# Nash equilibrium payoffs for nonzero-sum stochastic differential games

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**Abstract :** Existence and characterization of Nash equilibrium payoffs are proved for stochastic nonzero-sum differential games.

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## 1 Introduction

In this paper, we investigate the notion of Nash equilibrium payoff for nonzero-sum stochastic two players differential games. Our main result is an existence theorem for such equilibrium payoffs. We also give a characterization of these payoffs.

Let us denote by $X_s^{t,x,u,v}$ the solution of the following equation:

$$dX_s = f(s, X_s, u_s, v_s)ds + \sigma(s, X_s, u_s, v_s)dB_s,\ t \leq s\ .$$

with initial condition

$$X_t = x$$

Here $B_\cdot$ is a $d-$dimensional standard Brownian motion, $u_\cdot$ and $v_\cdot$ are stochastic processes taking values in some compact subsets $U$ and $V$ of some finite dimensional spaces. Precise assumptions on $f : [0,T] \times I\!\!R^n \times U \times V \to I\!\!R^n$ and on $\sigma : [0,T] \times I\!\!R^n \times U \times V \to I\!\!R^{n\times d}$ are given in the next section.

The payoff of the players is a terminal payoff, given by $J_1(t,x,u,v) = E[g_1(X_T^{t,x,u,v})]$ for Player I and by $J_2(t,x,u,v) = E[g_2(X_T^{t,x,u,v})]$ for Player II. Loosely speaking, Player I aims at maximizing $J_1(t,x,u,v)$ while the goal of Player II is to maximize $J_2(t,x,u,v)$. As usual in differential game theory, the players do not play time-measurable controls but *strategies*. In order to avoid for the moment the technical details, we postpone the definition of the strategies to the next section. Here we only need to assume that for any strategy $\alpha$ of Player I and any strategy $\beta$ of Player II one can define a payoff $J_1(t,x,\alpha,\beta)$ for Player I and a payoff $J_2(t,x,\alpha,\beta)$ for Player II.

A particularly important notion for investigating nonzero-sum games is given by Nash equilibria. In our framework, a Nash equilibrium is a pair $(\bar\alpha, \bar\beta)$ of strategies such that, for any other pair $(\alpha, \beta)$ of strategies, we have

(1) $$J_1(t,x,\bar\alpha,\bar\beta) \geq J_1(t,x,\alpha,\bar\beta) \text{ and } J_2(t,x,\bar\alpha,\bar\beta) \geq J_2(t,x,\bar\alpha,\beta)\ .$$

The couple $(J_1(t,x,\bar{\alpha},\bar{\beta}), J_2(t,x,\bar{\alpha},\bar{\beta}))$ is called a Nash equilibrium payoff. In general, we do not expect Nash equilibria to exist, but only Nash equilibrium payoffs $(e_1,e_2)$ which can be approximated by the payoffs of strategies $(\bar{\alpha}_\epsilon,\bar{\beta}_\epsilon)$ for which the inequalities (1) only hold true up to some $\epsilon>0$ for any $(\alpha,\beta)$. (Remark also that, in general, Nash equilibrium payoffs are not unique.)

The main result of this paper (Theorem 2.9) states that Nash equilibrium payoffs exist for any initial position. Moreover we characterize these Nash equilibrium payoffs. In order to explain this characterization, we have to introduce the zero-sum games associated with the payoffs $J_1$ and $J_2$. Under Isaacs' condition (see (3) below), Fleming and Souganidis [2] (see also [7]) have proved that the zero-sum game, where Player I wants to maximize $J_1$ and Player II wants to minimize $J_1$, has a value, which will be denoted here by $W_1$:

$$W_1(t,x) = \inf_\beta \sup_\alpha J_1(t,x,\alpha,\beta) = \sup_\alpha \inf_\beta J_1(t,x,\alpha,\beta)\ .$$

In the same way, the zero-sum game, in which Player I aims at minimizing the payoff $J_2$ and Player II aims at maximizing it, has also a value, denoted $W_2$:

$$W_2(t,x) = \sup_\beta \inf_\alpha J_2(t,x,\alpha,\beta) = \inf_\alpha \sup_\beta J_2(t,x,\alpha,\beta)\ .$$

Our characterization result Theorem 2.10 loosely states (up to technical details) that a pair $(e_1,e_2)\in \mathbb{R}^2$ is a Nash equilibrium payoff for the initial position $(t,x)$ if and only if there is some pair $(u_., v_.) : [t,T]\to U\times V$ of adapted controls such that

i) for $j=1,2$, $E[g_j(X_T^{t,x,u,v})|\mathcal{F}_{t,s}] \geq W_j(s, X_s^{t,x,u,v})$ a.s. for any $s\in[t,T]$, where $\mathcal{F}_{t,s}$ is the $\sigma-$algebra generated by $\{B_u - B_t,\ u\in[t,s]\}$,

ii) for $j=1,2$, $e_j = J_j(t,x,u,v)$.

(In practice, the existence of such $(u_., v_.)$ is out of reach, and we only prove the existence, for any $\epsilon>0$, of some adapted controls $(u^\epsilon_., v^\epsilon_.)$ for which (i) holds true up to $\epsilon$ with a probability larger than $1-\epsilon$, and (ii) holds true up to $\epsilon$. However, this is enough for characterizing the Nash equilibrium payoffs.)

The controls $u_.$ and $v_.$ can be interpreted as follows: The Players agree at the beginning of the game to play respectively $u_.$ and $v_.$ . Condition (ii) then guaranties that their payoff is $(e_1,e_2)$ if they indeed play $u_.$ and $v_.$ up to the terminal time $T$. If on the contrary one of the players (say Player II) deviates at some time $t'\in(t,T)$, i.e., does not play $v_.$ on $[t',T]$, then Player I punishes Player II by playing some strategy which minimizes the expected payoff of Player II. Condition (i) guaranties that such a strategy exists, and that the resulting payoff of Player II is not larger than $e_2$. So Player II gains nothing at deviating.

In the deterministic case, the results presented in this paper have already been established by Kononenko in [6] and by Kleimenov in [5] in the framework of positional strategies, by Tolwinski, Haurie and Leitmann in [9] in the framework of Friedman strategies. Let us point out that the generalization to the stochastic case is far from being straightforward for at least two reasons: Firstly because of measurability issues, already encountered by Fleming and Souganidis when generalizing the existence of a value (and the dynamic programming) from deterministic zero-sum differential games to zero-sum stochastic differential games. Secondly, because the method used by Kononenko and Kleimenov - which makes an extensive use of the extremal aiming and of the existence of quasi-optimal positional strategies for some associated zero-sum differential games - does not apply to stochastic diffential games.

Let us finally recall another approach for the existence problem of Nash equilibrium payoffs: The dynamic programming approach. The idea is to find the Nash equilibrium payoff $(e_1,e_2)$ as a function of the initial position $(t,x)$: $(e_1,e_2)=(e_1(t,x),e_2(t,x))$. This function can be constructed as a solution of some system of parabolic p.d.e (as in [1] for instance), or by using backward or backward-forward stochastic differential equations (as in [3], [4]). Both methods rely heavily on a non degeneracy assumption on $\sigma$. In fact it can be proved (see [8]) that a payoff $(e_1(t,x),e_2(t,x))$ given by such a method is a Nash equilibrium payoff in our sense.

The paper is organized as follows: We first state the assumptions, notations and the main results of the paper. After we prove the characterization theorem, from which we derive the existence result. We complete the paper with some remarks on the notions of strategies.

## 2 Statements of the main results.

Let $T>0$ be a fixed finite time horizon. For $t\in[0,T]$, we consider the following doubly controlled stochastic system :

$$(2)\qquad \begin{array}{l} dX_s=f(s,X_s,u_s,v_s)ds+\sigma(s,X_s,u_s,v_s)dB_s,\ s\in[t,T],\\ X_t=x; \end{array}$$

where $B$ is a $d$-dimensional standard Brownian motion on the canonical Wiener space $(\Omega,\mathcal{F},P)$, i.e. $\Omega$ is the set of continuous functions from $[0,T]$ to $I\!R^d$ issued from 0, $\mathcal{F}$ the completed Borell $\sigma-$algebra over $\Omega$, $P$ the Wiener measure and $B$ the canonical process: $B_s(\omega)=\omega(s), s\in[0,T]$. The processes $u$ and $v$ are assumed to take their values in some compact metric spaces $U$ and $V$ respectively. We suppose that the functions $f:[0,T]\times I\!R^n\times U\times V\to I\!R^n$ and $\sigma:[0,T]\times I\!R^n\times U\times V\to I\!R^{n\times d}$ are measurable and satisfy the assumption (H):

(H) $f$ and $\sigma$ are bounded and Lipschitz continuous with respect to $(t,x)$, uniformly in $(u,v)\in U\times V$.

We should also assume that Isaacs' condition, i.e., that for all $(t,x)\in[0,T]\times I\!R^n$, $p\in I\!R^n$, and all $A\in\mathcal{S}_n$ (where $\mathcal{S}_n$ is the set of symmetric $n\times n$ matrices) holds:

$$(3)\qquad \begin{array}{c} \inf_u\sup_v\{<f(t,x,u,v),p>+\frac{1}{2}Tr(A\sigma(t,x,u,v)\sigma^*(t,x,u,v))\}=\\ \sup_v\inf_u\{<f(t,x,u,v),p>+\frac{1}{2}Tr(A\sigma(t,x,u,v)\sigma^*(t,x,u,v))\} \end{array}$$

We define the sets of admissible controls:

**Definition 2.1** *An admissible control process $u$ for player I (resp. II) on $[t,T]$ is a process taking values in $U$ (resp. $V$), progressively measurable with respect to the filtration $(\mathcal{F}_{t,s},s\geq t)$, where*

$$\mathcal{F}_{t,s}=\sigma\{B_r-B_t,r\in[t,s]\},\ s\in[t,T],$$

*augmented by all null-sets of $P$.*
*The set of admissible controls for player I (resp. II) on $[t,T]$ is denoted by $\mathcal{U}(t)$ (resp. $\mathcal{V}(t)$).*

We identify two processes $u$ and $\bar{u}$ in $\mathcal{U}(t)$ and write $u\equiv\bar{u}$, if $P\{u=\bar{u}$ a.e. in $[t,s]\}=1$.

Under assumption (H), for all $(t,x)\in[0,T]\times I\!R^n$ and $(u,v)\in\mathcal{U}(t)\times\mathcal{V}(t)$, there exists a unique solution to (2) that we denote by $X^{t,x,u,v}_{\cdot}$.

Now we have to define strategies. Let us first recall the definition of nonanticipative strategies.

**Definition 2.2** *A nonanticipative strategy for Player I on $[t,T]$ is a mapping $\alpha:\mathcal{V}(t)\to\mathcal{U}(t)$ such that, for any $s\in[t,T]$ and for any $v_1,v_2\in\mathcal{V}(t)$, if $v_1\equiv v_2$ on $[t,s]$, then $\alpha(v_1)\equiv\alpha(v_2)$ on $[t,s]$.*

Nonanticipative strategies for Player II are defined symmetrically.

For several reasons explained below, nonanticipative strategies are not the proper ones for nonzero-sum differential games. We merely use the notion of admissible strategies, whose definition needs some preliminary remarks:

For all $\bar{t}, t \in [0,T]$ with $\bar{t} \le t$, let $\Omega_{\bar{t},t}$ be the set of continuous functions from $[\bar{t}, t]$ to $I\!R^d$, issued from 0 and $P_{\bar{t},t}$ the Wiener-measure on $\Omega_{\bar{t},t}$ (in particular $\Omega_{0,T} = \Omega$ and $P_{0,T} = P$). If, for fixed $0 \le \bar{t} \le t \le T$, and $\omega \in \Omega_{\bar{t},T}$, we define $\pi(\omega) = (\omega_1, \omega_2)$ by

$$\begin{aligned} \omega_1 &= \omega|_{[\bar{t},t]} \\ \omega_2 &= (\omega - \omega(t))|_{[t,T]}, \end{aligned}$$

we can identify $\Omega_{\bar{t},T}$ with $\Omega_{\bar{t},t} \times \Omega_{t,T}$, and we have $P_{\bar{t},T} = P_{\bar{t},t} \otimes P_{t,T}$.
Furthermore, to every random variable $Y$ on $\Omega_{\bar{t},T}$ and all $\omega_1 \in \Omega_{\bar{t},t}$, we can associate a random variable $(Y(\omega_1))(\cdot)$ on $\Omega_{t,T}$, by setting $(Y(\omega_1))(\omega_2) = Y(\omega)$.
We also remark that, for all $(\mathcal{F}_{\bar{t},s}, s \ge \bar{t})$-progressively measurable process $(Y_s, s \ge \bar{t})$ and almost every $\omega_1 \in \Omega_{\bar{t},t}$, the process $(Y(\omega_1)_s, s \ge t)$ is $(\mathcal{F}_{t,s}, s \ge t)$-progressively measurable. This allows us to apply a nonanticipative strategy $\alpha$ defined on $[t,T]$ to controls living on the larger time interval $[\bar{t}, T]$: for $v \in \mathcal{V}(\bar{t})$, we define $\alpha(v|_{[t,T]})$ by

$$\alpha(v|_{[t,T]})(\omega)_s = (\alpha(v(\omega_1)))(\omega_2)_s, s \in [t,T] \tag{4}$$

(with a symmetric notation for strategies $\beta$ for player II).

**Definition 2.3** *An admissible strategy for Player I on $[t,T]$ is a mapping $\alpha : \mathcal{V}(t) \to \mathcal{U}(t)$ such that*

*i) $\alpha$ is a strongly nonanticipative strategy: Namely, for any $(\mathcal{F}_{t,s})_{s\in[t,T]}-$stopping time $S$ and any $v, \tilde{v} \in \mathcal{V}(t)$, if $v \equiv \tilde{v}$ on $[\![t,S]\!]$, then $\alpha(v) \equiv \alpha(\tilde{v})$ on $[\![t,S]\!]$*
*(with the notation $[\![t,S]\!] = \{(s,\omega) \in [0,T] \times \Omega, t \le s \le S(\omega)\}$)*

*ii) $\alpha$ is a nonanticipative strategy with delay: Namely, there is some partition $t = t_0 < t_1 < \ldots < t_m = T$ such that for all $v, \tilde{v} \in \mathcal{V}(t)$, $\alpha(v) = \alpha(\tilde{v})$ on $[t, t_1]$ and for any $i < n$, if $v \equiv \tilde{v}$ on $[t, t_i]$, then $\alpha(v) \equiv \alpha(\tilde{v})$ on $[t, t_{i+1}]$*

*iii) $\alpha$ is an r-strategy: Namely, for every $0 \le \bar{t} < t$ and $v \in \mathcal{V}(\bar{t})$ the process $\alpha(v|_{[t,T]})$ is $(\mathcal{F}_{\bar{t},s}, s \ge t)$-progressively measurable.*

*The set of all admissible strategies for Player I on $[t,T]$ is denoted by $\mathcal{A}(t)$. The set of admissible strategies $\beta : \mathcal{U}(t) \to \mathcal{V}(t)$ for Player II, which are defined symmetrically, is denoted by $\mathcal{B}(t)$.*

The r-strategies were introduced in [2], motivated by technical problems related to measurability issues. Not surprizingly, we have encountered the same kind of difficulties, hence the requirement for the admissible strategies to be r-strategies.

To the best of our knowledge, the notion of *strongly* nonanticipative strategies has never been introduced before. However it is, in our opinion, much more natural for stochastic differential games than that of standard nonanticipative strategies. Indeed strongly nonanticipative strategies formalize the fact that a player is only allowed to take into account the control of his opponent he observes in the present state of the world. In other words, if we want to make rigourous the following requirement: "if for some $\omega$, there exists a time $s \ge 0$ such that $v_1(\omega) = v_2(\omega)$ before $s$, then $\alpha(v_1(\omega)) = \alpha(v_2(\omega))$ before $s$", it becomes clear that $s$ depends from $\omega$, thus that the nonanticipativity has to involve random times.

The main reason for introducing nonanticipative strategies *with delay* is the following lemma:

**Lemma 2.4** *Let $\alpha \in \mathcal{A}(t)$ be an admissible strategy and $\beta : \mathcal{U}(t) \to \mathcal{V}(t)$ be nonanticipative. There is a unique control-pair $(u, v) \in \mathcal{U}(t) \times \mathcal{V}(t)$ such that*

$$\alpha(v) = u \text{ and } \beta(u) = v . \tag{5}$$

Of course, a symmetric result holds if $\alpha$ is nonanticipative and $\beta$ is admissible, or if both $\alpha$ and $\beta$ are admissible. Let us point out that one cannot omit one of the strategies to be with delay.

**Proof of the lemma:** Let $t = t_0 < t_1 < \ldots < t_m = T$ be a partition associated with the admissible strategy $\alpha$. We construct the controls $(u_., v_.)$ by induction on the interval $[t_k, t_{k+1})$.

For $k = 0$, we know that, for any $v' \in \mathcal{V}(t)$, the restriction of $u = \alpha(v')$ to the interval $[t, t_1)$ is independent of $v'$ since $\alpha$ admissible. Let us set $v = \beta(u)$ on $[t, t_1)$, which only depends on the values of $u$ on $[t, t_1)$ since $\beta$ is nonanticipative. Let us point out that this procedure uniquely defines $(u, v)$ on $[t, t_1)$.

Let us now assume that $u$ and $v$ are uniquely defined on $[t, t_k)$. Then the restriction of $u = \alpha(v')$ to the interval $[t_k, t_{k+1}]$ does not depend on the values of $v'$ on $[t_k, t_{k+1})$ provided that $v' = v$ on $[t, t_k)$, because $\alpha$ is admissible. This defines $u$ on $[t, t_{k+1})$. Then $v$ is uniquely defined on $[t, t_{k+1})$ by $v = \beta(u)$ since $\beta$ is nonanticipative.

This completes the proof by induction.

QED

We give several remarks and comments on strategies later on in Appendix. Here is an example of admissible strategy. This example is borrowed from [2].

**Example 2.5** Let $t_0 = t < t_1 < \ldots < t_m = T$ be a fixed partition of $[t, T]$, and, for $1 \le j \le m$, $(O_{ij})_{i \in \mathbb{N}}$ be a Borel partition of $\mathbb{R}^n$ and $u_{ij} \in U$, for $i \in \mathbb{N}$, be fixed. For any control $v \in \mathcal{V}(t)$, we define $\alpha(v)$ by induction on $[t, t_j)$ by setting:

$$\alpha(v)(s) = u_{10} \text{ on } [t, t_1) ,$$

and, if $\alpha(v)$ is built on $[t, t_j)$, we set

$$\alpha(v)(s) = \sum_i u_{ij} 1_{\{X_{t_j}^{t,x,\alpha(v),v} \in O_{ij}\}} \text{ on } [t_j, t_{j+1}) .$$

Then $\alpha$ is an admissible strategy.

The main result of [2] is that zero-sum stochastic games have a value when the players play nonanticipative strategies. A careful examination of the proof of [2] shows the following result:

**Theorem 2.6** *Let $g : \mathbb{R}^n \to \mathbb{R}$ be bounded and Lipschitz continuous and set*

$$\forall (u, v) \in \mathcal{U}(t) \times \mathcal{V}(t),\ J(t, x, u, v) = E[g(X_T^{t,x,u,v})] .$$

*Let $f$ and $\sigma$ satisfy the assumption (H) and Isaacs' condition (3). Then*

$$\inf_{\beta \in \mathcal{B}(t)} \sup_{u \in \mathcal{U}(t)} J(t, x, u, \beta(u)) = \sup_{\alpha \in \mathcal{A}(t)} \inf_{v \in \mathcal{V}(t)} J(t, x, \alpha(v), v) .$$

We have to explain briefly this result, which is a straightforward consequence of several results of [2]:

**Proof of Theorem 2.6:** Let us set

$$W^\sharp = \sup_{\alpha \in \mathcal{A}(t)} \inf_{v \in \mathcal{V}(t)} J(t, x, \alpha(v), v) \quad \text{and} \quad W^\flat = \inf_{\beta \in \mathcal{B}(t)} \sup_{u \in \mathcal{U}(t)} J(t, x, u, \beta(u)) .$$

For any $\epsilon > 0$, let us choose some admissible strategies $\alpha$ and $\beta$ such that

$$W^\sharp \leq \inf_{v\in\mathcal{V}(t)} J(t,x,\alpha(v),v) + \epsilon \quad \text{and} \quad W^\flat \geq \sup_{u\in\mathcal{U}(t)} J(t,x,u,\beta(u)) - \epsilon \ .$$

Using Lemma 2.4, there is some control-pair $(u,v) \in \mathcal{U}(t) \times \mathcal{V}(t)$ such that $\alpha(v) = u$ and $\beta(u) = v$ . Hence

$$W^\sharp \leq J(t,x,\alpha(v),v) + \epsilon = J(t,x,u,v) + \epsilon = J(t,x,u,\beta(u)) + \epsilon \leq W^\flat + 2\epsilon \ .$$

Therefore we have proved that $W^\sharp \leq W^\flat$.

For proving the reverse inequality, let us set

$$V^\flat = \inf_{\substack{\beta : \mathcal{U}(t) \to \mathcal{V}(t) \\ \text{nonanticipative}}} \sup_{u\in\mathcal{U}(t)} J(t,x,u,\beta(u)) \ .$$

Combining formula (2.4), Proposition 2.5 and Theorem 2.6 of [2] yields the existence of some nonanticipative strategy $\alpha$ such that

$$V^\flat \leq \inf_{v\in\mathcal{V}(t)} J(t,x,\alpha(v),v) + \epsilon \ .$$

A carefull examination of the proof of (2.4) also shows that the strategy $\alpha$ can be chosen from $\mathcal{A}(t)$. Indeed it is actually of the form of Example 2.5. This proves that $V^\flat \leq W^\sharp$. Using symmetric argument, one can prove that $V^\sharp \geq W^\flat$, where

$$V^\sharp = \sup_{\substack{\alpha : \mathcal{V}(t) \to \mathcal{U}(t) \\ \text{nonanticipative}}} \inf_{v\in\mathcal{V}(t)} J(t,x,\alpha(v),v) \ .$$

Hence we have already proved that

$$V^\flat \leq W^\sharp \leq W^\flat \leq V^\sharp \ .$$

Since, under Isaacs' condition (3), the game has a value, i.e., $V^\sharp = V^\flat$ (Theorem 2.6 of [2]), equality $W^\sharp = W^\flat$ holds.

QED

Now let $g_1 : I\!R^n \to I\!R$ and $g_2 : I\!R^n \to I\!R$ be two Lipschitz continuous functions bounded by some $C > 0$. For $(t,x) \in [0,T] \times I\!R^n$, $(u,v) \in \mathcal{U}(t) \times \mathcal{V}(t)$, set

$$J_1(t,x,u,v) = E[g_1(X_T^{t,x,u,v})] \text{ and } J_2(t,x,u,v) = E[g_2(X_T^{t,x,u,v})] \ .$$

In the sequel, for all couples of a nonanticipative and an admissible strategy $(\alpha,\beta)$, we will also use the following notation :

$$J_j(t,x,\alpha,\beta) = J_j(t,x,u,v) \qquad (\text{for } j = 1 \text{ or } j = 2)$$

where $(u,v)$ are associated to $(\alpha,\beta)$ by (5).
Recall that Player I wants to maximize $J_1(t,x,\alpha,\beta)$, while Player II wants to maximize $J_2(t,x,\alpha,\beta)$.

**Definition 2.7** *We say that a couple $(e_1,e_2) \in I\!R^2$ is a Nash equilibrium payoff at the point $(t,x)$ if, for any $\epsilon > 0$, there exist $(\alpha_\epsilon,\beta_\epsilon) \in \mathcal{A}(t) \times \mathcal{B}(t)$ such that*

(6) *for all* $(\alpha,\beta) \in \mathcal{A}(t) \times \mathcal{B}(t)$, *it holds that*
$J_1(t,x,\alpha_\epsilon,\beta_\epsilon) \geq J_1(t,x,\alpha,\beta_\epsilon) - \epsilon$ *and* $J_2(t,x,\alpha_\epsilon,\beta_\epsilon) \geq J_2(t,x,\alpha_\epsilon,\beta) - \epsilon$

*and*

(7) *for* $j = 1,2,\ |J_j(t,x,\alpha_\epsilon,\beta_\epsilon) - e_j| \leq \epsilon$ .

**Remarks :**

1. Condition (6) means that if one of the Players deviates from his strategy ($\alpha_\epsilon$ or $\beta_\epsilon$), then he cannot expect to get much more (less than $\epsilon$) than what he would have had by keeping his strategy.

2. The definition still makes sense if one uses the notion of nonanticipative strategies with delay instead of admissible strategies.

In the sequel, we shall often use an equivalent formulation of Condition (6) given by the following lemma.:

**Lemma 2.8** *Let $\epsilon > 0$ and $(\alpha_\epsilon, \beta_\epsilon) \in \mathcal{A}(t) \times \mathcal{B}(t)$. Condition (6) holds if and only if*

$$(8) \quad \begin{array}{ll} \textit{for any} & (u,v) \in \mathcal{U}(t) \times \mathcal{V}(t), \\ & J_1(t,x,\alpha_\epsilon,\beta_\epsilon) \geq J_1(t,x,u,\beta_\epsilon(u)) - \epsilon \text{ and } J_2(t,x,\alpha_\epsilon,\beta_\epsilon) \geq J_2(t,x,\alpha_\epsilon(v),v) - \epsilon \ . \end{array}$$

**Proof of the lemma:** Suppose that (8) holds and let $\alpha \in \mathcal{A}(t)$. By Lemma 2.4, there exists $(u,v) \in \mathcal{U}(t) \times \mathcal{V}(t)$ such that $\alpha(v) = u$ and $\beta_\epsilon(u) = v$. By (8) applied to this couple $(u,v)$, we get

$$J_1(t,x,\alpha_\epsilon,\beta_\epsilon) \geq J_1(t,x,u,\beta_\epsilon(u)) - \epsilon = J_1(t,x,\alpha,\beta_\epsilon) - \epsilon.$$

Repeating the same argument for some $\beta \in \mathcal{B}(t)$, we get Condition (6).

Conversely, for any fixed $u \in \mathcal{U}(t)$, we can define a strategy $\alpha \in \mathcal{A}(t)$ by setting $\alpha(v) = u$ for all $v \in \mathcal{V}(t)$. In particular, for $v = \beta_\epsilon(u)$, we have again $u = \alpha(v)$ and $v = \beta_\epsilon(u)$. It is then easy to deduce (8) from (6).

QED

From now on, we denote by $W_1$ and $W_2$ the value functions of the zero-sum games where Player I (resp. Player II) aims at maximizing $g_1$ (resp. $g_2$). According to Theorem 2.6, this means that

$$W_1(t,x) = \inf_{\beta \in \mathcal{B}(t)} \sup_{u \in \mathcal{U}(t)} J_1(t,x,u,\beta(u)) = \sup_{\alpha \in \mathcal{A}(t)} \inf_{v \in \mathcal{V}(t)} J_1(t,x,\alpha(v),v) \ .$$

and

$$W_2(t,x) = \sup_{\beta \in \mathcal{B}(t)} \inf_{u \in \mathcal{U}(t)} J_2(t,x,u,\beta(u)) = \inf_{\alpha \in \mathcal{A}(t)} \sup_{v \in \mathcal{V}(t)} J_2(t,x,\alpha(v),v) \ .$$

Now, once all notations and assumptions stated, we are able to announce the two results of this paper:

**Theorem 2.9** *(Existence) Under Isaacs' condition (3), for any initial position $(t,x) \in [0,T] \times \mathbb{R}^n$, there is some Nash equilibrium payoff at $(t,x)$.*

**Theorem 2.10** *(Characterization) Assume Isaacs' condition holds. Then a couple $(e_1,e_2) \in \mathbb{R}^2$ is a Nash equilibrium payoff at a point $(t,x)$ if and only if for any $\epsilon > 0$, there exists $(u^\epsilon, v^\epsilon) \in \mathcal{U}(t) \times \mathcal{V}(t)$ such that*

*i) for any $s \in [t,T]$ and $j = 1,2$,*

$$P\left\{E[g_j(X_T^\epsilon)|\mathcal{F}_{t,s}] \geq W_j(s,X_s^\epsilon) - \epsilon\right\} \ \geq \ 1 - \epsilon$$

*where $X_\cdot^\epsilon = X_\cdot^{t,x,u^\epsilon,v^\epsilon}$,*

*ii) and*

$$\textit{for } \ j = 1,2, \ |E[g_j(X_T^\epsilon)] - e_j| \leq \epsilon \ .$$

**Remarks :**

1. For proving that the conditions in Theorem 2.10 are necessary, we do not need the notion of admissible strategies: In fact, we only need that the strategies defining the Nash equilibrium payoff are nonanticipative with delay and satisfy some condition (**C**) introduced in Appendix ; the fact that the strategy is a strongly nonanticipative r-strategy is not needed. However, although the notion of Nash equilibrium payoff could still be defined by using the bigger class of nonanticipative strategies with delay, we do not know wether the characterization result remains true if one removes the requirement that the strategies satisfy condition (**C**). In other words, we do not know if this characterization holds if one allows the players to use the knowledge of the full control of his/her opponent (in any state of the world).

2. The generalization to the case of more than 2 players is not difficult (the idea is that each player can act as if he would be confrontated to one only opponent which activates the set of all the other controls). But since we could not avoid to reintroduce the hole definitions and notations and several preliminary results, we won't formulate it.

The sketch of the proof of the two previous results is the following: We first show the equivalence in Theorem 2.10, and, using this equivalence, we finally prove Theorem 2.9.

# 3 Proof of the characterization: sufficient condition.

The object of this section is the proof of the sufficient condition of Theorem 2.10.
We first point out a technical lemma, that will also be used in the proof of the other results.

**Lemma 3.1** *Fix $(t,x) \in [0,T] \times I\!R^n$ and $u \in \mathcal{U}(t)$.*
*a) For all $\theta \in [t,T]$ and $\epsilon > 0$, there exists a strongly nonanticipative r-strategy $\alpha : \mathcal{V}(t) \to \mathcal{U}(t)$, such that, for any $v \in \mathcal{V}(t)$,*

$$\text{(9)} \qquad \begin{aligned} &\alpha(v) \equiv u \quad \text{on } [t,\theta]; \\ &E[g_2(X_T^{t,x,\alpha(v),v})|\mathcal{F}_{t,\theta}] \le W_2(\theta, X_\theta^{t,x,\alpha(v),v}) + \epsilon, P\text{-a.s.} \ . \end{aligned}$$

*b) Let $B$ be a compact subset of $I\!R^n$. For all $\theta \in [t,T]$ and $\epsilon > 0$, there exists an admissible strategy $\alpha \in \mathcal{A}(t)$, such that, for any $v \in \mathcal{V}(t)$,*

$$\text{(10)} \qquad \begin{aligned} &\alpha(v) \equiv u \quad \text{on } [t,\theta]; \\ &E[g_2(X_T^{t,x,\alpha(v),v})|\mathcal{F}_{t,\theta}] \le W_2(\theta, X_\theta^{t,x,\alpha(v),v}) + \epsilon \quad P\text{-a.s. on } \{X_\theta^{t,x,\alpha(v),v} \in B\} \ . \end{aligned}$$

**Remark :** It can be proved similarly that, for all $\theta \in [t,T]$ and $\epsilon > 0$, there exists a strongly nonanticipative r-strategy $\alpha : \mathcal{V}(t) \to \mathcal{U}(t)$, such that, for any $v \in \mathcal{V}(t)$,

$$\alpha(v) \equiv u \ \text{ on } [t,\theta];$$

$$E[g_1(X_T^{t,x,\alpha(v),v})|\mathcal{F}_{t,\theta}] \ge W_1(\theta, X_\theta^{t,x,\alpha(v),v}) - \epsilon, P\text{-a.s.} \ .$$

**Proof of the lemma:** a) From the definition of the value function $W_2$, for any $y \in I\!R^n$, there is some admissible strategy $\alpha_y \in \mathcal{A}(\theta)$ such that,

$$\sup_{v \in \mathcal{V}(\theta)} E[g_2(X_T^{\theta,y,\alpha_y(v),v})] \le W_2(\theta,y) + \epsilon/2 \ .$$

Since $z \to W_2(\theta, z)$ and $z \to \sup_{v\in\mathcal{V}(\theta)} E[g_2(X_T^{\theta,z,\alpha_y(v),v})]$ are continuous, one can find a Borelian partition $(O_i,\ i=1,2,\ldots)$ of $I\!R^n$ such that, for any $i$, there is some $y_i \in O_i$ with

$$\forall z \in O_i,\ \sup_{v\in\mathcal{V}(\theta)} E[g_2(X_T^{\theta,z,\alpha_{y_i}(v),v})] \le W_2(\theta,z)+\epsilon. \tag{11}$$

Now we define the following strategy $\alpha$:

$$\forall v \in \mathcal{V}(t), \alpha(v)_s = \begin{cases} u_s & \text{for } s \in [t,\theta], \\ \alpha_{y_i}(v|_{[\theta,T]})_s & \text{for } s\in(\theta,T],\ \text{on } \{X_\theta^{t,x,u,v}\in O_i\}. \end{cases} \tag{12}$$

where the notation $\alpha_{y_i}(v|_{[\theta,T]})_s$ is defined by (4). By a tiresome but straightforward proof, we get that $\alpha$ is a nonanticipative r-strategy. The fact that it is strongly nonanticipative is proved in appendix, Lemma 6.1. It is clear that $\alpha(v)\equiv u$ on $[t,\theta]$.
Further, we obviously have (see also Lemma 1.11 in [2]), if we set $X = X^{t,x,\alpha(v),v}$,

$$\begin{aligned} E[g_2(X_T)|\mathcal{F}_{t,\theta}](\omega_1,\cdot) &= E_{\theta,T}[g_2(X_T^{\theta,X_\theta(\omega_1),\alpha(v(\omega_1)),v(\omega_1)})] \\ &= \textstyle\sum_{i\in N} 1_{\{X_\theta(\omega_1)\in O_i\}} E_{\theta,T}[g_2(X_T^{\theta,X_\theta(\omega_1),\alpha_{y_i}(v(\omega_1)),v(\omega_1)})]\ ,\ P_{t,\theta}(d\omega_1)-\text{a.s.}\ . \end{aligned} \tag{13}$$

Now (9) follows from (11).

b) Let $(O_i)_{i\in\{0,\ldots,m\}}$ be a finite Borelian partition of $I\!R^n$ and $y_i\in O_i, i\in\{0,\ldots k\}$ be such that $O_0 = B^c$ and, for $i\in\{1,\ldots,m\}$,(11) holds (indeed, since $B$ is compact, a finite partition of $B$ is sufficient to get (11)). Remark that there is no condition on $y_0\in B^c$.
Now let $\alpha$ be built like in (12). We already know that $\alpha(v)\equiv u$ on $[t,\theta]$ and that $\alpha$ is a strongly nonanticipative r-strategy. Let us prove that $\alpha$ is nonanticipative with delay. It is easy to see that $\alpha$ has delays corresponding to a partition that, from $\theta$ on, is a partition for all strategy $\alpha_{y_i}, i\in\{0,\ldots,m\}$. Since the number of strategies $\alpha_{y_i}$ involved in the construction of $\alpha$ is finite, this partition is also finite. It follows from the construction and from Proposition 6.3 that $\alpha\in\mathcal{A}(t)$.
Using again (11) and (13) and the choice of $O_1,\ldots,O_m$, one has that

$$1_{\{X_\theta\in B\}} E[g_2(X_T)|\mathcal{F}_{t,\theta}] \le 1_{\{X_\theta\in B\}}(W_2(\theta,X_\theta)+\epsilon)\ ,$$

for any $v\in\mathcal{V}(t)$, and relation (10) follows evidently.

QED

Now let us assume that $(e_1,e_2)$ satisfies conditions (i) and (ii) of Theorem 2.10: For any $\epsilon>0$, there is some control pair $(u^\epsilon,v^\epsilon)\in\mathcal{U}(t)\times\mathcal{V}(t)$ such that

i) for any $s\in[t,T]$ and $j=1,2$,

$$P\left\{E[g_j(X_T^\epsilon)|\mathcal{F}_{t,s}] \ge W_j(s,X_s^\epsilon)-\epsilon\right\} \ \ge\ 1-\epsilon \tag{14}$$

where $X_\cdot^\epsilon = X_\cdot^{t,x,u^\epsilon,v^\epsilon}$, and

ii)

$$\text{for } j=1,2,\ |E[g_j(X_T^\epsilon)]-e_j|\le\epsilon\ . \tag{15}$$

We have to prove that $(e_1,e_2)$ is a Nash equilibrium payoff for the initial position $(t,x)$.

For doing this, we are going to define, for any $\epsilon>0$, some strategies $(\alpha_\epsilon,\beta_\epsilon)\in\mathcal{A}(t)\times\mathcal{B}(t)$ satisfying (6) and (7). We only explain the construction of $\alpha_\epsilon$, the construction of $\beta_\epsilon$ being symmetric.

In order to simplify the notations, we assume throughout the proof that $g_j\ge 0$ for $j=1,2$. Let us point out that we can make this assumption without loss of generality since this only amounts to add some constant to the functions $g_j$. In particular, this assumption entails that $W_j \geq 0$. Recall that there exists also some $C > 0$ such that $|g_j| \leq C$.

Since the dynamic is bounded, it is easy to prove that, for all $(u,v),(u',v') \in \mathcal{U}(t)\times\mathcal{V}(t)$, for all $(\mathcal{F}_{t,s}, s\geq t)$-stopping time $S$ with $P[S\leq T]=1$ and such that $X_S^{t,x,u,v} = X_S^{t,x,u',v'}$, $P$-a.s., and for all $\tau \geq 0$,

$$E[\sup_{0\leq s\leq\tau} |X^{t,x,u,v}_{(S+s)\wedge T} - X^{t,x,u',v'}_{(S+s)\wedge T}|^2] \leq C_0\tau,$$

where the constant $C_0 > 0$ only depends on the dynamic. Thus, since $W_2(s,\cdot)$ is Lipschitz, uniformly in $s$, we can choose $\tau > 0$ such that, for all $(u,v),(u',v') \in \mathcal{U}(t)\times\mathcal{V}(t)$, for all $(\mathcal{F}_{t,s}, s\geq t)$-stopping time $S$ with $P[S\leq T]=1$ and such that $X_S^{t,x,u,v} = X_S^{t,x,u',v'}$, $P$-a.s.,

(16) $$E[\sup_{0\leq s\leq\tau} |W_2((S+s)\wedge T, X^{t,x,u,v}_{(S+s)\wedge T}) - W_2((S+s)\wedge T, X^{t,x,u',v'}_{(S+s)\wedge T})|^2] \leq (\epsilon/4)^2 .$$

Let us fix some partition $t_0 = t < t_1 < \ldots < t_m = T$ which satisfies $\sup_i |t_{i+1}-t_i| \leq \tau$.

We also fix some $M$ large enough such that

(17) $$\sup_{u\in\mathcal{U}(t)} \sup_{v\in\mathcal{V}(t)} P(\sup_{t\leq s\leq T} |X_s^{t,x,u,v}| > M) \leq \epsilon/(4C) .$$

Let us set

(18) $$\epsilon_0 = \frac{\epsilon}{4(2+mC)}$$

and let $(\bar u, \bar v) = (u^{\epsilon_0}, v^{\epsilon_0})$ satisfy (14) and (15) for $\epsilon = \epsilon_0$.

By Lemma 3.1b) applied to the closed ball $B$ in $\mathbb{R}^n$ with center 0 and radius $M$, to the control $\bar u$ and to $\theta = t_1,\ldots,t_m$, we get $m$ admissible strategies $\alpha_1 \in \mathcal{A}(t),\ldots,\alpha_m\in\mathcal{A}(t)$ such that, for any $v\in\mathcal{V}(t)$, for any $l\in\{1,\ldots,m\}$, $\alpha_j(v)\equiv \bar u$ on $[t,t_j]$ and

(19) $$1_{\{X^{\alpha_j}_{t_j}\in B\}} E[g_2(X_T^{\alpha_j})|\mathcal{F}_{t,t_j}] \leq 1_{\{X^{\alpha_j}_{t_j}\in B\}} W_2(t_j, X^{\alpha_j}_{t_j}) + \epsilon/4 ,$$

where we have set $X^{\alpha_j}_\cdot = X^{t,x,\alpha_j(v),v}_\cdot$.

For all $v\in\mathcal{V}(t)$, we introduce the stopping times

$$S^v = \inf\{s\geq t, v_s \neq \bar v_s\} \text{ and } t^v = \inf\{t_i > S^v,\ i\geq 1\},$$

with the convention $t^v = T$ if $v_s = \bar v_s$ on $[t,T]$.

We are now ready to define the admissible strategy $\alpha_\epsilon$ by setting

(20) $$\forall v\in\mathcal{V}(t), \alpha_\epsilon(v) = \begin{cases} \bar u & \text{on } [\![t,t^v]\!], \\ \alpha_j(v) & \text{on } (t_j,T]\times\{t^v = t_j\}. \end{cases}$$

It is easy to check that $\alpha_\epsilon$ is an admissible strategy.

Let $v\in\mathcal{V}(t)$ be fixed and let us set $X_\cdot = X_\cdot^{t,x,\alpha_\epsilon(v),v}$ . Let us notice that $\alpha_\epsilon(v)\equiv\bar u$ on $[\![t,t^v]\!]$ and that

$$X = \begin{cases} X^{t,x,\bar u,v} & \text{on } [\![t,t^v]\!]\ P-\text{a.s.} \\ \sum_j X^{t,x,\alpha_j(v),v} 1_{t^v=t_j} & \text{on } [\![t^v,T]\!]\ P-\text{a.s.} \end{cases}$$

Then using (19) we get:

(21) $$1_{\{X_{t^v}\in B\}} E[g_2(X_T)|\mathcal{F}_{t,t^v}] \leq 1_{\{X_{t^v}\in B\}} W_2(t^v, X_{t^v}) + \epsilon/4 .$$

We now claim that

$$(22) \qquad \forall v \in \mathcal{V}(t),\ J_2(t,x,\alpha_\epsilon(v),v) \le e_2+\epsilon \qquad \text{and} \qquad \alpha_\epsilon(\bar v)=\bar u\ .$$

Indeed, it follows from (21) that, for any $v \in \mathcal{V}(t)$, if we set $X_. = X_.^{t,x,\alpha_\epsilon(v),v}$, we have

$$(23) \qquad \begin{aligned} J_2(t,x,\alpha_\epsilon(v),v) &\le E[g_2(X_T)1_{\{|X_{t^v}|>M\}}]+E[W_2(t^v,X_{t^v})1_{\{|X_{t^v}|\le M\}}]+\epsilon/4 \\ &\le E[W_2(t^v,X_{t^v})]+\epsilon/2, \end{aligned}$$

where the last inequality comes from the choice of $M$ in (17).
Now set $\overline{X} = X^{t,x,\bar u,\bar v}$. Recall that, by the definition of $S^v$ and the fact that $\alpha_\epsilon(v)=\bar u$ on $[\![t,t^v]\!]$, we have $\overline{X}_s = X_s$ on $\{s \le S^v\}$.
Further we have $S^v \le t^v \le S^v+\tau$, thus, by (16), we get

$$\|W_2(t^v,X_{t^v}) - W_2(t^v,\overline{X}_{t^v})\|_2 \le \epsilon/4.$$

Accordingly,

$$(24) \qquad E[W_2(t^v,X_{t^v})] \le E[W_2(t^v,\overline{X}_{t^v})]+\epsilon/4\ .$$

Let us now denote by $\Omega_s$ (for $s\in[t,T]$), the set

$$\Omega_s = \left\{E[g_2(\overline{X}_T)|\mathcal{F}_{t,s}] \ge W_2(s,\overline{X}_s)-\epsilon_0\right\}\ ,$$

We recall that $P(\Omega_s)\ge 1-\epsilon_0$ thanks to (14). Thus

$$(25) \qquad \begin{array}{rcl} E[W_2(t^v,\overline{X}_{t^v})] &=& \sum_{i=1}^m E[W_2(t_i,\overline{X}_{t_i})1_{t^v=t_i}1_{\Omega_{t_i}}]+\sum_{i=1}^m E[W_2(t_i,\overline{X}_{t_i})1_{t^v=t_i}1_{\Omega^c_{t_i}}] \\ &\le& \sum_{i=1}^m E[(E[g_2(\overline{X}_T)|\mathcal{F}_{t,t_i}]+\epsilon_0)1_{t^v=t_i}1_{\Omega_{t_i}}]+\sum_{i=1}^m CP(\Omega^c_{t_i}\cap\{t^v=t^i\}) \\ &\le& E[g_2(\overline{X}_T)]+\epsilon_0+\sum_{i=1}^m CP(\Omega^c_{t_i}) \\ &\le& (e_2+2\epsilon_0)+mC\epsilon_0 \end{array}$$

where we have used in the last inequality on the one hand the fact that $g_2\ge 0$ and (15), and, on the other hand, the fact that $P(\Omega^c_{t_i})\le\epsilon_0$ for any $i$.

Putting (23), (24) and (25) together yields to:

$$J_2(t,x,\alpha_\epsilon(v),v)\ \le (e_2+2\epsilon_0)+mC\epsilon_0+\epsilon/4+\epsilon/2 \le e_2+\epsilon$$

from the choice of $\epsilon_0$. The last assertion of (22) being obvious, (22) is proved.

In the same way one can build an admissible strategy $\beta_\epsilon \in \mathcal{B}(t)$ such that

$$\forall u\in\mathcal{U}(t),\ J_1(t,x,u,\beta_\epsilon(u))\le e_1+\epsilon \text{ and } \beta_\epsilon(\bar u)=\bar v\ .$$

Combining (22), the previous assertion and Lemma 2.8 implies that $(\alpha_\epsilon,\beta_\epsilon)$ satisfies the two inequalities of the definition of a Nash equilibrium payoff.

QED

## 4 Proof of the characterization: necessary condition

Suppose that there is a Nash equilibrium payoff $(e_1, e_2) \in I\!R^2$ at a point $(t, x)$. For some fixed $\epsilon > 0$, let $(\alpha_\epsilon, \beta_\epsilon) \in \mathcal{A}(t) \times \mathcal{B}(t)$ be such that for any $(\alpha, \beta) \in \mathcal{A}(t) \times \mathcal{B}(t)$, the following inequalities hold:

$$(26) \qquad J_1(t,x,\alpha_\epsilon,\beta_\epsilon) \geq J_1(t,x,\alpha,\beta_\epsilon) - \epsilon^2/2 \ \text{ and } \ J_2(t,x,\alpha_\epsilon,\beta_\epsilon) \geq J_2(t,x,\alpha_\epsilon,\beta) - \epsilon^2/2$$

and

$$\text{for } j = 1,2, \ |J_j(t,x,\alpha_\epsilon,\beta_\epsilon) - e_j| \leq \epsilon^2/2 \ .$$

Thanks to Lemma 2.4, there exists a unique couple of controls $(u^\epsilon, v^\epsilon) \in \mathcal{U}(t) \times \mathcal{V}(t)$ such that

$$\alpha_\epsilon(v^\epsilon) = u^\epsilon \text{ and } \beta_\epsilon(u^\epsilon) = v^\epsilon \ .$$

Let us set $X^\epsilon_\cdot = X^{t,x,u^\epsilon,v^\epsilon}_\cdot$.

We argue by contradiction and assume that there is some time $\theta \in [t, T)$ and some $j = 1, 2$ (say $j = 1$ to fix the idea) such that

$$P\left\{E[g_1(X^\epsilon_T)|\mathcal{F}_{t,\theta}] < W_1(\theta, X^\epsilon_\theta) - \epsilon\right\} \ > \ \epsilon \ .$$

We set

$$(27) \qquad A = \left\{E[g_1(X^\epsilon_T)|\mathcal{F}_{t,\theta}] < W_1(\theta, X^\epsilon_\theta) - \epsilon\right\} \ .$$

By Lemma 3.1 (and the remark following the lemma) applied to $\theta$ and to the control $u^\epsilon$, there exists a nonanticipative strategy $\tilde{\alpha} : \mathcal{V}(t) \to \mathcal{U}(t)$ such that, for any $v \in \mathcal{V}(t)$, $\tilde{\alpha}(v) = u^\epsilon$ on $[t, \theta]$ and $P$-a.s.,

$$(28) \qquad E[g_1(X^{t,x,\tilde{\alpha}(v),v}_T)|\mathcal{F}_{t,\theta}] \geq W_1(\theta, X^{t,x,\tilde{\alpha}(v),v}_\theta) - \epsilon/2 \ .$$

Let $(u, v)$ be the unique couple associated with $(\tilde{\alpha}, \beta_\epsilon)$. Let us notice that $u \equiv u^\epsilon$ on $[t, \theta]$. We define a control $\bar{u}$ in the following way:

$$\bar{u} = u^\epsilon \text{ on } ([t,\theta) \times \Omega) \cup ([\theta, T] \times A^c), \ \bar{u} = u \text{ on } [\theta, T] \times A \ .$$

Since $\beta_\epsilon$ is *strongly* nonanticipative, Corollary 6.4 in the Appendix states that

$$\beta_\epsilon(\bar{u}) \equiv v^\epsilon \text{ on } [t,\theta) \text{ and } \beta_\epsilon(\bar{u})_s = \begin{cases} v_s & \text{on } A \\ v^\epsilon_s & \text{on } A^c \end{cases} \text{ for } s \in [\theta, T] \ .$$

Hence, we have:

$$X^{t,x,\bar{u},\beta_\epsilon(\bar{u})} \equiv X^\epsilon \text{ on } [t,\theta] \ \text{ and } X^{t,x,\bar{u},\beta_\epsilon(\bar{u})}_s = \begin{cases} X^{t,x,\tilde{\alpha}(v),v}_s & \text{on } A \\ X^\epsilon_s & \text{on } A^c \end{cases} \text{ for } s \in [\theta, T] \ .$$

Accordingly, by (28), we have:

$$(29) \qquad \begin{aligned} J_1(t,x,\bar{u},\beta_\epsilon(\bar{u})) &= E\left[g_1(X^\epsilon_T)1_{A^c}\right] + E\left[E[g_1(X^{t,x,\tilde{\alpha}(v),v}_T)|\mathcal{F}_{t,\theta}]1_A\right] \\ &\geq E\left[g_1(X^\epsilon_T)1_{A^c}\right] + E\left[W_1(\theta, X^\epsilon_\theta)1_A\right] - \tfrac{\epsilon}{2}P[A]. \end{aligned}$$

It follows from the definition (27) of $A$ that

$$J_1(t,x,\bar{u},\beta_\epsilon(\bar{u})) > E\left[g_1(X^\epsilon_T)\right] + \frac{\epsilon}{2}P(A) \geq J_1(t,x,\alpha_\epsilon,\beta_\epsilon) + \epsilon^2/2 \ .$$

This is in contradiction with (26) and the proof is complete.

QED

## 5 Proof of the existence

For proving Theorem 2.9, it is enough to show that, for any $\epsilon > 0$, there are some controls $u^\epsilon$ and $v^\epsilon$ satisfying the conditions (i) and (ii) of Theorem 2.10. In fact, we give a slightly stronger result:

**Proposition 5.1** *Suppose the assumptions of Theorem 2.9 hold. Then, for any $\epsilon > 0$, there is a control-pair $(u^\epsilon, v^\epsilon) \in \mathcal{U}(t) \times \mathcal{V}(t)$ such that, for any $t \le s_1 \le s_2 \le T$ and $j = 1, 2$,*

$$P\left\{E[W_j(s_2, X^\epsilon_{s_2})|\mathcal{F}_{t,s_1}] \ge W_j(s_1, X^\epsilon_{s_1}) - \epsilon\right\} \ge 1 - \epsilon$$

*where $X^\epsilon_\cdot = X^{t,x,u^\epsilon,v^\epsilon}_\cdot$.*

**Proof of Theorem 2.9 :** Combining the above Proposition applied to $s_1 = s$ and $s_2 = T$ with Theorem 2.10 gives the result for any $(e_1, e_2)$ which is an accumulation point of the payoff $(J_1(t, x, u^\epsilon, v^\epsilon), J_2(t, x, u^\epsilon, v^\epsilon))$ as $\epsilon \to 0^+$.

QED

The proof of Proposition 5.1 is splitted into several lemmata.

**Lemma 5.2** *For any $\epsilon > 0$, there is a couple $(u^\epsilon, v^\epsilon) \in \mathcal{U}(t) \times \mathcal{V}(t)$ such that, for any $t \le s \le T$ and $j = 1, 2$,*

$$E[W_j(s, X^\epsilon_s)] \ge W_j(t, x) - \epsilon$$

*where $X^\epsilon_\cdot = X^{t,x,u^\epsilon,v^\epsilon}_\cdot$.*

**Proof :** Let us choose $\alpha_\epsilon \in \mathcal{A}(t)$ and $\beta_\epsilon \in \mathcal{B}(t)$ such that $\alpha_\epsilon$ is $\epsilon/2$-optimal for $W_1(t, x)$ while $\beta_\epsilon$ is $\epsilon/2$-optimal for $W_2(t, x)$: Namely

(30) $$W_1(t, x) \le \inf_{v \in \mathcal{V}(t)} J_1(t, x, \alpha_\epsilon(v), v) + \epsilon/2 \text{ and } W_2(t, x) \le \inf_{u \in \mathcal{U}(t)} J_2(t, x, u, \beta_\epsilon(u)) + \epsilon/2 \ .$$

Let $(u^\epsilon, v^\epsilon)$ be the unique pair of controls such that

$$\alpha_\epsilon(v^\epsilon) = u^\epsilon \text{ and } \beta_\epsilon(u^\epsilon) = v^\epsilon \ .$$

We intend to prove that the couple $(u^\epsilon, v^\epsilon)$ satisfies the conclusion of the lemma. For this, we argue by contradiction and assume that there is some $\theta \in (t, T]$ and some $j = 1, 2$ (to fix the ideas, we suppose $j = 2$) such that

(31) $$E[W_2(\theta, X^\epsilon_\theta)] < W_2(t, x) - \epsilon \ .$$

By Lemma 3.1 applied to $\theta$ and to the control $u^\epsilon$, there exists a nonanticipative strategy $\alpha : \mathcal{U}(t) \to \mathcal{V}(t)$ such that, for any $v \in \mathcal{V}(t)$, $\alpha(v) \equiv u^\epsilon$ on $[t, \theta]$ and,

(32) $$E[g_2(X^{t,x,\alpha(v),v}_T)|\mathcal{F}_{t,\theta}] \le W_2(\theta, X^{t,x,\alpha(v),v}_\theta) + \epsilon/2, \ \ P \text{ a.s.}$$

By Lemma 2.4, there exists a unique couple of controls $(u, v) \in \mathcal{U}(t) \times \mathcal{V}(t)$ such that, $P$-a.s.

$$\alpha(v) = u \text{ and } \beta_\epsilon(u) = v.$$

Since $\alpha$ is nonanticipative and $\beta_\epsilon$ is admissible, and since $\alpha(v^\epsilon) \equiv u^\epsilon$ and $\beta_\epsilon(u^\epsilon) \equiv v^\epsilon$ on $[t, \theta]$, it is easy to check that

$$u \equiv u^\epsilon \text{ and } v \equiv v^\epsilon \text{ on } [t, \theta] \ .$$

Thus $X^{t,x,u,v}_\theta = X^{t,x,\alpha(v),v}_\theta = X^\epsilon_\theta$. It follows then from (31) and (32) that

$$\begin{aligned} J_2(t, x, u, \beta_\epsilon(u)) = J_2(t, x, \alpha(v), v) = & \ E[E(g_2(X^{t,x,\alpha(v),v}_T)|\mathcal{F}_{t,\theta})] \\ \le & \ E[W_2(\theta, X^{t,x,\alpha(v),v}_\theta)] + \epsilon/2 \\ < & \ W_2(t, x) - \epsilon/2, \end{aligned}$$

which is in contradiction with (30).

QED

**Lemma 5.3** *Let us fix $\epsilon > 0$ and $t_0 = t < t_1 < \ldots < t_m = T$. Then there is some $(u^\epsilon, v^\epsilon) \in \mathcal{U}(t) \times \mathcal{V}(t)$ such that, for any $i = 0, \ldots, (m-1)$, for $j = 1, 2$, $P$-a.s.,*

$$E[W_j(t_{i+1}, X^\epsilon_{t_{i+1}}) | \mathcal{F}_{t,t_i}] \geq W_j(t_i, X^\epsilon_{t_i}) - \epsilon \ ,$$

*where $X^\epsilon_\cdot = X^{t,x,u^\epsilon,v^\epsilon}_\cdot$.*

**Proof :** We construct $(u^\epsilon, v^\epsilon)$ by induction on the interval $[t_i, t_{i+1})$. Let us first notice that the result for $i = 0$ is given by Lemma 5.2.

Let us now assume that $(u^\epsilon, v^\epsilon)$ is constructed on $[t_0, t_i)$ and let us define it on $[t_i, t_{i+1})$. From Lemma 5.2, for any $y \in I\!\!R^n$, there is some $(u^y, v^y) \in \mathcal{U}(t_i) \times \mathcal{V}(t_i)$ such that for any $s \in [t_i, T]$ and for $j = 1, 2$,

$$E[W_j(s, X^{t_i,y,u^y,v^y}_s)] \geq W_j(t_i, y) - \epsilon/2 \ .$$

Using the continuity of $W_j(t_i, \cdot)$ and of $E[W_j(s, X^{t_i,\cdot,u^y,v^y}_s)]$, we can find a Borel partition $(O_l \ i = 1, 2, \ldots)$ of $I\!\!R^n$ such that, for any $l$, there is some $y_l \in O_l$ with, for $j = 1, 2$,

$$\forall z \in O_l, \ E[W_j(s, X^{t_i,z,u^{y_l},v^{y_l}}_s)] \geq W_j(t_i, z) - \epsilon \ .$$

Then we define, for any $z \in I\!\!R^n$, the control pair $(\tilde{u}(z), \tilde{v}(z))$ by

$$\forall s \geq t, \ \tilde{u}(z)_s = \sum_l 1_{O_l}(z) u^{y_l}_s \text{ and } \tilde{v}(z)_s = \sum_l 1_{O_l}(z) v^{y_l}_s \ ,$$

and we set

$$(u^\epsilon, v^\epsilon) = (\tilde{u}(X^{t,x,u^\epsilon,v^\epsilon}_{t_i}), \tilde{v}(X^{t,x,u^\epsilon,v^\epsilon}_{t_i})) \text{ on } [t_i, t_{i+1}) \ .$$

We have, for all $s \geq t_i$, $P$-a.s.,

$$E[W_j(s, X^\epsilon_s) | \mathcal{F}_{t,t_i}] \geq W_j(t_i, X^\epsilon_{t_i}) - \epsilon$$

where, as usual, $X^\epsilon_\cdot = X^{t,x,u^\epsilon,v^\epsilon}_\cdot$. Using the above inequality for $s = t_{i+1}$ completes the proof by induction.

QED

We are now ready to prove Proposition 5.1. Let us choose a partition $t_0 = t < t_1 < \ldots < t_m = T$ and let us set

$$\tau = \sup_i |t_{i+1} - t_i| \ .$$

Since $W_j(\cdot, y)$ are uniformly Hölder continuous and $W_j(s, \cdot)$ are uniformly Lipschitz continuous and since the dynamic is bounded, we can choose $\tau$ sufficiently small in such a way that, for all $k \in \{0, \ldots, m-1\}$ and all $s \in [t_k, t_{k+1})$,

$$\|W_j(t_{k+1}, X^{t,x,u,v}_{t_{k+1}}) - W_j(s, X^{t,x,u,v}_s)\|_2 \leq \gamma \tag{33}$$

where $\gamma = \epsilon^{\frac{3}{2}}/4$.
Let $(u, v) \in \mathcal{U}(t) \times \mathcal{V}(t)$ be defined by Lemma 5.3 for $\epsilon = \epsilon/(2m)$: for any $i = 0, \ldots, (m-1)$, for $j = 1, 2$, $P$-a.s.,

$$E[W_j(t_{i+1}, X_{t_{i+1}}) | \mathcal{F}_{t,t_i}] \geq W_j(t_i, X_{t_i}) - \epsilon/(2m) \tag{34}$$

where $X_\cdot = X^{t,x,u,v}_\cdot$. Let us now fix $t \leq s_1 < s_2 \leq T$. Let also $t_i$ and $t_k$ be such that $t_{i-1} \leq s_1 < t_i$ and $t_k < s_2 \leq t_{k+1}$. Then we have, thanks to (34), for $j = 1, 2$, $P$-a.s.,

$$E[W_j(t_{k+1}, X_{t_{k+1}}) | \mathcal{F}_{t,t_i}] \geq W_j(t_i, X_{t_i}) - \epsilon/2 \ .$$

Taking the conditionnal expectation with respect to $\mathcal{F}_{t,s_1}$ gives since $t_i \geq s_1$: $P$-a.s.,

$$E[W_j(t_{k+1}, X_{t_{k+1}})|\mathcal{F}_{t,s_1}] \geq E[W_j(t_i, X_{t_i})|\mathcal{F}_{t,s_1}] - \epsilon/2 \ . \tag{35}$$

Let us set

$$Z_1 = E[W_j(t_{k+1}, X_{t_{k+1}})|\mathcal{F}_{t,s_1}] - E[W_j(t_i, X_{t_i})|\mathcal{F}_{t,s_1}] + \epsilon/2$$

and

$$Z_2 = E[W_j(s_2, X_{s_2})|\mathcal{F}_{t,s_1}] - W_j(s_1, X_{s_1}) + \epsilon/2$$

From (33) and (35), we have $Z_1 \geq 0$ $P$-a.s., and $\|Z_2 - Z_1\|_2 \leq 2\gamma$. Therefore,

$$P[Z_2 < -\epsilon/2] \leq P[|Z_2 - Z_1| > \epsilon/2] \leq 4\gamma^2/(\epsilon/2)^2 = \epsilon \ ,$$

i.e.,

$$P\left\{E[W_j(s_2, X_{s_2})|\mathcal{F}_{t,s_1}] \geq W_j(s_1, X_{s_1}) - \epsilon\right\} \geq 1 - \epsilon \ .$$

Therefore the proof of Proposition 5.1 is complete.

QED

# 6 Appendix: On strategies

In the Appendix we bring together several technical facts on strongly nonanticipative strategies. Some of them (Lemma 6.1 and Corollary 6.4) are used in the proofs of Theorems 2.9 and 2.10, some others have seemed to us of general interest for a better understanding of the different notions of strategies.

In the following lemma we prove that the strategy built in the proof of Lemma 3.1 is strongly nonanticipative. Let us first recall the construction of the strategy $\alpha$: Let $\theta \in (t, T)$ be a fixed time, $(O_i)_{i \in \mathbb{N}}$ be a Borel partition of $\mathbb{R}^n$ and, for any $i \in \mathbb{N}$, $\alpha_i \in \mathcal{A}(\theta)$ be an admissible strategy. We also fix some control $u \in \mathcal{U}(t)$. The strategy $\alpha$ is defined by setting

$$\forall v \in \mathcal{V}(t), \alpha(v)_s = \begin{cases} u_s & \text{for } s \in [t, \theta], \\ \alpha_i(v|_{[\theta,T]})_s & \text{for } s \in (\theta, T], \text{ on } \{X_\theta^{t,x,u,v} \in O_i\}. \end{cases}$$

**Lemma 6.1** *The strategy $\alpha$ is strongly nonanticipative.*

**Proof :** Let $S$ be an $(\mathcal{F}_{t,s})_{s\in[t,T]}-$stopping time and let $v_1, v_2 \in \mathcal{V}(t)$ be such that $v_1 \equiv v_2$ on $[\![t, S]\!]$. We have to prove that $\alpha(v_1) \equiv \alpha(v_2)$ on $[\![t, S]\!]$.

Let us notice that the result is obvious on $\{S \leq \theta\}$, because we have $\alpha(v_1) = \alpha(v_2) = u$ on $[t, \theta]$. Let us now set $B = \{S > \theta\}$ and $B_i = B \cap \{X_\theta^{t,x,u,v} \in O_i\}$. We can assume that $B$ is not neglectable, since otherwise there is nothing to prove. Let us now prove that $\alpha(v_1) \equiv \alpha(v_2)$ on $[\![\theta, S]\!] \cap ([t, T] \times B)$.

For this, let us first recall the identifications $\Omega_{t,T} = \Omega_{t,\theta} \times \Omega_{\theta,T}$ and $P_{t,T} = P_{t,\theta} \otimes P_{\theta,T}$. We know that, for almost all $\omega_1 \in \Omega_{t,\theta}$, the controls $v_1(\omega_1)$ and $v_2(\omega_1)$ belong to $\mathcal{V}(\theta)$. Let us finally introduce, for any $0 \leq r \leq s \leq T$, $\mathcal{F}^0_{r,s} = \sigma\{B_\tau - B_r,\ \tau \in [r, s]\}$ ; let us underline that $\mathcal{F}_{r,s}$ is nothing but $\mathcal{F}^0_{r,s}$ augmented by all null-sets of $P$. Using the identification $\mathcal{F}^0_{t,T} = \mathcal{F}^0_{t,\theta} \otimes \mathcal{F}^0_{\theta,T}$ and the fact that the section $N(\omega_1) = \{\omega_2 \in \Omega_{\theta,T},\ (\omega_1, \omega_2) \in N\}$ of a $P_{t,T}-$neglectable set $N \in \mathcal{F}_{t,T}$ is $P_{\theta,T}-$neglectable for almost all $\omega_1 \in \Omega_{t,\theta}$, it can also be proved that, for almost all $\omega_1 \in \Omega_{t,\theta}$, $S(\omega_1)$ is an $(\mathcal{F}_{\theta,s})_{s\in[\theta,T]}-$stopping time. From the definition of the strategy $\alpha$, we have, for any $i \in \mathbb{N}$ and for almost all $\omega_1 \in B_i$: $\alpha(v_1)(\omega_1, \cdot) = \alpha_i(v_1(\omega_1)_{[\theta,T]})$ and $\alpha(v_2)(\omega_1, \cdot) = \alpha_i(v_2(\omega_1)_{[\theta,T]})$ on $[\theta, T]$. Since $\alpha_i$ is strongly nonanticipative and since $v_1(\omega_1)_{[\theta,T]} \equiv v_2(\omega_1)_{[\theta,T]}$ on $[\![\theta, S(\omega_1)]\!]$ for almost all $\omega_1 \in B_i$, we have for almost all $\omega_1 \in B_i$,

$$\alpha_i(v_1(\omega_1)_{[\theta,T]}) \equiv \alpha_i(v_2(\omega_1)_{[\theta,T]}) \text{ on } [\![\theta, S(\omega_1)]\!] \ .$$

Therefore $\alpha(v_1)(\omega_1,\cdot) \equiv \alpha(v_2)(\omega_1,\cdot)$ on $[\![\theta, S(\omega_1)]\!]$ for almost all $\omega_1 \in B$. This completes the proof of the lemma.

QED

Let us now introduce the following notion which appears naturally in the proof of Theorem 2.10.

**Definition 6.2 (Condition (C))** *Let $\alpha : \mathcal{V}(t) \to \mathcal{U}(t)$ be a nonanticipative strategy. We say that $\alpha$ satisfies condition (**C**) if for any (deterministic) $\theta \in [t,T)$, for any control $v, \bar{v} \in \mathcal{V}(t)$ with $v \equiv \bar{v}$ on $[t,\theta]$ and for any $A \in \mathcal{F}_{t,\theta}$, we have*

$$\alpha(v\mathbf{1}_A + \bar{v}\mathbf{1}_{A^c}) \equiv \alpha(v)\mathbf{1}_A + \alpha(\bar{v})\mathbf{1}_{A^c} \ ,$$

*where $v\mathbf{1}_A + \bar{v}\mathbf{1}_{A^c}$ denotes for simplicity the control equal to $v \equiv \bar{v}$ on $[t,\theta] \times \Omega$, to $v$ on $[\theta,T] \times A$ and to $\bar{v}$ on $[\theta,T] \times A^c$.*

**Remark :** Although condition (**C**) is weaker than the assumption of being strongly nonanticipative for a strategy (cf Corollary 6.4 below), the main results of this paper, Theorems 2.9 and 2.10, remain unchanged if condition (i) in the definition of admissible strategies is replaced by condition (**C**). However, as we shall also see below, condition (**C**) does not seem to be the right definition for modelizing the fact that a player can only take into account the observation of the control played by his opponent.

Here is an equivalent formulation of condition (**C**):

**Proposition 6.3** *Let $\alpha : \mathcal{V}(t) \to \mathcal{U}(t)$ be a nonanticipative strategy. The following assertions are equivalent:*

1. *$\alpha$ satisfies condition (**C**),*
2. *for all stopping time $S$ taking a finite number of values in $[t,T]$, if $v \equiv \tilde{v}$ on $[\![t,S]\!]$, then $\alpha(v) \equiv \alpha(\tilde{v})$ on $[\![t,S]\!]$.*

This result implies that condition (**C**) is weaker than the notion of strongly nonanticipative strategies:

**Corollary 6.4** *A strongly nonanticipative strategy $\alpha$ satisfies condition (**C**).*

Indeed, a strongly nonanticipative strategy $\alpha$ obviously satisfies the second condition of the Proposition.

**Proof of Proposition 6.3 :** Let us first assume that $\alpha$ satisfies the second condition. For proving that $\alpha$ satisfies condition (**C**), let us fix $\theta \in [t,T)$, $A \in \mathcal{F}_{t,\theta}$, and $v, \bar{v}$ in $\mathcal{V}(t)$ such that $v \equiv \bar{v}$ on $[t,\theta]$. We set $v_1 = v1_A + \bar{v}1_{A^c}$. We want to prove that $\alpha(v_1) = \alpha(v)1_A + \alpha(\bar{v})1_{A^c}$ on $[\theta,T]$.

For this let us introduce the stopping time $S$ defined by

$$S = \theta \text{ on } A^c \text{ and } S = T \text{ on } A\ .$$

Then $v_1 \equiv v$ on $[\![t,S]\!]$. Since $\alpha$ satisfies 2, $\alpha(v_1) \equiv \alpha(v)$ on $[\![t,S]\!]$. This implies that $\alpha(v_1) \equiv \alpha(v)$ on $[\theta,T] \times A$. We can prove in the same way (using the stopping time $S' = \theta$ on $A$ and $S' = T$ on $A^c$) that $\alpha(v_1) \equiv \alpha(\bar{v})$ on $[\theta,T] \times A^c$. Therefore $\alpha$ satisfies condition (**C**).

Let us now assume that $\alpha$ satisfies condition (**C**). Let $S$ be a stopping time taking values in $t_0 = t < t_1 < \ldots < t_m = T$, and $v, \bar{v}$ be such that $v \equiv \bar{v}$ on $[\![t, S]\!]$. We have to prove that $\alpha(v) \equiv \alpha(\bar{v})$ on $[\![t, S]\!]$. For this, let us set, for any $j \in \{0, \ldots, m\}$,

$$A_j = \{S \le t_j\} \text{ and } v_j = \begin{cases} v & \text{in } ([t, t_j) \times \Omega) \cup ([t_j, T] \times A_j) \\ \bar{v} & \text{in } [t_j, T] \times A_j^c \end{cases}$$

We are going to prove by induction that

$$(36) \qquad \alpha(v_j) \equiv \alpha(\bar{v}) \text{ in } [t_j, T] \times A_j^c \text{ and } \alpha(v) \equiv \alpha(\bar{v}) \text{ in } [\![t, S \wedge t_j]\!]$$

For $j = m$, this shows the desired result: $\alpha(v) = \alpha(\bar{v})$ on $[\![t, S]\!]$.

For $j = 0$, we have $A_0 = \{S = t\} \in \mathcal{F}_{t,t} = \{\emptyset, \Omega\}$ $P$-a.s. and thus either $v_0 = v$ a.s. (if $A_0 = \Omega$) or $v_0 = \bar{v}$ a.s. (if $A_0 = \emptyset$). In both cases, equalities in (36) are clear.

Let us assume that (36) holds for some $j$. Let us first prove that $\alpha(\bar{v}) = \alpha(v)$ in $[\![t, S \wedge t_{j+1}]\!]$. For that purpose, let us notice that $v_j \equiv v$ on $[t, t_{j+1})$, because $v \equiv \bar{v}$ on $[t_j, t_{j+1}) \times A_j^c$, since $A_j^c = \{S \ge t_{j+1}\}$ and $v \equiv \bar{v}$ on $[\![t, S]\!]$. Since $\alpha$ is nonanticipative, we have therefore that $\alpha(v_j) \equiv \alpha(v)$ on $[t, t_{j+1})$. Using assumption (36), which states that $\alpha(v_j) \equiv \alpha(\bar{v})$ in $[t_j, T] \times A_j^c$, we deduce that $\alpha(\bar{v}) = \alpha(v)$ in $[t_j, t_{j+1}] \times A_j^c$. Let us now notice that

$$[\![t, S \wedge t_{j+1}]\!] = [\![t, S \wedge t_j]\!] \cup ([t_j, t_{j+1}] \times A_j^c) \ .$$

From assumption (36) we know that $\alpha(\bar{v}) = \alpha(v)$ in $[\![t, S \wedge t_j]\!]$ and we have just proved that $\alpha(\bar{v}) = \alpha(v)$ in $[t_j, t_{j+1}] \times A_j^c$. Therefore we have established that $\alpha(\bar{v}) = \alpha(v)$ in $[\![t, S \wedge t_{j+1}]\!]$.

It remains to show that $\alpha(v_{j+1}) \equiv \alpha(\bar{v})$ in $[t_{j+1}, T] \times A_{j+1}^c$. Let us first notice that $v_{j+1} = v_j 1_{A_{j+1}^c} + v 1_{A_{j+1}}$. Then, since $v_j \equiv v$ on $[t, t_{j+1}]$ and since $A_{j+1} \in \mathcal{F}_{t, t_{j+1}}$, we have, from assumption (**C**),

$$(37) \qquad \alpha(v_{j+1}) = \alpha(v_j 1_{A_{j+1}^c} + v 1_{A_{j+1}}) \equiv \alpha(v_j) 1_{A_{j+1}^c} + \alpha(v) 1_{A_{j+1}} \text{ on } [t, T] \ .$$

In particular, $\alpha(v_{j+1}) \equiv \alpha(v_j)$ on $[t, T] \times A_{j+1}^c$. Moreover, from (36), $\alpha(v_j) \equiv \alpha(\bar{v})$ on $[t_{j+1}, T] \times A_{j+1}^c$, because $A_{j+1}^c \subset A_j^c$. Thus

$$\alpha(v_{j+1}) \equiv \alpha(v_j) \equiv \alpha(\bar{v}) \text{ on } [t_{j+1}, T] \times A_{j+1}^c \ .$$

By induction the proof is now complete.

QED

We complete this appendix by showing that condition (**C**) is not equivalent with the notion of strongly nonanticipative strategies. More precisely we build a strategy which is nonanticipative with delay, an r-strategy, satisfies condition (**C**), but is not strongly nonanticipative.
Let us suppose that each of the spaces in which the controls take their values has only two elements: $U = \{u_1, u_2\}$ and $V = \{v_1, v_2\}$. Let $t \in [0, T]$. We define the strategy $\alpha : \mathcal{V}(t) \to \mathcal{U}(t)$ in the following way:
Let $t = t_0 < t_1 < t_2 = T$ be a partition of $[t, T]$. For any $v \in \mathcal{V}(t)$, we set $\alpha(v)_s = u_1$ for $s \in [t, T]$, if $v$ satisfies the following property:

$$(38) \qquad \exists \epsilon > 0 \text{ such that } v \equiv v_1 \text{ on } [t, t + \epsilon].$$

Otherwise we set $\alpha(v)_s = u_1$ for $s \in [t, t_1]$ and $\alpha(v)_s = u_2$ for $s \in (t_1, T]$.
It is easy to check that $\alpha$ is a nonanticipative strategy with delay. Let us now prove that $\alpha$ is an r-strategy: If $t > 0$, for $\bar{t} \in [0, t)$ and $v \in \mathcal{V}(\bar{t})$, define the process $(\tilde{u}_s = \alpha(v|_{[t,T]})_s, s \in [t, T])$ (using

the notations of Chapter 2. This process satisfies $\tilde{u} \equiv u_1$ on $[t, t_1]$, then is constant on $(t_1, T]$ equal to $u_1$ or $u_2$, with

$$\{\tilde{u}_s = u_1, s \in (t_1, T]\} = \bigcap_{\epsilon > 0 \text{ rationnal}} \{v \equiv v_1 \text{ in } [t, t+\epsilon]\} \in \mathcal{F}_{\bar{t}, t+} \subset \mathcal{F}_{\bar{t}, t_1}.$$

Thus $\tilde{u}$ is adapted to the filtration $(\mathcal{F}_{\bar{t},s})_{s \geq t}$. Since, moreover, its paths are left continuous, it follows that $\tilde{u}$ is $(\mathcal{F}_{\bar{t},s})_{s \geq t}$-progressively measurable.
Let us now prove that $\alpha$ satisfies the condition (**C**): Let us point out that $\mathcal{F}_{t,t} = \{\emptyset, \Omega\}$ $P-$a.s. . Thus (**C**) is trivially satisfied for $\theta = t$. Further, by the construction of $\alpha$, if for $v, \overline{v} \in \mathcal{V}(t)$, we have $v \equiv \overline{v}$ on some time interval $[t, \theta], \theta > t$, then $\alpha(v) = \alpha(\overline{v})$ on $[t, T]$. In particular, $\alpha$ satisfies (**C**).
We finally show that $\alpha$ is not strongly anticipative: Let $S$ be an $(\mathcal{F}_{t,s})_{s \in [t,T]}$-stopping time such that $P[S > t_1] > 0$ and, for all $\epsilon > 0, P[S \leq t + \epsilon] > 0$, say $S = \inf\{s \geq t, B_s - B_t = 1\} \wedge T$. We define $v$ and $\overline{v} \in \mathcal{V}(t)$ by

$$v \equiv v_1 \text{ on } [t, T],$$
$$\overline{v} \equiv v_1 \text{ on } [\![t, S]\!] \text{ and } \overline{v} \equiv v_2 \text{ on } ]\!]S, T]\!].$$

It holds that $v \equiv \overline{v}$ on $[\![t, S]\!]$. But the strategy $\alpha$ applied to the two controls gives

$$\alpha(v) \equiv u_1 \text{ and } \alpha(\overline{v}) \equiv u_2 \text{ on } (t_1, T].$$

**Remark 6.5** Remark also that in Assertion 2. of Proposition 6.3 we have found an example of a property which holds for every stopping time taking a finite number of values, but which cannot be generalised to all stopping times.

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