0, t_{\bar{k}}] \\ \alpha^{i,\bar{j},\bar{k}}((u^{-i})_{|[t_{\bar{k}}, T]})_t & \text{if } t \in [t_{\bar{k}}, T] \end{cases}$$

Note that $\bar{\alpha}^i$ is a delay strategy because the (finite number of strategies) $\alpha^{i,\bar{j},k}$ only involve a finite number of delays.

Let us show that the $(\bar{\alpha}^i)$ satisfy (4.12). Let $u^i \in \mathcal{U}^i(t_0)$ and If $u^i = \bar{u}^i$ a.e. on $[t_0, t_{n-1}]$, then, for $j \neq i$, $\bar{\alpha}^j((u^{-j,-i}, u^i)) = \bar{u}^j$ and, since $X_t = X_t^{t_0,x_0,(\bar{u}^{-i}),u^i}$ on $[t_0, t_{n-1}]$, we have

$$|\mathcal{J}_i(t_0, x_0, (\bar{u}^{-i}), \bar{u}^i) - \mathcal{J}_i(t_0, x_0, (\bar{u}^{-i}), u^i)| = |g_i(X_T) - g_i(X_T^{t_0,x_0,(\bar{u}^{-i}),u^i})|$$
$$\leq \mathrm{Lip}(g_i)|X_T - X_T^{t_0,x_0,(\bar{u}^{-i}),u^i}| \leq \mathrm{Lip}(g_i)\|f\|_\infty(T - t_{n-1}) = \mathrm{Lip}(g_i)\|f\|_\infty(T - t_0)/n .$$

So, if we choose n in such that $\mathrm{Lip}(g_i)\|f\|_\infty(T - t_0)/n \leq \epsilon$, then (4.12) holds. If equality if $u^i = \bar{u}^i$ a.e. on $[t_0, t_{n-1}]$ does not hold, let

$$\bar{k} = \sup\left\{k \in \{1, \ldots, n-1\}\ ,\ u^i = \bar{u}^i \text{ a.e. on } [t_0, t_{k-1}]\right\}\ ,$$

Then we have $X_{t_{\bar{k}}-1} = X_{t_{\bar{k}}-1}^{t_0,x_0,(\bar{\alpha}^{-i}),u^i}$, so that

$$|X_{t_{\bar{k}}}^{t_0,x_0,(\bar{\alpha}^{-i}),u^i} - X_{t_{\bar{k}}}| \leq \|f\|_\infty(T - t_0)/n .$$

So, if we choose n such that $\|f\|_\infty(T - t_0)/n \leq \eta$, we get from (4.13) and then from (4.10)

$$\mathcal{J}_i(t_0, x_0, (\bar{\alpha}^{-i}), u^i) = g_i\left(X_T^{t_{\bar{k}}, X_{t_{\bar{k}}}^{t_0,x_0,(\bar{u}^{-i}),u^i}, (\alpha^{j,i,\bar{k}})_{j\neq i}, u^i_{|[t_{\bar{k}},T]}}\right) \leq \bar{V}_i^+(t_{\bar{k}}, X_{t_{\bar{k}}}) + \frac{\epsilon}{2}\ \leq\ e'_i + \epsilon\ .$$

This completes the proof of Theorem 4.4. □

4.3 Existence of Nash equilibrium payoffs

Nash equilibrium payoffs exist, at least if Isaacs' condition holds:

Theorem 4.5 *If assumption (4.1) on f and the g_i and Isaacs' condition (4.5) hold, then, for any $(t_0, x_0) \in [0, T) \times \mathbb{R}^N$, there is at least one Nash equilibrium payoff at (t_0, x_0).*

Before starting the proof, we need two Lemmas.

Lemma 4.6 *For any $(t_0, x_0) \in [0, T) \times \mathbb{R}^N$ and any $\epsilon > 0$, there is a $I-$tuple of controls $(\bar{u}^i) \in \mathcal{U}^1(t_0) \times \ldots \times \mathcal{U}^I(t_0)$ such that*

$$\forall i \in \{1, \ldots, I\},\ \forall t \in [t_0, T],\ \mathbf{V}_i^+(t, X^{t_0,x_0,(\bar{u}^i)}) \geq \mathbf{V}_i^+(t_0, x_0) - \epsilon\ . \tag{4.14}$$

Proof : For any $i \in \{1, \ldots, I\}$ let $\bar{\alpha}^i \in \mathcal{A}_d^i(t_0)$ be a delay strategy which is ϵ-optimal for $\mathbf{V}_i^+(t_0, x_0)$:

$$\inf_{(u^{-i})\in\mathcal{U}^{-i}(t_0)} \mathcal{J}_i(t_0, x_0, (u^{-i}), \bar{\alpha}^i) \geq \mathbf{V}_1^+(t_0, x_0) - \epsilon\ . \tag{4.15}$$

Let $(\bar{u}^i)$ the unique $I-$tuple of controls such that

$$\bar{\alpha}^i((\bar{u}^{-i})) = \bar{u}^i \qquad \forall i \in \{1, \ldots, I\}\ .$$

Let us set $X_t = X_t^{t_0,x_0,(\bar{u}^i)}$. We claim that (4.14) holds for $(\bar{u}^i)$. Indeed, let us fix $i \in \{1, \ldots, I\}$ and $t_1 \in (t_0, T)$ and let us define the new delay strategy $\alpha^i \in \mathcal{A}_d^i(t_1)$ by

$$\alpha^i((u^{-i})) = \bar{\alpha}(\tilde{u}^{-i})_{|[t_1,T]} \text{ where } (\tilde{u}^{-i})(t) = \begin{cases} (\bar{u}^{-i})(t) & \text{if } t \in [t_0, t_1) \\ (u^{-i})(t) & \text{if } t \in [t_1, T] \end{cases} \qquad \forall (u^{-i}) \in \mathcal{U}^{-i}(t_1)\ .$$

Then

$$\mathbf{V}_i^+(t_1, X_{t_1}) \geq \inf_{(u^{-i})\in\mathcal{U}^{-i}(t_1)} \mathcal{J}_i(t_1, X_{t_1}, \alpha^i, (u^{-i})) \geq \inf_{(v^{-i})\in\mathcal{U}^{-i}(t_0)} \mathcal{J}_i(t_0, x_0, \bar{\alpha}^i, (v^{-i}))$$

because, if $(v^{-i}) = (\bar{u}^{-i})$ on $[t_0, t_1)$, then

$$X_T^{t_0,x_0,\bar{\alpha}^i,(v^{-i})} = X_T^{t_1,x_1,\alpha^i,(v^{-i})_{|[t_1,T]}}$$

where $(v^{-i})_{|[t_1,T]}$ stands for the restriction of v^{-i} to the interval $[t_1, T]$. From (4.15) we get

$$\mathbf{V}_1^+(t_1, X_{t_1}) \geq \inf_{(v^{-i})\in\mathcal{U}^{-i}(t_0)} \mathcal{J}_1(t_0, x_0, \bar{\alpha}^i, (v^{-i})) \geq \mathbf{V}_1^+(t_0, x_0) - \epsilon$$

which is the desired result. □

Lemma 4.7 *For any $(t_0, x_0) \in [0, T) \times \mathbb{R}^N$ and any $\epsilon > 0$, there is a $I-$tuple of controls $(u^i) \in \mathcal{U}^1(t_0) \times \ldots \times \mathcal{U}^I(t_0)$ such that*

$$\mathbf{V}_i^+(s, X_s^{t_0,x_0,(u^i)}) \leq \mathbf{V}_i^+(t, X_t^{t_0,x_0,(u^i)}) + \epsilon \qquad \forall t_0 \leq s \leq t \leq T, \forall i \in \{1, \ldots, I\} \ . \tag{4.16}$$

Proof : Let us fix $n > 1$ large and let us set $t_k = t_0 + \frac{(T-t_0)k}{n}$. Thanks to Lemma 4.6, we can construct by induction on the interval $[t_k, t_{k+1})$ a $I-$tuple of measurable controls $(u_n^i) : [t_k, t_{k+1}) \to U^1 \times \ldots \times U^I$ such that

$$\mathbf{V}_i^+(t, X_t^{t_0,x_0,(u_n^i)}) \geq \mathbf{V}_i^+(t_k, X_{t_k}^{t_0,x_0,(u_n^i)}) - \frac{1}{n^2} \qquad \forall t \in [t_k, t_{k+1}],\ \forall i \in \{1, \ldots, I\} \ .$$

Let us now prove that (4.16) holds for n large enough. Indeed, for any $t_0 \leq s \leq t \leq T$, we can find t_{k_1} and t_{k_2} such that $t_{k_1} \leq s < t_{k_1+1}$ and $t_{k_2} \leq t < t_{k_2+1}$. Since f is bounded and the $\mathbf{V}_i^+$ are $C-$Lipschitz continuous for some constant C, we have, for all i,

$$|\mathbf{V}_i^+(t, X_t^n) - \mathbf{V}_i^+(t_{k_2}, X_{t_{k_2}}^n)| \leq C(\|f\|_\infty + 1)/n$$

and

$$|\mathbf{V}_i^+(s, X_s^n) - \mathbf{V}_i^+(t_{k_1}, X_{t_{k_1}}^n)| \leq C(\|f\|_\infty + 1)/n \ .$$

So

$$\begin{array}{rcl} \mathbf{V}_i^+(t, X_t^n) - \mathbf{V}_i^+(s, X_s^n) & \geq & \mathbf{V}_i^+(t_{k_2}, X_{t_{k_2}}^n) - \mathbf{V}_i^+(t_{k_1+1}, X_{t_{k_1+1}}^n) - 2C(\|f\|_\infty + 1)/n \\ & \geq & \sum_{j=k_1+1}^{k_2-1} (\mathbf{V}_i^+(t_{j+1}, X_{t_{j+1}}^n) - \mathbf{V}_i^+(t_j, X_{t_j}^n)) - 2C(\|f\|_\infty + 1)/n \\ & \geq & -1/n - 2C(\|f\|_\infty + 1)/n \ \geq \ -\epsilon \end{array}$$

for a suitable choice of n. □

We are now ready to prove Theorem 4.5:

Proof of Theorem 4.5 : Let (u_n^i) be the control given by Lemma 4.7 for $\epsilon = 1/n$ and let $X^n = X^{t_0,x_0,(u_n^i)}$. The sequence $(X^n)_n$ being uniformly continuous on $[t_0, T]$, we can find a subsequence, relabelled again $(X^n)_n$, which converges to some continuous trajectory X uniformly on $[t_0, T]$. Then, from the continuity of the $\mathbf{V}_i^+$ and the construction of (u_n^i), we have

$$\mathbf{V}_i^+(s, X_s) \leq \mathbf{V}_i^+(t, X_t) \qquad \forall t_0 \leq s \leq t \leq T,\ \forall i \in \{1, \ldots, I\} \ . \tag{4.17}$$

Let us set $(e_i) = (g_i(X_T))$. We now prove that (e_i) belongs to $\mathcal{R}(t_0, x_0)$. For this we use the characterization Theorem 4.4 which states that it is enough to prove that (e_i) is a reachable and consistent payoff. From (4.17), we know that $t \to \mathbf{V}_i^+(t, X_t)$ is nondecreasing on $[t_0, T]$ for all i. Thus,

$$\mathbf{V}_i^+(t, X_t) \leq \mathbf{V}_i^+(T, X_T) = e_i \qquad \forall t \in [t_0, T],\ \forall i \in \{1, \ldots, I\} \ .$$

Since the $X^n = X^{t_0,x_0,(u_n^i)}$ uniformly converge to X and since the $\mathbf{V}_i^+$ are continuous, we can find for any positive ϵ some n such that

$$|e_i - g_i(X_T^{t_0,x_0,(u_n^i)})| \leq \epsilon$$

and such that

$$\mathbf{V}_i^+(t, X_t^{t_0,x_0,(u_n^i)}) \leq e_i + \epsilon \qquad \forall t \in [t_0, T],\ \forall i \in \{1, \ldots, I\} \ .$$

□

4.4 Exercises

Exercice 4.1 Show that the set-valued map which associates to an initial position (t, x) the set of Nash equilibrium payoff has a closed graph. In other words, if the sequence (t_n, x_n) converges to (t, x), if (e_i^n) are Nash equilibrium payoffs for the initial position (t_n, x_n) and if (e_i^n) converges to some $(e_i) \in \mathbb{R}^I$, then (e_i) is a Nash equilibrium payoff for the initial position (t, x).

4.5 Comments

The results of this section are due to Kononenko [138] (see also Kleimenov [137], Tolwinski-Haurie-Leitmann [205], Gaitsgory-Nitzan [113], Coulomb-Gaitsgory [81]). It is the counterpart of the so-called Folk Theorem in repeated game theory [167]. It has been extended to differential games played with random strategies by Souquière [194] and to stochastic differential games by Buckdhan-Cardaliaguet-Rainer [56] (see also Rainer [178]).

The main open issue for nonzero-sum differential games is the existence of subgame perfect equilibrium: these equilibria are (at least heuristically) given by feedback strategies and should be solutions of a system of Hamilton-Jacobi equations. In this framework, a verification theorem, analogous to the one given in Chapter 1, has been obtained by Case [79] (see also the monograph Friedman [111] and the references therein) and successfully applied to linear-quadratic differential games (see again Case [79], but also Starr-Ho [196], etc...). This later class of differential games is probably the class which has been the most investigated, because in particular the numerous applications (see for instance the monograph [90] and the references therein).

Beside the linear quadratic case, very little is known. Results on the system of Hamilton-Jacobi equations formally derived for the Nash equilibria are very sparse. Actually recent papers on the subject seem to indicate that this system is ill-defined and highly unstable. In a series of articles, Bressan and his co-authors (Bressan-Chen [50], [51], Bressan-Priuli [52], Bressan [54], [55]) show that, in space dimension 1, the system can be recasted in terms of a system of conservation laws which turns out to be ill-posed in general. Cardaliaguet-Plaskacz [68], Cardaliaguet [70] also investigate a game in dimension 1 and show that the Nash equilibria in feedback strategies is highly unstable.

This is in sharp contrast with what happens for stochastic differential games (with non degenerate viscosity term), which have been investigated either via P.D.E methods by Bensoussan-Frehse [36], [37], or by the use of backward stochastic differential equations methods by Hamadène [126].

In any case, very little is known about uniqueness of Nash equilibria and the selection of such equilibria is a challenging issue. One of the ways to overcome this problem is to consider the case with infinitely many players, and where none of these players has a strong influence on the game. In this situation, one can expect that the limit system has a unique solution. This approach has very recently been developed in a sequence of papers by Lasry-Lions [145, 146, 147, 148, 157] under the terminology of mean-field games (see also, in the framework of backward-stochatic differential equation, Buckdahn and al. [60, 61]).

Chapter 5

Differential games with incomplete information

In this chapter we investigate a two-player zero-sum differential game of Bolza type in which the players have a private information on the payoff. More precisely, we assume that the running payoff ℓ_{ij} and terminal payoff g_{ij} depend on some indices $i \in \{1, \ldots, I\}$ and $j \in \{1, \ldots, J\}$. At the initial time t_0 of the game, the pair (i, j) is chosen randomly according to some probability law $p \otimes q$ on $\{1, \ldots, I\} \times \{1, \ldots, J\}$. The index i is told to Player 1, but not to Player 2, while the index j is told to Player 2, and not to Player 1. Then the game runs as usual, Player 1 trying to minimize its cost given by

$$\mathcal{J}_{ij}(t_0, x_0, u, v) = \int_{t_0}^{T} \ell_{ij}(s, X_s, u_s, v_s)ds + g_{ij}(X_T)$$

while Player 2 aims at maximizing that same quantity.

Note that the players do not know which payoff they are actually optimizing, because they only have a part of the information on the pair (i, j). The interesting point is that, since the players observe their opponent's control, they can nevertheless try to guess their missing information by observing what their opponent is playing.

We will first introduce the definition of the value functions: this definition requires the introduction of random strategies, because the players need to hide their private information by randomizing their behavior. Then we will show that the value functions, which depend on the probability $p \otimes q$, have some convexity properties with respect to this probability. This allows us to introduce a new game—the so-called dual game–, the value function of which enjoys a sub-dynamic property. Thanks to this we will complete the proof of the existence of a value and characterize this value in terms of viscosity solutions of some "double obstacle" Hamilton-Jacobi equation.

5.1 Definition of the value functions

Since the players need to hide a part of their information, they have to play random strategies. For this reason the definition of the value function is a little involved and takes some time.

Let us start with some notations. Since most functions involved in this chapter depend on many variables (typically on (t, x, p, q)), we keep the notation ∂_t for the time derivative, but we specify D_x or D^2_{xx} for the first or second order derivative with respect to the space variable x, D_p or D^2_{pp} for the first or second order derivative with respect to the variable p, etc...

The dynamics of the game is as usual given by:

(5.1) $$\begin{cases} X'_t = f(t, X_t, u_t, v_t) , \qquad u_t \in U,\ v_t \in V \\ X_{t_0} = x_0 \end{cases}$$

Throughtout the chapter we assume the following conditions on the dynamics and the payoff:

$$(5.2)\quad \begin{cases} i) & U \text{ and } V \text{ are compact subsets of some finite dimensional spaces,} \\ ii) & f:[0,T]\times\mathbb{R}^N\times U\times V\to\mathbb{R}^N \text{ is bounded, continuous, uniformly} \\ & \text{Lipschitz continuous with respect to the } x \text{ variable,} \\ iii) & \text{for } i=1,\ldots,I \text{ and } j=1,\ldots,J,\ \ell_{ij}:[0,T]\times\mathbb{R}^N\times U\times V\to\mathbb{R} \text{ is continuous, bounded} \\ & \text{and uniformly Lipschitz continuous with respect to } x \\ iv) & \text{for } i=1,\ldots,I \text{ and } j=1,\ldots,J,\ g_{ij}:\mathbb{R}^N\to\mathbb{R} \text{ is Lipschitz} \\ & \text{continuous and bounded.} \end{cases}$$

Controls : For any $t_0 < t_1 \leq T$, the set of open-loop controls for Player 1 on $[t_0, t_1]$ is defined by

$$\mathcal{U}(t_0, t_1) = \{u : [t_0, t_1] \mapsto U \text{ Lebesgue measurable}\} .$$

If $t_1 = T$, we simply set $\mathcal{U}(t_0) := \mathcal{U}(t_0, T)$. Open-loop controls on the interval $[t_0, t_1]$ for Player 2 are defined symmetrically and denoted by $\mathcal{V}(t_0, t_1)$ (and by $\mathcal{V}(t_0)$ if $t_1 = T$).

If $u \in \mathcal{U}(t_0)$ and $t_0 \leq t_1 < t_2 \leq T$, we denote by $u_{|[t_1,t_2]}$ the restriction of u to the interval $[t_1, t_2]$. We note that $u_{|[t_1,T]}$ belongs to $\mathcal{U}(t_1)$.

For any $(u, v) \in \mathcal{U}(t_0) \times \mathcal{V}(t_0)$ and any initial position $x_0 \in \mathbb{R}^N$, we denote by $t \mapsto X_t^{t_0,x_0,u,v}$ the unique solution to (5.1).

Strategies : Next we introduce the notions of pure and random strategies (see also Chapter 2). The definition of random strategies involves a set $\mathcal{S}$ of (non trivial) probability spaces, which has to be stable by finite product. For simplicity we will assume that $\mathcal{S}$ contains the probability space $([0,1], B([0,1]), \mathcal{L}^1)$, where $B([0,1])$ is the class of Borel sets on $[0,1]$ and $\mathcal{L}^1$ is the Lebesgue measure on $[0,1]$. For instance we can choose

$$\mathcal{S} = \{([0,1]^n, B([0,1]^n), \mathcal{L}^n), \text{ for some } n \in \mathbb{N}^*\} ,$$

where $B([0,1]^n)$ is the class of Borel sets and $\mathcal{L}^n$ is the Lebesgue measure on $\mathbb{R}^n$.

A *pure strategy* for Player 1 at time t_0 is a map $\alpha : \mathcal{V}(t_0) \mapsto \mathcal{U}(t_0)$ which satisfies the following conditions:

(i) α is a measurable map from $\mathcal{V}(t_0)$ to $\mathcal{U}(t_0)$ where $\mathcal{U}(t_0)$ and $\mathcal{V}(t_0)$ are endowed with the Borel $\sigma-$field associated with the L^1 distance,

(ii) α is nonanticipative with delay, i.e., there is a delay $\tau > 0$ such that, for any $v_1, v_2 \in \mathcal{V}(t_0)$ and any $t \in (t_0, T]$,

$$v_1 \equiv v_2 \text{ on } [t_0, t] \quad \Rightarrow \quad \alpha(v_1) \equiv \alpha(v_2) \text{ on } [t_0, (t + \tau) \wedge T] .$$

(Note that the only difference with delay strategies introduced in Chapter 2 is that we require here the map α to be measurable; this will be more convenient later on).

A *random strategy* for Player 1 is a pair $((\Omega_\alpha, \mathcal{F}_\alpha, \mathbb{P}_\alpha), \alpha)$, where $(\Omega_\alpha, \mathcal{F}_\alpha, \mathbb{P}_\alpha)$ belongs to the set of probability spaces $\mathcal{S}$ and $\alpha : \Omega_\alpha \times \mathcal{V}(t_0) \mapsto \mathcal{U}(t_0)$ satisfies

(i) α is a measurable map from $\Omega_\alpha \times \mathcal{V}(t_0)$ to $\mathcal{U}(t_0)$, with Ω_α endowed with the $\sigma-$field $\mathcal{F}_\alpha$ and $\mathcal{U}(t_0)$ and $\mathcal{V}(t_0)$ with the Borel $\sigma-$field associated with the L^1 topology (see section 2.2),

(ii) there is a delay $\tau > 0$ such that, for any $v_1, v_2 \in \mathcal{V}(t_0)$, any $t \in (t_0, T]$ and any $\omega \in \Omega_\alpha$,

$$v_1 \equiv v_2 \text{ on } [t_0, t] \quad \Rightarrow \quad \alpha(\omega, v_1) \equiv \alpha(\omega, v_2) \text{ on } [t_0, (t + \tau) \wedge T] .$$

We denote by $\mathcal{A}_d(t_0)$ the set of pure strategies and by $\mathcal{A}_r(t_0)$ the set of random delay strategies for Player 1. By abuse of notations, an element of $\mathcal{A}_r(t_0)$ is simply noted α (instead of $((\Omega_\alpha, \mathcal{F}_\alpha, \mathbb{P}_\alpha), \alpha)$), the underlying probability space being always denoted by $(\Omega_\alpha, \mathcal{F}_\alpha, \mathbb{P}_\alpha)$. Let us point out the inclusion $\mathcal{A}_d(t_0) \subset \mathcal{A}_r(t_0)$.

To take into account the fact that Player 1 knows the index i, a strategy for Player 1 is actually a $I-$tuple $\hat{\alpha} = (\alpha_1, \ldots, \alpha_I) \in (\mathcal{A}_r(t_0))^I$.

Pure and random strategies for Player 2 are defined symmetrically and the set of pure and random delay strategies for Player 2 are denoted by $\mathcal{B}_d(t_0)$ and $\mathcal{B}_r(t_0)$ respectively. Elements of $\mathcal{B}_r(t_0)$ are denoted simply by β, and the underlying probability space by $(\Omega_\beta, \mathcal{F}_\beta, \mathbb{P}_\beta)$.

Since Player 2 knows the index j, a strategy for Player 2 is a $J-$tuple $\hat{\beta} = (\beta_1, \dots, \beta_J) \in (\mathcal{B}_r(t_0))^J$.

One of the main interests of delay strategies is the following property, proved in Chapter 2, Lemma 2.5: For any pair $(\alpha, \beta) \in \mathcal{A}_r(t_0) \times \mathcal{B}_r(t_0)$ and any $\omega := (\omega_1, \omega_2) \in \Omega_\alpha \times \Omega_\beta$, there is a unique pair $(u_\omega, v_\omega) \in \mathcal{U}(t_0) \times \mathcal{V}(t_0)$, such that

$$\alpha(\omega_1, v_\omega) = u_\omega \text{ and } \beta(\omega_2, u_\omega) = v_\omega \ . \tag{5.3}$$

Furthermore the map $\omega \to (u_\omega, v_\omega)$ is measurable from $\Omega_\alpha \times \Omega_\beta$ endowed with $\mathcal{F}_\alpha \otimes \mathcal{F}_\beta$ into $\mathcal{U}(t_0) \times \mathcal{V}(t_0)$ endowed with the Borel $\sigma-$field associated with the L^1 distance.

Expectation with respect to the strategies: Given any pair $(\alpha, \beta) \in \mathcal{A}_r(t_0) \times \mathcal{B}_r(t_0)$, we denote by $(X_t^{t_0,x_0,\alpha,\beta})$ the map $(t, \omega) \mapsto X_t^{t_0,x_0,u_\omega,v_\omega}$ defined on $[t_0, T] \times \Omega_\alpha \times \Omega_\beta$, where (u_ω, v_ω) satisfies (5.3). We also define the expectation $\mathbb{E}_{\alpha\beta}$ as the integral over $\Omega_\alpha \times \Omega_\beta$ against the probability measure $\mathbb{P}_\alpha \otimes \mathbb{P}_\beta$. In particular, if $\phi : \mathbb{R}^N \to \mathbb{R}$ is some bounded continuous map and $t \in (t_0, T]$, we have

$$\mathbb{E}_{\alpha\beta}\left[\phi\left(X_t^{t_0,x_0,\alpha,\beta}\right)\right] := \int_{\Omega_\alpha \times \Omega_\beta} \phi\left(X_t^{t_0,x_0,u_\omega,v_\omega}\right) d\mathbb{P}_\alpha \otimes \mathbb{P}_\beta(\omega) \ , \tag{5.4}$$

where (u_ω, v_ω) is defined by (5.3). Note that (5.4) makes sense because the map $(u, v) \mapsto X_t^{t_0,x_0,u,v}$ being continuous in L^1, the map $\omega \mapsto \phi\left(X_t^{t_0,x_0,u_\omega,v_\omega}\right)$ is measurable in $\Omega_\alpha \times \Omega_\beta$ and bounded. If either α or β is a pure strategy, then we simply drop α or β in the expectation $\mathbb{E}_{\alpha\beta}$, which then becomes $\mathbb{E}_\beta$ or $\mathbb{E}_\alpha$.

Probability measures on $\{1, \dots, I\}$ and $\{1, \dots, J\}$: For a fixed integer $I \geq 1$, the set $\Delta(I)$ denotes the set of probability measures on $\{1, \dots, I\}$, always identified with the simplex of $\mathbb{R}^I$:

$$p = (p_1, \dots, p_I) \in \Delta(I) \quad \Leftrightarrow \quad \sum_{i=1}^{I} p_i = 1 \text{ and } p_i \geq 0 \text{ for } i = 1, \dots I \ .$$

In the same way, for a fixed integer $J \geq 1$, the set $\Delta(J)$ is the set of probability measures on $\{1, \dots, J\}$.

Definition of the payoff: Let $(p, q) \in \Delta(I) \times \Delta(J)$, $(t_0, x_0) \in [0, T) \times \mathbb{R}^N$, $\hat{\alpha} = (\alpha_i)_{i=1,\dots,I} \in (\mathcal{A}_r(t_0))^I$ and $\hat{\beta} = (\beta_j) \in (\mathcal{B}_r(t_0))^J$. We set

$$\mathcal{J}_{ij}(t_0, x_0, \alpha_i, \beta_j) = \mathbb{E}_{\alpha_i\beta_j}\left[\int_{t_0}^{T} \ell_{ij}(s, X_s^{t_0,x_0,\alpha_i,\beta_j}, \alpha_i(s), \beta_j(s))ds + g_{ij}\left(X_T^{t_0,x_0,\alpha_i,\beta_j}\right)\right] , \tag{5.5}$$

and

$$\mathcal{J}(t_0, x_0, \hat{\alpha}, \hat{\beta}, p, q) = \sum_{i=1}^{I}\sum_{j=1}^{J} p_i q_j \mathcal{J}_{ij}(t_0, x_0, \alpha_i, \beta_j) \ , \tag{5.6}$$

where $\mathbb{E}_{\alpha_i\beta_j}$ is defined by (5.4). Note that $\hat{\alpha}$ does not depend on j, while $\hat{\beta}$ does not depend on i. This modelizes the fact that Player 1 knows i but not j, while Player 2 knows j and not i.

Definition of the value functions: The upper value function is given by

$$\mathbf{V}^+(t_0, x_0, p, q) = \inf_{\hat{\alpha} \in (\mathcal{A}_r(t_0))^I} \sup_{\hat{\beta} \in (\mathcal{B}_r(t_0))^J} \mathcal{J}(t_0, x_0, \hat{\alpha}, \hat{\beta}, p, q)$$

while the lower value function is defined by

$$\mathbf{V}^-(t_0, x_0, p, q) = \sup_{\hat{\beta} \in (\mathcal{B}_r(t_0))^J} \inf_{\hat{\alpha} \in (\mathcal{A}_r(t_0))^I} \mathcal{J}(t_0, x_0, \hat{\alpha}, \hat{\beta}, p, q) \ .$$

In particular

$$\mathbf{V}^-(t_0, x_0, p, q) \leq \mathbf{V}^+(t_0, x_0, p, q) \qquad \forall (t_0, x_0, p, q) \in [0, T] \times \mathbb{R}^N \times \Delta(I) \times \Delta(J) \ .$$

Hamiltonian and Isaacs' condition: Throughout the chapter we assume that the following Isaacs' condition holds:

$$
\begin{aligned}
H(t,x,\xi,p,q) &:= \inf_{u\in U}\sup_{v\in V}\left\{\langle f(t,x,u,v),\xi\rangle + \sum_{i,j} p_i q_j \ell_{ij}(t,x,u,v)\right\} \\
&= \sup_{v\in V}\inf_{u\in U}\left\{\langle f(t,x,u,v),\xi\rangle + \sum_{i,j} p_i q_j \ell_{ij}(t,x,u,v)\right\}
\end{aligned}
\tag{5.7}
$$

for any $(t,x,\xi,p,q)\in[0,T]\times\mathbb{R}^N\times\mathbb{R}^N\times\Delta(I)\times\Delta(J)$.

5.2 Convexity properties of the value functions

The main result of this section is Lemma 5.2 which states that the value functions $\mathbf{V}^+$ and $\mathbf{V}^-$ are convex in p and concave in q. We also investigate some regularity properties of the value functions.

Lemma 5.1 (Regularity of $\mathbf{V}^+$ and $\mathbf{V}^-$) *Under assumption (5.2),* $\mathbf{V}^+$ *and* $\mathbf{V}^-$ *are Lipschitz continuous.*

Proof : We first note that the Lipschitz continuity of $\mathbf{V}^-$ and $\mathbf{V}^+$ with respect to p and q just comes from the boundedness of the ℓ_{ij} and g_{ij}. Using standard arguments, one easily shows that, for any $t_0\in[0,T]$, $(u,v)\in\mathcal{U}(t_0)\times\mathcal{V}(t_0)$, the maps

$$x\to\int_{t_0}^T \ell_{ij}(s,X_s^{t_0,x,u,v},u(s),v(s))ds \text{ and } x\to g_{ij}\left(X_T^{t_0,x,u,v}\right)$$

are Lipschitz continuous with a Lipschitz constant independent of $t_0\in[0,T]$. Hence for any pair of strategies $(\hat\alpha,\hat\beta)\in(\mathcal{A}_r(t_0))^I\times(\mathcal{B}_r(t_0))^J$ the map

$$x\to\mathcal{J}_{ij}(t,x,\hat\alpha,\hat\beta,p,q)$$

is $C-$Lipschitz continuous for some constant C independent of $t\in[0,T]$, of $p\in\Delta(I)$ and of $q\in\Delta(J)$. From this one easily deduces that $\mathbf{V}^+$ and $\mathbf{V}^-$ are $C-$Lipschitz continuous with respect to x.

We now consider the time regularity of $\mathbf{V}^-$ and $\mathbf{V}^+$. We only do the proof for $\mathbf{V}^-$, since the case of $\mathbf{V}^+$ can be treated similarly. Let $x_0\in\mathbb{R}^N$, $(p,q)\in\Delta(I)\times\Delta(J)$ and $t_0<t_1<T$ be fixed. Let $\hat\beta=(\beta_j)\in(\mathcal{B}_r(t_0))^J$ and $\bar u\in U$ be fixed. Let us define a new strategy $(\tilde\beta_j)\in(\mathcal{B}_r(t_1))^J$ by setting

$$\tilde\beta_j(\omega,u)=\beta_j(\omega,\tilde u)_{|[t_1,T]} \text{ where } \tilde u(t)=\begin{cases}\bar u & \text{if } t\in[t_0,t_1)\\ u & \text{otherwise}\end{cases}$$

for any $\omega\in\Omega_{\tilde\beta_j}:=\Omega_{\beta_j}$ and $u\in\mathcal{U}(t_1)$. For $\epsilon>0$ let now $(\tilde\alpha_i)\in(\mathcal{A}_r(t_1))^I$ be $\epsilon-$optimal against $(\tilde\beta_j)$ at (t_1,x_0,p,q):

$$\mathcal{J}(t_1,x_0,(\tilde\alpha_i),(\tilde\beta_j),p,q)\le\mathbf{V}^-(t_1,x_0,p,q)+\epsilon\ .$$

We come back to $[t_0,T]$ by defining a new strategy $\hat\alpha=(\alpha_i)\in(\mathcal{A}_r(t_0))^I$ as

$$\alpha_i(\omega,v)=\begin{cases}\bar u & \text{if } t\in[t_0,t_1)\\ \alpha_i(\omega,v_{|[t_1,T]}) & \text{otherwise}\end{cases}\qquad\forall\omega\in\Omega_{\alpha'}:=\Omega_\alpha,\ \forall v\in\mathcal{V}(t_0)\ .$$

Then

$$\mathcal{J}(t_0,x_0,\hat\alpha,\hat\beta,p,q)\le C(t_1-t_0)+\sum_{i,j}p_iq_j\mathbb{E}_{\alpha_i,\beta_j}\left[\int_{t_1}^T\ell_{ij}(s,X_s^{t_0,x_0,\alpha_i,\beta_j}\tilde\alpha_i(s),\tilde\beta_j(s))ds+g_{ij}(X_T^{t_0,x_0,\alpha_i,\beta_j})\right]\ ,$$

where, since

$$\sup_{s\in[t_1,T]}\left|X_s^{t_0,x_0,\alpha_i,\beta_j}-X_s^{t_1,x_0,\tilde\alpha_i,\tilde\beta_j}\right|\le C(t_1-t_0)\ ,$$

we have

$$\sum_{i,j} p_i q_j \mathbb{E}_{\alpha_i,\beta_j}\left[\int_{t_1}^T \ell_{ij}(s, X_s^{t_0,x_0,\alpha_i,\beta_j}\tilde{\alpha}_i(s), \tilde{\beta}_j(s))ds + g_{ij}(X_T^{t_0,x_0,\alpha_i,\beta_j})\right] \leq C(t_1-t_0)+\mathcal{J}(t_1,x_0,(\tilde{\alpha}_i),(\tilde{\beta}_j),p,q) \ .$$

Using the $\epsilon-$optimality of $(\tilde{\alpha}_i)$ this implies that

$$\mathcal{J}(t_0,x_0,\hat{\alpha},\hat{\beta},p,q) \leq C(t_1-t_0) + \mathbf{V}^-(t_1,x_0,p,q) + \epsilon \ .$$

Taking the infimum over $\hat{\alpha} \in (\mathcal{A}_r(t_0))^I$ and then the supremum over $\hat{\beta} \in (\mathcal{B}_r(t_0))^J$ we get

$$\mathbf{V}^-(t_0,x_0,p,q) \leq C(t_1-t_0) + \mathbf{V}^-(t_1,x_0,p,q) + \epsilon \ ,$$

which shows that

$$\mathbf{V}^-(t_0,x_0,p,q) \leq C(t_1-t_0) + \mathbf{V}^-(t_1,x_0,p,q)$$

since ϵ is arbitrary.

The reverse inequality can be proved in a similar way: we choose some $\epsilon-$optimal strategy $\hat{\beta} = (\beta_j) \in (\mathcal{B}_r(t_1))^J$ for $\mathbf{V}^-(t_1,x_0,p,q)$ and we extend it to a strategy $(\tilde{\beta}_j) \in (\mathcal{B}_r(t_0))^J$ by setting (for some $\bar{v} \in V$ fixed)

$$\tilde{\beta}_j(\omega,u) = \begin{cases} \bar{v} & \text{if } t \in [t_0,t_1) \\ \beta_j(\omega, u_{|[t_1,T]}) & \text{otherwise} \end{cases} \qquad \forall \omega \in \Omega_{\tilde{\beta}_j} := \Omega_{\beta_j}, \ \forall u \in \mathcal{U}(t_0) \ .$$

Similar estimates as above then show that, for any $\hat{\alpha} \in (\mathcal{A}_r(t_0))^I$ we have

$$\mathcal{J}(t_0,x_0,\hat{\alpha},(\tilde{\beta}_j),p,q) \ \geq \mathbf{V}^-(t_1,x_0,p,q) - \epsilon - C|t_0-t_1|$$

from the $\epsilon-$optimality of $\hat{\beta}$ for $\mathbf{V}^-(t_1,x_0,p,q)$. Then we get

$$\mathbf{V}^-(t_0,x_0,p,q) \geq \mathbf{V}^-(t_1,x_0,p,q) - C|t_0-t_1| \ .$$

□

Lemma 5.2 (Convexity properties of $\mathbf{V}^-$ and $\mathbf{V}^+$) *For any $(t,x) \in [0,T) \times \mathbb{R}^N$, the maps $\mathbf{V}^+ = \mathbf{V}^+(t,x,p,q)$ and $\mathbf{V}^- = \mathbf{V}^-(t,x,p,q)$ are convex with respect to p and concave with respect to q on $\Delta(I)$ and $\Delta(J)$ respectively.*

Remark 5.3 This result is well-known for repeated games with lack of information. The procedure we use in the proof is usually called "splitting": see [189] for instance.

Proof of Lemma 5.2: We only do the proof for $\mathbf{V}^+$, the proof for $\mathbf{V}^-$ can be achieved by reversing the roles of the players. One first easily checks that

$$\mathbf{V}^+(t_0,x_0,p,q) = \inf_{(\alpha_i)\in(\mathcal{A}_r(t_0))^I} \sum_{j=1}^J q_j \sup_{\beta\in\mathcal{B}_r(t_0)} \left[\sum_{i=1}^I p_i \mathcal{J}_{ij}(t_0,x_0,\alpha_i,\beta_j)\right] \ .$$

Hence $q \to \mathbf{V}^+(t,x,p,q)$ is concave for any (t,x,p) as the infimum of concave functions.

We now prove the convexity of $\mathbf{V}^+$ with respect to p. Let $(t,x,q) \in [0,T) \times \mathbb{R}^N \times \Delta(J)$, $p^0,p^1 \in \Delta(I)$, $\lambda \in (0,1)$. Let us set $p^\lambda = (1-\lambda)p^0 + \lambda p^1$. We can assume without loss of generality that $p_i^\lambda \neq 0$ for any i because if $p_i^\lambda = 0$, then $p_i^0 = p_i^1 = 0$, so that this index i plays no role in our computation. For $\epsilon > 0$ let $\hat{\alpha}^0 = (\alpha_i^0) \in (\mathcal{A}_r(t))^I$ and $\hat{\alpha}^1 = (\alpha_i^1) \in (\mathcal{A}_r(t))^I$ be $\epsilon-$optimal for $\mathbf{V}^+(t,x,p^0,q)$ and $\mathbf{V}^+(t,x,p^1,q)$ respectively. We now define the strategy $\hat{\alpha}^\lambda = (\alpha_i^\lambda) \in (\mathcal{A}_r(t))^I$ by setting

$$\Omega_{\alpha_i^\lambda} = [0,1] \times \Omega_{\alpha_i^0} \times \Omega_{\alpha_i^1}, \ \mathcal{F}_{\alpha_i^\lambda} = B([0,1]) \otimes \mathcal{F}_{\alpha_i^0} \otimes \mathcal{F}_{\alpha_i^1}, \ \mathbb{P}_{\alpha_i^\lambda} = \mathcal{L}^1 \otimes \mathbb{P}_{\alpha_i^0} \otimes \mathbb{P}_{\alpha_i^1} \ ,$$

and

$$\alpha_i^\lambda(\omega_1,\omega_2,\omega_3,v) = \begin{cases} \alpha_i^0(\omega_2,v) & \text{if } \omega_1 \in [0, \frac{(1-\lambda)p_i^0}{p_i^\lambda}) \\ \alpha_i^1(\omega_3,v) & \text{if } \omega_1 \in [\frac{(1-\lambda)p_i^0}{p_i^\lambda}, 1] \end{cases}$$

for any $(\omega_1, \omega_2, \omega_3) \in \Omega_{\alpha_i^\lambda}$ and $v \in \mathcal{V}(t)$. We note that $(\Omega_{\alpha_i^\lambda}, \mathcal{F}_{\alpha_i^\lambda}, \mathbb{P}_{\alpha_i^\lambda})$ belongs to the set of probability spaces $\mathcal{S}$ and that α_i^λ belongs to $\mathcal{A}_r(t_0)$ for any $i = 1, \ldots, I$.

The interpretation of the strategy $\hat{\alpha}^\lambda$ is the following: if the index i is choosen according to the probability p^λ, then Player 1 chooses α_i^0 with probability $\frac{(1-\lambda)p_i^0}{p_i^\lambda}$ and α_i^1 with probability $1 - \frac{(1-\lambda)p_i^0}{p_i^\lambda} = \frac{\lambda p_i^1}{p_i^\lambda}$. Hence the probability for the strategy α_i^0 to be chosen is $p_i^\lambda \frac{(1-\lambda)p_i^0}{p_i^\lambda} = (1-\lambda)p_i^0$, while the strategy α_i^1 appears with probability $p_i^\lambda \frac{\lambda p_i^1}{p_i^\lambda} = \lambda p_i^1$. Therefore

$$
\begin{array}{rl}
\sup_{\hat{\beta} \in (\mathcal{B}_d(t_0))^J} & \mathcal{J}(t, x, \hat{\alpha}^\lambda, \hat{\beta}, p^\lambda, q) = \sum_{j=1}^J q_j \sup_{\beta \in \mathcal{B}_d(t_0)} \sum_{i=1}^I p_i^\lambda \mathcal{J}_{ij}(t, x, \alpha_i^\lambda, \beta) \\
= & \sum_{j=1}^J q_j \sup_{\beta \in \mathcal{B}_d(t_0)} \sum_{i=1}^I p_i^\lambda \left[\frac{(1-\lambda)p_i^0}{p_i^\lambda} \mathcal{J}_{ij}(t, x, \alpha_i^0, \beta) + \frac{\lambda p_i^1}{p_i^\lambda} \mathcal{J}_{ij}(t, x, \alpha_i^1, \beta) \right] \\
\leq & (1-\lambda) \sum_{j=1}^J q_j \sup_{\beta \in \mathcal{B}_d(t_0)} \sum_{i=1}^I p_i^0 \mathcal{J}_{ij}(t, x, \alpha_i^0, \beta) + \lambda \sum_{j=1}^J q_j \sup_{\beta \in \mathcal{B}_d(t_0)} \sum_{i=1}^I p_i^1 \mathcal{J}_{ij}(t, x, \alpha_i^1, \beta) \\
\leq & (1-\lambda) \mathbf{V}^+(t, x, p^0, q) + \lambda \mathbf{V}^+(t, x, p^1, q) + \epsilon
\end{array}
$$

because $\hat{\alpha}^0$ and $\hat{\alpha}^1$ are $\epsilon-$ optimal for $\mathbf{V}^+(t, x, p^0, q)$ and $\mathbf{V}^+(t, x, p^1, q)$ respectively. Therefore

$$
\begin{array}{rcl}
\mathbf{V}^+(t, x, p^\lambda, q) & \leq & \sup_{\hat{\beta}} \mathcal{J}(t, x, \hat{\alpha}^\lambda, \hat{\beta}, p^\lambda, q) \\
& \leq & (1-\lambda) \mathbf{V}^+(t, x, p^0) + \lambda \mathbf{V}^+(t, x, p^1) + \epsilon \, ,
\end{array}
$$

which proves the desired claim because ϵ is arbitrary. $\square$

Since the value functions are convex with respect to p we are naturally lead to consider their Fenchel conjugates with respect to p (see for instance [181]). Let $w : [0, T] \times \mathbb{R}^N \times \Delta(I) \times \Delta(J) \mapsto \mathbb{R}$ be some function. We denote by w^* its convex conjugate with respect to variable p:

$$
w^*(t, x, \hat{p}, q) = \sup_{p \in \Delta(I)} \langle \hat{p}, p \rangle - w(t, x, p, q) \qquad \forall (t, x, \hat{p}, q) \in [0, T] \times \mathbb{R}^N \times \mathbb{R}^I \times \Delta(J) \, .
$$

In particular $\mathbf{V}^{-*}$ and $\mathbf{V}^{+*}$ denote the convex conjugate with respect to the $p-$variable of the functions $\mathbf{V}^-$ and $\mathbf{V}^+$. Note that, since we take the supremum over $\Delta(I)$, this implicitely means that we extend w by $+\infty$ outside of $\Delta(I)$.

For a function $w = w(t, x, \hat{p}, q)$ defined on the dual space $[0, T] \times \mathbb{R}^N \times \mathbb{R}^I \times \Delta(J)$ we also denote by w^* its convex conjugate with respect to $\hat{p}$ defined on $[0, T] \times \mathbb{R}^N \times \Delta(I) \times \Delta(J)$:

$$
w^*(t, x, p, q) = \sup_{\hat{p} \in \mathbb{R}^I} \langle p, \hat{p} \rangle - w(t, x, p, q) \qquad \forall (t, x, p, q) \in [0, T] \times \mathbb{R}^N \times \Delta(I) \times \Delta(J) \, .
$$

In a symmetric way, we denote by $w^\sharp = w^\sharp(t, x, p, \hat{q})$ the concave conjugate with respect to q of w:

$$
w^\sharp(t, x, p, \hat{q}) = \inf_{q \in \Delta(J)} \langle \hat{q}, q \rangle - w(t, x, p, q) \qquad \forall (t, x, p, \hat{q}) \in [0, T] \times \mathbb{R}^N \times \Delta(I) \times \mathbb{R}^J \, .
$$

Again, taking the infimum over $\Delta(J)$ implicitely means that we extend w by $-\infty$ outside of $\Delta(J)$. However there will never be a contradiction with the convention for w^* since we will never consider at the same time the convex and the concave conjugates.

If $w : [0, T] \times \mathbb{R}^N \times \Delta(I) \times \Delta(J) \mapsto \mathbb{R}$ is convex with respect to p and concave with respect to q, we denote by $\partial_p^- w(t, x, p, q)$ and $\partial_q^+ w(t, x, p, q)$ the convex and concave sub- and super-differential of w at (t, x, p, q) with respect to p and q. Namely

$$
\partial_p^- w(t, x, p, q) = \left\{ \hat{p} \in \mathbb{R}^I \, , \; w(t, x, p, q) + \langle \hat{p}, p' - p \rangle \leq w(t, x, p', q) \quad \forall p' \in \Delta(I) \right\}
$$

and

$$
\partial_q^+ w(t, x, p, q) = \left\{ \hat{q} \in \mathbb{R}^J \, , \; w(t, x, p, q) + \langle \hat{q}, q' - q \rangle \geq w(t, x, p, q') \quad \forall q' \in \Delta(J) \right\} \, .
$$

5.3 The subdynamic programming

The main result of this section is that $\mathbf{V}^{-*}$ is a subsolution of some Hamilton-Jacobi equation while $\mathbf{V}^{+\sharp}$ is a supersolution of another Hamilton-Jacobi equation. To fix the ideas, we do the analysis for the lower value functions, and deduce at the very end of the section the symmetric results for upper value function. The result is proved in three steps. We first reformulate $\mathbf{V}^{-*}$. Then we deduce from this reformulation that $\mathbf{V}^{-*}$ satisfies a subdynamic programming principle. We finally deduce from this subdynamic programming that $\mathbf{V}^{-*}$ satisfies some differential inequality.

Lemma 5.4 (Reformulation of $\mathbf{V}^{-*}$) *We have, for any $(t,x,\hat{p},q)\in[0,T]\times\mathbb{R}^N\times\mathbb{R}^I\times\Delta(J)$,*

$$\mathbf{V}^{-*}(t,x,\hat{p},q) \;=\; \inf_{(\beta_j)\in(\mathcal{B}_r(t_0))^J}\ \sup_{\alpha\in\mathcal{A}_d(t_0)}\ \max_{i\in\{1,\ldots,I\}}\left\{\hat{p}_i-\sum_{j=1}^J q_j\mathcal{J}_{ij}(t,x,\alpha,\beta_j)\right\}\ . \tag{5.8}$$

Remark 5.5 $\mathbf{V}^{-*}$ can be viewed as the value function of a dual game. The striking point of this game is that Player 1 no longer hides his information. This turns out to be extremely useful for establishing a subdynamic programming.

Proof of Lemma 5.4: Let us recall for later use that

$$\mathbf{V}^{-}(t,x,p,q) \;=\; \sup_{(\beta_j)\in(\mathcal{B}_r(t_0))^J}\sum_{i=1}^I p_i \inf_{\alpha\in\mathcal{A}_d(t_0)}\sum_{j=1}^J q_j\mathcal{J}_{ij}(t,x,\alpha,\beta_j)\ . \tag{5.9}$$

Let us denote by $z=z(t,x,\hat{p},q)$ the right-hand side of equality (5.8). We first claim that

$$z \text{ is convex with respect to } \hat{p}. \tag{5.10}$$

Proof of (5.10): The proof mimics the proof of the convexity of $\mathbf{V}^{+}$. Let $(t,x,q)\in[0,T)\times\mathbb{R}^N\times\Delta(J)$, $\hat{p}^0,\hat{p}^1\in\mathbb{R}^I$, $\lambda\in(0,1)$ and $(\beta_j^0)\in(\mathcal{B}_r(t))^J$ and $(\beta_j^1)\in(\mathcal{B}_r(t))^J$ be $\epsilon-$optimal for $z(t,x,\hat{p}^0,q)$ and $z(t,x,\hat{p}^1,q)$ respectively ($\epsilon>0$). Let us set $\hat{p}^\lambda=(1-\lambda)\hat{p}^0+\lambda\hat{p}^1$. We define the strategies $\beta_j^\lambda\in\mathcal{B}_r(t)$ by setting

$$\Omega_{\beta_j^\lambda}=[0,1]\times\Omega_{\beta_j^0}\times\Omega_{\beta_j^1},\ \mathcal{F}_{\beta_j^\lambda}=B([0,1])\otimes\mathcal{F}_{\beta_j^0}\otimes\mathcal{F}_{\beta_j^1},\ \mathbb{P}_{\beta_j^\lambda}=\mathcal{L}^1\otimes\mathbb{P}_{\beta_j^0}\otimes\mathbb{P}_{\beta_j^1}\ ,$$

and

$$\beta_j^\lambda(\omega_1,\omega_2,\omega_3,u)=\begin{cases}\beta_j^0(\omega_2,u) & \text{if } \omega_1\in[0,(1-\lambda))\\ \beta_j^1(\omega_3,u) & \text{if } \omega_1\in[(1-\lambda),1]\end{cases}$$

for any $(\omega_1,\omega_2,\omega_3)\in\Omega_{\beta_j^\lambda}$ and $u\in\mathcal{U}(t)$. Then $(\Omega_{\beta_j^\lambda},\mathcal{F}_{\beta_j^\lambda},\mathbb{P}_{\beta_j^\lambda})$ belongs to $\mathcal{S}$ and $(\beta_j^\lambda)\in(\mathcal{B}_r(t_0))^J$. For any $\alpha\in\mathcal{A}_d(t)$, we have, by using the convexity of the map $(s_i)\mapsto\max_i\{s_i\}$:

$$\begin{aligned}
&\max_{i=1,\ldots,I}\left\{\hat{p}_i^\lambda-\sum_{j=1}^J q_j\mathcal{J}_{ij}(t,x,\alpha,\beta_j^\lambda)\right\}\\
&=\ \max_{i=1,\ldots,I}\left\{(1-\lambda)(\hat{p}_i^0-\sum_{j=1}^J q_j\mathcal{J}_{ij}(t,x,\alpha,\beta_j^0))+\lambda(\hat{p}_i^1-\sum_{j=1}^J q_j\mathcal{J}_{ij}(t,x,\alpha,\beta_j^1))\right\}\\
&\le\ (1-\lambda)\sup_{\alpha\in\mathcal{A}_d(t_0)}\max_{i=1,\ldots,I}\left\{\hat{p}_i^0-\sum_{j=1}^J q_j\mathcal{J}_{ij}(t,x,\alpha,\beta_j^0)\right\}\\
&\qquad\qquad +\lambda\sup_{\alpha\in\mathcal{A}_d(t_0)}\max_{i=1,\ldots,I}\left\{\hat{p}_i^1-\sum_{j=1}^J q_j\mathcal{J}_{ij}(t,x,\alpha,\beta_j^1)\right\}\\
&\le\ (1-\lambda)z(t,x,\hat{p}^0,q)+\lambda z(t,x,\hat{p}^1,q)+\epsilon
\end{aligned}$$

because β^0 and β^1 are $\epsilon-$optimal for $z(t,x,\hat{p}^0,q)$ and $z(t,x,\hat{p}^1,q)$ respectively. Hence

$$\begin{aligned}
z(t,x,\hat{p}^\lambda,q)\ &\le\ \sup_{\alpha\in\mathcal{A}_d(t_0)}\max_{i=1,\ldots,I}\left\{\hat{p}_i^\lambda-\sum_{j=1}^J q_j\mathcal{J}_{ij}(t,x,\alpha,\beta_j^\lambda)\right\}\\
&\le\ (1-\lambda)z(t,x,q^0)+\lambda z(t,x,q^1)+\epsilon\ ,
\end{aligned}$$

which proves (5.10) because ϵ is arbitrary.

Next we show that $\mathbf{V}^{-*} = z$. Indeed we have by definition of z:

$$\begin{aligned} z^*(t,x,p,q) &= \sup_{\hat{p}\in\mathbb{R}^I} \left\{ \langle p,\hat{p}\rangle - \inf_{(\beta_j)\in(\mathcal{B}_r(t_0))^J} \max_{i=1,\ldots,I} \left\{ \hat{p}_i - \inf_{\alpha\in\mathcal{A}_d(t_0)} \sum_{j=1}^J q_j \mathcal{J}_{ij}(t,x,\alpha,\beta_j) \right\}\right\} \\ &= \sup_{(\beta_j)\in(\mathcal{B}_r(t0))^J} \sup_{\hat{p}\in\mathbb{R}^I} \min_{i=1,\ldots,I} \left\{ \langle p,\hat{p}\rangle - \hat{p}_i + \inf_{\alpha\in\mathcal{A}_d(t_0)} \sum_{j=1}^J q_j \mathcal{J}_{ij}(t,x,\alpha,\beta_j) \right\} \end{aligned}$$

In this last expression, the $\sup_{\hat{p}}$ is attained by

$$\hat{p}_i = \inf_{\alpha\in\mathcal{A}_d(t_0)} \sum_{j=1}^J q_j \mathcal{J}_{ij}(t,x,\alpha,\beta_j) \ ,$$

for which all the arguments of the $\min_{i=1,\ldots,I}$ are equal. Hence

$$z^*(t,x,p,q) = \sup_{(\beta_j)\in(\mathcal{B}_r(t_0))^J} \sum_{i=1}^I p_i \inf_{\alpha\in\mathcal{A}_d(t_0)} \sum_{j=1}^J q_j \mathcal{J}_{ij}(t,x,\alpha,\beta_j) = \mathbf{V}^-(t,x,p,q)$$

because of (5.9). Since we have proved that z is convex with respect to $\hat{p}$, we get by duality

$$\mathbf{V}^{-*} = z^{**} = z \ .$$

□

Lemma 5.6 (Sub-dynamic principle for $\mathbf{V}^{-*}$) *We have for any $(t_0,x_0,\hat{p},q)\in[0,T)\times\mathbb{R}^N\times\mathbb{R}^I\times\Delta(J)$ and any $h\in(0,T-t_0]$,*

$$\mathbf{V}^{-*}(t_0,x_0,\hat{p},q) \leq \inf_{\beta\in\mathcal{B}_d(t_0)} \sup_{\alpha\in\mathcal{A}_d(t_0)} \mathbf{V}^{-*}(t_0+h, X_{t_0+h}^{t_0,x_0,\alpha,\beta}, \hat{p}(t_0+h), q) \ , \tag{5.11}$$

where

$$(\hat{p}(t_0+h))_i = \hat{p}_i - \sum_{j=1}^J q_j \int_{t_0}^{t_0+h} \ell_{ij}(s, X_s^{t_0,x_0,\alpha,\beta}, \alpha(s), \beta(s))ds \qquad \forall i\in\{1,\ldots,I\} \ .$$

Proof : We do the proof in two steps. First we prove the result when $\ell_{ij}=0$ for any (i,j). Then we complete the proof of the general case by a reduction argument.

Proof of (5.11) when $\ell_{ij}=0$: Let us denote by $V_1^{-*}(t_0,t_0+h,x_0,\hat{p},q)$ the right-hand side of (5.11) (where $\hat{p}(t_0+h)=\hat{p}$). Arguing as in Lemma 5.1 one can prove that $\mathbf{V}_1^{-*}$ is Lipschitz continuous with respect to x.

Let $\epsilon>0$ and $\beta^0\in\mathcal{B}_d(t_0)$ be some pure $\epsilon-$optimal strategy for $V_1^{-*}(t_0,t_0+h,x_0,\hat{p},q)$. For any $x\in\mathbb{R}^N$, we can find some $\epsilon-$optimal strategy $\hat{\beta}^x=(\beta_j^x)\in\mathcal{B}_r(t_0+h)$ for Player 2 in the game $\mathbf{V}^{-*}(t_0+h,x,\hat{p},q)$. From the Lipschitz continuity of the map

$$y \to \sup_{\alpha\in\mathcal{A}_d(t)} \max_{i\in\{1,\ldots,I\}} \left\{ \hat{p}_i - \sum_j q_j \mathcal{J}_{ij}(t_0+h,y,\alpha,\beta_j^x) \right\} \ ,$$

and of the map $y\to\mathbf{V}^{-*}(t_0+h,y,\hat{p},q)$, the strategy β^x is also $(2\epsilon)-$optimal for $\mathbf{V}^{-*}(t_0+h,y,\hat{p},q)$ if $y\in B(x,r)$ for some radius $r>0$.

Let $M=\|f\|_\infty$ and let us set $R=MT+|x_0|$. Then we choose $(x_l)_{l=1,\cdots,l_0}$ such that $\bigcup_{l=1}^{l_0} B(x_l,r/2)$ contains the ball $B(0,R)$. Let us set $E_1=B(x_1,r/2)$, and, for any $l=2,\ldots,l_0$, $E_l=B(x_l,r/2)\backslash\bigcup_{l'<l}B(x_{l'},r/2)$. Then $(E_l)_{l=1,\ldots,l_0}$ be a Borel partition of $B(0,R)$ such that, for any l, $E_l\subset B(x_l,r/2)$. We set

$$\beta_j^l=\beta_j^{x_l},\ \Omega_j^l=\Omega_{\beta_j^l},\ \mathcal{F}_j^l=\mathcal{F}_{\beta_j^l} \text{ and } \mathbb{P}_j^l=\mathbb{P}_{\beta_j^l}$$

for $j = 1, \ldots, J$ and $l = 1, \ldots, l_0$. Let τ be a delay common to β^0 and all the β_j^l. Without loss of generality we can also assume that τ is smaller than $r/(2M)$ and than h.

Let us now define a new strategy $\hat{\beta} = (\beta_j) \in (\mathcal{B}_r(t_0))^J$ in the following way: set

$$\Omega_{\beta_j} = \Pi_{l=1}^{l_0} \Omega_j^l,\ \mathcal{F}_{\beta_j} = \mathcal{F}_j^1 \otimes \ldots \otimes \mathcal{F}_j^{l_0} \text{ and } \mathbb{P}_{\beta_j} = \mathbb{P}_j^1 \otimes \ldots \otimes \mathbb{P}_j^{j_0}$$

and, for any $\omega = (\omega^1, \ldots, \omega^{l_0}) \in \Omega_{\beta_j}$ and $u \in \mathcal{U}(t_0)$,

$$\beta_j(\omega, u)(t) = \begin{cases} \beta^0(u)(t) & \text{if } t \in [t_0, t_0 + h) \\ \beta_j^l(\omega^l, u_{|[t_0+h,T]})(t) & \text{if } t \in [t_0 + h, T] \text{ and } X_{t_0+h-\tau}^{t_0,x_0,u,\beta^0(u)} \in E_l \end{cases}$$

Note that $X_{t_0+h-\tau}^{t_0,x_0,u,\beta^0(u)}$ always belongs to one and only one E_l thanks to the choice of R. Then $(\Omega_{\beta_j}, \mathcal{F}_{\beta_j}, \mathbb{P}_{\beta_j})$ belongs to $\mathcal{S}$ and $\hat{\beta} = (\beta_j) \in (\mathcal{B}_r(t_0))^J$.

For any pure strategy $\alpha \in \mathcal{A}(t_0)$, we have:

$$g_{ij}(X_T^{t_0,x_0,\alpha,\beta_j}) = \sum_{l=1}^{l_0} g_{ij}\left(X_T^{t_0+h, X_{t_0+h}^{t_0,x_0,\alpha,\beta^0}, \tilde{\alpha}, \beta_j^l}\right) \mathbf{1}_{\{X_{t_0+h-\tau}^{t_0,x_0,\alpha,\beta^0} \in E_l\}}$$

where $\tilde{\alpha} \in \mathcal{A}(t_0 + h)$ is a restriction of α to the time interval $[t_0 + h, T]$ defined by

$$\tilde{\alpha}(v) = \alpha(v')_{|[t_0+h,T]} \qquad \forall v \in \mathcal{V}(t_0 + h) \text{ where } v'(t) = \begin{cases} \bar{v}(t) & \text{if } t \in [t_0, t_0 + h] \\ v(t) & \text{otherwise} \end{cases}$$

the controls $(\bar{u}, \bar{v})$ being the pair associated with (α, β^0) as in (5.3). Note that, if $X_{t_0+h-\tau}$ belongs to some E_l, then, by definition of E_l and τ, X_{t_0+h} belongs to the ball $B(x_l, r)$ and therefore (β_j^l) is $(2\epsilon)-$optimal for $\mathbf{V}^{-*}(t_0 + h, X_{t_0+h}, \hat{p}, q)$. Hence

$$\max_{i \in \{1,\ldots,I\}} \left\{ \hat{p}_i - \sum_j q_j \mathbb{E}_{\beta_j}\left[g_{ij}(X_T^{t_0,x_0,\alpha,\beta_j})\right] \right\} =$$

$$\max_{i \in \{1,\ldots,I\}} \left\{ \hat{p}_i - \sum_j q_j \sum_{l=1}^{l_0} \left(\int_{\Omega_j^l} g_{ij}\left(X_T^{t_0+h, X_{t_0+h}^{t_0,x_0,\alpha,\beta^0}, \tilde{\alpha}, \beta_j^l}\right) d\mathbb{P}_j^l(\omega^l) \right) \mathbf{1}_{O^l} \right\}$$

(where we have set $O^l = \{X_{t_0+h-\tau}^{t_0,x_0,\alpha,\beta^0} \in E_l\}$)

$$\leq \sum_{l=1}^{l_0} \sup_{\alpha' \in \mathcal{A}(t_0+h)} \max_{i \in \{1,\ldots,I\}} \left\{ \hat{p}_i - \sum_j q_j \left(\int_{\Omega_j^l} g_{ij}\left(X_T^{t_0+h, X_{t_0+h}^{t_0,x_0,\alpha,\beta^0}, \alpha', \beta_j^l}\right) d\mathbb{P}_j^l(\omega^l) \right) \right\} \mathbf{1}_{O^l}$$

(because of the convexity of the map $s = (s_i) \mapsto \max\{s_i\}$)

$$\leq \sum_{l=1}^{l_0} \left(\mathbf{V}^{-*}\left(t_0 + h, X_{t_0+h}^{t_0,x_0,\alpha,\beta^0}, \hat{p}, q\right) + 2\epsilon \right) \mathbf{1}_{O^l}$$

(because, on O^l, (β_j^l) is $(2\epsilon)-$optimal for $\mathbf{V}^{-*}\left(t_0 + h, X_{t_0+h}^{t_0,x_0,\alpha,\beta^0}, \hat{p}, q\right)$)

$$= \mathbf{V}^{-*}\left(t_0 + h, X_{t_0+h}^{t_0,x_0,\alpha,\beta^0}, \hat{p}, q\right) + 2\epsilon$$

$$\leq \mathbf{V}_1^{-*}(t_0, t_0 + h, x_0, \hat{p}, q) + 3\epsilon\ ,$$

because β^0 is $\epsilon-$optimal for $\mathbf{V}_1^{-*}(t_0, t_0 + h, x_0, \hat{p}, q)$. Therefore

$$\sup_{\alpha \in \mathcal{A}(t_0)} \max_{i \in \{1,\ldots,I\}} \left\{ \hat{p}_i - \sum_j q_j \mathbb{E}_{\beta_j}\left[g_{ij}(X_T^{t_0,x_0,\alpha,\beta_j})\right] \right\} \leq \mathbf{V}_1^{-*}(t_0, t_0 + h, x_0, \hat{p}, q) + 3\epsilon\ ,$$

which implies that

$$\mathbf{V}^{-*}(t_0,x_0,\hat{p},q) \le \ \mathbf{V}_1^{-*}(t_0,t_0+h,x_0,\hat{p},q) \ .$$

Proof in the general case : The idea is to reduce the Bolza problem to a Mayer problem thanks to an extension of the state. Namely, let us consider the extended dynamics (in $\mathbb{R}^N \times \mathbb{R}^{I\times J}$)

$$\left\{ \begin{array}{ll} x'(t) = f(t,x(t),u(t),v(t)) \ , & u(t)\in U, \ v(t)\in V \\ \rho_{ij}'(t) = \ell_{ij}(t,x(t),u(t),v(t)) & \\ x(t_0)=x_0, \ \rho_{ij}(t_0)=\rho_{ij,0} & \end{array}\right.$$

We denote by $\tilde{X}^{t_0,\rho_0,u,v}$ the solution of the above system for fixed $(u,v)\in\mathcal{U}(t_0)\times\mathcal{V}(t_0)$ and set

$$\tilde{\mathcal{J}}_{ij}(t_0,x_0,\rho_0,\alpha_i,\beta_j) = \mathbb{E}_{\alpha_i\beta_j}\left[\tilde{g}_{ij}(\tilde{X}^{t_0,x_0,\rho_0,\alpha_i,\beta_j})\right]$$

where $\tilde{g}_{ij}(x,\rho)=g_{ij}(x)+\rho_{ij}$. We also denote by $\tilde{\mathbf{V}}^{\pm}$ the upper and lower value functions associated to this Mayer problem. One easily checks that

$$\tilde{\mathbf{V}}^{\pm}(t_0,x_0,\rho_0,p,q) = \sum_{ij} p_iq_j\rho_{ij,0} + \mathbf{V}^{\pm}(t_0,x_0,p,q) \ .$$

Therefore

$$\begin{array}{rcl} \mathbf{V}^{-*}(t_0,x_0,\hat{p},q) & = & \sup_{p\in\Delta(I)}\left\{\langle \hat{p},p\rangle - \tilde{\mathbf{V}}^{-}(t_0,x_0,\rho_0,p,q) + \sum_{ij} p_iq_j\rho_{ij,0}\right\} \\ & = & \tilde{\mathbf{V}}^{-*}(t_0,x_0,\rho_0,\hat{p}+(\sum_j q_j\rho_{ij,0})_i,q) \end{array} \tag{5.12}$$

Note for later use that, since $\mathbf{V}^{-*}(t_0,x_0,\hat{p},q)$ is independent of ρ_0, we have

$$\tilde{\mathbf{V}}^{-*}(t_0,x_0,\rho_0,\hat{p}+(\sum_j q_j\rho_{ij,0})_i,q) = \tilde{\mathbf{V}}^{-*}(t_0,x_0,\rho_1,\hat{p}+(\sum_j q_j\rho_{ij,1})_i,q) \tag{5.13}$$

for any $\rho_0,\rho_1\in\mathbb{R}^{IJ}$. From the subdynamic programming for Mayer problems proved above, we have

$$\tilde{\mathbf{V}}^{-*}(t_0,x_0,\rho_0,\hat{p}+(\sum_j q_j\rho_{ij,0})_i,q) \le \inf_{\beta\in\mathcal{B}_d(t_0)} \sup_{\alpha\in\mathcal{A}_d(t_0)} \tilde{\mathbf{V}}^{-*}(t_0+h,\tilde{X}_{t_0+h}^{t_0,x_0,\rho_0,\alpha,\beta},\hat{p}+(\sum_j q_j\rho_{ij,0})_i,q) \ .$$

For fixed $(\alpha,\beta)\in\mathcal{A}_d(t_0)\times\mathcal{B}_d(t_0)$, let us set $\tilde{X}_t^{t_0,x_0,\rho_0,\alpha,\beta}=(x(t),\rho(t))$. Then

$$\rho_{ij}(t) = \rho_{ij,0} + \int_{t_0}^t \ell_{ij}(s,x(s),\alpha(s),\beta(s))ds \ ,$$

so that, from (5.13),

$$\begin{array}{l} \tilde{\mathbf{V}}^{-*}(t_0+h,x(t_0+h),\rho(t_0+h),\hat{p}+(\sum_j q_j\rho_{ij,0})_i,q) \\ \quad = \ \tilde{\mathbf{V}}^{-*}(t_0+h,x(t_0+h),\rho_0,\hat{p}+(\sum_j q_j(\rho_{ij,0}-\int_{t_0}^t \ell_{ij}(s,x(s),\alpha(s),\beta(s))ds))_i,q) \\ \quad = \ \mathbf{V}^{-*}(t_0+h,x(t_0+h),\hat{p}(t_0+h),q) \end{array}$$

where $\hat{p}(t_0+h)=\hat{p}-(\sum_j q_j\int_{t_0}^t \ell_{ij}(s,x(s),\alpha(s),\beta(s))ds)_i$. Therefore

$$\mathbf{V}^{-*}(t_0,x_0,\hat{p},q) \le \inf_{\beta\in\mathcal{B}_d(t_0)} \sup_{\alpha\in\mathcal{A}_d(t_0)} \mathbf{V}^{-*}(t_0+h,x(t_0+h),\hat{p}(t_0+h),q) \ .$$

□

As usual, the dynamic programming property has very much to do with Hamilton-Jacobi equations. In order to describe the Hamilton-Jacobi equation associated with the problem, we introduce some notations: first recall that $\mathcal{S}(I)$ denotes the set of $I\times I$ real symmetric matrices. For any $(X,p)\in\mathcal{S}(I)\times\Delta(I)$, we set

$$\lambda_{\min}(X,p) = \min\{\langle Xz,z\rangle \ ; \ z\in T_p\Delta(I), \ |z|=1\}$$

and

$$\lambda_{\max}(X,p) = \max\{\langle Xz,z\rangle \ ; \ z\in T_p\Delta(I), \ |z|=1\} \ ,$$

where $T_p\Delta(I)$ is the tangent cone to $\Delta(I)$ at p: namely

$$T_p\Delta(I) = \{z=(z_i)_{i\in\{1,\ldots,I\}}\in\mathbb{R}^I \ ; \ z_i<0 \ \Rightarrow \ p_i>0 \ \forall i\in\{1,\ldots,I\} \text{ and } \sum_i z_i = 0\} \ .$$

Proposition 5.7 ($\mathbf{V}^-$ is a supersolution of HJ) *For any fixed $\bar{q} \in \Delta(J)$, the map $(t, x, p) \to \mathbf{V}^-(t, x, p, \bar{q})$ satisfies:*

$$\min\left\{\lambda_{\min}\left(D^2_{pp}V(t,x,p,\bar{q}), p\right) \; ; \, \partial_t V(t,x,p,\bar{q}) + H(t,x,D_xV(t,x,p,\bar{q}),p,\bar{q})\right\} \leq 0 \tag{5.14}$$

in the viscosity sense with state constraints in $(0,T)\times\mathbb{R}^N\times\Delta(I)$, where H is defined by (5.7): namely for any fixed $\bar{q} \in \Delta(J)$ and test function $\phi \in \mathcal{C}^2((0,T)\times\mathbb{R}^N\times\mathbb{R}^I)$ such that $(t,x,p)\to \mathbf{V}^-(t,x,p,\bar{q}) - \phi(t,x,p)$ has a local minimum on $(0,T)\times\mathbb{R}^N\times\Delta(I)$ at some point $(\bar{t},\bar{x},\bar{p})$, one has

$$\min\left\{\lambda_{\min}\left(D^2_{pp}\phi(\bar{t},\bar{x},\bar{p}), \bar{p}\right) \; ; \, \partial_t \phi(\bar{t},\bar{x},\bar{p}) + H(\bar{t},\bar{x},D_x\phi(\bar{t},\bar{x},\bar{p}),\bar{p},\bar{q})\right\} \leq 0 \; .$$

Before starting the proof, let us give a Lemma which will be needed repeatedly.

Lemma 5.8 *Let $V = V(t,x,p)$ be continuous in $(0,T)\times\mathbb{R}^N\times\Delta(I)$ and convex with respect to p, and $\phi \in \mathcal{C}^2((0,T)\times\mathbb{R}^N\times\mathbb{R}^I)$ be a test function such that $V-\phi$ has a local minimum on $(0,T)\times\mathbb{R}^N\times\Delta(I)$ at some point $(\bar{t},\bar{x},\bar{p})$. If*

$$\lambda_{\min}\left(D^2_{pp}\phi(\bar{t},\bar{x},\bar{p}), \bar{p}\right) \; > \; 0 \; , \tag{5.15}$$

then there are some $\delta,\eta > 0$ such that

$$V(t,x,p) \geq \phi(t,x,\bar{p}) + \langle D_p\phi(t,x,\bar{p}), p-\bar{p}\rangle + \frac{\delta}{2}|p-\bar{p}|^2$$

for any $(t,x)\in B((\bar{t},\bar{x}),\eta)$ and $p\in\Delta(I)$.

Proof of Lemma 5.8 : Because of (5.15), there are some $\eta,\gamma > 0$ such that

$$\langle D^2_{pp}\phi(t,x,p)z,z\rangle \geq \gamma|z|^2 \qquad \forall z\in T_{\bar{p}}\Delta(I) \text{ and } \forall (t,x,p)\in B((\bar{t},\bar{x},\bar{p}),\eta) \; .$$

Therefore

$$V(t,x,p) \geq \phi(t,x,p) \geq \phi(t,x,\bar{p}) + \langle D_p\phi(t,x,\bar{p}), p-\bar{p}\rangle + \frac{\gamma}{2}|p-\bar{p}|^2 \tag{5.16}$$

for any $(t,x,p)\in B((\bar{t},\bar{x},\bar{p}),\eta)$ with $p\in\Delta(I)$, because $p-\bar{p}\in T_{\bar{p}}\Delta(I)$.

We also note that, for any $(t,x)\in B((\bar{t},\bar{x}),\eta)$ and for any $p\in\Delta(I)\backslash \mathrm{Int}(B(\bar{p},\eta))$, we have

$$V(t,x,p) \; \geq \; \phi(t,x,\bar{p}) + \langle D_p\phi(t,x,\bar{p}), p-\bar{p}\rangle + \frac{\gamma}{2}\eta^2 \; . \tag{5.17}$$

Indeed, let us set $p_1 = \bar{p} + \frac{p-\bar{p}}{|p-\bar{p}|}\eta$ and let $\hat{p}'\in\partial_p^- V(t,x,p_1)$. Then we have

$$\begin{array}{rcl} V(t,x,p) & \geq & V(t,x,p_1) + \langle \hat{p}', p-p_1\rangle \\ & \geq & \phi(t,x,\bar{p}) + \langle D_p\phi(t,x,\bar{p}), p_1-\bar{p}\rangle + \frac{\gamma}{2}\eta^2 + \langle \hat{p}', p-p_1\rangle \\ & \geq & \phi(t,x,\bar{p}) + \langle D_p\phi(t,x,\bar{p}), p-\bar{p}\rangle + \langle \hat{p}' - D_p\phi(t,x,\bar{p}), p-p_1\rangle + \frac{\gamma}{2}\eta^2 \end{array}$$

where

$$\langle \hat{p}' - D_p\phi(t,x,\bar{p}), p-p_1\rangle \geq 0$$

because V is convex, $\hat{p}'\in\partial_p^- V(t,x,p_1)$, $D_p\phi(t,x,\bar{p})\in\partial_p^- V(t,x,\bar{p})$ and $p-p_1 = \sigma(p_1-\bar{p})$ for some $\sigma>0$. So (5.17) holds. Let us now choose $\delta\in(0,\gamma)$ such that $\max_{p\in\Delta(I)}\delta|p-\bar{p}|^2 \leq \gamma\eta^2$. Then combining (5.16) and (5.17) readily gives the desired result. □

Proof of Proposition 5.7: Assume that the test function $\phi\in\mathcal{C}^2((0,T)\times\mathbb{R}^N\times\mathbb{R}^I)$ is such that $(t,x,p)\to\mathbf{V}^-(t,x,p,\bar{q}) - \phi(t,x,p)$ has a local minimum on $(0,T)\times\mathbb{R}^N\times\Delta(I)$ at some point $(\bar{t},\bar{x},\bar{p})$. Without loss of generality we suppose that $\mathbf{V}^-(\bar{t},\bar{x},\bar{p},\bar{q}) = \phi(\bar{t},\bar{x},\bar{p})$ and that $\bar{p}_i>0$ for any $i\in\{1,\ldots,I\}$: otherwise we just restrict the functions of the indices i for which $\bar{p}_i>0$. Let us furthermore suppose that

$$\lambda_{\min}\left(D^2_{pp}\phi(\bar{t},\bar{x},\bar{p}), \bar{p}\right) \; > \; 0 \; .$$

Then we have to prove that

$$\partial_t\phi(\bar{t},\bar{x},\bar{p}) + H(\bar{t},\bar{x},D_x\phi(\bar{t},\bar{x},\bar{p},\bar{q}),\bar{p},\bar{q}) \leq 0 \; . \tag{5.18}$$

According to Lemma 5.8, there are some $\delta,\eta>0$ such that

$$\mathbf{V}^-(t,x,p,\bar{q}) \geq \phi(t,x,\bar{p}) + \langle D_p\phi(t,x,\bar{p}), p-\bar{p}\rangle + \frac{\delta}{2}|p-\bar{p}|^2 \tag{5.19}$$

for any $(t,x)\in B((\bar t,\bar x),\eta)$ and $p\in\Delta(I)$.

Let us set $\hat p = D_p\phi(\bar t,\bar x,\bar p)$. From (5.19), we have, for any $(t,x,\hat p',p)$ with $(t,x)\in B((\bar t,\bar x),\eta)$, $p\in\Delta(I)$ and $\hat p'\in\mathbb{R}^I$,

$$\langle p,\hat p'\rangle - \mathbf{V}^-(t,x,p,\bar q) \le -\phi(t,x,\bar p) - \langle D_p\phi(t,x,\bar p)-\hat p', p-\bar p\rangle + \langle \bar p,\hat p'\rangle - \frac{\delta}{2}|p-\bar p|^2 \; . \tag{5.20}$$

Maximizing both sides of the inequality with respect to $p\in\Delta(I)$ we get

$$\begin{aligned}\mathbf{V}^{-*}(t,x,\hat p',\bar q) \;&\le\; -\phi(t,x,\bar p) + \sup_{p\in\mathbb{R}^I}\left\{-\langle D_p\phi(t,x,\bar p)-\hat p', p-\bar p\rangle - \frac{\delta}{2}|p-\bar p|^2\right\} + \langle\bar p,\hat p'\rangle \\ &\le\; -\phi(t,x,\bar p) + \frac{1}{2\delta}\left|D_p\phi(t,x,\bar p)-\hat p'\right|^2 + \langle\bar p,\hat p'\rangle \; .\end{aligned}\tag{5.21}$$

Note that, since $\mathbf{V}^-$ is convex with respect to p, inequality (5.19) implies that the vector $\hat p$ belongs to $\partial_p^- V(\bar t,\bar x,\bar p,\bar q)$, so that

$$\langle\bar p,\hat p\rangle - \mathbf{V}^{-*}(\bar t,\bar x,\hat p,\bar q) = \mathbf{V}^-(\bar t,\bar x,\bar p,\bar q) = \phi(\bar t,\bar x,\hat p) \; .\tag{5.22}$$

Let us now choose $h>0$ small enough and apply the sub-dynamic property (Lemma 5.6) of $\mathbf{V}^{-*}$. In view of (5.21) we have

$$\begin{aligned}&\mathbf{V}^{-*}(\bar t,\bar x,\hat p,\bar q)\\ &\le \inf_{\beta\in\mathcal{B}_d(\bar t)}\sup_{\alpha\in\mathcal{A}_d(\bar t)} \mathbf{V}^{-*}(\bar t+h, X_{\bar t+h}^{\bar t,\bar x,\alpha,\beta},\hat p(\bar t+h),\bar q)\\ &\le \inf_{\beta\in\mathcal{B}_d(\bar t)}\sup_{\alpha\in\mathcal{A}_d(\bar t)} -\phi(\bar t+h, X_{\bar t+h}^{\bar t,\bar x,\alpha,\beta},\bar p) + \frac{1}{2\delta}\left|D_p\phi(\bar t+h,X_{\bar t+h}^{\bar t,\bar x,\alpha,\beta},\bar p)-\hat p(\bar t+h)\right|^2 + \langle\bar p,\hat p(\bar t+h)\rangle\end{aligned}\tag{5.23}$$

where

$$\hat p(\bar t+h)_i = \hat p_i - \sum_{j=1}^J \bar q_j\int_{\bar t}^{\bar t+h}\ell_{ij}(s,X_s^{\bar t,\bar x,\alpha,\beta},\alpha(s),\beta(s))ds \qquad \forall i\in\{1,\dots,I\}\; .\tag{5.24}$$

In particular

$$\lim_{h\to0^+}\frac{1}{h}\left|D_p\phi(\bar t+h,X_{\bar t+h}^{\bar t,\bar x,\alpha,\beta},\bar p)-\hat p(\bar t+h)\right|^2 = 0\; .$$

Putting together (5.23), (5.22) and (5.24) gives

$$\begin{aligned}\sup_{\beta\in\mathcal{B}_d(\bar t)}\inf_{\alpha\in\mathcal{A}_d(\bar t)}\Big\{ &\phi(\bar t+h,X_{\bar t+h}^{\bar t,\bar x,\alpha,\beta},\bar p)-\phi(\bar t,\bar x,\bar p)\\ &-\frac{1}{2\delta}\left|D_p\phi(\bar t+h,X_{\bar t+h}^{\bar t,\bar x,\alpha,\beta},\bar p)-\hat p(\bar t+h)\right|^2 + \sum_{i,j}\bar p_i\bar q_j\int_{\bar t}^{\bar t+h}\ell_{ij}(s,X_s^{\bar t,\bar x,\alpha,\beta},\alpha(s),\beta(s))ds\Big\}\le 0\; .\end{aligned}$$

Dividing by $h>0$ and letting $h\to0$ gives (5.18) by the arguments of Chapter 3, Lemma 3.15. □

To state the symmetric results for $\mathbf{V}^+$, we only need to note that

$$(-\mathbf{V}^+)(t,x,p,q) = \sup_{\hat\alpha\in(\mathcal{A}_r(t_0))^I}\inf_{\hat\beta\in(\mathcal{B}_r(t_0))^J}\sum_{i=1}^I\sum_{j=1}^I p_iq_j(-\mathcal{J}_{ij}(t_0,x_0,\alpha_i,\beta_j))\; ,$$

which is of the same form as $\mathbf{V}^-$ when one switches the roles of the Players and the sign of the payoffs. From this we easily deduce:

Proposition 5.9 ($\mathbf{V}^+$ is a subsolution of some HJ) *For any fixed $\bar p\in\Delta(I)$, the map $(t,x,q)\to\mathbf{V}^+(t,x,\bar p,q)$ satisfies in $(0,T)\times\mathbb{R}^N\times\Delta(J)$:*

$$\max\left\{\lambda_{\max}\left(D^2_{qq}V(t,x,\bar p,q),q\right)\;;\;\partial_tV(t,x,\bar p,q)+H(t,x,D_xV(t,x,\bar p,q),\bar p,q)\right\}\ge0\;,\tag{5.25}$$

in the viscosity sense with state constraints. Namely for any fixed $\bar p\in\Delta(I)$ and for any test function $\phi\in\mathcal{C}^2((0,T)\times\mathbb{R}^N\times\mathbb{R}^J)$ such that $(t,x,q)\to\mathbf{V}^+(t,x,\bar p,q)-\phi(t,x,q)$ has a local maximum on $(0,T)\times\mathbb{R}^N\times\Delta(J)$ at some point $(\bar t,\bar x,\bar q)$, one has

$$\max\left\{\lambda_{\max}\left(D^2_{qq}\phi(\bar t,\bar x,\bar q),\bar q\right)\;;\;\partial_t\phi(\bar t,\bar x,\bar q)+H(\bar t,\bar x,D_x\phi(\bar t,\bar x,\bar q,\bar p,\bar q)\right\}\ge0\;.$$

Remark : We use here Isaacs' assumption (5.7). Indeed, $\mathbf{V}^-$ is actually a supersolution of the Hamilton-Jacobi equation (5.14) with

$$H^-(t,x,\xi,p,q) = \inf_u \sup_v \left\{ \langle f(t,x,u,v), \xi \rangle + \sum_{i,j} p_i q_j \ell_{ij}(t,x,u,v) \right\} ,$$

while $\mathbf{V}^+$ is a subsolution of (5.25) with a Hamiltonian H^+ defined by

$$H^+(t,x,\xi,p,q) = \sup_v \inf_u \left\{ \langle f(t,x,u,v), \xi \rangle + \sum_{i,j} p_i q_j \ell_{ij}(t,x,u,v) \right\} .$$

5.4 Comparison principle and existence of the value

In this section we prove that our game has a value: $\mathbf{V}^+ = \mathbf{V}^-$. This value can be characterized in terms of viscosity solutions of some Hamilton-Jacobi equations with a double obstacle.

The key argument for this is a comparison principle, that we state for later use for a general Hamiltonian H. We assume that $H : (0,T) \times \mathbb{R}^N \times \mathbb{R}^N \times \Delta(I) \times \Delta(J) \to \mathbb{R}$ is continuous and that there is a constant C such that, for any $(p,q) \in \Delta(I) \times \Delta(J)$, any $(t_1,x_1),(t_2,x_2) \in (0,T) \times \mathbb{R}^N$ and any $\xi \in \mathbb{R}^N$,

$$|H(t_1,x_1,\xi,p,q) - H(t_2,x_2,\xi,p,q)| \le C|(t_1,x_1)-(t_2,x_2)|(1+|\xi|) \tag{5.26}$$

while, for any $(p,q) \in \Delta(I) \times \Delta(J)$, any $(t,x) \in (0,T) \times \mathbb{R}^N$ and any $\xi_1, \xi_2 \in \mathbb{R}^N$,

$$|H(t,x,\xi_1,p,q) - H(t,x,\xi_2,p,q)| \le C|\xi_1 - \xi_2| \tag{5.27}$$

Let us point out that the map H defined by (5.7) satisfies the above assumptions under conditions (5.2) on the dynamics.

We now consider a double obstacle Hamilton-Jacobi equation in $(0,T) \times \mathbb{R}^N \times \Delta(I) \times \Delta(J)$ which is formally defined as

$$\min\left\{\lambda_{\min}(D^2_{pp}z,p) \,;\, \max\left\{\lambda_{\max}(D^2_{qq}z,q) \,;\, \partial_t z + H(t,x,D_x z,p,q)\right\}\right\} = 0 , \tag{5.28}$$

or, equivalently, for functions which are convex with respect to p and concave with respect to q, as

$$\max\left\{\lambda_{\max}(D^2_{qq}z,q) \,;\, \min\left\{\lambda_{\min}(D^2_{pp}z,p) \,;\, \partial_t z + H(t,x,D_x z,p,q)\right\}\right\} = 0 .$$

Definition 5.10 *We say that a function $w : [0,T] \times \mathbb{R}^N \times \Delta(I) \times \Delta(J) \mapsto \mathbb{R}$ is a subsolution of (5.28) if w is Lipschitz continuous, convex with respect to p and concave with respect to q and if, for any test function $\phi \in \mathcal{C}^2((0,T) \times \mathbb{R}^N \times \mathbb{R}^J)$ such that the map*

$$(t,x,q) \to w(t,x,\bar{p},q) - \phi(t,x,q)$$

has a local maximum at some point $(\bar{t},\bar{x},\bar{q}) \in (0,T) \times \mathbb{R}^N \times \Delta(J)$ for some $\bar{p} \in \Delta(I)$, one has

$$\max\left\{\lambda_{\max}(D^2_{qq}\phi,\bar{q}) \,;\, \partial_t \phi + H(\bar{t},\bar{x},D_x\phi,\bar{p},\bar{q})\right\} \ge 0 \qquad \text{at} \qquad (\bar{t},\bar{x},\bar{p},\bar{q}) .$$

In a symmetric way, w is a supersolution of the Hamilton-Jacobi equation (5.28) if w is Lipschitz continuous, convex with respect to p and concave with respect to q and if, for any test function $\phi \in \mathcal{C}^2((0,T) \times \mathbb{R}^N \times \mathbb{R}^I)$ such that the map

$$(t,x,p) \to w(t,x,p,\bar{q}) - \phi(t,x,p)$$

has a local minimum at some point $(\bar{t},\bar{x},\bar{p}) \in (0,T) \times \mathbb{R}^N \times \Delta(I)$ for some $\bar{q} \in \Delta(J)$, one has

$$\min\left\{\lambda_{\min}(D^2_{pp}\phi,\bar{p}) \,;\, \partial_t \phi + H(\bar{t},\bar{x},D_x\phi,\bar{p},\bar{q})\right\} \le 0 \qquad \text{at} \qquad (\bar{t},\bar{x},\bar{p},\bar{q}) .$$

Finally we say that w is a solution of (5.28) if w is at the same time a dual subsolution and a dual supersolution of (5.28).

Remark 5.11 In the case $J = 1$ (when only Player 1 has a private information), a solution of (5.28) does not depend on q and the definition of subsolution reduces to the usual inequality

$$\partial_t w(t,x,p) + H(t,x,D_x w(t,x,p),p) \ \geq\ 0 \qquad \text{in } (0,T)\times\mathbb{R}^N\ .$$

Theorem 5.12 (Comparison principle) *Let $w_1, w_2 : [0,T]\times\mathbb{R}^N\times\Delta(I)\times\Delta(J)\mapsto\mathbb{R}$ be respectively a subsolution and a supersolution of the Hamilton-Jacobi equation (5.28). We assume that for any $(x,p,q)\in\mathbb{R}^N\times\Delta(I)\times\Delta(J)$, $w_1(T,x,p,q)\leq w_2(T,x,p,q)$. Then $w_1\leq w_2$ in $[0,T]\times\mathbb{R}^N\times\Delta(I)\times\Delta(J)$.*

The comparison principle is proved at the end of the section. Let us now state the main result of this chapter:

Theorem 5.13 (Existence and characterization of the value) *Assume that conditions (5.2) on f and on the g_i hold and that Isaacs' assumption (5.7) is satisfied. Then we have*

$$\mathbf{V}^+(t,x,p,q) = \mathbf{V}^-(t,x,p,q) \qquad \forall (t,x,p,q)\in[0,T]\times\mathbb{R}^N\times\Delta(I)\times\Delta(J)\ .$$

Furthermore the value function $\mathbf{V} := \mathbf{V}^+ = \mathbf{V}^-$ is the unique solution of the Hamilton-Jacobi equations (5.28), where H is defined by (5.7), such that $V(T,x,p,q) = \sum_{ij} p_i q_j g_{ij}(x)$.

Proof of Theorem 5.13: From Lemma 5.1 $\mathbf{V}^-$ and $\mathbf{V}^+$ are Lipschitz continuous. From Lemma 5.2, we know that $\mathbf{V}^+$ and $\mathbf{V}^-$ are convex with respect to p and concave with respect to q. Proposition 5.7 states that $\mathbf{V}^-$ is a supersolution of (5.28), where H is defined by (5.7), while Proposition 5.9 states that $\mathbf{V}^+$ is a subsolution of (5.28). Since $\mathbf{V}^+(T,\cdot,p,q) = \mathbf{V}^-(T,\cdot,p,q) = \sum_{i,j} p_i q_j g_{ij}(\cdot)$, the comparison principle states that $\mathbf{V}^+\leq\mathbf{V}^-$. But the reverse inequality always holds. Hence $\mathbf{V}^- = \mathbf{V}^+$ and the game has a value. □

The proof requires some localization argument which can be established exactly as in Lemma 3.40 of Chapter 3:

Lemma 5.14 *Assume that H satisfies (5.26) and (5.27). If w is a subsolution of (5.28) on $(0,T)\times\mathbb{R}^N$ (resp. a supersolution of (5.28) on $(0,T)\times\mathbb{R}^N$), then, for any $(t_0,x_0)\in[0,T)\times\mathbb{R}^N$, w is still a subsolution (resp. supersolution) in the set $C(t_0,x_0)\times\Delta(I)\times\Delta(J)$, where*

(5.29)
$$C_{t_0,x_0} = \{(t,x)\in[t_0,T]\times\mathbb{R}^N\ ,\ |x-x_0|\leq C(t-t_0)\}\ .$$

For the subsolution for instance, this means that, if a $\mathcal{C}^2$ test function $\phi=\phi(t,x,q)$ is such that $(t,x,q)\to w(t,x,\bar p,q)-\phi(t,x,q)$ has a local maximum on $C_{t_0,x_0}\times\Delta(J)$ at some point $(\bar t,\bar x,\bar q)$ with $\bar t<T$ for some $\bar p\in\Delta(I)$, then

$$\max\left\{\lambda_{\max}(D^2_{qq}\phi,\bar q)\ ;\ \partial_t\phi + H(\bar t,\bar x,D_x\phi,\bar p,\bar q)\right\}\geq 0 \qquad \text{at} \qquad (\bar t,\bar x,\bar p,\bar q)\ .$$

Proof of Theorem 5.12 : We now start the proof of the inequality $w_1\leq w_2$ in the usual way, by assuming that

$$\sup_{(t,x,p,q)\in(0,T)\times\mathbb{R}^N\times\Delta(I)\times\Delta(J)} (w_1-w_2) > 0\ .$$

Then, for $\sigma,\beta>0$ sufficiently small, there is some (t_0,x_0) such that

$$M := \sup_{(t,x,p,q)\in C_{t_0,x_0}\times\Delta(I)\times\Delta(J)} w_1(t,x,p,q)-w_2(t,x,p,q)-\sigma(T-t)+\beta(|p|^2+|q|^2)\ >\ 0\ ,$$

where C_{t_0,x_0} is defined by (5.29). We now use the separation of variables technique: for $\epsilon>0$ we set

$$\Phi_\epsilon((t,x),(s,y),p,q) = w_1(t,x,p,q)-w_2(s,y,p,q)-\frac{1}{2\epsilon}|(s,y)-(t,x)|^2-\sigma(T-s)+\beta(|p|^2+|q|^2)$$

and consider the problem

$$M_\epsilon := \sup_{(t,x),(s,y)\in C_{t_0,x_0},\ (p,q)\in\Delta(I)\times\Delta(J)} \Phi_\epsilon((t,x),(s,y),p,q)\ .$$

Note that $M_\epsilon\geq M$ and that the above problem has a maximum point $((t_\epsilon,x_\epsilon),(s_\epsilon,y_\epsilon))$. As in Lemma 3.41 of Chapter 3 we have the following estimates on $(t_\epsilon,x_\epsilon),(s_\epsilon,y_\epsilon),p_\epsilon,q_\epsilon$.

Lemma 5.15 *(i)* $\lim_{\epsilon\to 0^+} M_\epsilon = M$,

(ii) $\lim_{\epsilon\to 0^+} \frac{1}{\epsilon}|(t_\epsilon, x_\epsilon) - (s_\epsilon, y_\epsilon)|^2 = 0$

(iii) for $\epsilon > 0$ *and* $\beta > 0$ *small enough,* $t_\epsilon < T$ *and* $s_\epsilon < T$.

We now complete the proof of the Theorem: since the map $(t, x, q) \to \Phi_\epsilon((t, x), (s_\epsilon, y_\epsilon), p_\epsilon, q_\epsilon)$ has a maximum at the point $(t_\epsilon, x_\epsilon, q_\epsilon)$ on $C_{t_0,x_0} \times \Delta(J)$, we have, for any $(t, x, q) \in C_{t_0,x_0} \times \Delta(J)$ and any $\hat{q}_\epsilon \in \partial^+_q w_2(s_\epsilon, y_\epsilon, p_\epsilon, q_\epsilon)$,

$$\begin{array}{rcl} w_1(t,x,p_\epsilon,q) & \leq & w_1(t_\epsilon, x_\epsilon, p_\epsilon, q_\epsilon) + w_2(s_\epsilon, y_\epsilon, p_\epsilon, q) - w_2(s_\epsilon, y_\epsilon, p_\epsilon, q_\epsilon) \\ & & +\dfrac{1}{2\epsilon}\left[|(s_\epsilon, y_\epsilon) - (t,x)|^2 - |(s_\epsilon, y_\epsilon) - (t_\epsilon, x_\epsilon)|^2\right] - \beta(|q|^2 - |q_\epsilon|^2) \\ & \leq & w_1(t_\epsilon, x_\epsilon, p_\epsilon, q_\epsilon) + \langle \hat{q}_\epsilon, q - q_\epsilon\rangle \\ & & +\dfrac{1}{2\epsilon}\left[|(s_\epsilon, y_\epsilon) - (t,x)|^2 - |(s_\epsilon, y_\epsilon) - (t_\epsilon, x_\epsilon)|^2\right] - \beta(|q|^2 - |q_\epsilon|^2) \end{array}$$

Let us denote by $\phi(t, x, q)$ the right-hand side of the above inequality. Then ϕ is a smooth function such that $w_1(t_\epsilon, x_\epsilon, p_\epsilon, q_\epsilon) = \phi(t_\epsilon, x_\epsilon, q_\epsilon)$. In particular the map $(t, x, q) \to w_1(t, x, p_\epsilon, q) - \phi(t, x, q)$ has a maximum at the point $(t_\epsilon, x_\epsilon, q_\epsilon)$ on $C_{t_0,x_0} \times \Delta(J)$, with $t_\epsilon < T$. Since w_1 is a subsolution and $D^2_{qq}\phi(t_\epsilon, x_\epsilon, q_\epsilon) = -2\beta I_J < 0$, Lemma 5.14 implies that

$$\frac{t_\epsilon - s_\epsilon}{\epsilon} + H\left(t_\epsilon, x_\epsilon, \frac{x_\epsilon - y_\epsilon}{\epsilon}, p_\epsilon, q_\epsilon\right) \geq 0\ . \tag{5.30}$$

In a symmetric way, since the map $(s, y, p) \to \Phi_\epsilon((t_\epsilon, x_\epsilon), (s, y), p, q_\epsilon)$ has a maximum at $(s_\epsilon, y_\epsilon, p_\epsilon)$ on $C_{t_0,x_0} \times \Delta(I)$, one has, for any $(s, y, p) \in C_{t_0,x_0} \times \Delta(I)$ and any $\hat{p}_\epsilon \in \partial^-_p w_1(t_\epsilon, x_\epsilon, p_\epsilon, q_\epsilon)$,

$$\begin{array}{rcl} w_2(s,y,p,q_\epsilon) & \geq & w_2(s_\epsilon, y_\epsilon, p_\epsilon, q_\epsilon) - \langle \hat{p}_\epsilon, p - p_\epsilon\rangle - \dfrac{1}{2\epsilon}\left[|(s,y) - (t_\epsilon, x_\epsilon)|^2 - |(s_\epsilon, y_\epsilon) - (t_\epsilon, x_\epsilon)|^2\right] \\ & & +\sigma(s - s_\epsilon) + \beta(|p|^2 - |p_\epsilon|^2) \end{array}$$

and, since w_2 is a supersolution of (5.28) we obtain, again thanks to Lemma 5.14,

$$\frac{t_\epsilon - s_\epsilon}{\epsilon} + \sigma + H\left(s_\epsilon, y_\epsilon, \frac{x_\epsilon - y_\epsilon}{\epsilon}, p_\epsilon, q_\epsilon\right) \leq 0\ . \tag{5.31}$$

Computing the difference between (5.30) and (5.31) gives

$$-\sigma + H\left(t_\epsilon, x_\epsilon, \frac{x_\epsilon - y_\epsilon}{\epsilon}, p_\epsilon, q_\epsilon\right) - H\left(s_\epsilon, y_\epsilon, \frac{x_\epsilon - y_\epsilon}{\epsilon}, p_\epsilon, q_\epsilon\right) \geq 0\ .$$

We now use assumption (5.26) on H:

$$-\sigma + C\left[1 + \frac{|x_\epsilon - y_\epsilon|}{\epsilon}\right]|x_\epsilon - y_\epsilon| \geq 0\ .$$

Letting finally $\epsilon \to 0^+$ and using Lemma 5.15 we get a contradiction since σ is positive. □

5.5 Comments

The game studied in this chapter is strongly inspired by *repeated games* with lack of information introduced by Aumann and Maschler: see the monographs by Aumann and Maschler [12] and by Sorin [189] for a general presentation. Repeated games with lack of information on one side (i.e., $I = 1$ or $J = 1$) or on both sides (i.e., $I, J \geq 2$) have a value [12], [163], in the sense that the averaged $n-$stage games converge to a limit as $n \to +\infty$. This value can be characterized in terms of the value of the "non revealing game" via the convexification operator (for $I = 1$ or $J = 1$) or the Mertens-Zamir operator (when $I, J \geq 2$).
There are several proofs of Aumann-Maschler's result (see [189]). The most convenient for our purpose—the *dual approach*—was initiated by De Meyer in [87] and later developped by De Meyer and Rosenberg [88] and by Laraki [144]. It is this approach which can be extended to differential games.

Now let us turn to the litterature on differential games with incomplete/imperfect information: several papers analyse differential games where the players do not share the same information on the game. In most of these papers one tries to build a strategy for a nonfully informed controller, the other player being seen as a disturbance: see for instance the monograph by Baçsar and Bernhard [14] and the papers by Bernhard [42], by Baras and Patel [16], Baras and James [17]. So in terms of game this means that one looks at a kind of worse case design. In contrast, few works are dedicated to the existence of a value for this class of games: Rapaport and Bernhard [180], on the one hand, and Petrosjan [171], on the other hand, analyse this question through some examples. Cardaliguet and Quincampoix [71] consider a general class of differential games where the players only know that the initial position of the game is distributed according to some probability.

The first adaptation of the Aumann-Maschler's theory to differential games goes back to Cardaliaguet [69], which deals with deterministic differential games with a terminal payoff, and with games where there is some private information on the initial position of the system. It is generalized to stochastic differential games and to games with running payoffs in Cardaliaguet-Rainer [74]. The infinite horizon problem is considered in As Soulaimani [8]. Examples of such games are analysed in Cardaliaguet [72], Cardaliaguet-Rainer [76] and Souquière [193], while the construction of optimal strategies and approximations are carried out in Cardaliaguet [75] and Souquière [193].

Appendix A

Complement on ordinary differential equations

Throughout this chapter $T > 0$ denotes a fixed horizon. Our aim is to recall some basic properties of differential equations of the form

$$X'_t = f(t, X_t)$$

where $f : [0,T] \times \mathbb{R}^N \to \mathbb{R}^N$ is measurable, locally Lipschitz continuous with respect to the x variable. To do so we first recall the notion of absolutely continuous maps. Then we state and prove the Cauchy-Lipschitz Theorem in this framework.

A.1 Absolutely continuous maps

Let us first recall that a map $X : [0,T] \to \mathbb{R}^N$ is absolutely continuous if, for $\epsilon > 0$, there is some $\eta > 0$ such that, for any collection $([a_n, b_n])_{n\in\mathbb{N}}$ of disjoint subintervals of $[0,T]$,

$$\sum_{n=0}^{\infty}(b_n - a_n) \ < \ \eta \quad \Rightarrow \quad \sum_{n=0}^{\infty}|X_{b_n} - X_{a_n}| \ < \ \epsilon \ .$$

Note that, if X is absolutely continuous, then X is continuous. It is well-known (see for instance [112]) that X is absolutely continuous if and only if there is some $Z \in L^1([0,T], \mathbb{R}^N)$ such that

$$X_t - X_0 = \int_0^t Z_s ds \qquad \forall t \in [0,T] \ . \tag{A.1}$$

Moreover Z is uniquely defined by the above equality.

The following Lemma states that the map Z is the derivative of X at almost every point of $(0,T)$:

Lemma A.1 *If X is absolutely continuous on $[0,T]$, then there is a set S of full Lebesgue measure on $[0,T]$ such that, for any $t \in S$, X has a derivative with $X'_t = Z_t$.*

Notation : From now on, we denote by X' the unique map $Z \in L^1([0,T], \mathbb{R}^N)$ such that (A.1) holds.

Remark A.2 If $X' \in L^\infty([0,T], \mathbb{R}^N)$, then X is Lipschitz continuous because

$$|X_{t_1} - X_{t_2}| = \left|\int_{t_1}^{t_2} Z(s)ds\right| \le \left|\int_{t_1}^{t_2} |Z(s)|ds\right| \le \|Z\|_\infty |t_2 - t_1|$$

for any $t_1, t_2 \in [0,T]$. The converse also holds: if X is Lipschitz continuous, then X is clearly absolutely continuous with $X' \in L^\infty([0,1], \mathbb{R}^N)$.

Proof of Lemma A.1: Let $S \subset (0,T)$ be the set of Lebesgue points of Z: namely $t \in S$ if

$$\lim_{h\to 0} \frac{1}{2h}\int_{t-h}^{t+h} |Z_s - Z_t| ds = 0 \ .$$

It is well-known that S has a full measure in $(0,T)$. Then

$$\left|\frac{1}{h}(X_{t+h} - X_t - hZ_t)\right| \le \frac{1}{|h|}\left|\int_0^h |Z_s - Z_t| ds\right| \to 0 \text{ as } h \to 0 \ .$$

□

A.2 Ordinary differential equations

Let $f : [0,T] \times \mathbb{R}^N \to \mathbb{R}^N$ be a Borel measurable map. In this section we recall some well-known results on the differential equation

$$X'_t = f(t, X_t) \tag{A.2}$$

We assume the following conditions on f: f is locally Lipschitz continuous with respect to the x variable: for any $R > 0$ there is a map $m_R \in L^1([0,T], \mathbb{R}^+)$ such that

$$|f(t,x) - f(t,y)| \le m_R(t)|x - y| \qquad \forall x, y \in B(0,R), \text{ for a.e. } t \in [0,T] \tag{A.3}$$

and f has at most a linear growth: there are $a, b \in L^1([0,T], \mathbb{R}_+)$ such that

$$|f(t,x)| \le a(t)|x| + b(t) \qquad \forall x \in \mathbb{R}^N, \text{ for a.e. } t \in [0,T] \ . \tag{A.4}$$

Definition A.3 *A solution of equation (A.2) with initial condition $(t_0, x_0) \in [0,T] \times \mathbb{R}^N$ is an absolutely continuous map $X : [0,T] \to \mathbb{R}^N$ such that $X_{t_0} = x_0$ and which satisfies*

$$X'_t = f(t, X_t) \qquad \text{for almost every } \ t \in [0,T] \ .$$

Theorem A.4 *Under the above assumptions, for any initial condition $(t_0, x_0) \in [0,T] \times \mathbb{R}^N$ there is a unique absolutely continuous solution to (A.2) such that $X_{t_0} = x_0$.*

Moreover this solution satifies the bounds

$$|X_t| \le e^{A(t)}|x_0| + \int_{t_0}^t e^{A(t)-A(s)} b(s) ds \qquad \forall t \in [t_0, T] \ , \qquad \text{where } A(t) = \int_{t_0}^t a(s) ds \ .$$

In the proof of the Theorem we shall need the following result:

Lemma A.5 (Gronwall Lemma) *Let $a, b \in L^1([0,T], \mathbb{R}_+)$ and $\rho : [0,T] \to \mathbb{R}_+$ be continuous and such that*

$$\rho(t) \le \int_0^t (a(s)\rho(s) + b(s))\, ds + \rho(0) \qquad \forall t \in [0,T] \ .$$

Then

$$\rho(t) \le e^{A(t)}\rho(0) + \int_0^t e^{A(t)-A(s)} b(s) ds \qquad \forall t \in [0,T] \qquad \text{where } A(t) = \int_0^t a(s) ds \ . \tag{A.5}$$

Proof of Lemma A.5: Let $\theta(t)$ denote the right-hand side of inequality (A.5) and $\theta_\epsilon(t) = \theta(t) + \epsilon e^{A(t)}$. We note that θ_ϵ satisfies

$$\theta_\epsilon(t) = \int_0^t (a(s)\theta_\epsilon(s) + b(s))\, ds + \rho(0) + \epsilon \qquad \forall t \in [0,T] \ .$$

Let us assume for a while that $\max_{[0,T]}(\rho(t) - \theta_\epsilon) \ge 0$ and let

$$t^* = \inf\{t \ge 0 \ ; \ \rho(t) \ge \theta_\epsilon(t)\} \ .$$

Since ρ and ρ_ϵ are continuous we have $t^* > 0$ and $\rho(t^*) = \rho_\epsilon(t^*)$. Moreover $\rho(t) < \theta_\epsilon(t)$ for any $t \in [0, t^*)$. Therefore

$$\rho(t^*) \ \le \ \int_0^{t^*} (a(s)\rho(s) + b(s))\, ds + \rho(0) \ < \ \int_0^{t^*} (a(s)\theta_\epsilon(s) + b(s))\, ds + \rho(0) + \epsilon = \theta_\epsilon(t^*) = \rho(t^*) \ .$$

This is impossible. So $\rho \le \theta_\epsilon$ on $[0,T]$, which gives the result by letting $\epsilon \to 0$. $\Box$

Proof of Theorem A.4: To simplify the notations we assume without loss of generality that $t_0 = 0$. Let $A(t) = \int_0^t a(s)ds$ and

$$\rho(t) = e^{A(t)}|x_0| + \int_0^t e^{A(t)-A(s)} b(s)ds$$

and

$$\mathcal{X} = \left\{ X \in \mathcal{C}^0([0,T], \mathbb{R}^N) \ , \ |X_t| \le \rho(t) \qquad \forall t \in [0,T] \right\} \ .$$

Note for later use that

$$\rho'(t) = a(t)\rho(t) + b(t) \quad \text{for almost all } t \in [0,T] \ , \qquad \rho(0) = |x_0| \ ,$$

so that ρ is nondecreasing and

$$\rho(t) = |x_0| + \int_0^t (a(s)\rho(s) + b(s))ds \qquad \forall t \in [0,T] \ .$$

For $X \in \mathcal{X}$, let $\Phi(X) : [0,T] \to \mathbb{R}^N$ be defined by

$$\Phi(X)_t = x_0 + \int_0^t f(s, X_s)ds \qquad \forall t \in [0,T] \ .$$

We claim that Φ maps $\mathcal{X}$ into itself. Indeed let us first note that, if $X \in \mathcal{X}$,

$$|f(s, X_s)| \le \alpha(s)|X_s| + b(s) \le \alpha(s)\rho(T) + b(s)$$

where the right-hand side belongs to $L^1([0,T])$ by assumption. So $\Phi(X)$ is well-defined and absolutely continuous, and thus continuous. Moreover, from assumption (A.4) and the definition of $\mathcal{X}$ and ρ, we have

$$|\Phi(X)_t| \le |x_0| + \int_0^t |f(s, X_s)|ds \le |x_0| + \int_0^t (a(s)\rho(s) + b(s))ds = \rho(t) \ .$$

Therefore $\Phi(X) \in \mathcal{X}$. Set now $R = \rho(T)$. Recalling the definition of m_R in (A.3), we endow $\mathcal{X}$ with the distance

$$d(X,Y) = \max_{t\in[0,T]} |X_t - Y_t| e^{-2M_R(t)} \quad \forall X, Y \in \mathcal{X} \qquad \text{where } M_R(t) = \int_0^t m_R(s)ds \ .$$

Note that

$$|X_t - Y_t| \le d(X,Y) e^{2M_R(t)} \qquad \forall t \in [0,T] \ .$$

Since $m_R \in L^1$, the above distance is equivalent with the usual L^∞ distance, so that $(\mathcal{X}, d)$ is a complete metric space. Let us show that Φ is contracting for this metric. Let $X, Y \in \mathcal{X}$ and $t \in [0,T]$. Then

$$\begin{aligned} |\Phi(X)_t - \Phi(Y)_t| \ &\le \ \int_0^t |f(s,X_s) - f(s,Y_s)|ds \ \le \ \int_0^t m_R(s)|X_s - Y_s|ds \\ &\le \ \int_0^t m_R(s) e^{2M_R(s)} d(X,Y)ds \ \le \ \frac{1}{2} e^{2M(t)} d(X,Y) \end{aligned}$$

so that

$$d(\Phi(X), \Phi(Y)) = \max_{t\in[0,T]} |\Phi(X)_t - \Phi(Y)_t| e^{-2M(t)} \le \frac{1}{2} d(X,Y) \ .$$

Hence Φ is contracting. Since $(\mathcal{X}, d)$ is complete, Φ has a unique fixed point X, which is clearly a solution of (A.2) with initial condition $X_0 = x_0$.

Let us assume that Y is another solution. Then

$$|Y_t| \le |x_0| + |\int_0^t f(s, Y_s)ds| \le |x_0| + \int_0^t a(s)|Y_s| + b(s)ds$$

so that, by Gronwall Lemma, $|Y_t| \le \rho(t)$. Hence $Y \in \mathcal{X}$ and, since Y is a solution, it is also a fixed point of Φ. Therefore $Y = X$, which proves the uniqueness. $\Box$

Lemma A.6 (Lipschitz estimate of the solution map) *Let us fix* $R_0 > 0$ *and let*

$$R = e^{A(T)} R_0 + \int_{t_0}^{T} e^{A(T)-A(s)} b(s) ds \qquad \text{where } A(t) = \int_{t_0}^{t} a(s) ds \; .$$

Let $x_0, y_0 \in \mathbb{R}^N$ *with* $|x_0|, |y_0| \leq R_0$ *and* X *and* Y *be the solution of (A.2) starting from* (t_0, x_0) *and* (t_0, y_0) *respectively. Then*

$$|X_t - Y_t| \leq e^{M(t)} |x_0 - y_0| \qquad \forall t \in [t_0, T] \qquad \text{where } M(t) = \int_{t_0}^{t} m_R(s) ds \; ,$$

where m_R *is defined by (A.3).*

Proof : From Theorem A.4 we know that $|X_t|, |Y_t| \leq R$ for any $t \in [0, T]$. Let $\rho(t) = |X_t - Y_t|$. Then, using assumption (A.3), we have

$$\rho(t) \leq |x_0 - y_0| + \int_{t_0}^{t} |f(s, X_s) - f(s, Y_s)| ds \leq |x_0 - y_0| + \int_{t_0}^{t} m_R(s) \rho(s) ds \; .$$

So, by Gronwall Lemma, $\rho(t) \leq \rho(t_0) e^{M(t)}$. □

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