Viscosity solutions of Hele-Shaw moving boundary problem for power-law fluid*

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Abstract : Existence and uniqueness of solutions for Hele-shaw moving boundary problem for power-law fluid is established in the framework of viscosity solutions.
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1 Introduction

The aim of the work is to investigate a mathematical model describing the Hele-Shaw approximation of the injection of a power-law fluid between two closely situated plates. The fluid is supposed to be surrounded by another fluid with small viscosity, so we consider a one-phase moving boundary problem. Let us denote by S the source of the injection, by $\Omega(t)$ the portion of space occupied by the fluid at time t, by $\Sigma(t)$ the moving boundary. Then, according to [2], [3] and [18], $\Sigma(t)$ evolves with a normal velocity $V_{t,x}$ given at each point $x \in \Sigma(t)$ by the quasi-static equation

$$V_{t,x} = |\nabla u(t,x)|^{p-1}$$

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where $u(t,x)$ satisfies at any time $t > 0$ the $p-$Laplace equation (with $p > 1$)

$$\begin{cases} -\text{div}\,\left(|\nabla u(t,x)|^{p-2}\nabla u(t,x)\right) = f(x) & \text{in } \Omega(t)\backslash S \\ u(t,x) = 0 & \text{on } \Sigma(t) \\ u(t,x) = g(x) & \text{on } S \end{cases}$$

where f and g are some positive functions.

When $p = 2$, this is the well-known Hele-Shaw problem, which has been investigated by many authors. In particular, if the initial data is smooth, the evolution equation admits a smooth solution for a short period of time [11], but singularities appear in general in finite time. In order to define the solutions after the onset of singularities, various notions of generalized solutions have been introduced. For instance the Hele-Shaw problem is reformulated in terms of variationnal inequalities via the Baiocchi transform in [10] while Kim proposes in [14] a definition of viscosity solution for this problem.

To the best of our knowledge, the case of $p \neq 2$ has never been studied up to now. Our aim is to define a notion of generalized solution, to prove its uniqueness and its stability.

The key feature of the moving boundary problem we are investigating is that it preserves, at least formally, the inclusion: Namely, if $\Omega_1(t)$ and $\Omega_2(t)$ are two families of solutions, with $\Omega_1(0) \subset \Omega_2(0)$, then this inclusion should be preserved along the time. This is just because the velocity is increasing with respect to the set. The main result of the paper (Theorem 3.1) is that this inclusion principle holds true even for weak solutions.

The inclusion principle is the key feature for building generalized solutions of other front propagation problems: It has extensively been used in the construction of viscosity solutions for the mean curvature motion (see in particular [12], [9] for the so-called level-set approach and [19], [4], [5] for related but more geometric approaches). Similar viscosity solutions have also been introduced for the porous-medium equation [6] and for a free boundary problem motivated by combustion [15]. In [14] Kim proved the inclusion principle for the viscosity solutions of the Hele-Shaw problem when $p = 2$, $f \equiv 0$ and a particular source S.

In order to prove the inclusion principle for power-law fluids, the method of [14] seems no longer applicable, because it involves the construction of rather tricky super and sub-solutions, which could hardly be possible in a more general context. Here we use instead several ideas introduced by the first author in [8] for simpler moving boundary problems. In particular, we use two basic ingredients of [8]: An equivalent definition of solutions (Proposition 2.7) and Ilmanen interposition Lemma [13]. However the proof differs substancially from that of [8] because the main feature of the problem studied in [8] is an invariance by translation which is no longer satisfied for the Hele-Shaw problem.

We now briefly explain the organization of the paper. We first introduce the notion of viscosity solutions for the Hele-Shaw problem for power-law fluids, and investigate the main properties of the velocity. Next we state and prove the inclusion principle. We finally apply this inclusion principle in the last part to derive existence, uniqueness and stability of solutions.

2 Definitions and preliminary results

2.1 Definition of the solutions

Let us first fix some notations: throughout the paper $|\cdot|$ denotes the euclidean norm (of $\mathbb{R}^N$ or $\mathbb{R}^{N+1}$, depending on the context). If K is a subset of $\mathbb{R}^N$ and $x \in \mathbb{R}^N$, then $d_K(x)$ denotes the usual distance from x to K: $d_K(x) = \inf_{y\in K} |y-x|$. Finally we denote by $B(x,R)$ the open ball centered at x and of radius R.

We intend to study the evolution of compact hypersurfaces $\Sigma(t) = \partial\Omega(t)$ of $\mathbb{R}^N$, where $\Omega(t)$ is an open set, evolving with the following law:

$$\forall t \geq 0\ ,\ x \in \Sigma(t),\ \ V_{t,x} = h(x, \Omega(t)) \tag{1}$$

where $V_{t,x}$ is the normal velocity of $\Omega(t)$ at the point x, $h = h(x,\Omega)$ is given, for any set Ω with smooth boundary by

$$h(x,\Omega) = |\nabla u(x)|^{p-1}\ . \tag{2}$$

In the previous equation, $u : \Omega \to \mathbb{R}$ is the solution of the following p.d.e.

$$\left\{\begin{array}{lrl} i) & -div(|\nabla u|^{p-2}\nabla u) = f & \text{in } \Omega\backslash S \\ ii) & u = g & \text{on } \partial S \\ iii) & u = 0 & \text{on } \partial\Omega \end{array}\right. \tag{3}$$

and (i) is understood in the sense of distributions. The set S is a fixed source and we always assume above that $S \subset\subset \Omega(t)$. If $S = \emptyset$, then we omit condition (ii) in (3). Here and throughout the paper, we assume that

$$\left\{\begin{array}{rl} i) & S \subset \mathbb{R}^N \text{ is bounded and equal to the closure of an open set} \\ & \text{with a } \mathcal{C}^2 \text{ boundary} \\ ii) & f : \mathbb{R}^N \to \mathbb{R} \text{ is continuous and bounded and} \\ & \text{- either } f > 0 \text{ on } \mathbb{R}^N \text{ and } f \text{ is locally Lipschitz continuous} \\ & \text{- or } S \neq \emptyset \text{ and } f = 0 \text{ on } \mathbb{R}^N \\ iii) & g : S \to (0, +\infty) \text{ is } \mathcal{C}^{1,\beta} \text{ (for some } \beta \in (0,1)) \end{array}\right. \tag{4}$$

Remark 2.1 *Following [16] $h(x, \Omega)$ is well defined as soon as Ω has a "smooth" boundary. More precisely, it is proved in [16] that, if Ω has a $\mathcal{C}^{1,\beta}$ boundary and if $S \subset\subset \Omega$, then the solution u is $\mathcal{C}^{1,\alpha}$ for some $\alpha \in (0, \beta)$. Moreover the $\mathcal{C}^{1,\alpha}$ norm of u is bounded by a constant which depends only on $\|f\|_\infty$, $|g|_{1,\beta}$, p and on the $\mathcal{C}^{1,\beta}$ norm of the mapping which locally flattens the boundary of $\Omega \backslash S$.*

In the sequel, we set

$$\mathcal{D} = \{K \subset \mathbb{R}^N \ , \ K \text{ bounded and } S \subset Int(K)\} \ , \tag{5}$$

where $Int(K)$ denotes the interior of K.

From now on, we consider the graph of the evolving sets $\Omega(t)$ and we denote it $\mathcal{K}$. This set $\mathcal{K}$ is a subset of $\mathbb{R}^+ \times \mathbb{R}^N$. Formally, with the notations above

$$\mathcal{K} = \{(t, x) \text{ such that } x \in \Omega(t)\} \ .$$

The set $\mathcal{K}$ is our main unknown. We denote by (t, x) an element of such a set, where $t \in \mathbb{R}^+$ denotes the time and $x \in \mathbb{R}^N$ denotes the space. We set

$$\mathcal{K}(t) \ = \ \left\{x \in \mathbb{R}^N \mid (t, x) \in \mathcal{K}\right\} \ .$$

The closure of the set $\mathcal{K}$ in $\mathbb{R}^{N+1}$ is denoted by $\overline{\mathcal{K}}$. The closure of the complementary of $\mathcal{K}$ is denoted $\widehat{\mathcal{K}}$:

$$\widehat{\mathcal{K}} = \overline{(\mathbb{R}^+ \times \mathbb{R}^N) \backslash \mathcal{K}}$$

and we set

$$\widehat{\mathcal{K}}(t) = \left\{x \in \mathbb{R}^N \mid (t, x) \in \widehat{\mathcal{K}}\right\} \ .$$

Let us continue with the terminology: If $\mathcal{K}$ is a subset of $[0, +\infty) \times \mathbb{R}^N$, we say that

- $\mathcal{K}$ is a *tube*, if $\forall T \geq 0$, $\overline{\mathcal{K}} \cap ([0,T] \times \mathbb{R}^N)$ is a compact subset of $\mathbb{R}^{N+1}$.
- $\mathcal{K}$ is *non decreasing* if $\mathcal{K}(s) \subset \mathcal{K}(t)$ for any $0 \leq s \leq t$.
- $\mathcal{K}$ is *left lower semi-continuous* if

$$\forall t > 0,\ \forall x \in \mathcal{K}(t),\ \text{if } t_n \to t^-,\ \ \exists x_n \in \mathcal{K}(t_n) \text{ such that } x_n \to x\ .$$

- $\mathcal{K}_r$ is a *smooth tube* if $\mathcal{K}_r$ is closed in $I \times \mathbb{R}^N$ (where I is some open interval), has a non empty interior and $\partial\mathcal{K}_r \cap (I \times \mathbb{R}^N)$ is a $\mathcal{C}^{1,1}$ submanifold of $\mathbb{R}^{N+1}$, such that at any point $(t,x) \in \mathcal{K}_r$ the outward normal (ν_t, ν_x) to $\mathcal{K}_r$ at (t,x) satisfies $\nu_x \neq 0$. In this case the normal velocity $V^{\mathcal{K}_r}_{(t,x)}$ of $\mathcal{K}_r$ at the point $(t,x) \in \partial\mathcal{K}_r$ is given by $V^{\mathcal{K}_r}_{(t,x)} = -\nu_t/|\nu_x|$, where (ν_t, ν_x) is the outward normal to $\mathcal{K}_r$ at (t,x).

We use smooth tubes as "test sets". Namely, we say that the smooth tube $\mathcal{K}_r$ is *externally tangent* to a tube $\mathcal{K}$ at $(t,x) \in \partial\mathcal{K}$ if $\mathcal{K}_r$ is defined on some open interval I containing t, and if

$$\mathcal{K}(s) \subset \mathcal{K}_r(s) \quad \forall s \in I \quad \text{and} \quad (t,x) \in \partial\mathcal{K}_r\ .$$

In the same way, the smooth tube $\mathcal{K}_r$ is said to be *internally tangent* to $\mathcal{K}$ at $(t,x) \in \partial\widehat{\mathcal{K}}$ if $\mathcal{K}_r(s)$ is defined on some open interval I containing t, and if

$$\mathcal{K}_r(s) \subset \mathcal{K}(s) \quad \forall s \in I \text{ and } (t,x) \in \partial\mathcal{K}_r\ .$$

We are now ready to define the viscosity solutions of (1). Recall that the set $\mathcal{D}$ is defined by (5).

Definition 2.2 *Let $\mathcal{K}$ be a tube and $K_0 \in \mathcal{D}$ be an initial position.*

1. *$\mathcal{K}$ is* a viscosity subsolution *to the front propagation problem (1) if $\mathcal{K}$ is non decreasing, left lower semi-continuous and $\mathcal{K}(0) \in \mathcal{D}$, and if, for any smooth tube $\mathcal{K}_r$ externally tangent to $\mathcal{K}$ at some point (t,x), with $\mathcal{K}_r(t) \in \mathcal{D}$ and $t > 0$, we have*

$$V^{\mathcal{K}_r}_{(t,x)} \leq h(x, \mathcal{K}_r(t))$$

where $V^{\mathcal{K}_r}_{(t,x)}$ is the normal velocity of $\mathcal{K}_r$ at (t,x).

We say that $\mathcal{K}$ is a subsolution to the front propagation problem with initial position K_0 if $\mathcal{K}$ is a subsolution and if $\overline{\mathcal{K}}(0) \subset \overline{K_0}$.

2. $\mathcal{K}$ *is a* viscosity supersolution *to the front propagation problem if* $\mathcal{K}$ *is non decreasing and* $\mathcal{K}(0) \subset \mathcal{D}$ *and if, for any smooth tube* $\mathcal{K}_r$ *internally tangent to* $\mathcal{K}$ *at some point* (t,x)*, with* $\mathcal{K}_r(t) \in \mathcal{D}$ *and* $t > 0$*, we have*

$$V_{(t,x)}^{\mathcal{K}_r} \geq h(x, \mathcal{K}_r(t)) \ .$$

   We say that $\mathcal{K}$ *is a supersolution to the front propagation problem with initial position* K_0 *if* $\mathcal{K}$ *is a supersolution and if* $\widehat{\mathcal{K}}(0) \subset \overline{\mathbb{R}^N \backslash K_0}$*.*

3. *Finally, we say that a tube* $\mathcal{K}$ *is a* viscosity solution *to the front propagation problem (with initial position* K_0*) if* $\mathcal{K}$ *is a sub- and a supersolution to the front propagation problem (with initial position* K_0*).*

Let us point out that any classical solution is a viscosity solution. The previous definition has been introduced in [1] and was also used in [8].

2.2 Regularity properties of the velocity h

We now investigate the main regularity properties of the map h defined by (2) and (3). Recall that the set S and the function f and g satisfy assumptions (4).

The following Proposition is straightforward application of the maximum principle:

Proposition 2.3 *The function* h *is non negative and non decreasing with respect to the inclusion. Namely,*

$$\begin{array}{c} h(x,K) \geq 0 \text{ for any closed subset } K \in \mathcal{D} \text{ of } \mathbb{R}^N \\ \text{with } \mathcal{C}^{1,1} \text{ boundary and any } x \in \partial K \end{array} \tag{6}$$

and

$$\begin{array}{c} \text{if } K_1 \in \mathcal{D} \text{ and } K_2 \in \mathcal{D} \text{ are closed and with a } \mathcal{C}^{1,1} \text{ boundary,} \\ \text{if } K_1 \subset K_2 \text{ and if } x \in K_1 \cap \partial K_2, \\ \text{then } h(x,K_1) \leq h(x,K_2) \end{array} \tag{7}$$

In order to describe the continuity properties of h, let us first recall that, if K is a closed set with $\mathcal{C}^{1,1}$ boundary, then the signed distance $\mathbf{d}$ defined by

$$\mathbf{d}(x) = \left\{ \begin{array}{ll} d_K(x) & \text{if } x \notin K \\ -d_{\partial K}(x) & \text{otherwise} \end{array} \right.$$

is $\mathcal{C}^{1,1}$ in a neighborhood of ∂K. We say that a sequence K_n of sets with $\mathcal{C}^{1,1}$ boundary converges to some set K with $\mathcal{C}^{1,1}$ boundary if K_n converges to K and ∂K_n converges to ∂K for the Hausdorff distance, and if there is an open neighborhood $\mathcal{O}$ of ∂K such that, if $\mathbf{d}$ (resp. $\mathbf{d}_n$) is the signed distance to K (resp. to K_n), then $(\mathbf{d}_n)$ and $(\nabla \mathbf{d}_n)$ converge uniformly to $\mathbf{d}$ and $\nabla \mathbf{d}$ on $\mathcal{O}$ and if the L^∞ norms of $(\nabla^2 \mathbf{d}_n)$ are uniformly essentially bounded on $\mathcal{O}$.

Proposition 2.4 *The velocity h is sequentially continuous with respect to its arguments, i.e.,*

$$\begin{array}{c}\text{if } K_n \text{ and } K \in \mathcal{D} \text{ are closed subsets of } I\!\!R^N \text{ with } \mathcal{C}^{1,1} \text{ boundary,} \\ \text{such that } K_n \text{ converge to } K, \text{ if } x_n \in \partial K_n \text{ converge to } x \in \partial K, \\ \text{then } \lim_n h(x_n, K_n) = h(x, K)\end{array} \quad (8)$$

Proof of Proposition 2.4: Note that, if $K \in \mathcal{D}$, then, for n large enough, the sets K_n also belongs to $\mathcal{D}$. The rest of the Proposition is an straightforward application of the regularity results of [16] recalled in Remark 2.1. QED

We now state some estimates on the variations of the mapping $v \to h(x+v, K+v)$ for a set K with a smooth boundary and $x \in \partial K$. The key point is that such an estimate has to be independent of the regularity of K. Here and below, we set

$$S_r = \{x \in I\!\!R^N \ , \ d_S(x) \leq r\} \ .$$

Proposition 2.5 *Let $R > 0$ be some large constant and $r > 0$ be sufficiently small so that S_r has a $\mathcal{C}^2$ boundary. There is a constant $\lambda > 1/r$, such that, for any compact set K with $\mathcal{C}^{1,1}$ boundary such that $S_r \subset Int(K)$ and $K \subset B(0, R-r)$, for any $v \in I\!\!R^N$ with $|v| < 1/\lambda$ and any $x \in \partial K$, we have*

$$h(x+v, K+v) \geq (1 - \lambda|v|) h(x, K)$$

Proof of Proposition 2.5: We do the proof in the case $S \neq \emptyset$ and $f > 0$, the proof for the other cases being similar. Since $f > 0$ and f Lipschitz continuous, we can find a constant $C_1 > 0$ such that

$$\forall x, y \in I\!\!R^N, \text{ with } |x|, |y| \leq R, \ f(x) \geq f(y)(1 - C_1|x-y|) \ . \quad (9)$$

Let u^+ and u_r^- be respectively the solutions of

$$\left\{\begin{array}{ll} -div(|\nabla u^+|^{p-2} \nabla u^+) = f & \text{in } B(0,R) \backslash S \\ u^+ = g & \text{on } \partial S \\ u^+ = 0 & \text{on } \partial B(0,R) \end{array}\right.$$

and

$$\begin{cases} -div(|\nabla u_r^-|^{p-2}\nabla u_r^-) = f & \text{in } Int(S_r)\backslash S \\ u_r^- = g & \text{on } \partial S \\ u_r^- = 0 & \text{on } \partial S_r \end{cases}$$

Since g is $\mathcal{C}^{1,\beta}$, S, S_r and $B(0,R)$ have a $\mathcal{C}^2$ boundary, the functions u^+ and u_r^- belongs to $\mathcal{C}^{1,\alpha}(\overline{\Omega})$ (for some $\alpha \in (0,\beta)$) and there is some constant $C_2 > 0$ such that u^+ and u_r^- are C_2-Lipschitz continuous. Whence

$$\forall x \in S_r\backslash S\ ,\ u_r^-(x) \geq u^+(x) - 2C_2 d_S(x)\ , \tag{10}$$

because $u^+ = u_r^-$ on ∂S. We now choose $\lambda = \max\{6C_2(p-1)/m, C_1, 2/r\}$ where $m = \min_{S_r\backslash S} u^+$. Note that m is positive.

Let $K \subset I\!R^N$ be some compact set with $\mathcal{C}^{1,1}$ boundary, such that $S_r \subset Int(K)$ and $K \subset B(0, R-r)$ and let $v \in I\!R^N$ with $|v| < 1/\lambda$. Let u and u_v be the solution of (3) with K and $K+v$ respectively in place of Ω. From the maximum principle, we have $u_r^- \leq u \leq u^+$ and $u_r^- \leq u_v \leq u^+$ on $S_r\backslash S$ because $S_r \subset\subset K \subset\subset B(0,R)$ and $S_r \subset\subset K+v \subset\subset B(0,R)$ from the choice of $\lambda > 1/r$ and v.

Since $S_r \subset K$ and $|v| < r$, we have $S \subset Int(K+v)$. We claim that

$$\frac{1}{(1-\lambda|v|)^{1/(p-1)}} u_v(x+v) \geq u(x) \qquad \forall x \in K\backslash S_{|v|}\ . \tag{11}$$

For proving this claim, let us set $w(x) = \frac{1}{(1-\lambda|v|)^{1/(p-1)}} u_v(x+v)$ for $x \in K\backslash S_{|v|}$ and let us show that

$$w \geq u \text{ on } \partial S_{|v|} \quad \text{and} \quad -div(|\nabla w|^{p-2}\nabla w) \geq f \text{ on } K\backslash S_{|v|}\ . \tag{12}$$

We have, for any $x \in \partial S_{|v|}$,

$$(1-\lambda|v|)^{1/(p-1)} w(x) = u_v(x+v) \geq u_r^-(x+v) \geq u_r^-(x) - C_2|v| \geq u^+(x) - 3C_2|v|$$

because u_r^- is C_2-Lipschitz continuous and thanks to (10). Thus

$$(1-\lambda|v|)^{1/(p-1)} w(x) \geq u^+(x) - 3C_2|v| \geq (1-\lambda|v|)^{1/(p-1)} u^+(x)$$

because $u^+ \geq m$ in $S_r\backslash S$, $|v| < 1/\lambda$ and $\lambda \geq 6C_2(p-1)/m$. So we have proved that $w(x) \geq u^+(x) \geq u(x)$ for any $x \in \partial S_{|v|}$.

From the definition of C_1 in (9) and from the choice of λ, we have, for any $x \in K\backslash S_{|v|}$,

$$-div(|\nabla u_v(x+v)|^{p-2}\nabla u_v(x+v)) = f(x+v) \geq f(x)(1-\lambda|v|)\ .$$

Thus

$$-div(|\nabla w(x)|^{p-2}\nabla w(x)) \geq f(x) \qquad \forall x \in K\backslash S_{|v|} \ .$$

So (12) is proved, which entails (11). In particular, at any point $x \in \partial K$, we have, since $w \geq u$ in $K\backslash S_{|v|}$ and $w = u$ in ∂K,

$$\begin{aligned} h(x,K) &= |\nabla u(x)|^{p-1} \leq |\nabla w(x)|^{p-1} \\ &= \frac{1}{1-\lambda|v|}|\nabla u_v(x+v)|^{p-1} = \frac{1}{1-\lambda|v|}h(x+v,K+v) \ . \end{aligned}$$

QED

In order to prove the global existence of solution, we need below to control the growth of h:

Proposition 2.6 *There are constants $r_0 > 0$ and $\sigma > 0$ such that*

$$\forall r \geq r_0,\ \forall x \in \partial B(0,r), \qquad h(x,B(0,r)) \leq \sigma r \ . \tag{13}$$

Moreover, the constants r_0 and σ only depend on p, S, $\|f\|_\infty$ and $\|g\|_\infty$.

Proof of Proposition 2.6: Let us fix $r_0 > 0$ such that $S \subset\subset B(0, r_0/2^{\frac{(p-1)}{p}})$ and $\kappa = \max\{\frac{2\|g\|_\infty}{r_0^{p/(p-1)}}, \frac{\|f\|_\infty^{1/(p-1)}(p-1)}{N^{1/(p-1)}p}\}$. Let $r \geq r_0$ and u be the solution to

$$\left\{ \begin{array}{rl} -div(|\nabla u|^{p-2}\nabla u) = f & \text{in } B(0,r)\backslash S \\ u = g & \text{on } \partial S \\ u = 0 & \text{on } \ \partial B(0,r) \end{array} \right.$$

We claim that $u \leq w$ on $B(0,r)\backslash S$, where $w(x) = -\kappa|x|^{p/(p-1)} + \kappa r^{p/(p-1)}$ and $u = w$ on $\partial B(0,r)$. Indeed, $-div(|\nabla w|^{p-2}\nabla w) = \kappa^{p-1}N[p/(p-1)]^{p-1} \geq f$ in $B(0,r)\backslash S$. Since $S \subset B(0, r_0/2^{(p-1)/p})$ we also have

$$\forall x \in \partial S, \quad w(x) \geq -\kappa\frac{r_0^{p/(p-1)}}{2} + \kappa r_0^{p/(p-1)} \geq \|g\|_\infty \geq g(x) \ .$$

Finally, $u = w = 0$ on $\partial B(0,r)$ by construction. So $u \leq w$ on $B(0,r)\backslash S$ and $u = w$ on $\partial B(0,r)$. This entails that $h(x,B(0,r)) = |\nabla u|^{p-1} \leq |\nabla w|^{p-1} = \kappa^{p-1}[p/(p-1)]^{p-1}r$ for any $x \in \partial B(0,r)$. Whence the result with $\sigma = \kappa^{p-1}[p/(p-1)]^{p-1}$.

QED

2.3 A preliminary result

We need below an equivalent definition for the solutions. This formulation is introduced in [8].

Let us set, for any compact set K, $x \in \partial K$ and $\nu \in I\!\!R^N$, $\nu \neq 0$,

$$h^\sharp(x, K, \nu) = \inf\{h(x, K')\} \tag{14}$$

where the infimum is taken over the sets K' with $\mathcal{C}^{1,1}$ boundary, such that $K \subset K'$, $K' \in \mathcal{D}$, $x \in \partial K'$ and ν is an outward normal to K' at x. In the same way, we set

$$h^\flat(x, K, \nu) = \sup\{h(x, K')\} \tag{15}$$

where the supremum is taken over the sets K' with $\mathcal{C}^{1,1}$ boundary, such that $K' \subset K$, $K' \in \mathcal{D}$, $x \in \partial K'$ and ν is an outward normal to K' at x.

We set $h^\sharp(x, K, \nu) = +\infty$ or $h^\flat(x, K, \nu) = -\infty$ if there is no set K' with the required properties.

If A is a subset of some finite dimension space and x belongs to A, we say that a vector ν is *a proximal normal* to A at x if the distance of $x + \nu$ to A is equal to $|\nu|$.

Proposition 2.7 *Let $\mathcal{K}$ be a nondecreasing tube with $\mathcal{K}(0) \in \mathcal{D}$.*

Then $\mathcal{K}$ is a subsolution of the front propagation problem for h if and only if $\overline{\mathcal{K}}$ is left lower semi-continuous and if, for any $(t, x) \in \mathcal{K}$ with $t > 0$, for any proximal normal (ν_t, ν_x) to $\mathcal{K}$ at (t, x) such that $\nu_x \neq 0$, we have

$$-\frac{\nu_t}{|\nu_x|} \leq h^\sharp(x, \mathcal{K}(t), \nu_x) \; .$$

In the same way, the tube $\mathcal{K}$ is a supersolution of the front propagation problem for h if and only if, for any $(t, x) \in \widehat{\mathcal{K}}$ with $t > 0$, for any proximal normal (ν_t, ν_x) to $\widehat{\mathcal{K}}$ at (t, x) such that $\nu_x \neq 0$, we have

$$\frac{\nu_t}{|\nu_x|} \geq h^\flat(x, \mathcal{K}(t), -\nu_x) \; .$$

Remark : Proposition 2.7 also holds true for any velocity h satisfying (6), (7) and (8).

Proof of Proposition 2.7: The proof is completely similar to that of Proposition 2.2 of [8]. The only difference yields in the construction of some approximation of the tube $\mathcal{K}$. We only give the main arguments for the case of subsolutions, the case of supersolutions being symetric. For any $\epsilon > 0$, let us set

$$\mathcal{K}^\epsilon = \{(t,x) \in I\!R^+ \times I\!R^N \mid \exists (s,y) \in \overline{\mathcal{K}} \text{ with } (t-s)^2 + |x-y|^2 \le \epsilon^2\}$$

In [8], since h was translation invariant, $\mathcal{K}^\epsilon$ was also a subsolution of the front propagation problem. Instead here we have:

Lemma 2.8 *If $\mathcal{K}$ is a subsolution of the front propagation problem for h, then for any $\epsilon > 0$, $\mathcal{K}^\epsilon$ is a subsolution for h^ϵ on the time interval $[\epsilon, +\infty)$, where*

$$h^\epsilon(x,K) = \sup_{|v|\le\epsilon} h(x+v, K+v)$$

is defined for any set K with $\mathcal{C}^{1,1}$ boundary such that $S_\epsilon \subset\subset K$ and $x \in \partial K$.

Once Lemma 2.8 establish, we can complete the proof of the Proposition as in [8] by noticing that $h^\epsilon \to h$ as $\epsilon \to 0^+$, thanks to (8). QED

Proof of Lemma 2.8 : A straightforward application of the definition of $\mathcal{K}^\epsilon$ shows that $\mathcal{K}^\epsilon$ is non decreasing and left lower semicontinuous. Moreover, since $S \subset\subset Int(\mathcal{K}(0)) \subset Int(\mathcal{K}(\epsilon))$, we have $S_\epsilon \subset\subset \mathcal{K}^\epsilon(\epsilon)$.

Let $\mathcal{K}_r$ be a smooth tube which is externally tangent to $\mathcal{K}^\epsilon$ at some point (t,x), with $t > \epsilon$. Since (t,x) belongs to the boundary of $\mathcal{K}^\epsilon$, there is some $(s,y) \in \overline{\mathcal{K}}$ such that

$$(t-s)^2 + |x-y|^2 = \epsilon^2 \; .$$

Let us notice that, since $t > \epsilon$, we have $s > 0$. Now it is easy to check that the tube $\mathcal{K}_r - ((t,x)-(s,y))$ is externally tangent to $\mathcal{K}$ at (s,y). Since $\mathcal{K}$ is a subsolution, we have

$$V_{(s,y)}^{\mathcal{K}_r - ((t,x)-(s,y))} \le h(y, \mathcal{K}_r(t) - (x-y))$$

where $V_{(s,y)}^{\mathcal{K}_r - ((t,x)-(s,y))}$ is the outward normal velocity of the smooth tube $\mathcal{K}_r - ((t,x) - s,y))$ at (s,y). Using the definition of h^ϵ, this leads to

$$V_{(t,x)}^{\mathcal{K}_r} = V_{(s,y)}^{\mathcal{K}_r - ((t,x)-s,y))} \le h(y, \mathcal{K}_r(t) - (x-y)) \le h^\epsilon(x, \mathcal{K}_r(t))$$

because $|x-y| \le \epsilon$. QED

3 The inclusion principle

The main result of this paper is the following:

Theorem 3.1 (Inclusion principle) *Let $\mathcal{K}_1$ be a subsolution of the front propagation problem on the interval $[0,T)$ for some $T > 0$ and $\mathcal{K}_2$ be a supersolution on $[0,T)$. If $\mathcal{K}_1(t)$ and $\mathcal{K}_2(t)$ are non empty for $t \in [0,T)$ and if*

$$\overline{\mathcal{K}_1}(0) \cap \widehat{\mathcal{K}_2}(0) = \emptyset \ ,$$

then

$$\forall t \in [0,T), \qquad \overline{\mathcal{K}_1}(t) \cap \widehat{\mathcal{K}_2}(t) = \emptyset \ .$$

Remarks :

1. Let K_1 and K_2 be bounded subsets of $I\!\!R^N$ such that $\overline{K_1} \subset Int(K_2)$. If $\mathcal{K}_1$ is a subsolution with initial condition K_1 and $\mathcal{K}_2$ is a supersolution with initial position K_2, then the assumption of the Theorem holds: $\overline{\mathcal{K}_1}(0) \cap \widehat{\mathcal{K}_2}(0) = \emptyset$.
2. The statement $\overline{\mathcal{K}_1}(t) \cap \widehat{\mathcal{K}_2}(t) = \emptyset$ implies that $\overline{\mathcal{K}_1}(t) \subset Int(\mathcal{K}_2(t))$.
3. Theorem 3.1 holds true for any velocity h satisfying (6), (7), (8) and the conclusion of Proposition 2.5.

The rest of this section is devoted to the proof of Theorem 3.1.

From the definition of subsolutions, we can assume that $\mathcal{K}_1$ has a closed graph: $\overline{\mathcal{K}_1} = \mathcal{K}_1$. The main step of the proof amounts to show that, for any $\gamma > 1$, such that $S_{\gamma-1} \subset Int(\mathcal{K}_1(0))$,

$$\forall t \in [0,T), \qquad \mathcal{K}_1(t) \cap \widehat{\mathcal{K}_2}(\gamma t) = \emptyset \ . \tag{16}$$

We explain how to obtain Theorem 3.1 from (16) at the very end of the proof.

For showing (16), we argue by contradiction, by assuming that there is some $\gamma > 1$ with $S_{\gamma-1} \subset Int(\mathcal{K}_1(0))$ and some $T^* \in [0,T)$ such that

$$\mathcal{K}_1(T^*) \cap \widehat{\mathcal{K}_2}(\gamma T^*) \neq \emptyset \ . \tag{17}$$

Since $\mathcal{K}_1(0) \cap \widehat{\mathcal{K}_2}(0) = \emptyset$, we have $T^* > 0$. We now introduce several notations: Let $R > 0$ be sufficiently large so that $\mathcal{K}_1(T) \subset B(0, R-(\gamma-1))$ and $\mathcal{K}_2(T) \subset B(0, R-(\gamma-1))$. We denote by λ the constant defined in

Proposition 2.5 for R and $r := \gamma - 1 > 0$. Let us recall that $\lambda > 1/r$ and that, for any compact set K with $\mathcal{C}^{1,1}$ boundary such that $S_r \subset Int(K)$ and $K \subset B(0, R-r)$, for any $v \in I\!R^N$ with $|v| < 1/\lambda$ and any $x \in \partial K$, we have

$$h(x+v, K+v) \geq (1-\lambda|v|)h(x,K) \tag{18}$$

For any $\epsilon \in (0, \tau_0)$ and any $\sigma \in (0,1]$, we set

$$\mathcal{K}_1^{\epsilon,\sigma} = \{(t,x) \in I\!R^+ \times I\!R^N \mid \exists (s,y) \in \mathcal{K}_1 \text{ with } \frac{1}{\sigma^2}(t-s)^2 + |x-y|^2 \leq \epsilon^2 e^{-2s}\} \tag{19}$$

and

$$\widehat{\mathcal{K}}_2^{\epsilon,\sigma} = \{(t,x) \in I\!R^+ \times I\!R^N \mid \exists (s,y) \in \widehat{\mathcal{K}_2} \text{ with } \frac{1}{\sigma^2}(t-s)^2 + |x-y|^2 \leq \epsilon^2\} \tag{20}$$

and

$$T^{\epsilon,\sigma,\gamma} = \min\{t \geq \epsilon \mid \mathcal{K}_1^{\epsilon,\sigma}(t) \cap \widehat{\mathcal{K}}_2^{\epsilon,\sigma}(\gamma t) \neq \emptyset\}\ .$$

Let us point out that

$$T^{\epsilon,\sigma,\gamma} \leq T^* \tag{21}$$

because assumption (17) implies that $\mathcal{K}_1(T^*) \cap \widehat{\mathcal{K}_2}(\gamma T^*) \neq \emptyset$ and $\mathcal{K}_1(t) \subset \mathcal{K}_1^{\epsilon,\sigma}(t)$ and $\widehat{\mathcal{K}_2}(t) \subset \widehat{\mathcal{K}}_2^{\epsilon,\sigma}(t)$.

Let us define Π_1^σ and Π_2^σ the projections on the sets $\mathcal{K}_1$ and $\widehat{\mathcal{K}_2}$ as: $\forall \sigma \in (0,1]$,

$$\Pi_1^\sigma(t,x) = \left\{ (s_1,y_1) \in \mathcal{K}_1 \mid \begin{array}{l} \frac{1}{\sigma^2}(t-s_1)^2 + |x-y_1|^2 = \\ \inf_{(s,y)\in\mathcal{K}_1} \frac{1}{\sigma^2}(t-s)^2 + |x-y|^2 \end{array} \right\}$$

and

$$\Pi_2^\sigma(t,x) = \left\{ (s_2,y_2) \in \widehat{\mathcal{K}_2} \mid \begin{array}{l} \frac{1}{\sigma^2}(t-s_2)^2 + |x-y_2|^2 = \\ \inf_{(s,y)\in\widehat{\mathcal{K}_2}} \frac{1}{\sigma^2}(t-s)^2 + |x-y|^2 \end{array} \right\}\ .$$

Proposition 3.2 *One can choose ϵ and σ sufficiently small so that, for any $x \in \mathcal{K}_1^{\epsilon,\sigma}(T^{\epsilon,\sigma,\gamma}) \cap \widehat{\mathcal{K}}_2^{\epsilon,\sigma}(\gamma T^{\epsilon,\sigma,\gamma})$, for any $(s_1,y_1) \in \Pi_1^\sigma(T^{\epsilon,\sigma,\gamma}, x)$ and any $(s_2,y_2) \in \Pi_2^\sigma(\gamma T^{\epsilon,\sigma,\gamma}, x)$, we have $y_1 \neq x$, $y_2 \neq x$, $s_1 > 0$ and $s_2 > 0$.*

Proof of Proposition 3.2 : Let us first prove that there is some positive ϵ_0 such that

$$\text{for any } \epsilon \in (0,\epsilon_0) \text{ and any } \sigma \in (0,1], \text{ we have } T^{\epsilon,\sigma,\gamma} > \epsilon. \tag{22}$$

Since $\mathcal{K}_1(0) \cap \widehat{\mathcal{K}_2}(0) = \emptyset$ and the sets $\mathcal{K}_1$ and $\widehat{\mathcal{K}_2}$ are closed in $I\!R^+ \times I\!R^N$, there is some $\tau > 0$ such that

$$\forall 0 \le s, t \le \tau,\ \mathcal{K}_1(s) \cap \widehat{\mathcal{K}_2}(t) = \emptyset \ .$$

Set

$$\theta = \min\{|y_1 - y_2| \mid y_1 \in \mathcal{K}_1(s),\ y_2 \in \widehat{\mathcal{K}_2}(t),\ 0 \le s, t \le \tau\} \ . \tag{23}$$

Then $\theta > 0$. Set $\epsilon_0 = \min\{\frac{\theta}{2}, \frac{\tau}{1+\gamma}\}$. We claim that, for any $\epsilon \in (0, \epsilon_0)$ and for any $\sigma \in (0,1]$, we have

$$\mathcal{K}_1^{\epsilon,\sigma}(\epsilon) \cap \widehat{\mathcal{K}}_2^{\epsilon,\sigma}(\gamma\epsilon) = \emptyset \ . \tag{24}$$

It is clearly enough to prove the result for $\sigma = 1$. We argue by contradiction. Suppose that, contrary to our claim, there is some $x \in \mathcal{K}_1^{\epsilon,1}(\epsilon) \cap \widehat{\mathcal{K}}_2^{\epsilon,1}(\gamma\epsilon)$. Then there is some $(s_1, y_1) \in \mathcal{K}_1$ and some $(s_2, y_2) \in \widehat{\mathcal{K}_2}$ such that

$$|(\epsilon, x) - (s_1, y_1)| \le \epsilon e^{-s_1} \text{ and } |(\gamma\epsilon, x) - (s_2, y_2)| \le \epsilon \ .$$

This implies, on the one hand, that $|y_1 - y_2| \le 2\epsilon < \theta$ and, on another hand, that

$$0 \le s_1 \le 2\epsilon < \tau \text{ and } 0 \le s_2 \le (1+\gamma)\epsilon < \tau \ ,$$

which is in contradiction with the definition of θ in (23). Thus (24) is proved, which obviously implies that $T^{\epsilon,\sigma,\gamma} > \epsilon$, i.e., (22) holds.

From now on we fix $\epsilon \in (0, \epsilon_0)$. Let us first notice that $T^{\epsilon,\sigma,\gamma}$ is non decreasing with respect to σ and is bounded by T^* thanks to (21). Let us set $\bar{t} = \lim_{\sigma \to 0^+} T^{\epsilon,\sigma,\gamma}$. Let us also define

$$\mathcal{K}_1^{\epsilon,0} = \{(t,x) \in I\!R^+ \times I\!R^N \mid \exists y \in \mathcal{K}_1(t) \text{ with } |x - y| \le \epsilon e^{-t}\}$$

and

$$\widehat{\mathcal{K}}_2^{\epsilon,0} = \{(t,x) \in I\!R^+ \times I\!R^N \mid \exists y \in \widehat{\mathcal{K}_2}(t) \text{ with } |x - y| \le \epsilon\}$$

It is easy checked that

$$\bigcap_{\sigma \in (0,1]} \mathcal{K}_1^{\epsilon,\sigma} = \mathcal{K}_1^{\epsilon,0} \qquad \text{and} \qquad \bigcap_{\sigma \in (0,1]} \widehat{\mathcal{K}}_2^{\epsilon,\sigma} = \widehat{\mathcal{K}}_2^{\epsilon,0} \ . \tag{25}$$

Moreover, $\mathcal{K}_1^{\epsilon,0}$ and $\widehat{\mathcal{K}}_2^{\epsilon,0}$ are closed since so are $\mathcal{K}_1$ and $\widehat{\mathcal{K}_2}$. Hence, from the definition of $T^{\epsilon,\sigma,\gamma}$ and of $\bar{t}$, we have:

$$\mathcal{K}_1^{\epsilon,0}(\bar{t}) \cap \widehat{\mathcal{K}}_2^{\epsilon,0}(\gamma\bar{t}) \neq \emptyset \ .$$

Let us also point out that, for any $t \in (0, \bar{t})$,

$$\mathcal{K}_1^{\epsilon,0}(t) \cap \widehat{\mathcal{K}}_2^{\epsilon,0}(\gamma t) = \emptyset \tag{26}$$

because, $\mathcal{K}_1^{\epsilon,0}(t) \subset \mathcal{K}_1^{\epsilon,\sigma}(t)$ and $\widehat{\mathcal{K}}_2^{\epsilon,0}(\gamma t) \subset \widehat{\mathcal{K}}_2^{\epsilon,\sigma}(\gamma t)$, and $\mathcal{K}_1^{\epsilon,\sigma}(t) \cap \widehat{\mathcal{K}}_2^{\epsilon,\sigma}(\gamma t) = \emptyset$ as soon as $T^{\epsilon,\sigma,\gamma} > t$.

The next step of the proof amounts to show that,

$$\forall x \in \mathcal{K}_1^{\epsilon,0}(\bar{t}) \cap \widehat{\mathcal{K}}_2^{\epsilon,0}(\gamma \bar{t}),\ d_{\mathcal{K}_1(\bar{t})}(x) = \epsilon e^{-\bar{t}} \text{ and } d_{\widehat{\mathcal{K}_2}(\gamma\bar{t})}(x) = \epsilon\ . \tag{27}$$

For proving this, we argue by contradiction, by assuming (for instance) that $d_{\mathcal{K}_1(\bar{t})}(x) < \epsilon$. Then there is some $y_1 \in \mathcal{K}_1(\bar{t})$ such that $|y_1 - x| < \epsilon e^{-\bar{t}}$. Let also $y_2 \in \widehat{\mathcal{K}_2}(\gamma\bar{t})$ be such that $|y_2 - x| \leq \epsilon$. Since $\mathcal{K}_1$ is a subsolution, it is left lower semi-continuous. Thus, for any sequence $t_k \to \bar{t}^-$, there is some $y_1^k \to y_1$ with $y_1^k \in \mathcal{K}_1(t_k)$. In the same way, since $\mathcal{K}_2$ is a supersolution, $\widehat{\mathcal{K}_2}$ is left lower semi-continuous, and there is a sequence $y_2^k \in \widehat{\mathcal{K}_2}(\gamma t_k)$ which converges to y_2. Since $|y_1 - y_2| < \epsilon(1 + e^{-\bar{t}})$, for k large enough we still have $|y_1^k - y_2^k| \leq \epsilon(1 + e^{-t_k})$. Then it is easy to find some point $x_k \in [y_1^k, y_2^k]$ such that $|y_1^k - x_k| \leq \epsilon e^{-t_k}$ and $|y_2^k - x_k| \leq \epsilon$, i.e., $x_k \in \mathcal{K}_1^{\epsilon,0}(t_k) \cap \widehat{\mathcal{K}}_2^{\epsilon,0}(\gamma t_k)$. This is in contradiction with (26). Hence claim (27) is proved.

From this claim we deduce that, for $\epsilon' = \epsilon/4 < \epsilon$, we have

$$\mathcal{K}_1^{\epsilon',0}(\bar{t}) \cap \widehat{\mathcal{K}}_2^{\epsilon,0}(\gamma\bar{t}) = \emptyset\ .$$

Hence, there is some $\sigma_0 \in (0,1)$ such that

$$\mathcal{K}_1^{\epsilon',\sigma_0}(\bar{t}) \cap \widehat{\mathcal{K}_2}^{\epsilon,\sigma_0}(\gamma\bar{t}) = \emptyset\ ,$$

because of (25). Since $\mathcal{K}_1^{\epsilon',\sigma_0}$ and $\widehat{\mathcal{K}_2}^{\epsilon,\sigma_0}$ are closed and since $T^{\epsilon,\sigma,\gamma} \to \bar{t}$ as $\sigma \to 0^+$, we have, for any $\sigma > 0$ sufficiently small, that

$$\mathcal{K}_1^{\epsilon',\sigma_0}(T^{\epsilon,\sigma,\gamma}) \cap \widehat{\mathcal{K}_2}^{\epsilon,\sigma_0}(\gamma T^{\epsilon,\sigma,\gamma}) = \emptyset\ . \tag{28}$$

Let $x \in \mathcal{K}_1^{\epsilon,\sigma}(T^{\epsilon,\sigma,\gamma}) \cap \widehat{\mathcal{K}}_2^{\epsilon,\sigma}(T^{\epsilon,\sigma,\gamma})$. Then, from (28), $x \notin \mathcal{K}_1^{\epsilon',\sigma_0}(T^{\epsilon,\sigma,\gamma})$. Therefore, if $(s_1, y_1) \in \Pi_1^{\sigma}(T^{\epsilon,\sigma,\gamma}, x)$, we have

$$\frac{1}{\sigma^2}(T^{\epsilon,\sigma,\gamma} - s_1)^2 + |x - y_1|^2 \leq \epsilon^2 e^{-2s_1} \text{ and } \frac{1}{\sigma_0^2}(T^{\epsilon,\sigma,\gamma} - s_1)^2 + |x - y_1|^2 > (\epsilon')^2 e^{-2s_1}.$$

This implies that $x \neq y_1$ as soon as $\sigma < \sigma_0/2$ (recall that $\epsilon' = \epsilon/4$). So we have proved that, for any σ sufficiently small, for any $x \in \mathcal{K}_1^{\epsilon,\sigma}(T^{\epsilon,\sigma,\gamma}) \cap$

$\widehat{\mathcal{K}}_2^{\epsilon,\sigma}(T^{\epsilon,\sigma,\gamma})$ and for any $(s_1, y_1) \in \Pi_1^\sigma(T^{\epsilon,\sigma,\gamma}, x)$, we have $x \neq y_1$. We can prove in the same way that, for any $(s_2, y_2) \in \Pi_2^\sigma(T^{\epsilon,\sigma,\gamma}, x)$, we have $y_2 \neq x$.

Finally, s_1 is positive because the inequality $\frac{1}{\sigma^2}(T^{\epsilon,\sigma,\gamma} - s_1)^2 + |x - y_1|^2 \leq \epsilon e^{-2s_1}$ implies that

$$s_1 \geq T^{\epsilon,\sigma,\gamma} - \sigma\epsilon e^{-s_1} ,$$

where the right-hand side is positive thanks to (22). We can prove in the same way that $s_2 > 0$. QED

From now on we fix $\epsilon > 0$ and $\sigma > 0$ as in Proposition 3.2 and also sufficiently small so that

$$\epsilon < 1/(2\lambda) \quad \text{and} \quad \frac{1}{(1 - 2\lambda\epsilon)} \leq \gamma . \tag{29}$$

Recall that λ is defined at the begining of the proof. Let x, (s_1, y_1) and (s_2, y_2) be as in Proposition 3.2. For simplicity, we set $t^* = T^{\epsilon,\sigma,\gamma}$.

Let us define, for any two sets U and V, the minimal distance $d(U, V)$ between U and V by

$$d(U, V) = \inf_{x \in U, y \in V} |x - y| .$$

Proposition 3.3 *The point (t^*, x) belongs to the boundary of $\mathcal{K}_1^{\epsilon,\sigma}$ while the point $(\gamma t^*, x)$ belongs to the boundary of $\widehat{\mathcal{K}}_2^{\epsilon,\sigma}$. Moreover*

$$d(\mathcal{K}_1(s_1), \widehat{\mathcal{K}_2}(s_2)) = |y_1 - y_2| .$$

In particular, $y_1 \in \partial\mathcal{K}_1(s_1)$, $y_2 \in \partial\widehat{\mathcal{K}_2}(s_2)$ and

$$\frac{1}{\sigma^2}(t^* - s_1)^2 + |x - y_1|^2 = \epsilon^2 e^{-2s_1} \quad \text{and} \quad \frac{1}{\sigma^2}(\gamma t^* - s_2)^2 + |x - y_2|^2 = \epsilon^2 .$$

Proof of Proposition 3.3: For proving that the point (t^*, x) belongs to the boundary of $\mathcal{K}_1^{\epsilon,\sigma}$, we argue by contradiction by assuming that (t^*, x) belongs to the interior of $\mathcal{K}_1^{\epsilon,\sigma}$. Then, since $\widehat{\mathcal{K}_2}$ is left lower semicontinuous, so is $\widehat{\mathcal{K}}_2^{\epsilon,\sigma}$. Thus, for any $t_n \to (t^*)^-$, there is some $x_n \to x$ such that $(\gamma t_n, x_n) \in \widehat{\mathcal{K}}_2^{\epsilon,\sigma}$. But, since (t^*, x) belongs to the interior of $\mathcal{K}_1^{\epsilon,\sigma}$, (t_n, x_n) also belongs to $\mathcal{K}_1^{\epsilon,\sigma}$ for n large enough. This is in contradiction with the definition of t^*.

Symmetric arguments show that the point $(\gamma t^*, x)$ belongs to the boundary of $\widehat{\mathcal{K}}_2^{\epsilon,\sigma}$. Let us now prove that $d(\mathcal{K}_1(s_1), \widehat{\mathcal{K}_2}(s_2)) = |y_1 - y_2|$. Since

$y_1 \in \mathcal{K}_1(s_1)$ and $y_2 \in \widehat{\mathcal{K}_2}(s_2)$, we have $d(\mathcal{K}_1(s_1), \widehat{\mathcal{K}_2}(s_2)) \leq |y_1 - y_2|$. Assume for a while that $d(\mathcal{K}_1(s_1), \widehat{\mathcal{K}_2}(s_2)) < |y_1 - y_2|$. Let $z_1 \in \mathcal{K}_1(s_1)$ and $z_2 \in \widehat{\mathcal{K}_2}(s_2)$ be such that $|z_1 - z_2| < |y_1 - y_2|$. One can choose $\rho \in (0, 1)$ such that, if $x_\rho = \rho z_1 + (1-\rho) z_2$, then

$$|z_1 - x_\rho| < |y_1 - x| \text{ and } |z_2 - x_\rho| < |y_2 - x| ,$$

because $|z_1 - z_2| < |y_1 - y_2| \leq |y_1 - x| + |y_2 - x|$. Therefore,

$$\frac{1}{\sigma^2}(t^* - s_1)^2 + |z_1 - x_\rho|^2 \ < \ \frac{1}{\sigma^2}(t^* - s_1)^2 + |y_1 - x|^2 \ \leq \epsilon^2 e^{-2s_1}$$

and

$$\frac{1}{\sigma^2}(\gamma t^* - s_2)^2 + |z_2 - x_\rho|^2 \ < \ \frac{1}{\sigma^2}(\gamma t^* - s_2)^2 + |y_2 - x|^2 \ \leq \epsilon^2 .$$

So one can find some $t < t^*$ such that

$$\frac{1}{\sigma^2}(t - s_1)^2 + |z_1 - x_\rho|^2 \leq \epsilon^2 e^{-2s_1} \text{ and } \frac{1}{\sigma^2}(\gamma t - s_2)^2 + |z_2 - x_\rho|^2 \leq \epsilon^2 ,$$

which means that $x_\rho \in \mathcal{K}_1^{\epsilon,\sigma}(t) \cap \widehat{\mathcal{K}}_2^{\epsilon,\sigma}(\gamma t)$ and $t < t^*$. This is in contradiction with the definition of t^*. Therefore we have proved that $d(\mathcal{K}_1(s_1), \widehat{\mathcal{K}_2}(s_2)) = |y_1 - y_2|$. QED

Let us introduce two new notations:

$$(\nu_t^1, \nu_x^1) = (t^* - s_1 - \epsilon^2 \sigma^2 e^{-2s_1}, \sigma^2(x - y_1)) \text{ and } (\nu_t^2, \nu_x^2) = (\gamma t^* - s_2, \sigma^2(x - y_2)) . \tag{30}$$

Proposition 3.4 *There is some $\rho > 0$ such that the vectors $\rho(\nu_t^1, \nu_x^1)$ and $\rho(\nu_t^2, \nu_x^2)$ are proximal normals to $\mathcal{K}_1$ at (s_1, y_1) and to $\widehat{\mathcal{K}_2}$ at (s_2, y_2) respectively.*

Proof Proposition 3.4 : We only do the proof for (ν_t^1, ν_x^1), the proof for (ν_t^2, ν_x^2) being similar. From Proposition 3.3, (t^*, x) belongs to the boundary of $\mathcal{K}_1^{\epsilon,\sigma}$. Therefore, the set $\mathcal{K}_1$ is contained in the set

$$E = \{(s, y) \in \mathbb{R}^+ \times \mathbb{R}^N \ , \ \frac{1}{\sigma^2}(t^* - s)^2 + |y - x|^2 \geq \epsilon^2 e^{-2s}\} .$$

From Proposition 3.3 again, the point (s_1, y_1) belongs to the boundary of E. Moreover E has a smooth boundary in a neighbourhood of (s_1, y_1) because the gradient of the map $(s,y) \to \epsilon^2 e^{-2s} - \frac{1}{\sigma^2}(t^*-s)^2 - |y-x|^2$ at (s_1, y_1) is

$$(-2\epsilon^2 e^{-2s_1} - \frac{2(s_1-t^*)}{\sigma^2}, -2(y_1-x)) = \frac{2}{\sigma^2}(\nu_t^1, \nu_x^1) ,$$

which does not vanish since $y_2 \neq x$ from Proposition 3.2. Therefore this gradient is, up to some small positive multiplicative constant, a proximal normal to E at the point (s_1, y_1). Since $\mathcal{K}_1 \subset E$ with $(s_1, y_1) \in \mathcal{K}_1$, it is also a proximal normal to $\mathcal{K}_1$ at (s_1, y_1). Since (ν_t^1, ν_x^1) is proportional to this gradient, the proof is complete. QED

Since $\mathcal{K}_1$ is a subsolution and $\rho(\nu_t^1, \nu_x^1)$ is a proximal normal to $\mathcal{K}_1$ at (s_1, y_1), with $\nu_x^1 \neq 0$ and $s_1 > 0$ thanks to Proposition 3.2, Proposition 2.7 states that

$$-\frac{\nu_t^1}{|\nu_x^1|} = -\frac{t^* - s_1 - \epsilon^2\sigma^2 e^{-2s_1}}{\sigma^2|x-y_1|} \leq h^\sharp(y_1, \mathcal{K}_1(s_1), \nu_x^1) . \tag{31}$$

Similarly, since $\mathcal{K}_2$ is a supersolution and $\rho(\nu_t^2, \nu_x^2)$ is a proximal normal to $\widehat{\mathcal{K}_1}$ at (s_2, y_2), with $\nu_x^2 \neq 0$ and $s_2 > 0$, Proposition 2.7 also states that

$$\frac{\nu_t^2}{|\nu_x^2|} = \frac{\gamma t^* - s_2}{\sigma^2|x-y_2|} \geq h^\flat(y_2, \mathcal{K}_2(s_2), -\nu_x^2) . \tag{32}$$

To proceed, we need some relations between (ν_t^1, ν_x^1) and (ν_t^2, ν_x^2):

Proposition 3.5 *There is some $\theta > 0$ such that*

$$\nu_x^2 = -\theta\nu_x^1 \qquad \text{and} \qquad \nu_t^2 \leq -\theta(\nu_t^1 + \epsilon^2\sigma^2 e^{-2s_1})/\gamma .$$

Proof of Proposition 3.5 : From the definition of t^*, we know that

$$\forall \epsilon < s < t^*,\ \mathcal{K}_1^{\epsilon,\sigma}(s) \cap \widehat{\mathcal{K}}_2^{\epsilon,\sigma}(\gamma s) = \emptyset .$$

Let us now notice that the sets B_1 and B_2 defined by

$$B_1 = \{(s,y) \mid \frac{1}{\sigma^2}(s-s_1)^2 + |y-y_1|^2 \leq \epsilon^2 e^{-2s_1}\} ,$$

and

$$B_2 = \{(s,y) \mid \frac{1}{\sigma^2}(\gamma s - s_2)^2 + |y-y_2|^2 \leq \epsilon^2\} ,$$

are respectively subsets of $\mathcal{K}_1^{\epsilon,\sigma}$ and of the graph of $\widehat{\mathcal{K}}_2^{\epsilon,\sigma}(\gamma\cdot)$. Therefore

$$\forall (s,y),\ \text{if } (s,y) \in B_1 \cap B_2 \text{ , then } s \geq t^* . \tag{33}$$

From Proposition 3.3, the point (t^*, x) belongs to $\partial\mathcal{K}_1^{\epsilon,\sigma}$. Hence $(t^*, x) \in \partial B_1$. In the same way, $(\gamma t^*, x)$ belongs to $\partial\widehat{\mathcal{K}_2}^{\epsilon,\sigma}$, and so $(t^*, x) \in \partial B_2$. Therefore (33) states that (t^*, x) is a minimum in the following problem: Minimize s over the points $(s, y) \in B_1 \cap B_2$.

The necessary conditions for this problem (in term of extended lagrangian) state that there is some $(\lambda_1, \lambda_2, \lambda_3) \in I\!R_+^3$ with $(\lambda_1, \lambda_2, \lambda_3) \neq 0$, such that

$$\lambda_1(\frac{1}{\sigma^2}(t^* - s_1), x - y_1) + \lambda_2(\frac{\gamma}{\sigma^2}(\gamma t^* - s_2), x - y_2) + \lambda_3(1, 0) = 0 \ .$$

Using the notations (30), this is equivalent to:

$$\lambda_1(\nu_t^1 + \epsilon^2\sigma^2 e^{-2s_1}, \sigma^2\nu_x^1) + \lambda_2(\gamma\nu_t^2, \sigma^2\nu_x^2) + \lambda_3(\sigma^2, 0) = 0 \ .$$

Since, from Proposition 3.2, $\nu_x^1 \neq 0$ and $\nu_x^2 \neq 0$, we have $\lambda_1 \neq 0$ and $\lambda_2 \neq 0$. Setting $\theta = \lambda_1/\lambda_2 > 0$, we get $\nu_x^2 = -\theta\nu_x^1$ and

$$\nu_t^2 = -\frac{1}{\gamma}\left(\theta(\nu_t^1 + \epsilon^2\sigma^2 e^{-2s_1}) + \frac{\lambda_3\sigma^2}{\lambda_2}\right) \leq -\frac{\theta}{\gamma}(\nu_t^1 + \epsilon^2\sigma^2 e^{-2s_1}) \ .$$

QED

Let us now recall Ilmanen Interposition Lemma [13], which plays a crucial role in our study:

Lemma 3.6 (Ilmanen) *Let A and B be two disjoint subsets of $I\!R^N$, A being compact and B closed. Then there exists some closed set K_r with a $\mathcal{C}^{1,1}$ boundary, such that*

$$A \subset K_r \text{ and } K_r \cap B = \emptyset$$

and

$$d(A, B) = d(A, \partial K_r) + d(\partial K_r, B) \ .$$

Let us apply Lemma 3.6 to $A = \mathcal{K}_1(s_1)$ and $B = \widehat{\mathcal{K}_2}(s_2)$: There exists some set K_r with a $\mathcal{C}^{1,1}$ boundary such that

$$\mathcal{K}_1(s_1) \subset K_r \text{ and } \widehat{\mathcal{K}_2}(s_2) \cap K_r = \emptyset$$

and

$$d(\mathcal{K}_1(s_1), \widehat{\mathcal{K}_2}(s_2)) = d(\partial K_r, \mathcal{K}_1(s_1)) + d(\partial K_r, \widehat{\mathcal{K}_2}(s_2)) \ .$$

Let us set $\rho_1 = d(\partial K_r, \mathcal{K}_1(s_1))$, $\rho_2 = d(\partial K_r, \widehat{\mathcal{K}_2}(s_2))$ and $w = \frac{y_2 - y_1}{|y_2 - y_1|}$. Let us notice that $\nu_x^1 = |\nu_x^1| w$ while $\nu_x^2 = -|\nu_x^2| w$.

Proposition 3.7 *The smooth set $K_r - \rho_1 w$ is externally tangent to $\mathcal{K}_1(s_1)$ at the point y_1 and w is a normal to $K_r - \rho_1 w$ at y_1. Namely:*

$$\mathcal{K}_1(s_1) \subset (K_r - \rho_1 w) \quad \text{and} \quad y_1 \in \partial\mathcal{K}_1(s_1) \cap \partial(K_r - \rho_1 w) \tag{34}$$

In the same way, the smooth set $K_r + \rho_2 w$ is internally tangent to $\mathcal{K}_2(s_2)$ at the point y_2 and w is a normal to $K_r + \rho_2 w$ at y_2:

$$K_r + \rho_2 w \subset \widehat{\mathcal{K}_2}(s_2) \quad \text{and} \quad y_2 \in \partial(K_r + \rho_2 w) \cap \partial\widehat{\mathcal{K}_2}(s_2)\ . \tag{35}$$

Finally, $S \subset Int(K_r - \rho_1 w)$ and $S \subset Int(K_r + \rho_2 w)$.

Remark : The proposition states that we can estimate the quantity $h^\sharp(y_1, \mathcal{K}_1(s_1), \nu_x^1)$ by using the set $K_r - \rho_1 w$, while the estimate of $h^\flat(y_2, \mathcal{K}_2(s_2), -\nu_x^2)$ can be done by using $K_r + \rho_2 w$.

Proof of Proposition 3.7: For proving (34) and (35), let us first notice that the fact that $\mathcal{K}_1(s_1) \subset K_r$ and $d(\partial K_r, \mathcal{K}_1(s_1)) = \rho_1$ implies the inclusion $\mathcal{K}_1(s_1) \subset (K_r - \rho_1 w)$. In the same way, since $\widehat{\mathcal{K}_2}(s_2) \cap K_r = \emptyset$ and $d(\partial K_r, \widehat{\mathcal{K}_2}(s_2)) = \rho_2$, we have $K_r + \rho_2 w \subset \widehat{\mathcal{K}_2}(s_2)$.

Let $z = y_2 - \rho_2 w = y_1 + \rho_1 w$. We have $d_{\mathcal{K}_1(s_1)}(z) \le |y_1 - z| = \rho_1$ and $d_{\widehat{\mathcal{K}_2}(s_2)}(z) \le |y_2 - z| = \rho_2$. Since, from Propositions 3.3, $d(\mathcal{K}_1(s_1), \widehat{\mathcal{K}_2}(s_2)) = |y_2 - y_1|$, this leads to

$$\rho_1 + \rho_2 \ge d_{\mathcal{K}_1(s_1)}(z) + d_{\widehat{\mathcal{K}_2}(s_2)}(z) \ge d(\mathcal{K}_1(s_1), \widehat{\mathcal{K}_2}(s_2)) = |y_1 - y_2| = \rho_1 + \rho_2\ .$$

So

$$d_{\mathcal{K}_1(s_1)}(z) = |y_1 - z| = \rho_1 \quad \text{and} \quad d_{\widehat{\mathcal{K}_2}(s_2)}(z)|y_2 - z| = \rho_2\ . \tag{36}$$

This implies that $z \notin I\!R^N \backslash K_r$, since $d(\partial K_r, \mathcal{K}_1(s_1)) = \rho_1$, and that $z \notin Int(K_r)$, since $d(\partial K_r, \widehat{\mathcal{K}_2}(s_2)) = \rho_2$. Thus $z \in \partial K_r$, and $y_1 = z - \rho_1 w \in \partial\mathcal{K}_1(s_1) \cap \partial(K_r - \rho_1 w)$ while $y_2 = z + \rho_2 w \in \partial(K_r + \rho_2 w) \cap \partial\widehat{\mathcal{K}_2}(s_2)$. Moreover, using again (36) shows that $d(\mathcal{K}_1(s_1), \partial K_r) = |y_1 - z|$. Since K_r is smooth, this implies that w (which is proportional to $z - y_1$) is a normal to K_r at z. So (34) and (35) hold.

We now prove that $S \subset Int(K_r - \rho_1 w)$ and $S \subset Int(K_r + \rho_2 w)$. Indeed, since $S \subset Int(\mathcal{K}_1(0))$ and since $\mathcal{K}_1(0) \subset (K_r - \rho_1 w)$, we have that $S \subset Int(K_r - \rho_1 w)$. Moreover, since, from the choice of ϵ in (29),

$$S_{2\epsilon} \subset Int(S_{\gamma - 1}) \subset Int(\mathcal{K}_1(0)) \subset Int(\mathcal{K}_1(s_1)) \subset Int(K_r - \rho_1 w))$$

and since $\rho_1 + \rho_2 = |y_2 - y_1| \le 2\epsilon$, we have that $S \subset Int(K_r + \rho_2 w))$. QED

We are now ready to prove of main step:

Proof of (16) : Considering now the definition of $h^\sharp$ (introduced before Proposition 2.7), (34) and the fact that $\nu_x^1 = |\nu_x^1| w$ is a normal to K_r at y_1 and that $S \subset Int(K_r - \rho_1 w)$ yield to:

$$h^\sharp(y_1, \mathcal{K}_1(s_1), \nu_x^1) \leq h(y_1, K_r - \rho_1 w) \; . \tag{37}$$

In the same way, (35) together with the fact that $\nu_x^2 = -|\nu_x^2| w$ is a normal to K_r at y_2 and that $S \subset Int(K_r + \rho_2 w)$ implies that

$$h^\flat(y_2, \mathcal{K}_2(s_2), -\nu_x^2) \geq h(y_2, K_r + \rho_2 w) \; . \tag{38}$$

Using inequality (18), we can estimate the difference between the right-hand sides of the two previous inequalities:

$$h(y_2, K_r + \rho_2 w) \geq (1 - 2\lambda\epsilon) h(y_1, K_r - \rho_1 w) \tag{39}$$

because $y_2 - y_1 = (\rho_1 + \rho_2) w$ and $|y_1 - y_2| \leq 2\epsilon$ with $\epsilon < 1/(2\lambda)$. Using Proposition 3.5 and putting together (31), (32) and the three previous inequalities finally gives

$$\begin{aligned}
h(y_2, K_r + \rho_2 w) &\leq h^\flat(y_2, \mathcal{K}_2(s_2), -\nu_x^2) \quad \text{(from (38))} \\
&\leq \frac{\nu_t^2}{|\nu_x^2|} \quad \text{(from (32))} \\
&\leq -\frac{\nu_t^1 + \epsilon^2 \sigma^2 e^{-2s_1}}{\gamma |\nu_x^1|} \quad \text{(from Proposition 3.5)} \\
&\leq \frac{1}{\gamma} h^\sharp(y_1, \mathcal{K}_1(s_1), \nu_x^1) - \frac{\epsilon^2 \sigma^2 e^{-2s_1}}{\gamma |\nu_x^1|} \quad \text{(from (31))} \\
&\leq \frac{1}{\gamma} h(y_1, K_r - \rho_1 w) - \frac{\epsilon^2 \sigma^2 e^{-2s_1}}{\gamma |\nu_x^1|} \quad \text{(from (37))} \\
&\leq \frac{1}{\gamma(1 - 2\lambda\epsilon)} h(y_2, K_r + \rho_2 w) - \frac{\epsilon^2 \sigma^2 e^{-2s_1}}{\gamma |\nu_x^1|} \quad \text{(from (39))}
\end{aligned}$$

This is impossible since $h(y_2, K_r + \rho_2 w) \geq 0$ and we have chosen $\gamma \geq 1/(1 - 2\lambda\epsilon)$ in (29). So we have found a contradiction, and (16) is proved. QED

Proof of Theorem 3.1 : Since from our assumption $\overline{\mathcal{K}_1}(0)\cap\widehat{\mathcal{K}_2}(0)=\emptyset$, and since $\overline{\mathcal{K}_1}$ and $\widehat{\mathcal{K}_2}$ have a closed graph, one can find $\tau>0$ such that

$$\forall t\in[0,\tau], \qquad \overline{\mathcal{K}_1}(t)\cap\widehat{\mathcal{K}_2}(0)=\emptyset\ .$$

Let us now apply (16) to the subsolution $\mathcal{K}_{1,\delta}$ and the supersolution $\mathcal{K}_2$, where

$$\delta\in(0,\tau) \qquad \forall t\geq 0, \qquad \mathcal{K}_{1,\delta}(t)=\overline{\mathcal{K}_1}(t+\delta)\ .$$

Since $\mathcal{K}_{1,\delta}(0)\cap\widehat{\mathcal{K}_2}(0)=\emptyset$, we have, for any $\gamma>1$,

$$\forall t\geq 0,\ \mathcal{K}_{1,\delta}(t)\cap\widehat{\mathcal{K}_2}(\gamma t)=\emptyset$$

Applying this with $t-\delta$ (for $t>\delta$) and to $\gamma=t/(t-\delta)>1$ gives

$$\emptyset=\mathcal{K}_{1,\delta}(t-\delta)\cap\widehat{\mathcal{K}_2}(\gamma(t-\delta))=\mathcal{K}_1(t)\cap\widehat{\mathcal{K}_2}(t)$$

because $\gamma(t-\delta)=t$. Since we can choose $\delta>0$ arbitrary small, the proof of Theorem 3.1 is complete. QED

4 Existence, uniqueness and stability of solutions

In this section, we prove the existence of viscosity solutions for the front propagation problem. We also state some uniqueness and stability results.

4.1 Some preliminary estimates

Let us first give some estimates which are necessary in the sequel. The first one says that the left lower semicontinuity of the subsolution is somehow "uniform".

Lemma 4.1 *Let us fix $\epsilon>0$ and $\rho>0$ such that $\rho>\epsilon$. Then there is some constant $\eta>0$ such that, for any subsolution $\mathcal{K}$ of the front propagation problem, with $\overline{\mathcal{K}}(0)\subset B(0,\rho)$ and for any $x_0\in\mathbb{R}^N$ with $d_{\overline{\mathcal{K}}(0)}(x_0)\geq\epsilon$ and $|x_0|\leq\rho$, we have: $d_{\mathcal{K}}(0,x_0)\geq\eta$.*

Moreover, the constant $\eta>0$ only depends on ϵ, ρ and on the maximum of the velocity $h(y,K_r)$ for $y\in\partial K_r$ and for K_r belonging to the compact family of smooth sets:

$$\{B(z,R)\backslash B(z,r)\mid \frac{\epsilon}{4}\leq r\leq\frac{\epsilon}{2},\ 2\rho\leq R\leq 3\rho,\ |z|\leq\rho,\ d_S(z)\geq\epsilon\}\ .$$

Remarks :

1. Since the front propagation problem we are considering is invariant with respect to time translations, the above estimate also shows that, for any subsolution $\mathcal{K}$ of the front propagation problem, with $\overline{\mathcal{K}}(t) \subset B(0,\rho)$ and for any $x_0 \in I\!\!R^N$ with $d_{\overline{\mathcal{K}(t)}}(x_0) > \epsilon$ and $|x_0| \le \rho$, we have: $d_{\mathcal{K}}(t,x_0) \ge \eta$.

2. The symmetric estimates for supersolution (i.e., if $\mathcal{K}$ is a supersolution, $d_{\widehat{\mathcal{K}(0)}}(x_0) > \epsilon$ and $|x_0| \le \rho$ implies that $d_{\widehat{\mathcal{K}}}(0,x_0) \ge \eta$, for some η) clearly holds with $\eta = \epsilon$ because, since $\mathcal{K}$ being nondecreasing, $\widehat{\mathcal{K}}$ is nonincreasing.

Proof of Lemma 4.1: Let us denote by κ the maximum of the velocity $h(y,K_r)$ for $y \in \partial K_r$ and for K_r belonging to the compact family of smooth sets:

$$\{B(z,R)\backslash B(z,r) \mid \frac{\epsilon}{4} \le r \le \frac{\epsilon}{2},\ 2\rho \le R \le 3\rho,\ |z| \le \rho,\ d_S(z) \ge \epsilon\}\ .$$

Note that $\kappa < +\infty$ since the above family is compact and the velocity h is continuous (Proposition 2.4). Let us introduce the tube $\mathcal{K}_2$ defined by:

$$\mathcal{K}_2(t) = B(x_0, 2\rho + \kappa t)\backslash B(x_0, \frac{\epsilon}{2} - \kappa t)\ .$$

Then, from the definition of κ, $\mathcal{K}_2$ is a smooth supersolution of the Hele-Shaw problem on the time interval $[0,\tau]$, where $\tau = \min\{\frac{\epsilon}{4\kappa}, \frac{\rho}{\kappa}\}$, because, on this time interval, $2\rho \le 2\rho + \kappa t \le 3\rho$ and $\epsilon/4 \le \epsilon - \kappa t \le \epsilon/2$.

Let $\mathcal{K}$ be some subsolution of the Hele-Shaw problem, with $\overline{\mathcal{K}}(0) \subset B(0,\rho)$ and let $x_0 \in I\!\!R^N$ with $d_{\overline{\mathcal{K}(0)}}(x_0) > \epsilon$ and $|x_0| \le \rho$. Then we have $\overline{\mathcal{K}}(0) \subset B(x_0, 2\rho)\backslash B(x_0,\epsilon)$. Hence $\overline{\mathcal{K}}(0) \cap \widehat{\mathcal{K}_2}(0) = \emptyset$. Then the inclusion principle (Theorem 3.1) states that

$$\forall t \in [0,\tau],\ \overline{\mathcal{K}}(t) \cap \widehat{\mathcal{K}_2}(t) = \emptyset\ .$$

Therefore $d_{\mathcal{K}}(0,x_0) \ge \eta$, where $\eta = \min\{\epsilon/2, \tau\}$, because $B((0,x_0),\eta) \subset \widehat{\mathcal{K}_2}$. QED

Next we establish some estimates of the growth of the solutions. For this, we recall that, according to Proposition 2.6, there are constants $r_0 > 0$ and $\sigma > 0$ such that

$$\forall r \ge r_0,\ \forall x \in \partial B(0,r), \qquad h(x, B(0,r)) \le \sigma r\ .$$

Moreover, the constants r_0 and σ only depend on S, p, $\|f\|_\infty$ and $\|g\|_\infty$.

Lemma 4.2 *If $\mathcal{K}$ is a subsolution of the front propagation problem, then*

$$\forall t \geq 0, \quad \mathcal{K}(t) \subset B\left(0, \max\{|\mathcal{K}(0)|, r_0\}e^{\sigma t}\right) .$$

where $|\mathcal{K}(0)| = \sup_{y \in \mathcal{K}(0)} |y|$.

Proof of Lemma 4.2: From Proposition 2.6, for any $\epsilon > 0$, the tube

$$\mathcal{K}_2^\epsilon(t) = B\left(0, (\max\{|\mathcal{K}(0)|, r_0\} + \epsilon)e^{\sigma t}\right)$$

is a supersolution of the front propagation problem with $\overline{\mathcal{K}}(0) \cap \widehat{\mathcal{K}_2^\epsilon}(0) = \emptyset$. Hence, $\overline{\mathcal{K}}(t) \cap \widehat{\mathcal{K}_2^\epsilon}(t) = \emptyset$ for any $t \geq 0$, which entails the desired result when letting $\epsilon \to 0^+$. QED

4.2 Existence and uniqueness of solutions

Let us first give an existence result.

Theorem 4.3 *For any initial position K_0, with $S \subset Int(K_0)$ and K_0 bounded, there is (at least) one solution to the front propagation problem.*

Moreover, there is a largest solution, denoted by $S(K_0)$, and a smallest solution, denoted by $s(K_0)$, to this problem. The largest solution has a closed graph while the smallest solution has an open graph in $\mathbb{R}^+ \times \mathbb{R}^N$. The largest solution contains all the subsolutions of the front propagation problem with initial condition K_0, while the smallest solution is contained in any supersolution.

Remarks:

1. From the maximality property of the largest solution and the time invariance of the evolution law, the semi-group property holds for this solution:

$$\forall s \geq 0,\, t \geq 0, \quad S(\, S(K_0)(s)\,)(t) = S(K_0)(s+t) .$$

2. For general front propagation problems, one cannot expect the uniqueness of the solutions. Soner pointed out in [19] the existence of a maximal and a minimal solution for geometric flows of mean curvature type. This result has been generalized in [7] for some class of geometric flows with nonlocal terms.

Proof of Theorem 4.3: The proof is based on Perron's method. Since it is exactly the same as the proof of Theorem 4.1 and of Corollary 4.2 of [8], we omit it. QED

We say that the solution of our Hele-Shaw problem with initial position K_0 is *unique* if $\overline{s(K_0)} = S(K_0)$ or if, equivalently, $\widehat{S(K_0)} = \widehat{s(K_0)}$ (note that $\widehat{s(K_0)} = (\mathbb{R}^+ \times \mathbb{R}^N)\backslash s(K_0)$ since $s(K_0)$ has an open graph in $\mathbb{R}^+ \times \mathbb{R}^N$).

We have the following uniqueness result:

Theorem 4.4 *Assume that K_0 is the closure of an open, connected and bounded subset of $\mathbb{R}^N$ with $\mathcal{C}^2$ boundary and such that $S \subset Int(K_0)$. Then there is a unique viscosity solution to the Hele-Shaw problem.*

Remark : Some uniqueness criteria for geometric flows can be found in [19] and [4]. Our proof uses several arguments from these papers.

Proof of Theorem 4.4: Since K_0 is the closure of an open, connected and bounded subset of $\mathbb{R}^N$ with $\mathcal{C}^2$ boundary and such that $S \subset Int(K_0)$, Hopf maximum principle implies that there is a constant $\delta > 0$ such that $h(x, K_0) \geq 2\delta$ for any $x \in \partial K_0$. Let us set, for any $\sigma \in \mathbb{R}$, $K_\sigma = \{x \in K_0 |\ \mathbf{d}(x) \leq \sigma\}$, where $\mathbf{d}$ is the signed distance to the boundary of K_0 (negative in $Int(K_0)$). From the continuity of h (see (8)), there is some $\epsilon > 0$ such that $h(x, K_\sigma) \geq \delta$ for any $x \in \partial K_\sigma$ and for any σ such that $|\sigma| \leq \epsilon$. Hence the tube $\mathcal{K}(t) = K_{\delta t - \epsilon}$ is a subsolution of the Hele-Shaw problem starting from $K_{-\epsilon}$ on the time interval $[0, 2\epsilon/\delta]$, because it is smooth and has a normal velocity δ on this interval of time. In particular, $\mathcal{K}(t) \subset S(K_{-\epsilon})(t)$ on $[0, 2\epsilon/\delta]$, which proves that $\mathcal{K}(2\epsilon/\delta) = K_\epsilon \subset S(K_{-\epsilon})(2\epsilon/\delta)$.

Since $K_0 \subset Int(K_\epsilon) \subset Int(S(K_{-\epsilon})(2\epsilon/\delta))$, the inclusion principle (Theorem 3.1) combined with the semi-group property gives

$$S(K_0)(t) \subset S(S(K_{-\epsilon})(2\epsilon/\delta))(t) = S(K_{-\epsilon})(2\epsilon/\delta + t) \quad \forall t \geq 0 \ .$$

Moreover, since $K_{-\epsilon} \subset Int(K_0)$, the inclusion principle also states that $S(K_{-\epsilon})(t) \subset s(K_0)(t)$. Accordingly, we have, for all $t \geq 0$, $S(K_0)(t) \subset s(K_0)(2\epsilon/\delta + t)$. Letting $\epsilon \to 0^+$ gives the desired inclusion: $S(K_0)(t) \subset \overline{s(K_0)}(t)$. QED

4.3 Stability of the solutions

We are now investigating the stability of the flow under variations of the initial position and of the data f and g.

For this we first generalize the well-known stability result of viscosity solution to our framework.

Let us assume that we are given a family of maps $h_n = h_n(x,K)$ which are defined for any set K with $\mathcal{C}^{1,1}$ boundary and for any $x \in \partial K$, and continuous in the sense of (8). We also assume that h_n converges to a continuous map h, i.e., if a sequence of closed set K_n with $\mathcal{C}^{1,1}$ boundary converges to a closed set K with a $\mathcal{C}^{1,1}$ boundary for the $\mathcal{C}^{1,1}$ convergence, and if a sequence of points (x_n), with $x_n \in \partial K_n$, converges to some $x \in \partial K$, then $h_n(x_n, K_n)$ converges to $h(x,K)$.

Let us recall that the upper-limit of a sequence of sets A_n is the set of all limits of converging subsequences of sequences (x_n) with $x_n \in A_n$.

Proposition 4.5 *If $\mathcal{K}_n$ is a sequence of subsolutions for h_n, locally uniformly bounded w.r. to t, then $\mathcal{K}^*$, the upper limit of the $\mathcal{K}_n$, is also a subsolution for h.*

In a similar way, if $\mathcal{K}_n$ is a sequence of supersolutions for h_n, locally uniformly bounded w.r. to t, then $\mathcal{K}_*$, the complementary of the upper limit of $\widehat{\mathcal{K}_n}$, is also a supersolution for h.

Proof of Proposition 4.5: We only prove the statement for the subsolutions, the proof for the supersolutions being similar. Let us first prove that $\mathcal{K}^*$ is a left lower semicontinuous tube. Indeed $\mathcal{K}^*$ is a tube because the $\mathcal{K}_n$ are locally uniformly bounded. In order to show that $\mathcal{K}^*$ is left lower semicontinuous, it is enough to establish that, for any $T \geq 0$ and for any $\epsilon > 0$, there is some $\eta > 0$ such that

$$\forall t \in [0,T],\ \forall x \in I\!\!R^N, \quad d_{\mathcal{K}^*(t)}(x) > \epsilon \ \Rightarrow\ d_{\mathcal{K}^*}(t,x) \geq \eta\ .$$

For this, let us fix $T \geq 0$ and $\epsilon > 0$. Since the h_n converge to h, Lemma 4.1 states that there is some some $\eta > 0$ (independent of n and of $t \in [0,T]$) such that

$$\forall n \in \mathbb{N},\ \forall t \in [0,T],\ \forall x \in I\!\!R^N, \quad d_{\mathcal{K}_n(t)}(x) > \epsilon/2 \ \Rightarrow\ d_{\mathcal{K}_n}(t,x) \geq \eta\ .$$

Let us now assume that $d_{\mathcal{K}^*(t)}(x) > \epsilon$ for some $x \in I\!\!R^N$ and for some $t \in [0,T]$. Since $\mathcal{K}^*$ is equal to the upper limit of the $\mathcal{K}_n$, we have $d_{\mathcal{K}_n(t)}(x) > \epsilon/2$ for any n large enough, whence $d_{\mathcal{K}_n}(t,x) \geq \eta$. This implies that $d_{\mathcal{K}^*}(t,x) \geq \eta$. So $\mathcal{K}^*$ is a left lower semicontinuous tube.

Let us now check that $\mathcal{K}^*$ is a subsolution. For this, let us fix some smooth tube $\mathcal{K}_r$ which is externally tangent to $\mathcal{K}^*$ at some point (t,x). We denote by $\mathbf{d}$ the signed distance function to $\partial\mathcal{K}_r$. This function is $\mathcal{C}^{1,1}$ in a neighborhood $\mathcal{V} = \{|\mathbf{d}| < \eta\}$ of $\partial\mathcal{K}_r$.

Let us now consider the function $\mathbf{d}_\epsilon(s,y) = \mathbf{d}(s,y) - \epsilon|(s,y)-(t,x)|^2$, for $\epsilon > 0$. Let us underline that $\mathbf{d}_\epsilon$ has a unique maximum on $\mathcal{K}^*$ at the point (t,x). We can choose $\epsilon > 0$ sufficiently small in such a way that the set $\mathcal{K}_r^\epsilon = \{\mathbf{d}_\epsilon \leq 0\}$ has a boundary which is contained in $\mathcal{V}$ and $\nabla \mathbf{d}_\epsilon \neq 0$ on $\mathcal{V}$. Let us now consider a point (t_n, x_n) of maximum of $\mathbf{d}_\epsilon$ onto $\mathcal{K}_n$. Using standart argument, we can prove that a subsequence of (t_n, x_n) (again denoted (t_n, x_n)) converges to (t,x), because $\mathbf{d}_\epsilon$ has a unique maximum on $\mathcal{K}^*$ at the point (t,x). Hence the set $\mathcal{K}_r^n = \{\mathbf{d}_\epsilon \leq \mathbf{d}_\epsilon(t_n, x_n)\}$ is a smooth tube for n large enough, since, for n large enough, the boundary of $\mathcal{K}_r^n$ is in $\mathcal{V}$. Since $\mathcal{K}_n$ is a subsolution and since $\mathcal{K}_r^n$ is externally tangent to $\mathcal{K}_n$ at (t_n, x_n), we have

$$V_{(t_n,x_n)}^{\mathcal{K}_r^n} \leq h_n(x_n, \mathcal{K}_r^n(t_n)) \ .$$

The sequence of sets $\mathcal{K}_r^n$ converges to $\mathcal{K}_r^\epsilon$ for the $\mathcal{C}^{1,1}$ topology. So we get at the limit

$$V_{(t,x)}^{\mathcal{K}_r^\epsilon} \leq h(x, \mathcal{K}_r^\epsilon(t))$$

because h_n converges to h. Letting $\epsilon \to 0$ gives the desired result, since $\mathcal{K}_r^\epsilon$ converges to $\mathcal{K}_r$ for the $\mathcal{C}^{1,1}$ topology and h is continuous. QED

We finally investigate the stability of solutions with respect to the initial position K_0 and to the functions f and g.

Theorem 4.6 *Let (f_n, g_n) and (f,g) satisfy (4) for any n, with $f > 0$ and locally Lipschitz continuous. Let $K_n \in \mathcal{D}$ be a sequence of initial positions and $K_0 \in \mathcal{D}$.*

Let us assume that the f_n are globally bounded and converge to f locally uniformly, that the g_n converge to some $g > 0$ in $\mathcal{C}^{1,\beta}(\partial S)$ for some $\beta \in (0,1)$ and that the K_n converge to K_0, in the sense that the upper limit of the K_n is contained in K_0 and the upper limit of the $\overline{\mathbb{R}^N \backslash K_n}$ is contained in $\overline{\mathbb{R}^N \backslash K_0}$. Let us also suppose that there is a unique solution—denoted by $S(K_0)$—of the Hele-Shaw problem starting from K_0 with data f and g.

If $\mathcal{K}_n$ is a solution of the Hele-Shaw problem, with data f_n and g_n, starting from K_n, then the $\mathcal{K}_n$ converge to $S(K_0)$ in the following sense: The upper limit of the $\mathcal{K}_n$ is equal to $S(K_0)$, while the upper limit of the $\widehat{\mathcal{K}_n}$ is equal to $\widehat{S(K_0)}$.

Proof of Theorem 4.6: Let $\mathcal{K}^*$ be the upper limit of the $\mathcal{K}_n$ and $\mathcal{K}_*$ be the complementary of the upper limit of the $\widehat{\mathcal{K}_n}$.

Let us first prove that $\mathcal{K}^*$ is a subsolution to the front propagation problem with initial position K_0. According to Proposition 4.5, it is enough to prove that the $\mathcal{K}_n$ are locally uniformly bounded w.r. to t and the maps h_n defined, for any smooth set $K_r \in \mathcal{D}$ and any $x \in \partial K_r$ by $h_n(x, K_r) = |\nabla u(x)|^{p-1}$ where u is the solution to

$$\begin{cases} -div(|\nabla u|^{p-1}\nabla u) = f_n & \text{in } K_r \\ u = g_n & \text{on } \partial S \\ u = 0 & \text{on } \partial K_r \end{cases}$$

converge to h defined by (2). The $\mathcal{K}_n$ are locally uniformly bounded thanks to Lemma 4.2, because the f_n and the g_n are uniformly bounded. Moreover, the local uniform convergence of the h_n to h is a straightforward application of the estimates in [16]. Using Lemma 4.1, we can also show that $\mathcal{K}^*(0) \subset K_0$ (the arguments are similar to those developed for proving that $\mathcal{K}^*$ is left lower semicontinuous in Theorem 4.5). Hence $\mathcal{K}^*$ is a subsolution to the Hele-Shaw problem with initial position K_0. In particular, this implies that $\mathcal{K}^* \subset S(K_0)$, because $S(K_0)$ contains any subsolution.

In the same way, $\mathcal{K}_*$ is a supersolution for h, with $\widehat{\mathcal{K}_*}(0) \subset \overline{I\!R^N \backslash K_0}$. Hence $s(K_0) \subset \mathcal{K}_*$. So we have proved that

$$s(K_0) \subset \mathcal{K}_* \subset \mathcal{K}^* \subset S(K_0)\ . \tag{40}$$

From our assumption, the Hele-Shaw problem with initial position K_0 has a unique solution, i.e., $\overline{s(K_0)} = S(K_0)$. Combining this equality with (40) gives $\overline{s(K_0)} = \overline{\mathcal{K}_*} = \mathcal{K}^* = S(K_0)$, since $\mathcal{K}^*$ and $S(K_0)$ have a closed graph.

Taking the complementary in (40) also gives $\widehat{S(K_0)} \subset \widehat{\mathcal{K}^*} \subset \widehat{\mathcal{K}_*} \subset \widehat{s(K_0)}$. Since $\widehat{S(K_0)} = \widehat{s(K_0)}$ from the uniqueness of the solution, we finally have the equality $\widehat{\mathcal{K}_*} = \widehat{S(K_0)}$, which is the desired result since $\widehat{\mathcal{K}_*} = (I\!R^+ \times I\!R^N) \backslash \mathcal{K}_*$ is the upper limit of the $\widehat{\mathcal{K}_n}$. QED

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