Representations formulas for differential games with asymmetric information

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Abstract

We compute the value of several two-player zero-sum differential games in which the players have an asymmetric information on the random terminal payoff.

1 Introduction

In this paper we investigate several examples of a two-player zero-sum differential game in which the players have a private information on the random terminal payoff. The existence of a value for such game was proved in the companion paper [7]. Let us briefly recall the framework. The dynamics of the game is given by

$$\left\{ \begin{array}{ll} x'(t) = f(x(t), u(t), v(t)) \ , & u(t) \in U, \ v(t) \in V \\ x(t_0) = x_0 \end{array} \right. \tag{1}$$

where U and V are compact subsets of some finite dimensional spaces and $f : I\!\!R^N \times U \times V \to I\!\!R^N$ is Lipschitz continuous. The terminal time of the game is denoted by T. The game starts at time $t_0 \in [0, T]$ from the initial position x_0.

Let $g_{ij} : I\!\!R^N \to I\!\!R$ be $I \times J$ terminal payoffs (where $I, J \geq 1$), $p = (p_i)_{i=1,\ldots,I}$ belong to the set $\Delta(I)$ of probabilities on $\{1, \ldots, I\}$ and $q = (q_j)_{j=1,\ldots,J}$ belong to the set $\Delta(J)$ of probabilities on $\{1, \ldots, J\}$. At the initial time t_0, a pair (i, j) is chosen at random according to the probability

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$p \otimes q$, the index i is communicated to Player I only while the index j is communicated to Player II only. Then the players control system (1) in order, for Player I, to minimize the terminal payoff $g_{ij}(x(T))$, and for Player II to maximize it. Note that the player do not really know which terminal payoff they are actually optimizing. The key assumption is that both players observe their oponent's control, and they can try to guess their missing information by looking at their oponent's behavior.

In [7], we proved that this game has a value: namely the equality

$$\inf_{(\alpha_i)\in(\mathcal{A}_r(t_0))^I} \sup_{(\beta_j)\in(\mathcal{B}_r(t_0))^J} \sum_{i=1}^{I}\sum_{j=1}^{J} p_i q_j \mathbf{E}_{\alpha_i\beta_j}\left(g_{ij}\left(X_T^{t_0,x_0,\alpha_i,\beta_j}\right)\right) = \tag{2}$$

$$\sup_{(\beta_j)\in(\mathcal{B}_r(t_0))^J} \inf_{(\alpha_i)\in(\mathcal{A}_r(t_0))^I} \sum_{i=1}^{I}\sum_{j=1}^{J} p_i q_j \mathbf{E}_{\alpha_i\beta_j}\left(g_{ij}\left(X_T^{t_0,x_0,\alpha_i,\beta_j}\right)\right)$$

holds. In the above expressions, $\alpha_i \in \mathcal{A}_r(t_0)$ (for $i = 1, \ldots, I$) are I random strategies for Player I, $\beta_j \in \mathcal{B}_r(t_0)$ (for $j = 1, \ldots, J$) are J random strategies for Player II and $\mathbf{E}_{\alpha_i\beta_j}\left(g_{ij}\left(X_T^{t_0,x_0,\alpha_i,\beta_j}\right)\right)$ is the payoff associated with the pair of strategies (α_i, β_j) for the terminal payoff g_{ij}: these notions are explained in the next section. Player I chooses his strategies only according to the value of the index i, while Player II, on the contrary, plays a strategy β_j which depends only on j. This reflects the asymmetry of information of the players. The sum $\sum_i \sum_j p_i q_j \ldots$ is the expectation of the payoff when g_{ij} is chosen according to the probability $p \otimes q$. We denote by $\mathbf{V}(t_0, x_0, p, q)$ the value of the game given by (2).

The game studied in this paper has been introduced by Aumann and Maschler [2] in the framework of *repeated games* (see also [15] for a general presentation). Let us briefly recall their main results. When only Player I has some private information (i.e., $I \geq 2$ and $J = 1$), Aumann and Maschler prove that the game has a value, which is equal to the convex hull with respect to p of the game without information. In our framework of differential games and when $J = 1$, the game without information has a value given by

$$W(t_0, x_0, p) = \inf_{\alpha\in\mathcal{A}(t_0)} \sup_{\beta\in\mathcal{B}(t_0)} \sum_{i=1}^{I} p_i g_i\left(X_T^{t_0,x_0,\alpha,\beta}\right)$$

$$= \sup_{\beta\in\mathcal{B}(t_0)} \inf_{\alpha\in\mathcal{A}(t_0)} \sum_{i=1}^{I} p_i g_i\left(X_T^{t_0,x_0,\alpha,\beta}\right)$$

(since $J = 1$, we omit the index j and the dependence with respect to q). The statement corresponding to Aumann-Maschler result would be

$$\mathbf{V} = Vex(W) \tag{3}$$

where $Vex(W)$ is the convex hull of W with respect to p. The first aim of this paper is to give an example showing that such a statement is false in general (section 2). The interpretation is that there are some positions at which Player I has better to wait before revealing his information. In that respect, the game studied here is close to stochastic games with lack of information on one side (see [14]).

When both players have a private information (i.e., $I, J \geq 2$), Martens and Zamir have proved in [12] that the $n-$stage games have a limit, which can be characterized in terms of the value of the game without information, now given by

$$W(t_0, x_0, p, q) = \inf_{\alpha \in \mathcal{A}(t_0)} \sup_{\beta \in \mathcal{B}(t_0)} \sum_{i=1}^{I} \sum_{j=1}^{J} p_i q_j g_{ij} \left(X_T^{t_0, x_0, \alpha, \beta} \right)$$

$$= \sup_{\beta \in \mathcal{B}(t_0)} \inf_{\alpha \in \mathcal{A}(t_0)} \sum_{i=1}^{I} \sum_{j=1}^{J} p_i q_j g_{ij} \left(X_T^{t_0, x_0, \alpha, \beta} \right)$$

Differential games also have a value, but, of course, it cannot be characterized in terms of W in general. The second aim of this paper is to investigate two cases in which this characterization holds.

In the first case (section 4), we assume that Player II has in some sense more control on the system than Player I. By this we mean that H is convex in ξ. Then we prove that the value of the game can be represented as

$$\mathbf{V}(t_0, x_0, p, q) = \sum_{j=1}^{J} q_j W(t_0, x_0, p, e_j) = \mathrm{Cav}_q(W)(t_0, x_0, p, q) ,$$

where (e_j) is the standard basis of $\mathbb{R}^J$ and $Cav_q(W)$ is the concave hull of W with respect to q. The interpretation of this result is the following: Player II uses immediately his information, while Player I, on the contrary, never uses it.

For the second example (section 5), we assume that $J = 1$ (Player II has no private information) and that the following structure condition holds: the dynamics is independant of the state and the payoff functions are concave.

In this case we can use Lax representation formula to compute W and W^*, and we prove that $\mathbf{V} = Vex(W)$.

Through these examples, we also show that $\mathbf{V}$ is *neither* a supersolution *nor* a subsolution of the primal equation (11) in general, that $\mathbf{V}^*$ is *not* a supersolution of the dual equation, and that $\mathbf{V}^\sharp$ is *not* a subsolution of (10).

2 Existence and characterization of the value

Notations : Throughout the paper, $x.y$ denotes the scalar product in the space $\mathbb{R}^N$, $\mathbb{R}^I$ or $\mathbb{R}^J$ (depending on the context) and $|\cdot|$ the euclidean norm. The ball of center x and radius r will be denoted by $B_r(x)$. The set $\Delta(I)$ is the set of probabilities mesures on $\{1,\dots,I\}$, always identified with the simplex of $\mathbb{R}^I$:

$$p = (p_1,\dots,p_I) \in \Delta(I) \quad \Leftrightarrow \quad \sum_{i=1}^{I} p_i = 1 \text{ and } p_i \geq 0 \text{ for } i = 1,\dots I\ .$$

The set $\Delta(J)$ of probability measures on $\{1,\dots,J\}$ is defined symmetrically.

The dynamics of the game is given by:

$$\left\{\begin{array}{ll} x'(t) = f(x(t),u(t),v(t))\ , & u(t)\in U,\ v(t)\in V \\ x(t_0) = x_0 \end{array}\right. \tag{4}$$

Throughtout the paper we assume that

$$\left\{\begin{array}{rl} i) & U \text{ and } V \text{ are compact subsets of some finite dimensional spaces,} \\ ii) & f:\mathbb{R}^N\times U\times V\to\mathbb{R}^N \text{ is bounded, continuous, Lipschitz} \\ & \text{continuous with respect to the } x \text{ variable,} \\ iii) & \text{for } i=1,\dots,I \text{ and } j=1,\dots,J,\ g_{ij}:\mathbb{R}^N\to\mathbb{R} \text{ is Lipschitz} \\ & \text{continuous and bounded.} \end{array}\right. \tag{5}$$

We also assume that Isaacs condition holds, which allows us to define the Hamiltonian of our primal HJ equation:

$$H(x,\xi) := \inf_{u\in U}\sup_{v\in V} f(x,u,v).\xi = \sup_{v\in V}\inf_{u\in U} f(x,u,v).\xi \tag{6}$$

for any $(x,\xi)\in\mathbb{R}^N\times\mathbb{R}^N$.

For any $t_0 < T$, the set of open-loop controls for Player I is defined by

$$\mathcal{U}(t_0) = \{u : [t_0,T]\to U \text{ Lebesgue measurable}\}\ .$$

Open-loop controls for Player II are defined symmetrically and denoted by $\mathcal{V}(t_0)$. For any $(u,v) \in \mathcal{U}(t_0) \times \mathcal{V}(t_0)$ and any initial position $x_0 \in \mathbb{R}^N$, we denote by $t \to X_t^{t_0,x_0,u,v}$ the solution to (4).

A *pure* strategy for Player I at time t_0 is a map $\alpha : \mathcal{V}(t_0) \to \mathcal{U}(t_0)$ which is nonanticipative with delay, i.e., there is some $\tau > 0$ such that, for any $v_1, v_2 \in \mathcal{V}(t_0)$, if $v_1 \equiv v_2$ a.e. on $[t_0, t]$ for some $t \in (t_0, T)$, then $\alpha(v_1) \equiv \alpha(v_2)$ a.e. on $[t_0, t + \tau]$.

A random strategy for Player I is a pair $((\Omega_\alpha, \mathcal{F}_\alpha, \mathbf{P}_\alpha), \alpha)$, where $(\Omega_\alpha, \mathcal{F}_\alpha, \mathbf{P}_\alpha)$ is a probability space (chosen by Player I) and $\alpha : \Omega_\alpha \times \mathcal{V}(t_0) \to \mathcal{U}(t_0)$ satisfying

(i) α is measurable from $\Omega_\alpha \times \mathcal{V}(t_0)$ to $\mathcal{U}(t_0)$, with Ω_α endowed with the $\sigma-$field $\mathcal{F}_\alpha$ and $\mathcal{U}(t_0)$ and $\mathcal{V}(t_0)$ with the Borel $\sigma-$field associated with the L^1 distance,

(ii) there is some delay $\tau > 0$ such that, for any $v_1, v_2 \in \mathcal{V}(t_0)$ and any $t \in (t_0, T)$,

$$v_1 \equiv v_2 \text{ on } [t_0, t) \ \Rightarrow \ \alpha(\omega, v_1) \equiv \alpha(\omega, v_2) \text{ on } [t_0, t+\tau) \qquad \forall \omega \in \Omega_\alpha \ .$$

We denote by $\mathcal{A}(t_0)$ the set of pure strategies and by $\mathcal{A}_r(t_0)$ the set of random strategies for Player I. By abuse of notations, an element of $\mathcal{A}_r(t_0)$ is simply noted α—instead of $((\Omega_\alpha, \mathcal{F}_\alpha, \mathbf{P}_\alpha), \alpha)$—, the underlying probability space being always denoted by $(\Omega_\alpha, \mathcal{F}_\alpha, \mathbf{P}_\alpha)$.

In order to take into account the fact that Player I knows the index i of the terminal payoff, a strategy for Player I is actually a $I-$upplet $\hat{\alpha} = (\alpha_1, \ldots, \alpha_I) \in (\mathcal{A}_r(t_0))^I$.

Pure and random strategies for Player II are defined symmetrically; $\mathcal{B}(t_0)$ (resp. $\mathcal{B}_r(t_0)$) denotes the set of pure strategies (resp. random strategies). Generic elements of $\mathcal{B}_r(t_0)$ are denoted by β, with associated probablity space $(\Omega_\beta, \mathcal{F}_\beta, \mathbf{P}_\beta)$. Since Player II knows the index j of the terminal payoff, a strategy for Player II is actually a $J-$upplet $\hat{\beta} = (\beta_1, \ldots, \beta_J) \in (\mathcal{B}_r(t_0))^J$.

Lemma 2.1 (Lemma 2.2 of [7]) *For any pair $(\alpha, \beta) \in \mathcal{A}_r(t_0) \times \mathcal{B}_r(t_0)$ and any $\omega := (\omega_1, \omega_2) \in \Omega_\alpha \times \Omega_\beta$, there is a unique pair $(u_\omega, v_\omega) \in \mathcal{U}(t_0) \times \mathcal{V}(t_0)$, such that*

$$\alpha(\omega_1, v_\omega) = u_\omega \text{ and } \beta(\omega_2, u_\omega) = v_\omega \ . \tag{7}$$

Furthermore the map $\omega \to (u_\omega, v_\omega)$ is measurable from $\Omega_\alpha \times \Omega_\beta$ endowed with $\mathcal{F}_\alpha \otimes \mathcal{F}_\beta$ into $\mathcal{U}(t_0) \times \mathcal{V}(t_0)$ endowed with the Borel $\sigma-$field associated with the L^1 distance.

Notations : Given any pair $(\alpha, \beta) \in \mathcal{A}_r(t_0) \times \mathcal{B}_r(t_0)$, we denote by $(X_t^{t_0,x_0,\alpha,\beta})$ the map $(t,\omega) \to (X_t^{t_0,x_0,u_\omega,v_\omega})$ defined on $[t_0,T] \times \Omega_\alpha \times \Omega_\beta$, where (u_ω, v_ω) satisfies (7). The expectation $\mathbf{E}_{\alpha\beta}$ is the integral over $\Omega_\alpha \times \Omega_\beta$ against the probability measure $\mathbf{P}_\alpha \otimes \mathbf{P}_\beta$. In particular, if $\phi : I\!\!R^N \to I\!\!R$ is some bounded continuous map and $t \in (t_0, T]$, we have

$$\mathbf{E}_{\alpha\beta}\left(\phi\left(X_t^{t_0,x_0,\alpha,\beta}\right)\right) := \int_{\Omega_\alpha \times \Omega_\beta} \phi\left(X_t^{t_0,x_0,u_\omega,v_\omega}\right) d\mathbf{P}_\alpha \otimes \mathbf{P}_\beta(\omega) \ , \tag{8}$$

where (u_ω, v_ω) is defined by (7).

For $p \in \Delta(I)$, $q \in \Delta(J)$, $(t_0, x_0) \in [0,T) \times I\!\!R^N$, the payoff associated with a strategy $\hat{\alpha} = (\alpha_i)_{i=1,\ldots,I} \in (\mathcal{A}_r(t_0))^I$ of Player I and a strategy $\hat{\beta} = (\beta_j)_{j=1,\ldots,J} \in (\mathcal{B}_r(t_0))^J$ of Player II is defined by:

$$\mathcal{J}(t_0, x_0, \hat{\alpha}, \hat{\beta}, p, q) = \sum_{i=1}^{I}\sum_{j=1}^{J} p_i q_j \mathbf{E}_{\alpha_i\beta_j}\left(g_{ij}\left(X_T^{t_0,x_0,\alpha_i,\beta_j}\right)\right) \ , \tag{9}$$

where $\mathbf{E}_{\alpha_i\beta_j}$ is defined by (8).

Let us now recall the main result of [7]:

Theorem 2.2 (Existence of the value [7])
Assume that conditions (5) on f and on the g_{ij} hold and that Isaacs assumption (6) is satisfied. Then the following equality holds:

$$\inf_{\hat{\alpha} \in (\mathcal{A}_r(t_0))^I} \sup_{\hat{\beta} \in (\mathcal{B}_r(t_0))^J} \mathcal{J}(t_0, x_0, \hat{\alpha}, \hat{\beta}, p, q) \ =$$

$$\sup_{\hat{\beta} \in (\mathcal{B}_r(t_0))^J} \inf_{\hat{\alpha} \in (\mathcal{A}_r(t_0))^I} \mathcal{J}(t_0, x_0, \hat{\alpha}, \hat{\beta}, p, q) \ .$$

We denote by $\mathbf{V}(t_0, x_0, p, q)$ the common value of both expressions.

In order to compute our examples below, we now recall the characterization of $\mathbf{V}$. For this we need two notions of Fenchel conjugates. Let $w : [0,T] \times I\!\!R^N \times \Delta(I) \times \Delta(J) \to I\!\!R$ be some function. We denote by w^* its convex conjugate with respect to the p variable, and by $w^\sharp$ the concave conjugate with respect to the q variable:

$$w^*(t,x,\hat{p},q) = \sup_{p \in \Delta(I)} p.\hat{p} - w(t,x,p,q) \qquad \forall (t,x,\hat{p},q) \in [0,T] \times I\!\!R^N \times I\!\!R^I \times \Delta(J)$$

and

$$w^\sharp(t,x,p,\hat{q}) = \inf_{q\in\Delta(J)} q.\hat{q} - w(t,x,p,q) \qquad \forall (t,x,p,\hat{q}) \in [0,T]\times I\!\!R^N \times \Delta(I) \times I\!\!R^J \ .$$

In particular $\mathbf{V}^*$ and $\mathbf{V}^\sharp$ denote the convex and concave conjugates of $\mathbf{V}$ with respect to p and q respectively. If now w is defined on the dual space $[0,T]\times I\!\!R^N \times I\!\!R^I \times \Delta(J)$ (resp. $[0,T]\times I\!\!R^N \times \Delta(I) \times I\!\!R^J$) we also denote by w^* (resp. $w^\sharp$) its convex (resp. concave) conjugate with respect to $\hat{p}$ (resp. $\hat{q}$).

Proposition 2.3 (Characterization of the value, [7])
The value function $\mathbf{V}$ *is the unique function from* $[0,T]\times I\!\!R^N \times \Sigma(I) \times \Delta(J)$ *such that*

(i) $\mathbf{V}$ *is Lipschitz continuous in all its variables, convex in* p *and concave in* q*, and*

$$\mathbf{V}(T,x,p,q) = \sum_{i=1}^{I}\sum_{j=1}^{J} p_i q_j g_{ij}(x) \qquad \forall (x,p,q) \in I\!\!R^N \times \Delta(I) \times \Delta(J) \ ,$$

(ii) for any $(p,\hat{q}) \in \Delta(I) \times I\!\!R^J$*,* $(t,x) \to \mathbf{V}^\sharp(t,x,p,\hat{q})$ *is a viscosity supersolution of the dual HJ equation*

$$w_t + H^*(x, Dw) = 0 \text{ in } [0,T]\times I\!\!R^N \qquad (10)$$

where $H^*(x,\xi) = -H(x,-\xi)$ *and* H *is defined by (6),*

(iii) for any $(\hat{p},q) \in I\!\!R^I \times \Delta(J)$*,* $(t,x) \to \mathbf{V}^*(t,x,\hat{p},q)$ *is a viscosity subsolution of the dual HJ equation (10).*

We say that $\mathbf{V}$ *is the unique dual solution of the primal equation*

$$w_t + H(x, Dw) = 0 \text{ in } [0,T]\times I\!\!R^N \qquad (11)$$

with terminal condition $\mathbf{V}(T,x,p,q) = \sum_{i=1}^{I}\sum_{j=1}^{J} p_i q_j g_{ij}(x)$*.*

Remarks 2.4

1. We recall that the notion of viscosity solutions was introduced by Crandall-Lions in [8] and first used in the framework of differential games in [10]. The books [3], [5] are standard references on the subject. The idea of introducing the dual game and the Fenchel conjugate of the value functions comes back to De Meyer [9] for repeated games.

2. A function w satisfying condition (i) and (ii) only is called a dual subsolution of (11), whereas a function w satisfying (i) and (iii) only is called a dual supersolution of (11). The reason for this terminology is the following comparison principle given in [7] and nedeed below: if w_1 is a dual subsolution of (11) and w_2 is a dual supersolution of (11), then $w_1 \leq w_2$.

3. Let us also underline that in the case $J = 1$ (when only Player II has a private information), we can omit the dependence of $\mathbf{V}$ with respect to q and (ii) is equivalent to saying that, for any $p \in \Delta(I)$, $(t, x) \to \mathbf{V}(t, x, p)$ is a subsolution of the primal equation (11). In particular, since the value of the game without information is a solution of the primal equation, the usual comparison principle (see [3]) states that

$$V(t, x, p) \leq W(t, x, p) \qquad \forall (t, x, p) \in [0, T] \times I\!R^N \times \Delta(I) . \tag{12}$$

We need below the following reformulation of $\mathbf{V}^*$ (Lemma 4.2 of [7]):

$$\mathbf{V}^*(t, x, \hat{p}, q) = \inf_{\hat{\beta} \in (\mathcal{B}_r(t_0))^J} \sup_{\alpha \in \mathcal{A}_r(t_0)} \max_{i \in \{1,\dots,I\}} \left\{ \hat{p}_i - \sum_j q_j \mathbf{E}_{\alpha\beta_j} \left(g_{ij}(X_T^{t,x,\alpha,\beta_j}) \right) \right\} . \tag{13}$$

We finally introduce the game with symmetric lack of information. It is defined by one of the four equivalent expressions:

$$W(t_0, x_0, p, q) = \inf_{\alpha \in \mathcal{A}_r(t_0)} \sup_{\beta \in \mathcal{B}_r(t_0)} \sum_{i=1}^{I} \sum_{j=1}^{J} p_i q_j \mathbf{E}_{\alpha\beta} \left(g_{ij} \left(X_T^{t_0, x_0, \alpha, \beta} \right) \right) \tag{14}$$

$$= \sup_{\beta \in \mathcal{B}_r(t_0)} \inf_{\alpha \in \mathcal{A}_r(t_0)} \sum_{i=1}^{I} \sum_{j=1}^{J} p_i q_j \mathbf{E}_{\alpha\beta} \left(g_{ij} \left(X_T^{t_0, x_0, \alpha, \beta} \right) \right) \tag{15}$$

$$= \inf_{\alpha \in \mathcal{A}(t_0)} \sup_{\beta \in \mathcal{B}(t_0)} \sum_{i=1}^{I} \sum_{j=1}^{J} p_i q_j \left(g_{ij} \left(X_T^{t_0, x_0, \alpha, \beta} \right) \right) \tag{16}$$

$$= \sup_{\beta \in \mathcal{B}(t_0)} \inf_{\alpha \in \mathcal{A}(t_0)} \sum_{i=1}^{I} \sum_{j=1}^{J} p_i q_j \left(g_{ij} \left(X_T^{t_0, x_0, \alpha, \beta} \right) \right) . \tag{17}$$

The fact that these expressions are equal, and coincide with the unique *viscosity solution* of the primal HJ equation (11) is proved below.

Proof of equalities (14, 15, 16, 17): Let us denote by $W_1(t_0, x_0, p)$ the function in (14), by $W_2(t_0, x_0, p)$ the function in (14), and so on. The equality between W_3 and W_4 under Isaacs'assumption (6) is well-known, as well as the fact that these functions are the unique solutions to the primal HJ equation (11) (see [10]).

For strategies $\alpha \in \mathcal{A}_r(t_0)$ and $\beta \in \mathcal{B}_r(t_0)$, let us set

$$J_1(t_0, x_0, \alpha, \beta, p, q) = \sum_{i=1}^{I} \sum_{j=1}^{J} p_i q_j \mathbf{E}_{\alpha\beta} \left(g_{ij} \left(X_T^{t_0, x_0, \alpha, \beta} \right) \right)$$

(compare with (9)). Then one easily checks that

$$\begin{array}{rcl} W_1(t_0, x_0, p, q) & = & \inf_{\alpha \in \mathcal{A}_r(t_0)} \sup_{\beta \in \mathcal{B}(t_0)} J_1(t_0, x_0, \alpha, \beta, p, q) \\ & \leq & \inf_{\alpha \in \mathcal{A}(t_0)} \sup_{\beta \in \mathcal{B}(t_0)} J_1(t_0, x_0, \alpha, \beta, p, q) \\ & = & W_3(t_0, x_0, p, q) \end{array}$$

In the same way

$$\begin{array}{rcl} W_2(t_0, x_0, p, q) & = & \sup_{\beta \in \mathcal{B}_r(t_0)} \inf_{\alpha \in \mathcal{A}(t_0)} J_1(t_0, x_0, \alpha, \beta, p, q) \\ & \geq & \sup_{\beta \in \mathcal{B}(t_0)} \inf_{\alpha \in \mathcal{A}(t_0)} J_1(t_0, x_0, \alpha, \beta, p, q) \\ & = & W_4(t_0, x_0, p, q) \end{array}$$

Hence $W_1 \leq W_3 = W_4 \leq W_2$. Since the inequality $W_1 \geq W_2$ is obvious, the claim is proved.

QED

3 Counterexample to $\mathbf{V} = Vex(W)$

Throughout this section, we assume that $J = 1$, and omit the dependence in j and q of the various quantities.

In Aumann and Mashler's result for repeted games [2], the value of the game with asymmetric lack information is given by the convex hull with respect to p of the value of the game with symmetric lack of information. For differential games, this is no longer true, i.e., inequality (12) is not an equality in general. We explain this through the following counterexample.

In order to deal with computations as elementary as possible, we investigate a game with time dependent dynamics, discontinuous in time (but with only one discontinuity). It is not difficult to generalize the theory explained above on the existence and characterization of a value for this particular game. This example could also easily be transformed into a game with smooth and time-independant dynamics (by adding the time as an independant variable), but we shall not do so for simplicity.

Let $U = V = B_1(0)$ (where $B_1(0)$ is the unit ball of $I\!R^N$) and

$$f(x,u,v) = \begin{cases} u & \text{if } t \in [T/2, T] \\ v & \text{if } t \in [0, T/2] \end{cases}$$

We assume that $I = 2$. The payoffs g_1 and g_2 are given by

$$g_i(x) = \frac{1}{2}|x|^2 + a_i.x \qquad \forall x \in I\!R^N$$

where a_1 and a_2 belong to $I\!R^N$. To better match this situation, we slightly change the notations: we denote by p ($p \in [0,1]$)—instead of p_1—the probability of g_1 to be chosen and note that p_2 now simply writes $(1-p)$.

Proposition 3.1 *If $N \geq 3$ and a_1 and a_2 are linearly independant, there is some $(t_0, x_0) \in [0, T/2) \times I\!R^N$ such that*

$$\mathbf{V}(t_0, x_0, p) \; < \; Vex(W)(t_0, x_0, p) \qquad \forall p \in (0,1)\ .$$

Proof of Proposition 3.1 : Let us set $a_p = pa_1 + (1-p)a_2$ for $p \in [0,1]$. On $[T/2, T]$, the system is controlled by Player I only and W is given by the representation formula:

$$W(t,x,p) \; = \inf_{|y-x| \leq (T-t)} pg_1(y) + (1-p)g_2(y)\ .$$

Hence

$$W(t,x,p) \; = \; \begin{cases} \dfrac{1}{2}|a_p| & \text{if } |x + a_p| \leq (T-t) \\ \\ \dfrac{1}{2}|x|^2 + x.a_p - (T-t)|x + a_p| + \dfrac{1}{2}(T-t)^2 & \text{otherwise} \end{cases}$$

From the dynamic programming principle and the definition of the dynamics, we have for $t \in [0, T/2]$

$$W(t,x,p) = \sup_{|y-x| \leq T/2 - t} W(T/2, y) \qquad \forall x \in I\!R^N\ .$$

From now on, we compute $W(t,x,p)$ only when $|x|$ is large. The previous equality then becomes

$$W(t,x,p) = \frac{1}{2}|x|^2 + x.a_p - t|x+a_p| + \frac{1}{2}t^2 \ .$$

Since W is concave with respect to p we get

$$\begin{array}{rcl} Vex(W)(t,x,p) & = & pW(t,x,1) + (1-p)W(t,x,0) \\ & = & \frac{1}{2}|x|^2 + x.a_p - t\left(p|x+a_1| + (1-p)|x+a_2|\right) + \frac{1}{2}t^2 \ . \end{array}$$

We note that this function is smooth for $|x|$ large. Let us also choose x such that a_1, a_2 and x are linearly independant. We fix $p \in (0,1)$ and set $u(t,x) = Vex(W)(t,x,p)$. Then

$$u_t(t,x) = -\left(p|x+a_1| + (1-p)|x+a_2|\right) + t$$

and

$$\begin{array}{l} H(t,x,Du(t,x)) \ = \ |Du(t,x)| \\ \quad = \ |x + a_p - t\left(p(x+a_1)/|x+a_1| + (1-p)(x+a_2)/|x+a_2|\right)| \\ \quad < \ p\,|x+a_1 - t(x+a_1)/|x+a_1|| + (1-p)\,|x+a_1 - t(x+a_2)/|x+a_2|| \\ \quad < -u_t(t,x) \end{array}$$

because a_1, a_2 and x are linearly independant and $p \in (0,1)$. Hence

$$u_t(t,x) + H(t,x,Du(t,x)) < 0$$

which proves that $Vex(W) = u$ is *not* a subsolution of the primal HJ equation. Following Remark 2.4-3), $Vex(W)$ cannot be equal to $\mathbf{V}$ in a neighbourhood of x, which implies that $\mathbf{V}(t,x,p) < Vex(W)(t,x,p)$ because of (12).

QED

4 Case of convex H

In this example we come back to the game with lack of information on both sides. We assume that

$$H(x,\xi) \quad \text{is convex with respect to } \xi. \tag{18}$$

Theorem 4.1 *Under assumption (18), the value of the game is given by:*

$$\mathbf{V}(t_0, x_0, p, q) = \sum_{j=1}^{J} q_j W(t_0, x_0, p, e_j) = \mathrm{Cav}_q(W)(t_0, x_0, p, q) \tag{19}$$

for any $(t_0, x_0, p, q) \in [0,T] \times \mathbb{R}^N \times \Delta(I) \times \Delta(J)$, *where* (e_j) *is the standard basis of* $\mathbb{R}^J$ *and* $\mathrm{Cav}_q(W)$ *denotes the concave hull with respect to* q *of the function* W *given by (14).*

In this example, where the second Player has actually a strong control on the system, the first Player does not use its information at all, while Player II uses his information immediately.

Proof : Let us set, for $(t_0, x_0, p, q) \in [0,T] \times \mathbb{R}^N \times \Delta(I) \times \Delta(J)$,

$$\mathbf{V}_1(t_0, x_0, p, q) = \sum_{j=1}^{J} q_j W(t_0, x_0, p, e_j) \ .$$

Since H is convex with respect to ξ, there is some control system

$$x'(t) = f_1(x, b), \qquad b \in B \tag{20}$$

such that

$$H(x, \xi) = \sup_{b \in B} f_1(x, b).\xi \qquad \forall (x, \xi) \in \mathbb{R}^N \times \mathbb{R}^N \ .$$

Moreover, if we denote by $\mathcal{B}(t)$ the set of time measurable controls from $[t, T]$ to B, we have the representation formula for $W(\cdot, \cdot, p, q)$, which is the unique viscosity solution to the primal HJ equation (11) with terminal data $\sum_i p_i q_j g_{ij}$:

$$W(t, x, p, q) = \sup_{b \in \mathcal{B}(t)} \sum_i \sum_j p_i q_j g_{ij}(X_T^{t,x,b}) \tag{21}$$

where we have denoted here by $X_s^{t,x,b}$ the solution to (20) with initial condition $X_t^{t,x,b} = x$. In particular,

$$W(t, x, p, e_j) = \sup_{b \in \mathcal{B}(t)} \sum_i p_i g_{ij}(X_T^{t,x,b}) \ .$$

Therefore $\mathbf{V}_1$ is a convex function of p and a concave function of q. It is also clearly Lipschitz continuous, and thus satisfies (i) of Proposition 2.3.

Since H is convex and the $W(\cdot,\cdot,p,e_j)$ are the viscosity solutions of the primal HJ equation (11), $\mathbf{V}_1 = \sum_{j=1}^{J} q_j W(\cdot,\cdot,p,e_j)$ is a supersolution of the primal HJ equation (see [1]). Therefore, again by [1], $\mathbf{V}_1^*$ is a subsolution of the dual HJ equation (10). Thus $\mathbf{V}_1$ satisfies (iii) of Proposition 2.3.

Next we compute $\mathbf{V}_1^\sharp$:

$$\begin{array}{rcl} \mathbf{V}_1^\sharp(t,x,p,\hat{q}) & = & \inf_{q\in\Delta(J)} \{q.\hat{q} - \mathbf{V}_1(t,x,p,q)\} \\ & = & \min_j \{\hat{q}_j - W(t,x,p,e_j)\} \end{array}$$

The $W(\cdot,\cdot,p,e_j)$ being solutions of the primal HJ equation, $\hat{q}_j - W(\cdot,\cdot,p,e_j)$ are solutions of the dual one, and therefore $\mathbf{V}_1(\cdot,\cdot,p,\hat{q})$ is a supersolution of the dual equation (as minimum of solutions, see [3]). Therefore $\mathbf{V}_1$ satisfies (ii) of Proposition 2.3.

Since conditions (i), (ii) and (iii) of Proposition 2.3 characterize the value function $\mathbf{V}$, we have $\mathbf{V}_1 = \mathbf{V}$.

To complete the proof of (19), we note that, by (21), W is convex in q, and therefore its concave hull with respect to q is given by $\mathbf{V}_1$.

QED

5 Use of Hopf representation formula

In this last example we again assume that $J = 1$. Furthermore we suppose the following structure condition of H and g_i:

$$\begin{array}{c} H = H(\xi) \quad \text{does not depend on } x \text{ and that} \\ g_i \text{ are concave on } I\!\!R^N \text{ for any } i \;. \end{array} \tag{22}$$

Theorem 5.1 *Under assumption (22), we have*

$$\mathbf{V} = Vex(W) \;,$$

where $Vex(W)$ is the convex envelope of $W = W(t,x,p)$ with respect to p.

Remark : Here the g_i cannot be bounded. However, the result remain true in this case, because the dynamics is bounded and so we have a finite speed of propagation (see [3]).

Proof : Let us note that, the g_i being concave, we have

$$g_i(x) = \inf_{y \in R^N} \left\{ x.y - g_i^\sharp(x) \right\}$$

where $g_i^\sharp$ is the concave Fenchel conjugate of g_i with respect to the space variable x. Then the concave conjugate (with respect to x) of $\sum_{i=1}^I p_i g_i$ is given by

$$\left(\sum_{i=1}^I p_i g_i \right)^\sharp (x) = \sup_{z_i,\ \sum_i p_i z_i = x} \sum_{i=1}^I p_i g_i^\sharp(z_i) \tag{23}$$

(see [13]).

Let us now recall Hopf formula for solutions of HJ equations [4, 6, 11]. Let $h : I\!\!R^N \to I\!\!R$ be continuous and $g : I\!\!R^N \to I\!\!R$ be a continuous terminal data. If g is convex, then the unique solution to

$$\left\{ \begin{array}{l} w_t + H\left(Dw\right) = 0 \text{ in } [0,T] \times I\!\!R^N \\ w(T, \cdot) = g \text{ in } I\!\!R^N \end{array} \right. \tag{24}$$

is given by

$$w(t,x) = \sup_{y \in R^N} \left\{ (T-t)H(y) + x.y - g^*(y) \right\}$$

where g^* is the convex Fenchel conjugate of g. If g is concave, the above formula is still valid provided on replaces the “sup” by an “inf” and the convex conjugate by the concave one.

We first apply Hopf formula to W: since $H = H(\xi)$ and $x \to \sum_i p_i g_i(x)$ is concave, and since W the unique solution to the primal HJ equation (11) with terminal data $\sum_i p_i g_i$, W has the following representation

$$W(t,x,p) = \inf_{y \in R^N} \left\{ (T-t)H(y) + y.x - \sup_{z_i,\ \sum_i p_i z_i = y} \sum_{i=1}^I p_i g_i^\sharp(z_i) \right\} ,$$

according to the computation of the concave conjugate of $\sum_i p_i g_i$ in (23). Hence

$$\begin{array}{rcl} W(t,x,p) & = & \inf_{y \in R^N} \inf_{z_i,\ \sum_i p_i z_i = y} \left\{ (T-t)H(y) + y.x - \sum_i p_i g_i^\sharp(z_i) \right\} \\ & = & \inf_{z_i \in R^N} \left\{ (T-t)H(\sum_i p_i z_i) + \sum_i p_i (x.z_i - g_i^\sharp(z_i)) \right\} \end{array} \tag{25}$$

Therefore

$$W^*(t,x,q) \;=\; \sup_{p,\, z_i} \left\{ p.q - (T-t)H(\textstyle\sum_i p_i z_i) - \sum_i p_i(x.z_i - g_i^\sharp(z_i)) \right\} \;. \tag{26}$$

Next we compute the unique solution z of the dual HJ equation (10), again by using Hopf formula. This is possible because the Hamilonian $H^* = H^*(\xi)$ independant of x and the terminal data is given by $g := \max_i\{q_i - g_i(x)\}$ which is convex with respect to x. We first note that the convex conjugate of g is given by

$$g^*(x) \;=\; -\sup_{p,z_i,\, \sum_i p_i z_i = x} \left\{ \sum_i p_i(q_i + g_i^\sharp(-z_i)) \right\} \tag{27}$$

Indeed

$$\begin{array}{rcl} g^*(x) & = & \sup_y \{ y.x - \max_i\{q_i - g_i(y)\}\} \\ & = & \sup_y \inf_{p\in\Delta(I)} \{ y.x - \sum_i p_i(q_i - g_i(y))\} \\ & = & \inf_p \sup_y \{ y.x - \sum_i p_i(q_i - g_i(y))\} \end{array}$$

thanks to the min-max theorem. From the conjugate of a sum of convex functions we get:

$$g^*(x) \;=\; \inf_p \inf_{z_i,\, \sum_i p_i z_i = x} \left\{ \sum_i p_i(q_i - g_i)^*(z_i) \right\}$$

where $(q_i - g_i)^*$ is the *convex* Fenchel conjugate of $q_i - g_i$, i.e.,

$$(q_i - g_i)(z) = \sup_y\{z.y - q_i + g_i(y)\} = -\inf_y\{-y.x - g_i(y)\} - q_i = -g_i^\sharp(-z) - q_i \;.$$

We finally get formula (27) for g^*.

Applying Hopf formula, we obtain the following representation for $z(t,x,q)$:

$$\begin{array}{rcl} z(t,x,q) & = & \sup_y \{ (T-t)H^*(y) + x.y - g^*(y)\} \\ & = & \sup_{y,p,z_i,\, \sum_i p_i z_i = y} \left\{ -(T-t)H(-y) + x.y + \sum_i p_i(q_i + g_i^\sharp(-z_i)) \right\} \\ & = & \sup_{p,z_i} \left\{ p.q - (T-t)H(\sum_i p_i z_i) - \sum_i p_i(x_i.z_i - g_i^\sharp(z_i)) \right\} \\ & = & W^*(t,x,q) \end{array}$$

where the last equality comes from (26).

We finally note that, by (12) in Remark 2.4, $\mathbf{V} \le Vex(W)$. Moreover, since z is the solution of the dual HJ equation with terminal condition $\max_i\{p_i - g_i\}$, and since $\mathbf{V}^*$ is a subsolution of this equation, we have $\mathbf{V}^* \le z$. Hence $z^* \le \mathbf{V}^{**} = \mathbf{V}$. Therefore we have proved that

$$\mathbf{V} \le Vex(W) = W^{**} = z^* \le \mathbf{V} \;,$$

whence the equality in the Proposition.

QED

We note that W is neither convex nor concave with respect to p in general. For instance, if we assume that the g_i are linear, say $g_i(x) = a_i.x$, then formula (25) for W in the above proof becomes

$$W(t,x,p) \ = \ (T-t)H(\sum_i p_i a_i) + \sum_i p_i x.a_i$$

because $g_i^\sharp(x) = 0$ if $x = a_i$ and $+\infty$ otherwise. Hence

$$\mathbf{V}(t,x,p) = Vex(W)(t,x,p) = (T-t)Vex(h)(p) + \sum_i p_i x.a_i$$

where $h(p) = H(\sum_i p_i a_i)$ and $Vex(h)$ is the convex hull of h with respect to $p \in \Delta(I)$. Note that

$$\mathbf{V}_t + H(D\mathbf{V}) = -Vex(h)(p) + h(p) \leq 0 \ ,$$

with a strict inequality in general. In particular, $\mathbf{V}$ is not a supersolution of the primal HJ equation in general.

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