Differential games with asymmetric information

P. Cardaliaguet *

March 16, 2006

Abstract

We investigate a two-player zero-sum differential game in which the players have an asymmetric information on the random terminal payoff. We prove that the game has a value and characterize this value in terms of *dual* solutions of some Hamilton-Jacobi equation. We also explain how to adapt the results to differential games where the initial position is random.

1 Introduction

In this paper we investigate a two-player zero-sum differential game in which the player have an asymmetric information on the random terminal payoff. The dynamics of the game is given by

$$\begin{cases} x'(t) = f(x(t), u(t), v(t)) , & u(t) \in U,\ v(t) \in V \\ x(t_0) = x_0 \end{cases} \tag{1}$$

where U and V are compact subsets of some finite dimensional spaces, and $f : I\!\!R^N \times U \times V \to I\!\!R^N$ is Lipschitz continuous. We consider a finite horizon problem with a terminal time denoted by T. The game starts at time $t_0 \in [0, T]$ from the initial position x_0.

The description of the game involves $I \times J$ terminal payoffs (where $I, J \geq 1$): $g_{ij} : I\!\!R^N \to I\!\!R$ for $i = 1, \dots I$ and $j = 1, \dots, J$, a probability $p = (p_i)_{i=1,\dots,I}$ belonging to the set $\Delta(I)$ of probabilities on $\{1, \dots, I\}$ and a probability $q = (q_j)_{j=1,\dots,J}$ of the set $\Delta(J)$ of probabilities on $\{1, \dots, J\}$.

The game is played in two steps: at time t_0, a pair (i, j) is chosen at random among $\{1, \dots, I\} \times \{1, \dots, J\}$ according to the probability $p \otimes q$;

*Université de Bretagne Occidentale, Laboratoire de Mathématiques, UMR 6205, 6 Av. Le Gorgeu, BP 809, 29285 Brest

the choice of i is communicated to Player I only, while the choice of j is communicated to Player II only.

Then the players control system (1) in order, for Player I, to minimize the terminal payoff $g_{ij}(x(T))$, and for Player II to maximize it. We assume that both players observe their opponent's control. Note however that the players do not know which g_{ij} they are actually optimizing, because they only have a part of the information on the pair (i, j). They can nevertheless try to guess their missing information by observing what their oponent is doing. Indeed, in order to use his information a player necessarily reveals at least a part of it, and any piece of information he reveals can be later exploited by his oponent.

As usual we introduce two value functions associated to this game. We have here to take special care of the way we define the strategies of the players, since this definition has to represent the lack of symmetry in the knowledge of the players.

The upper-value is given by

$$V^+(t_0, x_0, p, q) = \inf_{(\alpha_i)\in(\mathcal{A}_r(t_0))^I} \sup_{(\beta_j)\in(\mathcal{B}_r(t_0))^J} \sum_{i=1}^{I}\sum_{j=1}^{J} p_i q_j \mathbf{E}_{\alpha_i\beta_j}\left(g_{ij}\left(X_T^{t_0,x_0,\alpha_i,\beta_j}\right)\right) ,$$

where the $\alpha_i \in \mathcal{A}_r(t_0)$ (for $i = 1, \ldots, I$) are I random strategies for Player I, the $\beta_j \in \mathcal{B}_r(t_0)$ (for $j = 1, \ldots, J$) are J random strategy for Player II and $\mathbf{E}_{\alpha_i\beta_j}\left(g_{ij}\left(X_T^{t_0,x_0,\alpha_i,\beta_j}\right)\right)$ is the payoff associated with the pair of strategies (α_i, β_j) for the terminal payoff g_{ij}: these notions are explained in the next section. The key point in the definition is that Player I chooses his strategy α_i $(i = 1, \ldots, I)$ according to the value of the index i only, while Player II has a strategy (β_j) which only depends upon the index j. This reflects the asymmetry of information of the players. The sum $\sum_i \sum_j p_i q_j \ldots$ is the expectation of the payoff when the pair (i, j) is chosen according to the probability $p \otimes q$, where $p = (p_1, \ldots, p_I)$ and $q = (q_1, \ldots, q_J)$.

The lower-value is defined by the symmetric formula:

$$V^-(t_0, x_0, p, q) = \sup_{(\beta_j)\in(\mathcal{B}_r(t_0))^J} \inf_{(\alpha_i)\in(\mathcal{A}_r(t_0))^I} \sum_{i=1}^{I}\sum_{j=1}^{J} p_i q_j \mathbf{E}_{\alpha_i\beta_j}\left(g_{ij}\left(X_T^{t_0,x_0,\alpha_i,\beta_j}\right)\right) .$$

Obviously we have

$$V^-(t_0, x_0, p, q) \leq V^+(t_0, x_0, p, q)$$

for any $(t_0, x_0) \in [0, T] \times I\!R^N$, any probability $p \in \Delta(I)$ on $\{1, \ldots, I\}$ and any probability $q \in \Delta(J)$ on $\{1, \ldots, J\}$. Our aim is to show that the equality holds, i.e., that the game has a value, and to provide a PDE characterization of the value.

The game studied in this paper is strongly inspired by *repeated games* with lack of information on one side and on both sides introduced by Aumann and Maschler : see [2], [21] for a general presentation. Repeated games with lack of information on one side (i.e., $I = 1$ or $J = 1$) or on both sides (i.e., $I, J \geq 2$) have a value [2], [17], in the sense that the averaged $n-$stage games converge to a limit as $n \to +\infty$. This value can be characterized in terms of the value of the game without information. In this paper, we prove the existence of a value for differential games with lack of information on both sides. However, we show in the companion paper [10] that the characterization in terms of game without information does not hold. In that respect, our game is close to stochastic games with incomplete information, as studied in [20] for instance. Although it is known that stochastic games with lack of information on one side have a value when the game is controlled by the informed player only [20], the general case is still open.

There are several proofs of Aumann and Maschler's result. In order to show that our game has a value, we use a strategy of proof initiated by De Meyer in [12] and later developed in [13, 14, 16]. We first note that the maps $V^+ = V^+(t, x, p, q)$ and $V^- = V^-(t, x, p, q)$ are convex in p and concave in q (Lemma 3.2). This leads us to introduce, for a generic map $w : [0, T] \times I\!R^N \times \Delta(I) \times \Delta(J) \to I\!R$, the convex Fenchel conjugate w^* of w with respect to the variable p and its concave conjugate $w^\sharp$ with respect to q: $\forall (t, x, \hat{p}, q) \in [0, T] \times I\!R^N \times I\!R^I \times \Delta(J)$

$$w^*(t, x, \hat{p}, q) = \sup_{p \in \Delta(I)} p.\hat{p} - w(t, x, p, q)$$

and, $\forall (t, x, p, \hat{q}) \in [0, T] \times I\!R^N \times \Delta(I) \times I\!R^J$

$$w^\sharp(t, x, p, \hat{q}) = \inf_{q \in \Delta(J)} q.\hat{q} - w(t, x, p, q) \ .$$

Then the proof of the equality $V^+ = V^-$ runs as follows: we first check (Lemma 4.2) that V^{-*} satisfies a subdynamic programming principle and thus (Corollary 4.3) that $(t, x) \to V^{-*}(t, x, \hat{p}, q)$ is a viscosity subsolution of the (dual) Hamilon-Jacobi (HJ) equation

$$w_t + H^* (x, Dw) = 0 \text{ in } [0, T] \times I\!R^N \qquad (2)$$

for any $(\hat{p}, q) \in I\!\!R^I \times \Delta(J)$. The map H^* is defined through the standard Hamiltonian H of the game

$$H(x, \xi) = \inf_{u \in U} \sup_{v \in V} f(x, u, v).\xi = \sup_{v \in V} \inf_{u \in U} f(x, u, v).\xi$$

via the relation by $H^*(x, \xi) = -H(x, -\xi)$. Note that we assume that Isaacs' condition holds. We recall that the notion of viscosity solutions was introduced by Crandall-Lions in [11] and first used in the framework of differential games in [15] (see also [3], [4] for a general presentation). We also establish a symmetric result for $V^{+\sharp}$ (Corollary 4.4): for any $(p, \hat{q}) \in \Delta(I) \times I\!\!R^J$, the map $(t, x) \to V^{+\sharp}(t, x, p, \hat{q})$ is a viscosity supersolution of the same equation (2). A new comparison principle (Theorem 5.1) then implies that $V^+ \leq V^-$. Since inequality $V^+ \geq V^-$ is obvious, the game has a value: $V^+ = V^-$. We also have the following characterization of this value: $\mathbf{V} := V^+ = V^-$ is the unique Lipschitz continuous function which is convex in p, concave in q, such that $(t, x) \to \mathbf{V}^*(t, x, \hat{p}, q)$ is a subsolution of the HJ equation (2) while $(t, x) \to \mathbf{V}^\sharp(t, x, p, \hat{q})$ is a supersolution of (2). We call such a function the *dual solution* to the Hamilton-Jacobi equation

$$\left\{ \begin{array}{ll} w_t + H(x, Dw) = 0 & \text{in } [0, T) \times I\!\!R^N \\ w(T, x) = \sum_{ij} p_i q_j g_{ij}(x) & \text{in } I\!\!R^N \end{array} \right.$$

We discuss this terminology below.

We explain in section 6 how to adapt our approach to differential games with lack of information on the initial positions. As previously, the game is played in two steps. At time t_0, the initial position of the game is chosen at random among $I \times J$ possible initial positions x^0_{ij} according to a probability $p \otimes q$ where $p \in \Delta(I)$ and $q \in \Delta(J)$; the index i is communicated to Player I while the index j is communicated to Player II. Then the players control system (1) in order, for Player I, to minimize a terminal payoff $g(x(T))$, and, for Player II, to maximize it. The key assumption is that the players observe their opponent's behaviour, but not the state of the system $x(\cdot)$. We prove that this game has a value, which can be characterized as the unique dual solution of some HJ equation in $[0, T] \times I\!\!R^{NIJ}$.

Although there has been several attemps to formalize differential games with lack of information [5, 6, 7, 8], there are only very few papers in which a game is proved to have a value: see in particular [18] and [19], which discuss interesting examples. In [9] we consider a game with lack of information on the current position, but with symmetric information. To the best of our knowledge, our result is the first one showing the existence of a value for differential games with asymmetry in the information in a general setting.

The kind of characterization proposed in this paper for the value function (as dual solution of some Hamilton-Jacobi equations) is also new. It relies upon a new comparison principle (Theorem 5.1) stating the following: assume that w_1 and w_2 defined on $[0,T] \times I\!\!R^N \times \Delta(I) \times \Delta(J)$ are convex in p, concave in q, that $(t,x) \to w_1^\sharp(t,x,p,\hat{q})$ is a supersolution of the dual HJ equation (2) for any $(p,\hat{q})$ and that $(t,x) \to w_2^*(t,x,\hat{p},q)$ is a subsolution of this HJ equation for any $(\hat{p},q)$. If futhermore $w_1(T,x,p,q) \leq w_2(T,x,p,q)$ for any (x,p,q), then $w_1 \leq w_2$.

Note that the fonction w_2 for instance is a kind of supersolution for our problem. For this reason we call it a dual supersolution of the orginal HJ equation

$$w_t + H\left(x, Dw\right) = 0 \text{ in } [0,T] \times I\!\!R^N \tag{3}$$

and we see the HJ equation (2) as a dual one. Let us recall that, although the Fenchel conjugate of a supersolution of (3) is a subsolution of the dual equation (2) (see [1]), the converse does not hold in general. In fact we show through several examples in [10] that the value function $\mathbf{V} := V^\sharp = V^-$ of our game is *not* a solution of the original HJ equation (3), nor are its Fenchel conjugates $\mathbf{V}^*$ and $\mathbf{V}^\sharp$ solutions of the dual one (2). The particular structure of our problem leads us to replace the classical notion of sub- and supersolutions by a weaker one, involving families of sub and supersolutions in some dual spaces (see also Lemma 5.4 where an equivalent definition for dual subsolution is discussed).

We complete this introduction by describing the organization of the paper. In section 2, we introduce the main notations: in particular we explain the notions of random strategies and define the value functions of our game. Section 3 is mainly devoted to the proof of the convexity properties of the value functions. In section 4 we show that V^{-*} satisfies a subdynamic programming principle and the dual HJ equation, and give the corresponding results for $V^{+\sharp}$. Section 5 is devoted to the comparison principle and to the existence of a value. In the last section, we extend our results to differential games with lack of information on the initial position.

2 Definitions of the value functions and notations

Notations : Throughout the paper, $x.y$ denotes the scalar product in the space $I\!R^N$, $I\!R^I$ or $I\!R^J$ (depending on the context) and $|\cdot|$ the euclidean norm. The ball of center x and radius r will be denoted by $B_r(x)$. If E is a set, then $\mathbf{1}_E$ is the indicatrix function of E (equal to 1 is E and to 0 outside of E). The set $\Delta(I)$ is the set of probabilities mesures on $\{1,\ldots,I\}$, always identified with the simplex of $I\!R^I$:

$$p=(p_1,\ldots,p_I)\in\Delta(I) \quad \Leftrightarrow \quad \sum_{i=1}^I p_i = 1 \text{ and } p_i\geq 0 \text{ for } i=1,\ldots I\ .$$

The set $\Delta(J)$ of probability measures on $\{1,\ldots,J\}$ is defined symmetrically.

The dynamics of the game is given by:

$$\left\{\begin{array}{ll} x'(t)=f(x(t),u(t),v(t))\ , & u(t)\in U,\ v(t)\in V\\ x(t_0)=x_0 \end{array}\right. \tag{4}$$

Throughtout the paper we assume that

$$\left\{\begin{array}{rl} i) & U \text{ and } V \text{ are compact subsets of some finite dimensional spaces,}\\ ii) & f: I\!R^N\times U\times V\to I\!R^N \text{ is bounded, continuous, Lipschitz}\\ & \text{ continuous with respect to the } x \text{ variable,}\\ iii) & \text{for } i=1,\ldots,I \text{ and } j=1,\ldots,J,\ g_{ij}:I\!R^N\to I\!R \text{ is Lipschitz}\\ & \text{continuous and bounded.} \end{array}\right. \tag{5}$$

We also assume that Isaacs condition holds:

$$H(x,\xi):=\inf_{u\in U}\sup_{v\in V} f(x,u,v).\xi = \sup_{v\in V}\inf_{u\in U} f(x,u,v).\xi \tag{6}$$

for any $(x,\xi)\in I\!R^N\times I\!R^N$. We note that the Hamilton-Jacobi equation naturally associated with the dynamics is the so-called primal Hamilton-Jacobi equation

$$w_t + H(x,Dw)=0 \qquad \text{in } [0,T)\times I\!R^N \tag{7}$$

For any $t_0<t_1\leq T$, the set of open-loop controls for Player I on $[t_0,t_1]$ is defined by

$$\mathcal{U}(t_0,t_1)=\{u:[t_0,t_1]\to U \text{ Lebesgue measurable}\}\ .$$

If $t_1=T$, we simply set $\mathcal{U}(t_0):=\mathcal{U}(t_0,T)$. Open-loop controls on the interval $[t_0,t_1]$ for Player II are defined symmetrically and denoted by $\mathcal{V}(t_0,t_1)$ (and by $\mathcal{V}(t_0)$ if $t_1=T$).

If $u \in \mathcal{U}(t_0)$ and $t_0 \leq t_1 < t_2 \leq T$, we denote by $u_{|_{[t_1,t_2]}}$ the restriction of u to the interval $[t_1, t_2]$. We note that $u_{|_{[t_1,T]}}$ belongs to $\mathcal{U}(t_1)$.

For any $(u, v) \in \mathcal{U}(t_0) \times \mathcal{V}(t_0)$ and any initial position $x_0 \in I\!\!R^N$, we denote by $t \to X_t^{t_0,x_0,u,v}$ the solution to (4).

Next we introduce the notions of pure and mixed strategies. The definition of mixed strategies involves a set $\mathcal{S}$ of (non trivial) probability spaces, which has to be stable by finite product. To fix the ideas we choose from now on

$$\mathcal{S} = \{([0,1]^n, B([0,1]^n), \mathcal{L}^n), \text{ for some } \ n \in I\!\!N^*\} \ ,$$

where $B([0,1]^n)$ is the class of Borel sets and $\mathcal{L}^n$ is the Lebesgue measure on $I\!\!R^n$. As the reader can easily check, the results presented in this paper do not depend on this particular choice of $\mathcal{S}$.

Definition 2.1 (Pure and random strategies)
A pure strategy for Player I at time t_0 is a map $\alpha : \mathcal{V}(t_0) \to \mathcal{U}(t_0)$ which is nonanticipative with delay, i.e., there is some $\tau > 0$ such that, for any $v_1, v_2 \in \mathcal{V}(t_0)$, if $v_1 \equiv v_2$ a.e. on $[t_0, t]$ for some $t \in (t_0, T - \tau)$, then $\alpha(v_1) \equiv \alpha(v_2)$ a.e. on $[t_0, t + \tau]$.

A random strategy for Player I is a pair $((\Omega_\alpha, \mathcal{F}_\alpha, \mathbf{P}_\alpha), \alpha)$, where $(\Omega_\alpha, \mathcal{F}_\alpha, \mathbf{P}_\alpha)$ belongs to the set of probability spaces $\mathcal{S}$ and $\alpha : \Omega_\alpha \times \mathcal{V}(t_0) \to \mathcal{U}(t_0)$ satisfies

(i) α is measurable from $\Omega_\alpha \times \mathcal{V}(t_0)$ to $\mathcal{U}(t_0)$, with Ω_α endowed with the $\sigma-$field $\mathcal{F}_\alpha$ and $\mathcal{U}(t_0)$ and $\mathcal{V}(t_0)$ with the Borel $\sigma-$field associated with the L^1 distance,

(ii) there is some delay $\tau > 0$ such that, for any $v_1, v_2 \in \mathcal{V}(t_0)$, any $t \in (t_0, T - \tau)$ and any $\omega \in \Omega_\alpha$,

$$v_1 \equiv v_2 \text{ on } [t_0, t) \ \Rightarrow \ \alpha(\omega, v_1) \equiv \alpha(\omega, v_2) \text{ on } [t_0, t + \tau) \ .$$

We denote by $\mathcal{A}(t_0)$ the set of pure strategies and by $\mathcal{A}_r(t_0)$ the set of random strategies for Player I. By abuse of notations, an element of $\mathcal{A}_r(t_0)$ is simply noted α—instead of $((\Omega_\alpha, \mathcal{F}_\alpha, \mathbf{P}_\alpha), \alpha)$—, the underlying probability space being always denoted by $(\Omega_\alpha, \mathcal{F}_\alpha, \mathbf{P}_\alpha)$.

In order to take into account the fact that Player I knows the index i of the terminal payoff, a strategy for Player I is actually a $I-$upplet $\hat{\alpha} = (\alpha_1, \ldots, \alpha_I) \in (\mathcal{A}_r(t_0))^I$.

Pure and random strategies for Player II are defined symmetrically: at time t_0, a pure strategy β is a nonanticipative map with delay from $\mathcal{U}(t_0)$

to $\mathcal{V}(t_0)$, while a random strategy is a map $\beta : \Omega_\beta \times \mathcal{U}(t_0) \to \mathcal{V}(t_0)$, where $(\Omega_\beta, \mathcal{F}_\beta, \mathbf{P}_\beta)$ belongs to $\mathcal{S}$, which satisfies the conditions:

(i) β is measurable from $\Omega_\beta \times \mathcal{U}(t_0)$ to $\mathcal{V}(t_0)$,

(ii) there is some delay $\tau > 0$ such that, for any $u_1, u_2 \in \mathcal{U}(t_0)$, any $t \in (t_0, T-\tau)$ and any $\omega \in \Omega_\beta$,

$$u_1 \equiv u_2 \text{ on } [t_0, t) \ \Rightarrow\ \beta(\omega, u_1) \equiv \beta(\omega, u_2) \text{ on } [t_0, t+\tau)\ .$$

The set of pure and random strategies for Player II are denoted $\mathcal{B}(t_0)$ and $\mathcal{B}_r(t_0)$ respectively. Elements of $\mathcal{B}_r(t_0)$ are denoted simply by β, and the underlying probability space by $(\Omega_\beta, \mathcal{F}_\beta, \mathbf{P}_\beta)$.
Since Player II knows the index j of the terminal payoff, a strategy for Player II is a $J-$upplet $\hat{\beta} = (\beta_1, \dots, \beta_J) \in (\mathcal{B}_r(t_0))^J$.

Lemma 2.2 *For any pair $(\alpha, \beta) \in \mathcal{A}_r(t_0) \times \mathcal{B}_r(t_0)$ and any $\omega := (\omega_1, \omega_2) \in \Omega_\alpha \times \Omega_\beta$, there is a unique pair $(u_\omega, v_\omega) \in \mathcal{U}(t_0) \times \mathcal{V}(t_0)$, such that*

$$\alpha(\omega_1, v_\omega) = u_\omega \text{ and } \beta(\omega_2, u_\omega) = v_\omega\ . \tag{8}$$

Furthermore the map $\omega \to (u_\omega, v_\omega)$ is measurable from $\Omega_\alpha \times \Omega_\beta$ endowed with $\mathcal{F}_\alpha \otimes \mathcal{F}_\beta$ into $\mathcal{U}(t_0) \times \mathcal{V}(t_0)$ endowed with the Borel $\sigma-$field associated with the L^1 distance.

Notations : Given any pair $(\alpha, \beta) \in \mathcal{A}_r(t_0) \times \mathcal{B}_r(t_0)$, we denote by $(X_t^{t_0,x_0,\alpha,\beta})$ the map $(t,\omega) \to (X_t^{t_0,x_0,u_\omega,v_\omega})$ defined on $[t_0,T]\times\Omega_\alpha\times\Omega_\beta$, where (u_ω, v_ω) satisfies (8). We also define the expectation $\mathbf{E}_{\alpha\beta}$ as the integral over $\Omega_\alpha \times \Omega_\beta$ against the probability measure $\mathbf{P}_\alpha \otimes \mathbf{P}_\beta$. In particular, if $\phi : I\!\!R^N \to I\!\!R$ is some bounded continuous map and $t \in (t_0, T]$, we have

$$\mathbf{E}_{\alpha\beta}\left(\phi\left(X_t^{t_0,x_0,\alpha,\beta}\right)\right) := \int_{\Omega_\alpha\times\Omega_\beta} \phi\left(X_t^{t_0,x_0,u_\omega,v_\omega}\right) d\mathbf{P}_\alpha \otimes \mathbf{P}_\beta(\omega)\ , \tag{9}$$

where (u_ω, v_ω) is defined by (8). Note that (9) makes sense because the map $(u,v) \to X_t^{t_0,x_0,u,v}$ being continuous in L^1, the map $\omega \to \phi\left(X_t^{t_0,x_0,u_\omega,v_\omega}\right)$ is measurable in $\Omega_\alpha\times\Omega_\beta$ and bounded. If either α or β is a pure strategy, then we simply drop α or β in the expectation $\mathbf{E}_{\alpha\beta}$, which then becomes $\mathbf{E}_\beta$ or $\mathbf{E}_\alpha$.

Proof of Lemma 2.2 : The existence of (u_ω, v_ω) is proved in [9]. We only show here the measurability of $\omega \to (u_\omega, v_\omega)$. For this we argue by induction by proving that $\omega \to (u_\omega, v_\omega)_{|_{[t_0,t_0+n\tau]}}$ from $\Omega_\alpha \times \Omega_\beta$ into $L^1([t_0, t_0 + n\tau])$ is measurable, where τ a the minimum of the delays for α and β (see condition (ii) in Definition 2.1).

Let us start with $n = 1$. It is enough to show that, for any Borel subsets B_1 and B_2 of $\mathcal{U}(t_0, t_0 + \tau)$ and $\mathcal{V}(t_0, t_0 + \tau)$, the set

$$\Omega := \{\omega \in \Omega_\alpha \times \Omega_\beta \mid (u_\omega, v_\omega)_{|_{[t_0,t_0+\tau)}} \in B_1 \times B_2\}$$

is measurable. Let us fix $\hat{u}$ and $\hat{v}$ in $\mathcal{U}(t_0)$ and $\mathcal{V}(t_0)$. Since $\alpha(\omega_1, \cdot)$ and $\beta(\omega_2, \cdot)$ are nonanticipative with delay τ, the restrictions of $\alpha(\omega_1, \hat{v})$ and $\beta(\omega_2, \hat{u})$ to $[t_0, t_0 + \tau]$ do not depend on $\hat{u}$ and $\hat{v}$. Hence $(u_\omega, v_\omega) \equiv (\alpha(\omega, \hat{v}), \beta(\omega, \hat{u}))$ a.e. in $[t_0, t_0 + \tau)$. Therefore

$$\Omega = \{\omega_1 \in \Omega_\alpha \mid \alpha(\omega_1, \hat{v})_{|_{[t_0,t_0+\tau)}} \in B_1\} \times \{\omega_2 \in \Omega_\beta \mid \beta(\omega_2, \hat{u})_{|_{[t_0,t_0+\tau)}} \in B_2\} \ ,$$

which is measurable since α and β are measurable. So the result holds true for $n = 1$.

Let us now assume that $\omega \to (u_\omega, v_\omega)_{|_{[t_0,t_0+n\tau]}}$ from $\Omega_\alpha \times \Omega_\beta$ into $L^1([t_0, t_0 + n\tau])$ is measurable, and let us show that this still holds true for $n + 1$. It is again enough to show that, for any Borel subsets B_1 and B_2 of $\mathcal{U}(t_0, t_0 + (n+1)\tau)$ and $\mathcal{V}(t_0, t_0 + (n+1)\tau)$, the set

$$\Omega := \{\omega \in \Omega_\alpha \times \Omega_\beta \mid (u_\omega, v_\omega)_{|_{[t_0,t_0+(n+1)\tau)}} \in B_1 \times B_2\}$$

is measurable. Let us fix again $\hat{u}$ and $\hat{v}$ in $\mathcal{U}(t_0)$ and $\mathcal{V}(t_0)$. For any $(u, v) \in \mathcal{U}(t_0, t_0+n\tau) \times \mathcal{V}(t_0, t_0+n\tau)$, we denote by $\tilde{u}$ and $\tilde{v}$ the maps equal to u and v on $[t_0, t_0 + n\tau]$ and to $\hat{u}$ and $\hat{v}$ on $[t_0 + n\tau, T]$. Note that $(u, v) \to (\tilde{u}, \tilde{v})$ is continuous from L^1 to L^1. Since α and β are nonanticipative with delay τ, $u_\omega \equiv \alpha(\omega_1, \widetilde{v_\omega})$ on $[t_0, t_0 + (n+1)\tau)$ and $v_\omega \equiv \beta(\omega_1, \widetilde{u_\omega})$ on $[t_0, t_0 + (n+1)\tau)$. Therefore Ω is the preimage of the set $B_1 \times B_2$ by the map $\omega \to (\alpha(\omega_1, \widetilde{v_\omega}), \beta(\omega_2, \widetilde{u_\omega}))$ which is measurable as the composition of the mesurable maps $\omega \to (u_\omega, v_\omega)_{|_{[t_0,t_0+n\tau]}}$, the map $(u, v) \to (\tilde{u}, \tilde{v})$ and the maps α and β. Hence Ω is measurable, and the result is proved.

QED

We now define the payoff associated with a strategy $\hat{\alpha}$ of Player I and a strategy $\hat{\beta}$ of Player II:

Definition of the payoff: Let $(p, q) \in \Delta(I) \times \Delta(J)$, $(t_0, x_0) \in [0, T) \times I\!\!R^N$, $\hat{\alpha} = (\alpha_i)_{i=1,\dots,I} \in (\mathcal{A}_r(t_0))^I$ and $\hat{\beta} = (\beta_j) \in (\mathcal{B}_r(t_0))^J$. We set

$$\mathcal{J}(t_0, x_0, \hat{\alpha}, \hat{\beta}, p, q) = \sum_{i=1}^{I} \sum_{j=1}^{J} p_i q_j \mathbf{E}_{\alpha_i \beta_j} \left(g_{ij} \left(X_T^{t_0,x_0,\alpha_i,\beta_j} \right) \right) \ , \qquad (10)$$

where $\mathbf{E}_{\alpha_i\beta_j}$ is defined by (9). Note that $\hat{\alpha}$ does not depend on j, while $\hat{\beta}$ does not depend on i, which formalizes the asymmetry of information.

Definition of the value functions: The upper value function is given by

$$V^+(t_0, x_0, p, q) = \inf_{\hat{\alpha}\in(\mathcal{A}_r(t_0))^I} \sup_{\hat{\beta}\in(\mathcal{B}_r(t_0))^J} \mathcal{J}(t_0, x_0, \hat{\alpha}, \hat{\beta}, p, q)$$

while the lower value function is defined by

$$V^-(t_0, x_0, p, q) = \sup_{\hat{\beta}\in(\mathcal{B}_r(t_0))^J} \inf_{\hat{\alpha}\in(\mathcal{A}_r(t_0))^I} \mathcal{J}(t_0, x_0, \hat{\alpha}, \hat{\beta}, p, q) .$$

Let us underline that, because of the special form of the payoff, the value functions defined above *cannot* be recasted in terms of usual value functions of a zero-sum differential game with perfect information. For instance they do not satisfy the standard dynamic programming principle, as we show in the companion paper [10].

3 Convexity properties of the value functions

The main result of this section is Lemma 3.2 which states that the value functions are convex in p and concave in q. We also investigate some regularity properties of the value functions.

Lemma 3.1 (Regularity of V^+ and V^-)
Under assumption (5), V^+ and V^- are Lipschitz continuous.

Proof : We first note that the Lipschitz continuity of V^- and V^+ with respect to p and q just comes from the boundness of the g_{ij}. Using standard arguments, one easily shows that, for any $t_0 \in [0, T]$, $(u, v) \in \mathcal{U}(t_0) \times \mathcal{V}(t_0)$, the map

$$x \to g_{ij}\left(X_T^{t_0,x,u,v}\right)$$

is Lipschitz continuous with a Lipschitz constant independant of $t_0 \in [0, T]$. Hence for any pair of strategies $(\hat{\alpha}, \hat{\beta}) \in (\mathcal{A}_r(t_0))^I \times (\mathcal{B}_r(t_0))^J$ the map

$$x \to \mathcal{J}(t, x, \hat{\alpha}, \hat{\beta}, p, q) = \sum_{i=1}^{I}\sum_{j=1}^{J} p_i q_j \mathbf{E}_{\alpha_i\beta_j}\left(g_{ij}(X_T^{t_0,x,\alpha_i,\beta_j})\right)$$

is $C-$Lipschitz continuous for some constant C independant of $t \in [0, T]$, of $p \in \Sigma(I)$ and of $q \in \Delta(J)$. From this one easily deduces that V^+ and V^-

are $C-$Lipschitz continuous with respect to x (see for instance [15]).

We now consider the time regularity of V^- and V^+. We only do the proof for V^-, since the case of V^+ can be treated similarly. Let $x_0 \in I\!\!R^N$, $(p,q) \in \Delta(I) \times \Delta(J)$ and $t_0 < t_1 < T$ be fixed. Let $\hat{\beta} = (\beta_j) \in (\mathcal{B}_r(t_0))^J$ be $\epsilon-$optimal for $V^-(t_0, x_0, p, q)$ and $\alpha \in \mathcal{A}_r(t_1)$. Let us define, for any $j = 1, \ldots, J$, $\tilde{\beta}_j \in \mathcal{B}_r(t_1)$ and $\alpha' \in \mathcal{A}_r(t_0)$ by setting (for some $\bar{u} \in U$ fixed)

$$\tilde{\beta}_j(\omega, u) = \beta_j(\omega, \tilde{u}) \text{ where } \tilde{u}(t) = \begin{cases} \bar{u} & \text{if } t \in [t_0, t_1) \\ u & \text{otherwise} \end{cases}$$

for any $\omega \in \Omega_{\tilde{\beta}_j} := \Omega_{\beta_j}$ and $u \in \mathcal{U}(t_1)$, and

$$\alpha'(\omega, v) = \begin{cases} \bar{u} & \text{if } t \in [t_0, t_1) \\ \alpha(\omega, v_{|_{[t_1,T]}}) & \text{otherwise} \end{cases} \qquad \forall \omega \in \Omega_{\alpha'} := \Omega_\alpha, \ \forall v \in \mathcal{V}(t_0)\ .$$

We note that, for any $\alpha \in \mathcal{A}_r(t_1)$ and $j = 1, \ldots, J$, we have

$$\left| X_t^{t_0, x_0, \alpha', \beta_j} - X_t^{t_1, x_0, \alpha, \tilde{\beta}_j} \right| \leq M|t_0 - t_1| e^{L(t - t_1)} \qquad \forall t \geq t_1\ ,$$

(where $M = \|f\|_\infty$ and f is $L-$Lipschitz continuous) because the pair (u_ω, v_ω) satisfying

$$\alpha'(\omega_1, v_\omega) = u_\omega \text{ and } \beta_j(\omega_2, u_\omega) = v_\omega$$

is given by $u_\omega = \bar{u}$ and $v_\omega = \beta_j(\omega_2, \bar{u})$ on $[t_0, t_1]$ and coincides on $[t_1, T]$ with the pair (u'_ω, v'_ω) satisfying

$$\alpha(\omega_1, v'_\omega) = u'_\omega \text{ and } \tilde{\beta}_j(\omega_2, u'_\omega) = v'_\omega \text{ on } [t_1, T]\ .$$

Therefore, for any $\hat{\alpha} = (\alpha_i) \in (\mathcal{A}_r(t_1))^I$, we have

$$\begin{aligned} &\mathcal{J}(t_1, x_0, \hat{\alpha}, (\tilde{\beta}_j), p, q) \\ &\qquad \geq\ \mathcal{J}(t_0, x_0, \hat{\alpha}', \hat{\beta}, p, q) - LM|t_0 - t_1|e^{L(T-t_1)} \\ &\qquad \geq\ \inf_{\hat{\alpha}" \in (\mathcal{A}_r(t_0))^I} \mathcal{J}(t_0, x_0, \hat{\alpha}", \hat{\beta}, p, q) - LM|t_0 - t_1|e^{L(T-t_1)} \\ &\qquad \geq\ V^-(t_0, x_0, p, q) - \epsilon - LM|t_0 - t_1|e^{L(T-t_1)} \end{aligned}$$

(where L is also a Lipchitz constant for the g_i), because $\hat{\beta}$ is $\epsilon-$optimal for $V^-(t_0, x_0, p, q)$. Since this holds for any $\hat{\alpha} = (\alpha_i) \in (\mathcal{A}_r(t_1))^I$ and any $\epsilon > 0$, we get

$$V^-(t_1, x_0, p, q) \geq V^-(t_0, x_0, p, q) - LM|t_0 - t_1|e^{L(T-t_1)}\ .$$

The reverse inequality can be proved in a similar way: we choose some $\epsilon-$optimal strategy $\hat{\beta} = (\beta_j) \in (\mathcal{B}_r(t_1))^J$ for $V^-(t_1, x_0, p, q)$ and we extend it to a strategy $(\tilde{\beta}_j) \in (\mathcal{B}_r(t_0))^J$ by setting (for some $\bar{v} \in V$ fixed)

$$\tilde{\beta}_j(\omega, u) = \begin{cases} \bar{v} & \text{if } t \in [t_0, t_1) \\ \beta_j(\omega, u_{|_{[t_1,T]}}) & \text{otherwise} \end{cases} \qquad \forall \omega \in \Omega_{\tilde{\beta}_j} := \Omega_{\beta_j},\ \forall u \in \mathcal{U}(t_0)\ .$$

Then similar estimates as above show that, for any $\hat{\alpha} \in (\mathcal{A}_r(t_0))^I$ we have

$$\mathcal{J}(t_0, x_0, \hat{\alpha}, (\tilde{\beta}_j), p, q) \geq V^-(t_1, x_0, p, q) - \epsilon - LM|t_0 - t_1|e^{L(T-t_1)}$$

from the $\epsilon-$optimality of $\hat{\beta}$ for $V^-(t_1, x_0, p, q)$. Then we get

$$V^-(t_0, x_0, p, q) \geq V^-(t_1, x_0, p, q) - LM|t_0 - t_1|e^{L(T-t_1)}\ .$$

QED

Lemma 3.2 (Convexity properties of V^- and V^+)
For any $(t, x) \in [0, T) \times I\!R^N$, the maps $V^+ = V^+(t, x, p, q)$ and $V^- = V^-(t, x, p, q)$ are convex in p and concave in q on $\Delta(I)$ and $\Delta(J)$ respectively.

Remark : This result is well-known for repeated games with lack of information. The procedure we use in the proof is usually called "splitting": see [21] for instance.

Proof of Lemma 3.2: We only do the proof for V^+, the proof for V^- can be achieved by reversing the roles of the players. One first easily checks that

$$V^+(t_0, x_0, p, q) = \inf_{(\alpha_i) \in (\mathcal{A}(t_0))^I} \sum_{j=1}^{J} q_j \sup_{\beta \in \mathcal{B}_r(t_0)} \left[\sum_{i=1}^{I} p_i \mathbf{E}_{\alpha_i \beta} \left(g \left(X_T^{t_0, x_0, \alpha_i, \beta} \right) \right) \right]\ .$$

Hence $q \to V^+(t, x, p, q)$ is concave for any (t, x, p).

We now prove the convexity of V^+ with respect to p. Let $(t, x, q) \in [0, T) \times I\!R^N \times \Delta(J)$, $p^0, p^1 \in \Delta(I)$, $\lambda \in (0, 1)$ and $\hat{\alpha}^0 = (\alpha_i^0) \in (\mathcal{A}_r(t))^I$ and $\hat{\alpha}^1 = (\alpha_i^1) \in (\mathcal{B}_r(t))^I$ be $\epsilon-$optimal for $V^+(t, x, p^0, q)$ and $V^+(t, x, p^1, q)$ respectively ($\epsilon > 0$). Let us set $p^\lambda = (1 - \lambda)p^0 + \lambda p^1$. We can assume without loss of generality that $p_i^\lambda \neq 0$ for any i (because $p_i^\lambda = 0$ implies that $p_i^0 = p_i^1 = 0$, so that this index i plays no role in our computation). We now define the strategy $\hat{\alpha}^\lambda = (\alpha_i^\lambda) \in (\mathcal{A}_r(t))^I$ by setting

$$\Omega_{\alpha_i^\lambda} = [0, 1] \times \Omega_{\alpha_i^0} \times \Omega_{\alpha_i^1},\ \mathcal{F}_{\alpha_i^\lambda} = B([0, 1]) \otimes \mathcal{F}_{\alpha_i^0} \otimes \mathcal{F}_{\alpha_i^1},\ \mathbf{P}_{\alpha_i^\lambda} = \mathcal{L}^1 \otimes \mathbf{P}_{\alpha_i^0} \otimes \mathbf{P}_{\alpha_i^1}\ ,$$

(where $B([0,1])$ is the Borel $\sigma-$field and $\mathcal{L}^1$ the Lebesgue measure on $[0,1]$) and

$$\alpha_i^\lambda(\omega_1,\omega_2,\omega_3,v)=\begin{cases}\alpha_i^0(\omega_2,v) & \text{if } \omega_1\in[0,\frac{(1-\lambda)p_i^0}{p_i^\lambda})\\ \alpha_i^1(\omega_3,v) & \text{if } \omega_1\in[\frac{(1-\lambda)p_i^0}{p_i^\lambda},1]\end{cases}$$

for any $(\omega_1,\omega_2,\omega_3)\in\Omega_{\alpha_i^\lambda}$ and $v\in\mathcal{V}(t)$. We note that $(\Omega_{\alpha_i^\lambda},\mathcal{F}_{\alpha_i^\lambda},\mathbf{P}_{\alpha_i^\lambda})$ belongs to the set of probability spaces $\mathcal{S}$ and that α_i^λ belongs to $\mathcal{A}_r(t_0)$ for any $i=1,\dots,I$.

The interpretation of the strategy $\hat\alpha^\lambda$ is the following: if the index i is choosen according to the probability p^λ, then Player I chooses α_i^0 with probability $\frac{(1-\lambda)p_i^0}{p_i^\lambda}$ and α_i^1 with probability $1-\frac{(1-\lambda)p_i^0}{p_i^\lambda}=\frac{\lambda p_i^1}{p_i^\lambda}$. Hence the probability for the strategy α_i^0 to be chosen is $p_i^\lambda\frac{(1-\lambda)p_i^0}{p_i^\lambda}=(1-\lambda)p_i^0$, while the strategy α_i^1 appears with probability $p_i^\lambda\frac{\lambda p_i^1}{p_i^\lambda}=\lambda p_i^1$. Therefore

$$\begin{aligned}
&\sup_{\hat\beta}\mathcal{J}(t,x,\hat\alpha^\lambda,\hat\beta)=\textstyle\sum_j q_j\sup_\beta\sum_i p_i^\lambda\mathbf{E}_{\alpha_i^\lambda,\beta}\left(g_{ij}(X_T^{t,x,\alpha_i^\lambda,\beta})\right)\\
&= \textstyle\sum_j q_j\sup_\beta\sum_i p_i^\lambda\left[\frac{(1-\lambda)p_i^0}{p_i^\lambda}\mathbf{E}_{\alpha_i^0,\beta}\left(g_{ij}(X_T^{t,x,\alpha_i^0,\beta})\right)+\frac{\lambda p_i^1}{p_i^\lambda}\mathbf{E}_{\alpha_i^1,\beta}\left(g_{ij}(X_T^{t,x,\alpha_i^1,\beta})\right)\right]\\
&\le (1-\lambda)\textstyle\sum_j q_j\sup_\beta\sum_i p_i^0\mathbf{E}_{\alpha_i^0,\beta}\left(g_{ij}(X_T^{t,x,\alpha_i^0,\beta})\right)+\\
&\qquad\qquad \lambda\textstyle\sum_j q_j\sup_\beta\sum_i p_i^1\mathbf{E}_{\alpha_i^1,\beta}\left(g_{ij}(X_T^{t,x,\alpha_i^1,\beta})\right)\\
&\le (1-\lambda)V^+(t,x,p^0,q)+\lambda V^+(t,x,p^1,q)+\epsilon
\end{aligned}$$

because $\hat\alpha^0$ and $\hat\alpha^1$ are $\epsilon-$ optimal for $V^+(t,x,p^0,q)$ and $V^+(t,x,p^1,q)$ respectively. Therefore

$$V^+(t,x,p^\lambda,q)\le\sup_{\hat\beta}\mathcal{J}(t,x,\hat\alpha^\lambda,\hat\beta)\le(1-\lambda)V^+(t,x,p^0)+\lambda V^+(t,x,p^1)+\epsilon\ ,$$

which proves the desired claim because ϵ is arbitrary.

QED

The convexity properties of the value functions leads naturally to consider their Fenchel conjugates. Let $w:[0,T]\times I\!\!R^N\times\Delta(I)\times\Delta(J)\to I\!\!R$ be some function. We denote by w^* its convex conjugate with respect to variable p:

$$w^*(t,x,\hat p,q)=\sup_{p\in\Delta(I)}\hat p.p-w(t,x,p,q)\qquad\forall(t,x,\hat p,q)\in[0,T]\times I\!\!R^N\times I\!\!R^I\times\Delta(J)\ .$$

For instance V^{-*} and V^{+*} denote the convex conjugate with respect to the $p-$variable of the functions V^- and V^+.

For a function $w = w(t,x,\hat{p},q)$ defined on the dual space $[0,T]\times I\!\!R^N \times I\!\!R^I \times \Delta(J)$ we also denote by w^* its convex conjugate with respect to $\hat{p}$ defined on $[0,T]\times I\!\!R^N \times \Delta(I)\times\Delta(J)$:

$$w^*(t,x,p,q) = \sup_{\hat{p}\in R^I} p.\hat{p} - w(t,x,p,q) \qquad \forall (t,x,p,q)\in[0,T]\times I\!\!R^N\times\Delta(I)\times\Delta(J)\ .$$

In a symmetric way, we denote by $w^\sharp = w^\sharp(t,x,p,\hat{q})$ the concave conjugate with respect to q of w:

$$w^\sharp(t,x,p,\hat{q}) = \inf_{q\in\Delta(J)} \hat{q}.q - w(t,x,p,q) \qquad \forall (t,x,p,\hat{q})\in[0,T]\times I\!\!R^N\times\Delta(I)\times I\!\!R^J\ .$$

4 The subdynamic programming

The main result of this section is that $V^{+\sharp}$ and V^{-*} are subsolution of the dual HJ equation. To fix the ideas, we study here the case of V^{-*}, and explain at the very end of the section how we deduce the symmetric results for $V^{+\sharp}$.

Lemma 4.1 (Reformulation of V^{-*})
We have

$$V^{-*}(t,x,\hat{p},q) = \inf_{(\beta_j)\in(\mathcal{B}_r(t_0))^J} \sup_{\alpha\in\mathcal{A}_r(t_0)} \max_{i\in\{1,\ldots,I\}} \left\{ \hat{p}_i - \sum_{j=1}^J q_j \mathbf{E}_{\alpha\beta_j}\left(g_{ij}(X_T^{t,x,\alpha,\beta_j})\right)\right\}\ . \tag{11}$$

Proof of Lemma 4.1: Let us denote by $z = z(t,x,\hat{p},q)$ the right-hand side of the equality. We first claim that

$$z \text{ is convex with respect to } p. \tag{12}$$

Proof of (12): The proof mimics the proof of the convexity of V^+. Let $(t,x,q)\in[0,T)\times I\!\!R^N\times\Delta(J)$, $\hat{p}^0,\hat{p}^1\in I\!\!R^I$, $\lambda\in(0,1)$ and $(\beta_j^0)\in(\mathcal{B}_r(t))^J$ and $(\beta_j^1)\in(\mathcal{B}_r(t))^J$ be $\epsilon-$optimal for $z(t,x,\hat{p}^0,q)$ and $z(t,x,\hat{p}^1,q)$ respectively ($\epsilon>0$). Let us set $\hat{p}^\lambda = (1-\lambda)\hat{p}^0+\lambda\hat{p}^1$. We define the strategies $\beta_j^\lambda\in\mathcal{B}_r(t)$ by setting

$$\Omega_{\beta_j^\lambda} = [0,1]\times\Omega_{\beta_j^0}\times\Omega_{\beta_j^1},\ \mathcal{F}_{\beta_j^\lambda} = B([0,1])\otimes\mathcal{F}_{\beta_j^0}\otimes\mathcal{F}_{\beta_j^1},\ \mathbf{P}_{\beta_j^\lambda} = \mathcal{L}^1\otimes\mathbf{P}_{\beta_j^0}\otimes\mathbf{P}_{\beta_j^1}\ ,$$

and

$$\beta_j^\lambda(\omega_1,\omega_2,\omega_3,u) = \left\{ \begin{array}{ll} \beta_j^0(\omega_2,u) & \text{if } \omega_1\in[0,(1-\lambda)) \\ \beta_j^1(\omega_3,u) & \text{if } \omega_1\in[(1-\lambda),1] \end{array}\right.$$

for any $(\omega_1,\omega_2,\omega_3)\in\Omega_{\beta_j^\lambda}$ and $u\in\mathcal{U}(t)$. Then $(\Omega_{\beta_j^\lambda},\mathcal{F}_{\beta_j^\lambda},\mathbf{P}_{\beta_j^\lambda})$ belongs to $\mathcal{S}$ and $(\beta_j^\lambda)\in(\mathcal{B}_r(t_0))^J$. For any $\alpha\in\mathcal{A}_r(t)$, we have by using the convexity of the map $(s_i)\to\max_i\{s_i\}$:

$$\begin{array}{rl} \multicolumn{2}{l}{\max_i\left\{\hat{p}_i^\lambda-\sum_j q_j\mathbf{E}_{\alpha,\beta_j^\lambda}\left(g_{ij}(X_T^{t,x,\alpha,\beta_j^\lambda})\right)\right\}} \\ = & \max_i\left\{(1-\lambda)(\hat{p}_i^0-\sum_j q_j\mathbf{E}_{\alpha\beta_j^0}\left(g_{ij}(X_T^{t,x,\alpha,\beta_j^0})\right)\right. \\ & \qquad\qquad \left. +\ \lambda(\hat{p}_i^1-\sum_j q_j\mathbf{E}_{\alpha\beta_j^1}\left(g_{ij}(X_T^{t,x,\alpha,\beta_j^1})\right)\right\} \\ \le & (1-\lambda)\sup_\alpha\max_i\left\{\hat{p}_i^0-\sum_j q_j\mathbf{E}_{\alpha\beta_j^0}\left(g_{ij}(X_T^{t,x,\alpha,\beta_j^0})\right)\right\} \\ & \qquad\qquad +\lambda\sup_\alpha\max_i\left\{\hat{p}_i^1-\sum_j q_j\mathbf{E}_{\alpha\beta_j^1}\left(g_{ij}(X_T^{t,x,\alpha,\beta_j^1})\right)\right\} \\ \le & (1-\lambda)z(t,x,\hat{p}^0,q)+\lambda z(t,x,\hat{p}^1,q)+\epsilon \end{array}$$

because β^0 and β^1 are $\epsilon-$optimal for $z(t,x,\hat{p}^0,q)$ and $z(t,x,\hat{p}^1,q)$ respectively. Hence

$$\begin{array}{l} z(t,x,\hat{p}^\lambda,q) \\ \qquad \le\ \sup_\alpha\max_i\left\{\hat{p}_i^\lambda-\sum_j q_j\mathbf{E}_{\alpha,\beta_j^\lambda}\left(g_{ij}(X_T^{t,x,\alpha,\beta_j^\lambda})\right)\right\} \\ \qquad \le\ (1-\lambda)z(t,x,q^0)+\lambda z(t,x,q^1)+\epsilon\ , \end{array}$$

which proves the desired claim because ϵ is arbitrary.

Next we show that $V^{-*}=z$. Indeed we have by definition of z:

$$\begin{array}{l} z^*(t,x,p,q) \\ = \displaystyle\sup_{\hat{p}} p.\hat{p}-\inf_{(\beta_j)}\max_i\left\{\hat{p}_i-\inf_\alpha\sum_j q_j\mathbf{E}_{\alpha\beta_j}\left(g_{ij}(X_T^{t,x,\alpha,\beta_j})\right)\right\} \\ = \displaystyle\sup_{(\beta_j)}\sup_{\hat{p}}\min_i\left\{p.\hat{p}-\hat{p}_i+\inf_\alpha\sum_j q_j\mathbf{E}_{\alpha\beta_j}\left(g_{ij}(X_T^{t,x,\alpha,\beta_j})\right)\right\} \end{array}$$

In this last expression, the $\sup_{\hat{p}}$ is attained by

$$\hat{p}_i=\inf_\alpha\sum_j q_j\mathbf{E}_{\alpha\beta_j}\left(g_{ij}(X_T^{t,x,\alpha,\beta_j}\right)\ ,$$

for which all the arguments of the $\min_i$ are equal. Hence

$$\begin{aligned} z^*(t,x,p,q) &= \sup_{\beta_j} \sum_i p_i \inf_\alpha \sum_j q_j \mathbf{E}_{\alpha\beta_j}\left(g_{ij}(X_T^{t,x,\alpha,\beta_j})\right) \\ &= V^-(t,x,p,q) \ . \end{aligned}$$

Since we have proved that z is convex with respect to $\hat{p}$, we get by duality $V^{-*} = z^{**} = z$.

QED

Lemma 4.2 (Sub-dynamic principle for V^{-*})
We have for any $(t_0, x_0, \hat{p}, q) \in [0,T) \times I\!R^N \times I\!R^I \times \Delta(J)$ and any $t_1 \in (t_0, T]$,

$$V^{-*}(t_0, x_0, \hat{p}, q) \leq \inf_{\beta \in \mathcal{B}(t_0)} \sup_{\alpha \in \mathcal{A}(t_0)} V^{-*}(t_1, X_{t_1}^{t_0,x_0,\alpha,\beta}, \hat{p}, q) \ .$$

Proof : Let us denote by $V_1^{-*}(t_0, t_1, x_0, \hat{p}, q)$ the right-hand side of the above inequality. Arguing as in Lemma 3.1 one can prove that V_1^{-*} is Lipschitz continuous with respect to x. We also note that Player I can play in pure strategies in V^{-*}: Namely

$$V^{-*}(t,x,\hat{p},q) = \inf_{(\beta_j) \in (\mathcal{B}_r(t))^J} \sup_{\alpha \in \mathcal{A}(t)} \max_{i \in \{1,\dots,I\}} \left\{ \hat{p}_i - \sum_j q_j \mathbf{E}_{\beta_j}\left[g_{ij}(X_T^{t,x,\alpha,\beta_j})\right] \right\} \tag{13}$$

for any $(t,x,\hat{p},q) \in [0,T) \times I\!R^N \times I\!R^I \times \Delta(J)$. Indeed, we have from Lemma 4.1 that

$$\begin{aligned} &V^{-*}(t,x,\hat{p},q) = \\ &\inf_{(\beta_j) \in (\mathcal{B}_r(t_0))^J} \sup_{\alpha \in \mathcal{A}_r(t_0)} \max_{i \in \{1,\dots,I\}} \left\{ \hat{p}_i - \sum_{j=1}^J q_j \mathbf{E}_{\alpha\beta_j}\left(g_{ij}(X_T^{t,x,\alpha,\beta_j})\right) \right\} \ . \end{aligned}$$

Hence the inequality "$\geq$" in (13) is obvious because $\mathcal{A}(t) \subset \mathcal{A}_r(t)$. To prove the reverse inequality we first note that, for any $\alpha \in \mathcal{B}_r(t_0)$ and for any $\omega_1 \in \Omega_\alpha$, $\alpha(\omega_1, \cdot)$ belongs to $\mathcal{A}(t_0)$. Let us fix $(\beta_j) \in (\mathcal{B}(t))^J$. We have, from the convexity of $(s_i) \to \max_i\{s_i\}$,

$$\begin{aligned} &\sup_{\alpha \in \mathcal{A}_r(t)} \max_i \left\{ \hat{p}_i - \textstyle\sum_j q_j \mathbf{E}_{\alpha\beta_j}(g_{ij}(X_T^{t,x,\alpha,\beta_j})) \right\} \\ &\quad \leq \sup_{\alpha \in \mathcal{A}_r(t)} \int_{\Omega_\alpha} \max_i \left\{ \hat{p}_i - \textstyle\sum_j q_j \mathbf{E}_{\beta_j}(g_{ij}(X_T^{t,x,\alpha(\omega_1,\cdot),\beta_j})) \right\} dP_\alpha(\omega_1) \\ &\quad \leq \sup_{\alpha \in \mathcal{A}_r(t)} \sup_{\omega_1 \in \Omega_\alpha} \max_i \left\{ \hat{p}_i - \textstyle\sum_j q_j \mathbf{E}_{\beta_j}(g_{ij}(X_T^{t,x,\alpha(\omega_1,\cdot),\beta_j})) \right\} \\ &\quad \leq \sup_{\alpha \in \mathcal{A}(t)} \max_i \left\{ \hat{p}_i - \textstyle\sum_j q_j \mathbf{E}_{\beta_j}(g_{ij}(X_T^{t,x,\alpha,\beta_j})) \right\} \end{aligned}$$

Taking the infimum over $(\beta_j) \in (\mathcal{B}(t))^J$ gives (13).

Let $\epsilon > 0$ and $\beta^0 \in \mathcal{B}(t_0)$ be some pure $\epsilon-$optimal strategy for $V_1^{-*}(t_0, t_1, x_0, \hat{p}, q)$. For any $x \in I\!R^N$, we can find some $\epsilon-$optimal strategy $\hat{\beta}^x = (\beta_j^x) \in \mathcal{B}_r(t_1)$ for Player II in the game $V^{-*}(t_1, x, \hat{p}, q)$. From the Lipschitz continuity of the map

$$y \to \sup_{\alpha \in \mathcal{A}(t)} \max_{i \in \{1,\ldots,I\}} \left\{ \hat{p}_i - \sum_j q_j \mathbf{E}_{\beta_j^x} \left[g_{ij}(X_T^{t,y,\alpha,\beta_j^x}) \right] \right\} ,$$

and of $y \to V^{-*}(t_1, y, \hat{p}, q)$, β^x is also $(2\epsilon)-$optimal for $V^{-*}(t_1, y, \hat{p}, q)$ if $y \in B_r(x)$ for some radius $r > 0$. Using the fact that f is bounded, one can show that the reachable states from (t_0, x_0) by using the differential equation (1) is bounded, and contained in some ball $B_R(0)$. Let us set $M = \|f\|_\infty$ and let us fix $\sigma > 0$ small such that $M\sigma \le r/2$. Then we chose $(x_l)_{l=1,\cdots,l_0}$ such that $\bigcup_{l=1}^{l_0} B_{r/2}(x_l)$ contains the ball $B_R(0)$. Let $(E_l)_{l=1,\ldots,l_0}$ be a Borel partition of $B_R(0)$ such that, for any l, $E_l \subset B_{r/2}(x_l)$. We set

$$\beta_j^l = \beta_j^{x_l},\ \Omega_j^l = \Omega_{\beta_j^l},\ \mathcal{F}_j^l = \mathcal{F}_{\beta_j^l} \text{ and } \mathbf{P}_j^l = \mathbf{P}_{\beta_j^l}$$

for $j = 1, \ldots, J$ and $l = 1, \ldots, l_0$. We choose some delay $\tau \in (0, \sigma]$ common to all the strategies β_j^l.

We note for later use that, if for some controls $(u, v) \in \mathcal{U}(t_0) \times \mathcal{V}(t_0)$ and for some l, we have $X_{t_1-\tau}^{t_0,x_0,u,v} \in E_l$, then

$$|X_{t_1-\tau}^{t_0,x_0,u,v} - X_{t_1}^{t_0,x_0,u,v}| \le \|f\|_\infty \tau \le M\sigma \le r/2 ,$$

so that $X_{t_1}^{t_0,x_0,u,v}$ belongs to $B_r(x_l)$. In particular $(\beta_j^l)_j$ is $(2\epsilon)-$optimal for V^+ at $(t_1, X_{t_1}^{t_0,x_0,u,v}, \hat{p}, q)$. To summerize

$$X_{t_1-\tau}^{t_0,x_0,u,v} \in E_l \ \Rightarrow\ (\beta_j^l)_j \text{ is } (2\epsilon)-\text{optimal for } V^{-*} \text{ at } (t_1, X_{t_1}^{t_0,x_0,u,v}, \hat{p}, q). \tag{14}$$

Let us now define a new strategy $\hat{\beta} = (\beta_j) \in (\mathcal{B}_r(t_0))^J$ in the following way: set

$$\Omega_{\beta_j} = \Pi_{l=1}^{l_0} \Omega_j^l,\ \mathcal{F}_{\beta_j} = \mathcal{F}_j^1 \otimes \ldots \otimes \mathcal{F}_j^{l_0} \text{ and } \mathbf{P}_{\beta_j} = \mathbf{P}_j^1 \otimes \ldots \otimes \mathbf{P}_j^{j_0}$$

and, for any $\omega = (\omega^1, \ldots, \omega^{l_0}) \in \Omega_{\beta_j}$ and $u \in \mathcal{U}(t_0)$,

$$\beta(\omega, u)(t) = \begin{cases} \beta^0(u)(t) & \text{if } t \in [t_0, t_1) \\ \beta_j^l(\omega^l, u_{|_{[t_1,T]}})(t) & \text{if } t \in [t_1, T] \text{ and } X_{t_1-\tau}^{t_0,x_0,u,\beta^0(u)} \in E_l \end{cases}$$

Then $(\Omega_{\beta_j}, \mathcal{F}_{\beta_j}, \mathbf{P}_{\beta_j})$ belongs to $\mathcal{S}$ and $\hat{\beta} = (\beta_j) \in (\mathcal{B}_r(t_0))^J$.

For any pure strategy $\alpha \in \mathcal{A}(t_0)$, we have:

$$g_{ij}(X_T^{t_0,x_0,\alpha,\beta_j}) = \sum_{l=1}^{l_0} g_{ij}\left(X_T^{t_1, X_{t_1}^{t_0,x_0,\alpha,\beta^0}, \tilde{\alpha}, \beta_j^l}\right) \mathbf{1}_{\{X_{t_1-\tau}^{t_0,x_0,\alpha,\beta^0} \in E_l\}}$$

where $\tilde{\alpha} \in \mathcal{A}(t_1)$ is defined by

$$\tilde{\alpha}(v) = \alpha(v') \qquad \forall v \in \mathcal{V}(t_1) \text{ where } v'(t) = \begin{cases} \bar{v}(t) & \text{if } t \in [t_0, t_1] \\ v(t) & \text{otherwise} \end{cases}$$

the controls $(\bar{u}, \bar{v})$ being the pair associated with (α, β^0) as in (8). Hence

$$\max_{i \in \{1,\ldots,I\}} \left\{ \hat{p}_i - \sum_j q_j \mathbf{E}_{\beta_j}\left(g_{ij}(X_T^{t_0,x_0,\alpha,\beta_j})\right)\right\} =$$

$$\max_{i \in \{1,\ldots,I\}} \left\{ \hat{p}_i - \sum_j q_j \sum_{l=1}^{l_0} \left(\int_{\Omega_j^l} g_{ij}\left(X_T^{t_1, X_{t_1}^{t_0,x_0,\alpha,\beta^0}, \tilde{\alpha}, \beta_j^l}\right) d\mathbf{P}_j^l(\omega^l)\right) \mathbf{1}_{O^l} \right\}$$

(where we have set $O^l = \{X_{t_1-\tau}^{t_0,x_0,\alpha,\beta^0} \in E_l\}$)

$$\leq \sum_{l=1}^{l_0} \sup_{\alpha' \in \mathcal{B}(t_1)} \max_{i \in \{1,\ldots,I\}} \left\{ \hat{p}_i - \sum_j q_j \left(\int_{\Omega_j^l} g_{ij}\left(X_T^{t_1, X_{t_1}^{t_0,x_0,\alpha,\beta^0}, \alpha', \beta_j^l}\right) d\mathbf{P}_j^l(\omega^l)\right)\right\} \mathbf{1}_{O^l}$$

(because of the convexity of the map $s = (s_i) \to \max\{s_i\}$)

$$\leq \sum_{l=1}^{l_0} \left(V^{-*}\left(t_1, X_{t_1}^{t_0,x_0,\alpha,\beta^0}, \hat{p}, q\right) + 2\epsilon \right) \mathbf{1}_{O^l}$$

(because of (14))

$$= V^{-*}\left(t_1, X_{t_1}^{t_0,x_0,\alpha,\beta^0}, \hat{p}, q\right) + 2\epsilon$$

$$\leq V_1^{-*}(t_0, t_1, x_0, \hat{p}, q) + 3\epsilon\ ,$$

because β^0 is $\epsilon-$optimal for $V_1^{-*}(t_0, t_1, x_0, \hat{p}, q)$.

From this we conclude easily that

$$V^{-*}(t_0, x_0, \hat{p}, q) \leq V_1^{-*}(t_0, t_1, x_0, \hat{p}, q)\ .$$

QED

Corollary 4.3 (V^{-*} is a subsolution of HJ)
For any $(\hat{p}, q) \in \mathbb{R}^I \times \Delta(J)$, the map $(t, x) \to V^{-*}(t, x, \hat{p}, q)$ is a viscosity subsolution of the dual Hamilton-Jacobi equation:

$$w_t + H^*(x, Dw) = 0 \text{ in } [0, T] \times \mathbb{R}^N \ . \tag{15}$$

where H is defined by (6) and $H^*(x, \xi) = -H(x, -\xi)$.

Remark : From the definition of H, we have

$$H^*(x, \xi) := \sup_{u \in U} \inf_{v \in V} f(x, u, v).\xi = \inf_{v \in V} \sup_{u \in U} f(x, u, v).\xi \tag{16}$$

Proof of Corollary 4.3: It is well-known that a function satisfying a subdynamic programming principle is a subsolution of the associated HJ equation when the game is played with classical nonanticipative strategies (see [15]). We give a short proof of this fact in the framework of nonanticipative strategies with delay. Let $(\hat{p}, q) \in \mathbb{R}^I \times \Delta(J)$ be fixed and let ϕ be a smooth test function such that

$$\phi(t, x) \geq V^{-*}(t, x, \hat{p}, q) \qquad \forall (t, x) \in [0, T] \times \mathbb{R}^N \ , \tag{17}$$

with an equality at (t_0, x_0), where $t_0 \in [0, T)$. For any $v \in V$, let us define the pure strategy $\beta \in \mathcal{B}(t_0)$ by setting

$$\beta(u)(t) = v \qquad \forall u \in \mathcal{U}(t_0), \ t \in [t_0, T] \ .$$

Let us fix $\epsilon > 0$ and $h > 0$ small.

Since V^{-*} satisfies the subdynamic programming principle of Lemma 4.2, there is some strategy $\alpha_h \in \mathcal{A}(t_0)$ such that

$$V^{-*}(t_0, x_0, \hat{p}, q) \leq V^{-*}(t_0 + h, X_{t_0+h}^{t_0, x_0, \alpha_h, \beta}, \hat{p}, q) + \epsilon h \ . \tag{18}$$

Let us set $u_h(s) = \alpha_h(v)(s)$ and $x_h(s) = X_s^{t_0, x_0, \alpha_h, \beta} = X_s^{t_0, x_0, u_h, v}$. Then

$$x_h(t_0+h) = x_0 + \int_{t_0}^{t_0+h} f(x_h(s), u_h(s), v) ds = x_0 + \int_{t_0}^{t_0+h} f(x_0, u_h(s), v) ds + h\epsilon(h)$$

where $\epsilon(h) \to 0$ as $h \to 0^+$. From (17) and (18) we have

$$\begin{array}{rcl} 0 & \leq & V^{-*}(t_0 + h, X_{t_0+h}^{t_0, x_0, \alpha_h, \beta}, \hat{p}, q) - V^{-*}(t_0, x_0, \hat{p}, q) + \epsilon h \\ & \leq & \phi(t_0 + h, x_0 + \int_{t_0}^{t_0+h} f(x_0, u_h(s), v) ds + h\epsilon(h)) ds - \phi(t_0, x_0) + \epsilon h \\ & \leq & h\phi_t(t_0, x_0) + \int_{t_0}^{t_0+h} D\phi(t_0, x_0).f(x_0, u_h(s), v) ds + h\epsilon_1(h) + \epsilon h \\ & \leq & h\phi_t(t_0, x_0) + h \sup_{u \in U} D\phi(t_0, x_0).f(x_0, u, v) + h\epsilon_1(h) + \epsilon h \end{array}$$

where $\epsilon_1(h) \to 0$ as $h \to 0^+$. Diviving the last inequality by $h > 0$ and letting $h \to 0^+$ gives

$$\phi_t(t_0, x_0) + \sup_{u \in U} D\phi(t_0, x_0).f(x_0, u, v) \geq -\epsilon \ .$$

Then we let $\epsilon \to 0^+$, take the minimum over $v \in V$ and use (16) to get the desired inequality:

$$\phi_t(t_0, x_0, p) + H^*(x_0, D\phi(t_0, x_0)) \geq 0 \ .$$

QED

To state the symmetric results for $V^{+\sharp}$, we only need to note that

$$-V^+(t, x, p, q) = \sup_{\hat{\alpha} \in (\mathcal{A}_r(t_0))^I} \inf_{\hat{\beta} \in (\mathcal{B}_r(t_0))^J} \sum_{i=1}^{I} \sum_{j=1}^{I} p_i q_j \mathbf{E}_{\alpha_i \beta_j} \left((-g_{ij}) \left(X_T^{t_0, x_0, \alpha_i, \beta_j} \right) \right) \ ,$$

which is of the same form as V^- when one changes the roles of the Players. In particular the convex Fenchel conjugate of $(-V^+)$ with respect to q, i.e., $-V^{+\sharp}(-\hat{q})$, satisfies a subdynamic programming principle and is therefore a subsolution of some associated Hamilton-Jacobi equation. From this we easily deduce the

Corollary 4.4 ($V^{+\sharp}$ is a supersolution of HJ)
For any $(t_0, t_1, x_0, p, \hat{q}) \in [0, T] \times [0, T] \times \mathbb{R}^N \times \Delta(I) \times \mathbb{R}^J$, we have

$$V^{+\sharp}(t_0, x_0, p, \hat{q}) \geq \sup_{\alpha \in \mathcal{A}(t_0)} \inf_{\beta \in \mathcal{B}(t_0)} V^{+\sharp}(t_1, X_{t_1}^{t_0, x_0, \alpha, \beta}, p, \hat{q}) \ .$$

Hence $V^{+\sharp}$ is a supersolution of the dual Hamilton-Jacobi equation (15).

Remark : We use here Isaacs assumption (6). Indeed, if V^{-*} is a subsolution of the HJ equation (15) with $H^*(x, \xi) = \inf_u \sup_v f(x, u, v).\xi$, $V^{+\sharp}$ is actually a supersolution of (15) with a Hamiltonian H^* defined by $H^*(x, \xi) = \sup_v \inf_u f(x, u, v).\xi$.

5 Existence of the value and solutions of the primal/dual HJ equations

In this section we prove that our game has a value: $V^+ = V^-$. This value can be characterized in terms of dual solutions of some HJ equations.

The key argument for this is the following comparison principle, that we state for later use for a general Hamiltonian H. We assume that $H : I\!R^N \times I\!R^N \to I\!R$ is continuous and we suppose that there is a constant C such that, for any $x_1, x_2 \in I\!R^N$ and $\theta \geq 0$,

$$|H(x_1, \theta(x_1 - x_2)) - H(x_2, \theta(x_1 - x_2))| \leq C|x_1 - x_2|(1 + \theta|x_1 - x_2|) \ . \tag{19}$$

Let us point out that the map H defined by (6) satisfies the above assumptions under conditions (5) on the dynamics.

Recall that, for any map $w = w(t, x, p, q)$ defined on $[0, T] \times I\!R^N \times \Delta(I) \times \Delta(J)$, w^* denotes the convex Fenchel conjugate of w with respect to p, while $w^\sharp$ denotes its concave Fenchel conjugate with respect to q.

We now consider a Hamilton-Jacobi equation of the form:

$$z_t + H(x, Dz) = 0 \ , \tag{20}$$

We say that a function $w : [0, T] \times I\!R^N \times \Delta(I) \times \Delta(J) \to I\!R$ is a dual subsolution of (20) if w is Lipschitz continuous, convex with respect to p and concave with respect to q and if, for any $(p, \hat{q}) \in \Delta(I) \times I\!R^J$, $(t, x) \to w^\sharp(t, x, p, \hat{q})$ is a supersolution of the dual HJ equation

$$z_t + H^*(x, Dz) = 0 \ , \tag{21}$$

where $H^*(x, \xi) = -H(x, -\xi)$. In a symmetric way, w is a dual supersolution of the HJ equation (20) if w is Lipschitz continuous, convex with respect to p and concave with respect to q and if, for any for any $(\hat{p}, q) \in I\!R^I \times \Delta(J)$, $(t, x) \to w_2^*(t, x, \hat{p}, q)$ is a subsolution of the dual HJ equation (21). We say that w is a dual solution of (20) if w is at the same time a dual subsolution and a dual supersolution of (20).

Theorem 5.1 (Comparison principle) *Let $w_1, w_2 : [0, T] \times I\!R^N \times \Delta(I) \times \Delta(J) \to I\!R$ be respectively a dual subsolution and a dual supersolutions of the HJ equation (20). We assume that for any $(x, p, q) \in I\!R^N \times \Delta(I) \times \Delta(J)$, $w_1(T, x, p, q) \leq w_2(T, x, p, q)$. Then $w_1 \leq w_2$ in $[0, T] \times I\!R^N \times \Delta(I) \times \Delta(J)$.*

Remarks:

1. We cannot compare $w_1^\sharp$ and w_2^* at time $t = T$. So this result is *not* an application of the classical comparison principle.

2. It is known that, if w_2 is a supersolution of the HJ equation (20), then w_2^* is a subsolution of the dual HJ equation (21) (see for instance

[1]. The converse does not hold true in general, and so we cannot rephrase the asumptions in term of sub- and supersolutions of (20) for w_1 and w_2. However it turns out that w_2 for instance is a supersolution at "some suitable points", related with its convexity property with respect to p. We explain this more precisely in Lemma 5.4 below.

3. The result can be extended to bounded uniformly continuous subsolutions by standard techniques (see [3] for instance).

The comparison principle is proved at the end of the section. Let us now state the main result of this paper:

Theorem 5.2 (Existence of the value)
Assume that conditions (5) on f and on the g_i hold and that Isaacs assumption (6) is satisfied. Then we have

$$V^+(t,x,p,q) = V^-(t,x,p,q) \qquad \forall (t,x,p) \in [0,T] \times \mathbb{R}^N \times \Delta(I) \times \Delta(J)\ .$$

Proof of Theorem 5.2: From Lemma 3.1 V^- and V^+ are Lipschitz continuous. From Lemma 3.2, we know that V^+ and V^- are convex with respect to p and concave with respect to q. Corollary 4.3 states that, for any $(\hat{p}, q) \in \mathbb{R}^I \times \Delta(J)$, $V^{-*}(\cdot,\cdot,\hat{p},q)$ is a subsolution of the dual HJ equation (15). Hence V^- is a dual supersolution of (7). Corollary 4.4 states that $V^{+\sharp}(\cdot,\cdot,p,\hat{q})$ is a supersolution of the HJ equation (15) for any $(p,\hat{q}) \in \Delta(I) \times \mathbb{R}^J$, and therefore a dual subsolution of (7). Since $V^+(T,\cdot,p,q) = V^-(T,\cdot,p,q) = \sum_{i,j} p_i q_j g_{ij}$, the comparison principle states that $V^+ \leq V^-$. But the reverse inequality always holds. Hence $V^- = V^+$ and the game has a value.

QED

The above proof also shows the

Corollary 5.3 (Characterization of the value)
Under the assumptions of Theorem 5.2, the value function $V := V^+ = V^-$ is the unique dual solution of the HJ equations (7), such that $V(T,x,p,q) = \sum_{ij} p_i q_j g_{ij}$.

We complete this section by an equivalent formulation of the notion of dual supersolution. Although the result is not needed in the rest of the text, we think that it can help to enlighten the notion.

Lemma 5.4 *Let $w : [0,T]\times \mathbb{R}^N \times \Delta(I)\times\Delta(J) \to \mathbb{R}$ be Lipschitz continuous, convex with respect to p and concave with respect to q. Then the following statements are equivalent:*

(i) w is a dual supersolution of (20).

(ii) for any $q \in \Delta(J)$, for any test function $\phi = \phi(t,x,p)$ which is $\mathcal{C}^1$ and convex in p, and such that

$$(t,x,p) \to w(t,x,p,q) - \phi(t,x,p)$$

has a strict global minimum at some point $(t_0,x_0,p_0) \in [0,T) \times \mathbb{R}^N \times \Delta(I)$, we have

$$\phi_t(t_0,x_0,p_0) + H(x_0, D\phi(t_0,x_0,p_0)) \leq 0 \ . \tag{22}$$

Remarks :

1. This result means that a dual supersolution of (20)—originaly defined in terms of subsolution of the dual HJ equation—is indeed a supersolution of the primal HJ equation (20) in weak sense. However it is not a classical supersolution. For instance, if $I = 1$, $f = f(u,v)$ and $g_j(x) = a_j.x$ for some $a_j \in \mathbb{R}^N$ $(j = 1,\ldots,J)$, then we prove in [10] that
$$V^+(t,x,p) = V^-(t,x,p) = (T-t)Cav(h)(p) + \sum_j p_j x.a_j$$
where $h(p) = H(\sum_j p_j a_j)$ and $Cav(h)$ is the concave hull of h with respect to $p \in \Delta(I)$. Then
$$V_t^- + H(DV^-) = -Cav(h)(p) + h(p) \geq 0 \ ,$$
with a strict inequality in general. In particular, V^- is not a classical supersolution of the primal HJ equation.

2. Note carefully that we require the minimum $w(t,x,p,q) - \phi(t,x,p)$ at (t_0,x_0,p_0) to be *strict*. This point is absolutely crucial for the equivalence. It is related with similar definition in repeated games, where some function has to be tested only at extreme points (see [16]). Let us point out that a general minimum of $w - \phi$ cannot not be made artificially strict by substracting $\epsilon|(t,x,p) - (t_0,x_0,p_0)|^2$ to ϕ (as is usually done in viscosity solutions) because one then looses the convexity of ϕ with respect to p.

3. A symmetric result holds for subsolutions: w is a dual subsolution of (20) if and only if, for any $p \in \Delta(I)$, for any test function $\phi = \phi(t, x, q)$ which is $\mathcal{C}^1$ and concave in q, and such that $w - \phi$ has a strict global maximum at some point $(t_0, x_0, q_0) \in [0, T) \times I\!R^N \times \Delta(J)$, we have

$$\phi_t(t_0, x_0, q_0) + H(x_0, D\phi(t_0, x_0, q_0)) \geq 0 \ .$$

Proof of Lemma 5.4 : Let us first assume that w is a dual supersolution of (20). Let $q \in \Delta(J)$, $\phi = \phi(t, x, p)$ be a test function which is $\mathcal{C}^1$ and convex in p, and such that $w - \phi$ has a strict global minimum at some point $(t_0, x_0, p_0) \in [0, T) \times I\!R^N \times \Delta(I)$. This means that

$$w(t, x, p, q) \leq \phi(t, x, p) + w(t_0, x_0, p_0, q) - \phi(t_0, x_0, p_0) \tag{23}$$

for any $(t, x, p) \in [0, T] \times I\!R^N \times \Delta(I)$, with an equality only at (t_0, x_0, p_0). By using the fact that the minimum of $w - \phi$ is strict and standard perturbation argument (consisting in replacing ϕ by $\phi + \epsilon |p|^2$ if necessary), we can assume that ϕ is strictly convex in p. Then ϕ^* is differentiable in t and x and one easily checks that

$$\phi^*_t(t_0, x_0, \hat{p}) = -\phi_t(t_0, x_0, p) \text{ and } D\phi^*(t_0, x_0, \hat{p}) = -D\phi(t_0, x_0, p) \ , \tag{24}$$

for any $\hat{p} \in I\!R^I$, p being the unique element of the subdifferential of $\phi^*(t_0, x_0, \cdot)$ at $\hat{p}$. Let $\hat{p}_0$ belong to the subdifferential with respect to p of w at (t_0, x_0, p_0). Then inequality (23) shows that $\hat{p}_0$ belongs to the subdifferential of ϕ with respect to p at (t_0, x_0, p_0). Since w and ϕ are convex in p, we have

$$w^*(t_0, x_0, \hat{p}_0, q) = p_0.\hat{p}_0 - w(t_0, x_0, p_0, q) \text{ and } \phi^*(t_0, x_0, \hat{p}_0) = p_0.\hat{p}_0 - \phi(t_0, x_0, p_0) \ .$$

Thus

$$w(t_0, x_0, p_0, q) - \phi(t_0, x_0, p_0) = w^*(t_0, x_0, \hat{p}_0, q) - \phi^*(t_0, x_0, \hat{p}_0) \ . \tag{25}$$

We note that (23) can be rewritten as

$$p.\hat{p}_0 - w(t, x, p, q) \geq p.\hat{p}_0 - \phi(t, x, p) - w(t_0, x_0, p_0, q) + \phi(t_0, x_0, p_0)$$

for all $(t, x, p) \in [0, T] \times I\!R^N \times \Delta(I)$. Taking the sup over $p \in \Delta(I)$ and taking into account (25) gives

$$w^*(t, x, \hat{p}_0, q) \geq \phi^*(t, x, \hat{p}_0) + w^*(t_0, x_0, \hat{p}_0, q) - \phi^*(t_0, x_0, \hat{p}_0) \ .$$

Therefore $(t,x) \to w^*(t,x,\hat{p}_0,q) - \phi^*(t,x,\hat{p}_0)$ has a maximum at (t_0,x_0). Since w^* is a subsolution of the dual HJ equation, we have

$$\phi_t^*(t_0,x_0,\hat{p}_0) + H^*(x_0, D\phi^*(t_0,x_0,\hat{p}_0)) \geq 0 ,$$

which implies the desired inequality (22) thanks to (24).

Conversely, let us assume that (ii) holds. Let ϕ be a $\mathcal{C}^1$ test function such that $(t,x) \to w^*(t,x,\hat{p}_0,q) - \phi(t,x)$ has a local minimum at (t_0,x_0) for some $(\hat{p}_0,q) \in \mathbb{R}^I \times \Delta(I)$. Without loss of generality, we can assume that this minimum is a global one and that $\phi(t_0,x_0) = w^*(t_0,x_0,\hat{p}_0,q)$ (see [3]). Let $\tilde{\phi}(t,x,\hat{p}) = \phi(t,x)$ if $\hat{p} = \hat{p}_0$ and $\tilde{\phi}(t,x,\hat{p}) = +\infty$ otherwise. Then $\tilde{\phi} \geq w^*(\cdot,\cdot,\cdot,q)$ on $[0,T] \times \mathbb{R}^N \times \mathbb{R}^I$, with an equality at $(t_0,x_0,\hat{p}_0)$. Thus, by duality,

$$p.\hat{p}_0 - \phi(t,x) = \tilde{\phi}^*(t,x,p) \leq w^{**}(t,x,p,q) = w(t,x,p,q)$$

for any $(t,x,p) \in [0,T] \times \mathbb{R}^N \times \Delta(I)$, with an equality at (t_0,x_0,p_0) for any $p_0 \in \partial w^*(t_0,x_0,\hat{p}_0,q)$ (where $\partial w^*(t_0,x_0,,\hat{p}_0,q)$ denotes the superdifferential of the convex function $\hat{p} \to w^*(t_0,x_0,\hat{p},q)$ at $\hat{p}_0$). Hence $(t,x,p) \to w(t,x,p,q) - (p.\hat{p}_0 - \phi(t,x))$ has a minimum at (t_0,x_0,p_0) for any $p_0 \in \partial w^*(t_0,x_0,\hat{p}_0,q)$. In order to get a *strict* minimum, we have to introduce some perturbation term. Let $\gamma > 0$, $\epsilon > 0$ and $(t_\epsilon,x_\epsilon,p_\epsilon)$ be a point of minimum of $w - \psi_{\epsilon,\gamma}$, where

$$\psi_{\epsilon,\gamma}(t,x,p) = p.\hat{p}_0 + \epsilon|p|^2 - \phi(t,x) - \gamma|(t,x) - (t_0,x_0)|^2 .$$

Then $(t_\epsilon,x_\epsilon,p_\epsilon)$ converges (up to some subsequence) to (t_0,x_0,p_0) for some $p_0 \in \partial w^*(t_0,x_0,\hat{p}_0,q)$ as $\epsilon \to 0^+$ (we use here the penalization term in γ). Moreover, we have

$$\begin{array}{rcl} \tilde{\psi}(t,x,p) & := & \psi_{\epsilon,\gamma}(t,x,p) - \epsilon|p-p_\epsilon|^2 - \epsilon|(t,x) - (t_\epsilon,x_\epsilon)|^2 \\ & < & \psi_{\epsilon,\gamma}(t,x,p) \\ & \leq & w(t,x,p) - w(t_\epsilon,x_\epsilon,p_\epsilon) + \tilde{\psi}(t_\epsilon,x_\epsilon,p_\epsilon) \end{array}$$

for any $(t,x,p) \neq (t_\epsilon,x_\epsilon,p_\epsilon)$, with an equality at $(t_\epsilon,x_\epsilon,p_\epsilon)$. This means that $w - \tilde{\psi}$ has a strict minimum at $(t_\epsilon,x_\epsilon,p_\epsilon)$. Since $\tilde{\psi}$ is still convex in p we get from assumption (ii) that

$$\tilde{\psi}_t(t_\epsilon,x_\epsilon,p_\epsilon) + H(x_\epsilon, D\tilde{\psi}(t_\epsilon,x_\epsilon,p_\epsilon)) \leq 0 .$$

Using the definition of $\tilde{\psi}$ and letting $\epsilon \to 0^+$, we then obtain

$$\phi_t(t_0,x_0) + H^*(x_0, D\phi(t_0,x_0)) \geq 0 ,$$

which proves that w is a dual supersolution of (20).

QED

Proof of Theorem 5.1: We follow the proof of Theorem 3.7 in [3]. Let us argue by contradiction, by assuming that there is some (t_1, x_1, p_1, q_1) such that $w_1(t_1, x_1, p_1, q_1) > w_2(t_1, x_1, p_1, q_1)$. This means that, for some $\sigma > 0$, we have

$$\sup_{t,x,p,q} w_1(t, x, p, q) - w_2(t, x, p, q) - \sigma(T - t) \ > \ 0 \ . \tag{26}$$

We now use the standard method of separation of variables. In order to avoid burdensome details, we do the proof under the additional assumption that there is some $R > 0$ such that $w_1(t, x, p, q) \le w_2(t, x, p, q)$ for any (t, x, p, q) with $|x| \ge R$. This assumption can be omitted by using penalization arguments at infinity (see [3] for the details). Let $\epsilon > 0$ be fixed. From our assumption, the map

$$(t, x, s, y, p, q) \to w_1(t, x, p, q) - w_2(s, y, p, q) - \frac{1}{\epsilon}|(t, x) - (s, y)|^2 - \sigma(T - t) \ . \tag{27}$$

has a maximum over $[0, T] \times I\!\!R^N \times \Delta(I) \times \Delta(J)$ and we denote by $(t_\epsilon, x_\epsilon, s_\epsilon, y_\epsilon, p_\epsilon, q_\epsilon)$ such a point of maximum. From usual arguments in [3], we have $t_\epsilon < T$ and $s_\epsilon < T$ for small ϵ because $w_1(T, x, p, q) \le w_2(T, x, p, q)$ and w_1 and w_2 are Lipschitz continuous. Moreover

$$\lim_{\epsilon \to 0^+} \frac{1}{\epsilon}|(t_\epsilon, x_\epsilon) - (s_\epsilon, y_\epsilon)|^2 = 0 \ . \tag{28}$$

Since, for $(s, y) = (s_\epsilon, y_\epsilon)$, $(t_\epsilon, x_\epsilon, q_\epsilon)$ is a maximum in (27) we have

$$\begin{array}{rl} w_1(t, x, p_\epsilon, q) \le & w_1(t_\epsilon, x_\epsilon, p_\epsilon, q_\epsilon) + w_2(s_\epsilon, y_\epsilon, p_\epsilon, q) - w_2(s_\epsilon, y_\epsilon, p_\epsilon, q_\epsilon) \\ & + \frac{1}{\epsilon}\left(|(t, x) - (s_\epsilon, y_\epsilon)|^2 - |(t_\epsilon, x_\epsilon) - (s_\epsilon, y_\epsilon)|^2\right) + \sigma(t_\epsilon - t) \end{array} \tag{29}$$

for any (t, x, q), with an equality at $(t_\epsilon, x_\epsilon, q_\epsilon)$. Let $\hat{q}_\epsilon$ belong to the superdifferential $\partial_q^+ w_2(s_\epsilon, y_\epsilon, p_\epsilon, q_\epsilon)$ of w_2 with respect to q at $(s_\epsilon, y_\epsilon, p_\epsilon, q_\epsilon)$. Then the above inequality shows that $\hat{q}_\epsilon \in \partial_q w_1(t_\epsilon, x_\epsilon, p_\epsilon, q_\epsilon)$. From the concavity of w_1 and w_2 with respect to q, we have

$$w_1^\sharp(t_\epsilon, x_\epsilon, p_\epsilon, \hat{q}_\epsilon) = q_\epsilon.\hat{q}_\epsilon - w_1(t_\epsilon, x_\epsilon, p_\epsilon, q_\epsilon)$$

and

$$w_2^\sharp(s_\epsilon, y_\epsilon, p_\epsilon, \hat{q}_\epsilon) = q_\epsilon.\hat{q}_\epsilon - w_2(s_\epsilon, y_\epsilon, p_\epsilon, q_\epsilon) \ ,$$

so that

$$w_1(t_\epsilon, x_\epsilon, p_\epsilon, q_\epsilon) - w_2(s_\epsilon, y_\epsilon, p_\epsilon, q_\epsilon) = w_2^\sharp(s_\epsilon, y_\epsilon, p_\epsilon, \hat{q}_\epsilon) - w_1^\sharp(t_\epsilon, x_\epsilon, p_\epsilon, \hat{q}_\epsilon) \ . \tag{30}$$

Combining (29) with (30) then gives

$$\begin{aligned}&q.\hat{q}_\epsilon - w_1(t,x,p_\epsilon,q) \geq \\ &\quad w_1^\sharp(t_\epsilon,x_\epsilon,p_\epsilon,\hat{q}_\epsilon) + q.\hat{q}_\epsilon - w_2(s_\epsilon,y_\epsilon,p_\epsilon,q) - w_2^\sharp(s_\epsilon,y_\epsilon,p_\epsilon,\hat{q}_\epsilon) \\ &\qquad -\tfrac{1}{\epsilon}\left(|(t,x)-(s_\epsilon,y_\epsilon)|^2 - |(t_\epsilon,x_\epsilon)-(s_\epsilon,y_\epsilon)|^2\right) - \sigma(t_\epsilon - t)\ .\end{aligned}$$

Taking the infimum over q in the above expression then gives

$$\begin{aligned}&w_1^\sharp(t,x,p_\epsilon,\hat{q}_\epsilon) \geq \\ &\quad w_1^\sharp(t_\epsilon,x_\epsilon,p_\epsilon,\hat{q}_\epsilon) - \tfrac{1}{\epsilon}\left(|(t,x)-(s_\epsilon,y_\epsilon)|^2 - |(t_\epsilon,x_\epsilon)-(s_\epsilon,y_\epsilon)|^2\right) - \sigma(t_\epsilon - t)\ .\end{aligned}$$

So $(t,x) \to w_1^\sharp(t,x,p_\epsilon,\hat{q}_\epsilon) - \left(-\frac{|(t,x)-(s_\epsilon,y_\epsilon)|^2}{\epsilon} + \sigma t\right)$ has a minimum at (t_ϵ,x_ϵ). Since $w_1^\sharp(\cdot,\cdot,p_\epsilon,\hat{q}_\epsilon)$ is a supersolution of the HJ equation (20), we get

$$\sigma + \frac{2}{\epsilon}(s_\epsilon - t_\epsilon) + H^*\left(x_\epsilon, \frac{2}{\epsilon}(y_\epsilon - x_\epsilon)\right) \leq 0\ . \tag{31}$$

We now argue in a symmetric way for w_2. Since $(s_\epsilon,y_\epsilon,p_\epsilon)$ is a maximum in (27), we have

$$\begin{aligned}w_2(s,y,p,q_\epsilon) \geq\ & w_2(s_\epsilon,y_\epsilon,p_\epsilon,q_\epsilon) + w_1(t_\epsilon,x_\epsilon,p,q_\epsilon) - w_1(t_\epsilon,x_\epsilon,p_\epsilon,q_\epsilon) \\ & -\tfrac{1}{\epsilon}\left(|(t_\epsilon,x_\epsilon)-(s,y)|^2 - |(t_\epsilon,x_\epsilon)-(s_\epsilon,y_\epsilon)|^2\right)\end{aligned} \tag{32}$$

for any $(s,y,p) \in [0,T] \times I\!\!R^N \times \Delta(I)$. Let $\hat{p}_\epsilon$ belong to the subdifferential $\partial_p^- w_1(t_\epsilon,x_\epsilon,p_\epsilon,q_\epsilon)$ of w_1 with respect to p at $(s_\epsilon,y_\epsilon,p_\epsilon,q_\epsilon)$. Then the above inequality shows that $\hat{p}_\epsilon \in \partial_p^- w_2(s_\epsilon,y_\epsilon,p_\epsilon,q_\epsilon)$. Therefore we have as above

$$w_2(s_\epsilon,y_\epsilon,p_\epsilon,q_\epsilon) - w_1(t_\epsilon,x_\epsilon,p_\epsilon,q_\epsilon) = w_1^*(t_\epsilon,x_\epsilon,\hat{p}_\epsilon,q_\epsilon) - w_2^*(s_\epsilon,y_\epsilon,\hat{p}_\epsilon,q_\epsilon)\ .$$

Then we get from (32):

$$w_2^*(s,y,\hat{p}_\epsilon,q_\epsilon) \leq w_2^*(s_\epsilon,y_\epsilon,\hat{p}_\epsilon,q_\epsilon) + \tfrac{1}{\epsilon}\left(|(t_\epsilon,x_\epsilon)-(s,y)|^2 - |(t_\epsilon,x_\epsilon)-(s_\epsilon,y_\epsilon)|^2\right)$$

for any $(s,y) \in [0,T] \times I\!\!R^N$, with an equality at (s_ϵ,y_ϵ). Since $w_2^*(\cdot,\cdot,\hat{p}_\epsilon,q_\epsilon)$ is a subsolution of the HJ equation (20), this gives

$$\frac{2}{\epsilon}(s_\epsilon - t_\epsilon) + H^*\left(y_\epsilon, \frac{2}{\epsilon}(y_\epsilon - x_\epsilon)\right) \geq 0\ . \tag{33}$$

Computing the difference between (31) and (33) and using the assumption (19) on H (recall that $H^*(x,\xi) = -H(x,-\xi)$) gives

$$-\sigma + C\frac{2|x_\epsilon - y_\epsilon|}{\epsilon}(1 + |x_\epsilon - y_\epsilon|) \geq 0\ ,$$

which is in contradiction with (28) as $\epsilon \to 0^+$.

QED

6 The case of lack of information on the initial position

In this section we investigate a two-player zero-sum differential game in which the Players have some private information on the random initial position. The dynamics of the game is still given by

$$x'(t) = f(x, u(t), v(t)) \ , \qquad u(t) \in U, \ v(t) \in V \tag{34}$$

where U, V and f satisfy (5). The terminal time of the game is denoted by T and the payoff is a terminal payoff $g(x(T))$ where $g : I\!R^N \to I\!R$ is Lipschitz continuous and bounded. The game starts at time $t_0 \in [0, T]$.

The description of the game involves $I \times J$ initial positions x_{ij}^0, $i = 1, \ldots, I$, $j = 1, \ldots, J$, a probability $p \in \Delta(I)$ and a probability $q \in \Delta(J)$. As before, the game is played in two steps: at time t_0, the pair (i, j) is chosen according to the probability $p \otimes q$, the index i is communicated to Player I only and the index j to Player II only.

Then the players control system (34) with initial position x_{ij}^0 in order, for Player I, to minimize the terminal payoff $g(x(T))$, and for Player II to maximize it. The players observe their oponent's behavior, and try to deduce from this behaviour their missing information. They cannot compute the actual position of the system in general.

As before we define the upper and lower value functions associated to this game. For this we introduce the new state of the system: $\mathbf{x} = (x_{ij})$, which denotes the $I \times J-$uplet of possible positions. The upper-value is given for $t_0 \in [0, T)$, $\mathbf{x}^0 = (x_{ij}^0) \in I\!R^{NIJ}$, $p \in \Delta(I)$ and $q \in \Delta(J)$, by

$$V^+(t_0, \mathbf{x}^0, p, q) = \inf_{(\alpha_i) \in (\mathcal{A}_r(t_0))^I} \ \sup_{(\beta_j) \in (\mathcal{B}_r(t_0))^J} \sum_{i=1}^{I} \sum_{j=1}^{J} p_i q_j \mathbf{E}_{\alpha_i \beta_j} \left(g \left(X_T^{t_0, x_{ij}^0, \alpha_i, \beta_j} \right) \right) \ ,$$

where $t \to X_t^{t_0, x_{ij}^0, \alpha_i, \beta_j}$ is the random solution to (34) with initial position x_{ij}^0 at time t_0 when the players play the random strategies α_i and β_j (see section 2). The lower-value is defined by the symmetric formula:

$$V^-(t_0, \mathbf{x}^0, p, q) = \sup_{(\beta_j) \in (\mathcal{B}_r(t_0))^J} \ \inf_{(\alpha_i) \in (\mathcal{A}_r(t_0))^I} \sum_{i=1}^{I} \sum_{j=1}^{J} p_i q_j \mathbf{E}_{\alpha_i \beta_j} \left(g \left(X_T^{t_0, x_{ij}^0, \alpha_i, \beta_j} \right) \right) \ .$$

Obviously we have

$$V^-(t_0, \mathbf{x}^0, p, q) \leq V^+(t_0, \mathbf{x}^0, p, q) \ \forall (t_0, \mathbf{x}^0, p, q) \in [0, T] \times I\!R^{NIJ} \times \Delta(I) \times \Delta(J) \ .$$

Our main result is that the equality holds:

Theorem 6.1 *Assume that f, U and V satisfy (5), that the payoff $g : I\!\!R^N \to I\!\!R$ is Lipschitz continuous and bounded and that the following generalized Isaacs condition holds:*

$$\mathbf{H}(\mathbf{x},\xi) = \inf_{u\in U}\sup_{v\in V}\sum_{i=1}^{I}\sum_{j=1}^{J} f(x_{ij},u,v).\xi_{ij} = \sup_{v\in V}\inf_{u\in U}\sum_{i=1}^{I}\sum_{j=1}^{J} f(x_{ij},u,v).\xi_{ij} \tag{35}$$

for any $\mathbf{x} = (x_{ij}) \in I\!\!R^{NIJ}$ and $\xi = (\xi_{ij}) \in I\!\!R^{NIJ}$.

Then the game has a value:

$$V^-(t_0,\mathbf{x}^0,p,q) = V^+(t_0,\mathbf{x}^0,p,q)\ \forall (t_0,\mathbf{x}^0,p,q) \in [0,T]\times I\!\!R^{NIJ}\times\Delta(I)\times\Delta(J)\ .$$

Furthermore this value is the dual solution of the HJ equation

$$\left\{\begin{array}{ll} z_t + \mathbf{H}(\mathbf{x},Dz) = 0 & \text{in } [0,T)\times I\!\!R^{NIJ} \\ z(T,\mathbf{x},p,q) = \sum_{i=1}^{I}\sum_{j=1}^{J} p_i q_j g(x_{ij}) & \text{for } \mathbf{x} = (x_{ij}) \in I\!\!R^{NIJ} \end{array}\right. \tag{36}$$

Proof of Theorem 6.1 : The proof is mainly the same as the proof of Theorem 5.2 and Corollary 5.3 and we only give an outline of it. We first note that V^+ and V^- are Lipschitz continuous in their arguments, convex in p, concave in q as in Lemma 3.1 and Lemma 3.2. Then, following Lemma 4.1, one proves that

$$V^{-*}(t,\mathbf{x}^0,\hat{p},q) = \inf_{(\beta_j)\in(\mathcal{B}_r(t_0))^J}\ \sup_{\alpha\in\mathcal{A}_r(t_0)}\ \max_{i=1,\dots,I}\left\{\hat{p}_i - \sum_j \mathbf{E}_{\alpha\beta_j}(g\left[X_T^{t,x^0_{ij},\alpha,\beta_j})\right]\right\}$$

for any $t\in[0,T]$, $\mathbf{x}^0 = (x^0_{ij}) \in I\!\!R^{NIJ}$, $\hat{p}\in I\!\!R^I$ and $q\in\Delta(J)$. Using this, one obtains as in Lemma 4.2 that V^{-*} satisfies the subdynamic programming principle

$$V^{-*}(t_0,\mathbf{x}^0,\hat{p},q) \le \inf_{\beta\in\mathcal{B}(t_0)}\ \sup_{\alpha\in\mathcal{A}(t_0)}\ V^{-*}(t_1,\mathbf{X}_{t_1}^{t_0,\mathbf{x}^0,\alpha,\beta},\hat{p},q)$$

for any $0\le t_0 < t_1 \le T$, $\mathbf{x}^0 \in I\!\!R^{NIJ}$, $\hat{p}\in I\!\!R^I$ and $q\in\Delta(J)$, where

$$\mathbf{X}_{t_1}^{t_0,\mathbf{x}^0,\alpha,\beta} = \left(X_{t_1}^{t_0,x^0_{ij},\alpha,\beta}\right)_{\substack{i=1,\dots,I\\ j=1,\dots,J}}\ .$$

Hence $V^{-*}(\cdot,\cdot,\hat{p},q)$ is a subsolution of the dual HJ equation

$$z_t + \mathbf{H}^*(x,Dz) = 0 \qquad \text{in } [0,T]\times I\!\!R^{NIJ}$$

for any $(\hat{p}, q)$, which means that V^- is a dual supersolution of (36). One proves in the same way that V^+ is a dual subsolution of (36). The comparison Theorem 5.1 then implies that $V^+ \leq V^-$. Since the inequality $V^- \leq V^+$ is obvious, we get the equality and the characterization of the value function.

QED

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