Université Paris-Dauphine
Master MASEF and Master MATH
2023-2024

Mean field games
(Very short list of) references and former exams

Some courses on the web

1. Pierre Cardaliaguet
   https://www.ceremade.dauphine.fr/ cardaliaguet/MFG20130420.pdf

2. Daniel Lacker
   - Short course = http://www.columbia.edu/ dl3133/IPAM-MFGCompactnessMethods.pdf
   - Rough notes of a PhD course = http://www.columbia.edu/ dl3133/MFGSpring2018.pdf

Two monographs on the subject:

1. Cardaliaguet, P., Delarue, F., Lasry, J. M., and Lions, P. L. (2019). The master equation and the convergence problem in mean field games:(ams-201). Princeton University Press.

2. Carmona, R., and Delarue, F. (2018). Probabilistic theory of mean field games with applications I-II. Switzerland: Springer Nature.

Exam of Mean Field Games — March 22, 2022
Master 2 MATH-MASEF
3h - Documents allowed: written notes and the notes on the course (on Teams).

The aim of this exam is to investigate a Mean Field Game (MFG) problem in which the player's cost depends on the control played by the other players.

The mean field game system. We are interested in a problem in which agents interact through the average $(\beta(t))_{0\leq t\leq T}$ of the optimal control. The MFG system takes the form

$$(MFG) \qquad \begin{cases} -\partial_t u - \Delta u + \frac{1}{2}|Du|^2 = F(x,\beta(t)) & \text{for any } (t,x)\in(0,T)\times\mathbb{R}^d \\ \partial_t m - \Delta m - \operatorname{div}(mDu) = 0 & \text{in } (0,T)\times\mathbb{R}^d \\ m(0)=m_0, \qquad u(T,x)=g(x) & \text{for any } x\in\mathbb{R}^d \\ \beta(t) = -\int_{\mathbb{R}^d} Du(t,x)m(t,dx) & \text{for any } t\in[0,T] \end{cases}$$

In the above system, the unknown are the deterministic maps $u:[0,T]\times\mathbb{R}^d\to\mathbb{R}$, $m\in C^0([0,T],\mathcal{P}_1(\mathbb{R}^d))$ and $\beta:[0,T]\to\mathbb{R}$. We denote by $Du(t,x)$, $D^2u(t,x)$ and $\Delta u(t,x)$ the gradient, the Hessian matrix and the Laplacian of u at $(t,x)\in[0,T]\times\mathbb{R}^d$ respectively. Finally, $|p|$ denote the euclidean norm of a vector $p\in\mathbb{R}^d$. The problem is on a finite horizon $T>0$ and the initial distribution of the players is the probability density m_0 on $\mathbb{R}^d$. We assume that $\int_{\mathbb{R}^d}|x|m_0(x)dx$ is finite. The continuous map $F:\mathbb{R}^d\times\mathbb{R}^d\to\mathbb{R}$ is the running cost while the continuous map $g:\mathbb{R}^d\to\mathbb{R}$ is the terminal cost of the problem.

Part 1. Computation on an elementary example (6 pts). Throughout this part *and in this part only* we assume that

$$F(x,b) = x\cdot b + \frac{1}{2}|x|^2, \qquad g(x) = \frac{1}{2}|x|^2 \qquad \forall (x,b)\in(\mathbb{R}^d)^2,$$

where $x\cdot b$ denote the usual scalar product between x and b.

1. Given a continuous map $\beta:[0,T]\to\mathbb{R}^d$, show that a solution to the Hamilton-Jacobi equation

$$\begin{cases} -\partial_t u(t,x) - \Delta u(t,x) + \frac{1}{2}|Du(t,x)|^2 = F(x,\beta(t)) & \text{for any } (t,x)\in(0,T)\times\mathbb{R}^d \\ u(T,x) = g(x) & \text{for any } x\in\mathbb{R}^d \end{cases}$$

   is given by $u(t,x) = \frac{1}{2}|x|^2 + B(t)\cdot x + C(t) \qquad \forall (t,x)\in[0,T]\times\mathbb{R}^d$, where $B:[0,T]\to\mathbb{R}^d$ and $C:[0,T]\to\mathbb{R}$ are the C^1 maps solving the equations

$$-B'(t)+B(t) = \beta(t), \qquad -C'(t) - d + \frac{1}{2}|B(t)|^2 = 0,\ \forall t\in(0,T), \qquad B(T)=0,\ C(T)=0.$$

2. Let $(\Omega,\mathcal{F},\mathbb{P})$ be a stochastic basis endowed with a filtration $(\mathcal{F}_t)_{t\geq 0}$ satisfying the usual assumptions and $(W_t)_{t\in[0,T]}$ be a $d-$dimensional $\mathcal{F}-$Brownian motion. We also fix $\bar{X}_0$ a $\mathcal{F}_0-$measurable random variable on $\mathbb{R}^d$, independent of (W_t) and of law m_0. We set $\bar{E}_0 := \mathbb{E}[X_0]$. Let u be the map given in the previous question. Show that the equation

$$X_t = \bar{X}_0 - \int_0^t Du(s,X_s)ds + \sqrt{2}W_t \qquad t\in[0,T]$$

   has a unique solution and that the law $m(t)=\mathcal{L}(X_t)$ solves in the sense of distribution the Kolmogorov equation

$$\begin{cases} \partial_t m - \Delta m - \operatorname{div}(mDu) = 0 & \text{in } (0,T)\times\mathbb{R}^d \\ m(0)=m_0 & \text{in } \mathbb{R}^d \end{cases}$$

3. Let β, u, X and m be defined as in the two previous questions. Explain why X_t is integrable for any $t \in [0,T]$ and prove that the map $t \to E(t) := \mathbb{E}[X_t]$ is of class C^2 and satisfies

$$E'(t) = -E(t) - B(t) \qquad \forall t \in (0,T), \qquad E(0) = \bar{E}_0.$$

4. Let β, u, X, m and E be defined as in the three previous questions. Show that (u, m, β) solves the MFG system (MFG) if and only if

$$B(t) = -E'(t) - E(t),\ C(t) = d(T-t) - \int_t^T \frac{1}{2}|B(s)|^2 ds,\ \beta(t) = -E(t) - B(t) \qquad \forall t \in (0,T)$$

and

$$E''(t) - E'(t) - E(t) = 0\ \ \forall t \in (0,T), \qquad E(0) = \bar{E}_0,\ E'(T) + E(T) = 0.$$

5. Infer from the previous questions that the MFG system (MFG) has at least one solution.

Part 2. An application to games with a finite number of players (6 pts). Throughout this part we assume that the maps $F : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ and $g : \mathbb{R}^d \to \mathbb{R}$ are globally Lipschitz continuous and bounded. We assume the existence of a classical solution (u, m, β) of system (MFG) (by classical, we mean that u is a solution of the Hamilton-Jacobi equation of class $C^{1,2}$, bounded with bounded derivatives, that $m \in C^0([0,T], \mathcal{P}_1(\mathbb{R}^d))$ solves the Kolmogorov equation in the sense of distribution and that β is of class C^1).

Given $(\Omega, \mathcal{F}, \mathbb{P})$, $(\mathcal{F}_t)_{t\geq 0}$, $(W_t)_{t\in[0,T]}$ and $\bar{X}_0$ as in question (2), we denote by $\mathcal{A}$ the set of progressively measurable processes $\alpha : [0,T] \times \Omega \to \mathbb{R}^d$ such that $\mathbb{E}\left[\int_0^T |\alpha_t|^2 dt\right]$ is finite. We set

$$J^\infty(\alpha) = \mathbb{E}\left[\int_0^T (\frac{1}{2}|\alpha_t|^2 + F(X_t^\alpha, \beta(t)))dt + g(X_T^\alpha)\right],$$

where $X_t^\alpha = \bar{X}_0 + \int_0^t \alpha_s ds + \sqrt{2} W_t \qquad \forall t \in [0,T]$.

6. Let X^* be the unique solution to

$$X_t^* = \bar{X}_0 - \int_0^t Du(s, X_s^*)ds + \sqrt{2} W_t \qquad t \in [0,T]$$

and $\alpha_t^* := -Du(t, X_t^*)$ for $t \in [0,T]$. Show that $\alpha^* \in \mathcal{A}$ and that

$$J^\infty(\alpha) \geq J^\infty(\alpha^*) \qquad \forall \alpha \in \mathcal{A}.$$

We now consider a game with a finite number of players. We denote by $N \geq 2$ the number of players. Let $(Z^i)_{i\geq 1}$ be a family of independent $\mathcal{F}_0-$measurable random variables with law m_0 and $(W^i)_{i\geq 1}$ be a family of independent $\mathcal{F}-$Brownian motions independent of the (Z^i). Given $(\alpha^{N,j})_{j=1,\dots,N}$ a family of elements of $\mathcal{A}$ and $\alpha \in \mathcal{A}$, we set

$$J^{N,i}(\alpha, ((\alpha^{N,j})_{j\neq i})) = \mathbb{E}\left[\int_0^T (\frac{1}{2}|\alpha_t|^2 + F(X_t^{i,\alpha}, \frac{1}{N-1}\sum_{j\in\{1,\dots,N\},\ j\neq i} \alpha_t^{N,j}))dt + g(X_T^{i,\alpha})\right],$$

where $X_t^{i,\alpha} = Z^i + \int_0^t \alpha_s ds + \sqrt{2} W_t^i$. Finally, for $j \geq 1$, we let $X^{*,j}$ be the solution to

$$X_t^{*,j} = Z^i - \int_0^t Du(s, X_s^{*,j})ds + \sqrt{2} W_t^j \qquad t \in [0,T]$$

and $\alpha_t^{*,j} := -Du(t, X_t^{*,j})$ for $t \in [0,T]$.

7. Show that, for any $\epsilon > 0$, there exists $N_\epsilon \in \mathbb{N}$ such that, if $N \geq N_\epsilon$ and $i \in \{1, \dots, N\}$, then

$$\int_0^T \mathbb{E}\left[\sup_{x\in\mathbb{R}^d}\left|F(x,\beta(t)) - F(x, \frac{1}{N-1}\sum_{j\in\{1,\dots,N\},\ j\neq i}\alpha_t^{*,j})\right|\right] \leq \epsilon/2.$$

8. Infer from this that, for any $N \geq N_\epsilon$, the family $(\alpha^{*,j})_{j=1,\dots,N}$ is an $\epsilon-$Nash equilibrium of the game: namely, for any $i \in \{1, \dots N\}$, we have

$$J^{N,i}(\alpha, ((\alpha^{*,j})_{j\neq i})) \geq J^{N,i}(\alpha^{*,i}, ((\alpha^{N,j})_{j\neq i})) - \epsilon \qquad \forall \alpha \in \mathcal{A}.$$

Part 3. Existence of a solution (12pts). Throughout this part we assume that the maps $F : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ and $g : \mathbb{R}^d \to \mathbb{R}$ are globally Lipschitz continuous and bounded. Our aim is to show the existence of a solution to (MFG).

9. Show that there exists a constant $C_0 > 0$ such that, for any continuous map $\beta : [0,T] \to \mathbb{R}^d$, the unique solution u^β to the Hamilton-Jacobi equation

$$\begin{cases} -\partial_t u^\beta(t,x) - \Delta u^\beta(t,x) + \frac{1}{2}|Du^\beta(t,x)|^2 = F(x,\beta(t)) \qquad \text{for any } (t,x) \in (0,T) \times \mathbb{R}^d \\ u^\beta(T,x) = g(x) \qquad \text{for any } x \in \mathbb{R}^d \end{cases}$$

satisfies

$$|u^\beta(t,x) - u^\beta(t,y)| \leq C_0|x-y| \qquad \forall (t,x,y) \in [0,T] \times \mathbb{R}^d \times \mathbb{R}^d.$$

We will admit that u^β is a classical solution of the Hamilton-Jacobi equation and that there exists a constant C_1, independent of β, such that

$$\sup_{(t,x)\in[0,T]\times\mathbb{R}^d} \{|\partial_t u^\beta(t,x)| + |D^2 u^\beta(t,x)|\} \leq C_1.$$

10. Prove that a $C^{1,2}$ map $w : [0,T] \times \mathbb{R}^d \to \mathbb{R}$ satisfying $\sup_{(t,x)\in[0,T]\times\mathbb{R}^d} \{|\partial_t w(t,x)| + |D^2 w(t,x)|\} \leq C_1$ also verifies

$$|Dw(t_1,x) - Dw(t_2,x)| \leq 2C_1|t_1 - t_2|^{1/2} \qquad \forall (t_1,t_2,x) \in [0,T]^2 \times \mathbb{R}^d.$$

Indication: notice that for any $(t_1,t_2,x) \in [0,T]^2 \times \mathbb{R}^d$, $z \in \mathbb{R}^d$ with $|z| \leq 1$ and $h > 0$,

$$\left|(Du(t_1,x) - Du(t_2,x)) \cdot z - h^{-1}\{u(t_1,x+hz) - u(t_1,x) - u(t_2,x+hz) + u(t_2,x)\}\right| \leq 2C_1h^2.$$

11. Given $(\Omega, \mathcal{F}, \mathbb{P})$, $(\mathcal{F}_t)_{t\geq 0}$, $(W_t)_{t\in[0,T]}$ and $\bar{X}_0$ as in question (2), let X be the unique solution to

$$X_t = \bar{X}_0 - \int_0^t Du(s,X_s)ds + \sqrt{2}W_t \qquad t \in [0,T]$$

Let us set $\Phi(\beta)(t) = -\mathbb{E}\left[Du^\beta(t,X_t)\right]$ for $t \in [0,T]$. Show that there is a constant C_2, independent of β, such that

$$\sup_{t\in[0,T]} |\Phi(\beta)(t)| \leq C_0 \qquad \text{and} \qquad |\Phi(\beta)(t_1) - \Phi(\beta)(t_2)| \leq C_2|t_1-t_2|^{1/2} \qquad \forall t_1,t_2 \in [0,T].$$

12. Infer the existence of a nonempty convex compact subset $\mathcal{K}$ of $C^0([0,T],\mathbb{R}^d)$ such that, if $\beta \in \mathcal{K}$, then $\Phi(\beta) \in \mathcal{K}$ (where $\Phi(\beta)$ is defined as above).

13. (difficult) Show that the map $\beta \to \Phi(\beta)$ is continuous in $C^0([0,T],\mathbb{R}^d)$.

14. Conclude that the system (MFG) has at least one classical solution.

Exam of Mean Field Games — March 8, 2021
Master 2 MATH-MASEF
3h - No document allowed.

The aim of this exam is to investigate stationary Mean Field Games in infinite horizon. Such problems often appear in models of heterogeneous agents in macroeconomy.

Let $\mathcal{P}_1(\mathbb{R}^d)$ be the set of Borel probability measures on $\mathbb{R}^d$ with finite first order moment, endowed with the Monge-Kantorovitch distance

$$\mathbf{d}_1(m,m') = \sup_{\phi\ 1-\text{Lipschitz}} \int_{\mathbb{R}^d} \phi(x)(m(dx)-m'(dx)) \qquad \forall m,m' \in \mathcal{P}_1(\mathbb{R}^d).$$

Let $F:\mathbb{R}^d\times\mathcal{P}_1(\mathbb{R}^d)\to\mathbb{R}$ be continuous map. We consider the MFG system (with unknown (u,m))

$$(MFG) \qquad \left\{\begin{array}{l} u(x)-\dfrac{1}{2}\Delta u(x)+\dfrac{1}{2}|Du(x)|^2 = F(x,m) \text{ in } \mathbb{R}^d \\ -\dfrac{1}{2}\Delta m(x) - \text{div}(m(x)Du(x)) = 0 \text{ in } \mathbb{R}^d \end{array}\right.$$

We say that (u,m) is a classical solution of the above system if the maps $u:\mathbb{R}^d\to\mathbb{R}$ and $m:\mathbb{R}^d\to\mathbb{R}$ are of class C^2, m being the density of a Borel probability measure in $\mathcal{P}_1(\mathbb{R}^d)$.

Throughout this part, we fix a stochastic basis $(\Omega,\mathcal{F},\mathbb{P})$ endowed with a filtration $(\mathcal{F}_t)_{t\geq 0}$ (satisfying the usual assumptions) and a $d-$dimensional Brownien motion $(B_t)_{t\geq 0}$. We denote by $\mathcal{A}$ the set of processes $\alpha:[0,+\infty)\times\Omega\to\mathbb{R}^d$, adapted to the filtration $(\mathcal{F}_t)_{t\geq 0}$, such that

$$\mathbb{E}\left[\int_0^{+\infty} e^{-t}|\alpha_t|^2 dt\right] < +\infty.$$

For $x_0\in\mathbb{R}^d$, we denote by $X^{x_0,\alpha}$ the solution to

$$dX_t^{x_0,\alpha} = \alpha_t dt + dB_t, \qquad X_0^{x_0,\alpha} = x_0.$$

Finally, we denote by $\mathcal{A}_{ad}$ the set of $\alpha\in\mathcal{A}$ such that $\mathbb{E}\left[\int_0^{+\infty} e^{-t}|X_t^{x_0,\alpha}|^2 dt\right] < +\infty$ for some (and thus for all) $x_0\in\mathbb{R}^d$.

Part1: Interpretation of the system (MFG)

In this question, we assume that (u,m) is a classical solution to (MFG) such that Du is globally Lipschitz continuous. We also suppose that there exists a constant $C_0>0$ such that

$$(*) \qquad C_0^{-1}|x|^2 - C_0 \leq F(x,m) \leq C_0(|x|^2+1) \qquad \forall (x,m)\in\mathbb{R}^d\times\mathcal{P}_1(\mathbb{R}^d)$$

and will admit that this implies that there exists a constant $C_1>0$ such that

$$(**) \qquad C_1^{-1}|x|^2 - C_1 \leq u(x) \leq C_1(|x|^2+1) \qquad \forall x\in\mathbb{R}^d.$$

(I-(i)) Fix $\alpha\in\mathcal{A}_{ad}$ and $x_0\in\mathbb{R}^d$. Check that, for any bounded stopping time $\tau>0$, one has

$$u(x_0) \leq \mathbb{E}\left[\int_0^{\tau} e^{-t}\left(\frac{1}{2}|\alpha_t|^2 + F(X_t^{x_0,\alpha},m)\right)dt + e^{-\tau}u(X_\tau^{x_0,\alpha})\right].$$

(I-(ii)) Using the fact that $\alpha\in\mathcal{A}_{ad}$ and inequalities (*) and (**), show that there exists a deterministic sequence $T_n\to+\infty$ such that $\mathbb{E}[e^{-T_n}|X_{T_n}^{x_0,\alpha}|^2]\to 0$ and derive from this that

$$u(x_0) \leq \mathbb{E}\left[\int_0^{+\infty} e^{-t}\left(\frac{1}{2}|\alpha_t|^2 + F(X_t^{x_0,\alpha},m)\right)dt\right].$$

(I-(iii)) Let $\bar\alpha_t = -Du(X_t)$ where (X_t) is the solution to

$$dX_t = -Du(X_t)dt + dB_t,\ t \geq 0, \qquad X_0 = x_0.$$

Show that, for any bounded stopping time $\tau > 0$, one has

$$u(x_0) = \mathbb{E}\left[\int_0^\tau e^{-t}\left(\frac{1}{2}|\bar\alpha_t|^2 + F(X_t^{x_0,\bar\alpha}, m)\right)dt + e^{-\tau}u(X_\tau^{x_0,\bar\alpha})\right].$$

(I-(iv)) Derive from the previous question and inequality (*) that $\bar\alpha \in \mathcal{A}_{ad}$ and that

$$u(x_0) = \mathbb{E}\left[\int_0^{+\infty} e^{-t}\left(\frac{1}{2}|\bar\alpha_t|^2 + F(X_t^{x_0,\bar\alpha}, m)\right)dt\right].$$

(I-(v)) Conclude that

$$u(x_0) = \inf_{\alpha\in\mathcal{A}_{ad}} \mathbb{E}\left[\int_0^{+\infty} e^{-t}\left(\frac{1}{2}|\alpha_t|^2 + F(X_t^{x_0,\alpha}, m)\right)dt\right].$$

(I-(vi)) Let $\bar X_0$ be a random variable with law m independent of B and (X_t) be the solution of the SDE

$$dX_t = -Du(X_t)dt + dB_t,\ t \geq 0, \qquad X_0 = \bar X_0.$$

Write the (time dependent) Kolmogorov equation satisfied by the law of (X_t) (in the sense of distributions) and, recalling that this equation has a unique solution, show that the law of X_t is constant and equal to m for any $t \geq 0$.
In other words, m is the invariant measure associated with the drift $-Du$.

Part II : Application to games with a large number of players

In this (short) question, we assume again that (u, m) is a classical solution to (MFG) such that Du is globally Lipschitz continuous and that (*) and (**) hold. In addition we suppose that $\int_{\mathbb{R}^d} |x|^2 m(dx) < +\infty$. Let $(B^i)_{i\in\mathbb{N}}$ be a family of independent, $d-$dimensional Brownian motions adapted to the filtration $(\mathcal{F}_t)$ and $(\bar X_0^i)_{i\in\mathbb{N}}$ be a family of independent random variable of law m, independent of the (B^i) and $\mathcal{F}_0-$measurable.

We fix $N \in \mathbb{N}$ a large number of players. Given $(\alpha^1, \dots, \alpha^N) \in (\mathcal{A}_{ad})^N$, the cost of player $i \in \{1, \dots, N\}$ is given by

$$J^{N,i}(\alpha^i, (\alpha^j)_{j\neq i}) = \mathbb{E}\left[\int_0^{+\infty} e^{-t}\left(\frac{1}{2}|\alpha_t^i|^2 + F(X_t^{i,\alpha^i}, m_{X_t}^N)\right)dt\right],$$

where $m_{X_t}^N = \dfrac{1}{N}\sum_{j=1}^N \delta_{X_t^{j,\alpha^j}}$ and where X^{i,α^i} solves

$$dX_t^{i,\alpha^i} = \alpha_t^i dt + dB_t^i,\ t \geq 0, \qquad X_0^{i,\alpha^i} = \bar X_0^i.$$

Finally let $(\tilde X^i)$ be the solution to

$$d\tilde X_t^i = -Du(\tilde X_t^i)dt + dB_t^i,\ t \geq 0, \qquad \tilde X_0^i = \bar X_0^i.$$

We set $\bar\alpha_t^i = -Du(\tilde X_t^i)$ and note that (by the previous part) $\bar\alpha^i \in \mathcal{A}_{ad}$ and that the $(\tilde X_t^i = X_t^{i,\bar\alpha^i})_{i=1,\dots,N}$ are independent and of law m.

(II-(i)) Check that, for any $i \in \mathbb{N}$,

$$\lim_{N\to+\infty} J^{N,i}(\bar\alpha^i, (\bar\alpha^j)_{j\neq i}) = \int_{\mathbb{R}^d} u(x)m(x)dx.$$

(II-(ii)) Check also that, for any $i \in \mathbb{N}$ and any $\alpha^i \in \mathcal{A}_{ad}$,

$$\liminf_{N\to+\infty} J^{N,i}(\alpha^i, (\bar{\alpha}^j)_{j\neq i}) \geq \int_{\mathbb{R}^d} u(x)m(x)dx.$$

These two results suggest (but do not prove) that $(\bar{\alpha}^1, \dots, \bar{\alpha}^N)$ is an approximate Nash equilibrium as N is large.

Part III : Construction of a solution to the system (MFG).

From now on, we assume that $F(x,m) = |x + G(m)|^2$, where $G : \mathcal{P}_1(\mathbb{R}^d) \to \mathbb{R}^d$ is continuous and bounded.

(III-(i)) Let $v : \mathbb{R}^d \to \mathbb{R}$ be C^2 and such that $x \to \exp\{-2v(x)\}$ is integrable. Show that $\mu(x) := \left(\int_{\mathbb{R}^d} \exp\{-2v(y)\}dy\right)^{-1} \exp\{-2v(x)\}$ solves the Kolmogorov equation

$$-\frac{1}{2}\Delta\mu(x) - \mathrm{div}(\mu(x)Dv(x)) = 0 \text{ in } \mathbb{R}^d$$

We will admit in the sequel that, if in addition Dv is globally Lipschitz continuous, then μ is the unique probability measure which satisfies the above Kolmogorov equation in the sense of distributions.

(III-(ii)) Check that, given $m \in \mathcal{P}_1(\mathbb{R}^d)$, there is a solution u to the Hamilton-Jacobi equation (the first equation) in (MFG) which is of the form $u(x) = \beta_1|x+a|^2 + \beta_2$, where $a \in \mathbb{R}^d$, $\beta_1 > 0$ and $\beta_2 \in \mathbb{R}$ are to be computed explicitly in function of $G(m)$ and d.

(III-(iii)) Let u be as in the previous question. Show that there is a constant $K > 0$, *independent of* m, such that $\mu(x) := (\int_{\mathbb{R}^d} \exp\{-2u(y)\}dy)^{-1} \exp\{-2u(x)\}$ satisfies $\int_{\mathbb{R}^d} |x|^2\mu(x)dx \leq K$.

(III-(iv)) Infer from the previous questions that there exists a solution (u,m) to the system (MFG) which satisfies (*) and (**) and is such that $\int_{R^d} |x|^2 m(x)dx < +\infty$.

Exam of "Mean field games" — 04/03/2018
Master 2 MASEF
Length 2h - all documents are allowed.

For $N \geq 2$, we consider the N player game in which each player controls her state (an element of $\mathbb{R}^d$, endowed with the euclidean distance $|\cdot|$) through the speed (which is bounded by 1) and where the cost depends on the position of the player and on the empirical distribution of the other players. The game is played in a finite horizon $T > 0$. We assume that all players start from the same initial point $x_0 \in \mathbb{R}^d$. We denote by $\mathcal{P}_1$ the set of Borel probability measures on $\mathbb{R}^d$ with a finite first order moment, endowed with the Monge-Kantorovitch distance $\mathbf{d}_1$. The aim of the exercise is to characterize the limit of this game, as $N \to +\infty$, as a mean field game.

To formalize this game, we define E as the set of 1−Lipschitz continuous maps[1] $\gamma : [0, T] \to \mathbb{R}^d$ such that $\gamma(0) = x_0$:

$$E := \{\gamma \in W^{1,\infty}([0, T], \mathbb{R}^d),\ \gamma(0) = x_0,\ \|\dot\gamma\|_{L^\infty([0,T])} \leq 1\}.$$

The action of each player $i \in \{1, \dots, N\}$ consists in choosing an element $\gamma^i \in E$. Given actions $\gamma^1, \dots, \gamma^N$ in E for each player, the associated cost for player $i \in \{1, \dots, N\}$ is

$$J^{N,i}(\gamma^i, (\gamma^j)_{j\neq i}) = \int_0^T F(\gamma^i(t), m^{N,i}_{\gamma(t)})dt,$$

where $F : \mathbb{R}^d \times \mathcal{P}_1 \to \mathbb{R}$ is a *continuous and bounded* map and

$$\gamma = (\gamma^1, \dots, \gamma^N) \qquad \text{and} \qquad m^{N,i}_{\gamma(t)} := \frac{1}{N-1}\sum_{j\neq i}\delta_{\gamma^j(t)},\ \forall t \in [0, T].$$

1. (Compactness of the action set) Check that E, endowed with the distance

   $$d_E(\gamma_1, \gamma_2) := \sup_{t\in[0,T]} |\gamma_1(t) - \gamma_2(t)| \qquad \forall \gamma_1, \gamma_2 \in E,$$

   is a compact set.

   We denote by $\mathcal{P}(E)$ the set of Borel probability measures on E (the convergence on $\mathcal{P}(E)$ is the narrow convergence, or, in an equivalent way, the associated Monge-Kantorovitch distance).

2. (Rewriting the cost function) For any $\eta \in \mathcal{P}(E)$ and any $t \in [0, T]$, we define the Borel probability measure $e_t\sharp\eta$ on $\mathbb{R}^d$ by: for any continuous and bounded map $\varphi : \mathbb{R}^d \to \mathbb{R}$,

   $$\int_{\mathbb{R}^d} \varphi(x) e_t\sharp\eta(dx) = \int_E \varphi(\gamma(t))\eta(d\gamma).$$

   (a) Show that $e_t\sharp\eta \in \mathcal{P}_1$.

   (b) Show that the map $t \to e_t\sharp\eta$ is continuous from $[0, T]$ to $\mathcal{P}_1$ for the distance $\mathbf{d}_1$.

   (c) Show that, if the sequence (η_n) of $\mathcal{P}(E)$ converges narrowly to $\eta \in \mathcal{P}(E)$, then, for any $t \in [0, T]$, the sequence $(e_t\sharp\eta_n)$ converges to $e_t\sharp\eta$ in $\mathcal{P}_1$ for the distance $\mathbf{d}_1$.

   (d) Let us define $J : E \times \mathcal{P}(E) \to \mathbb{R}$ by

   $$J(\gamma, \eta) := \int_0^T F(\gamma(t), e_t\sharp\eta)dt \qquad \forall (\gamma, \eta) \in E \times \mathcal{P}(E).$$

   Show that J is continuous on $E \times \mathcal{P}(E)$.

   (e) Finally show that, for any $(\gamma^1, \dots, \gamma^N) \in E^N$,

   $$J^{N,i}(\gamma^i, (\gamma^j)_{j\neq i}) = J(\gamma^i, \frac{1}{N-1}\sum_{j\neq i}\delta_{\gamma^j}).$$

[1] In other words, $\gamma \in W^{1,\infty}((0, T), \mathbb{R}^d)$ with $\|\dot\gamma\|_\infty \leq 1$

3. (The mean field limit) For any $N \geq 2$, let $\bar\gamma^N := (\bar\gamma^{N,1}, \dots \bar\gamma^{N,N})$ be a Nash equilibrium of the game: for any $i \in \{1, \dots, N\}$

$$J^{N,i}(\bar\gamma^{N,i}, (\bar\gamma^{N,j})_{j\neq i}) \leq J^{N,i}(\gamma^i, (\bar\gamma^{N,j})_{j\neq i}) \qquad \forall \gamma^i \in E.$$

We define η^N as the empirical measure associated with the actions $\bar\gamma^N$:

$$\eta^N := \frac{1}{N}\sum_{i=1}^N \delta_{\bar\gamma^{N,i}}.$$

By using a result in the course, show that there exists a subsequence (N_k) such that the sequence (η^{N_k}) converges narrowly to some $\bar\eta \in \mathcal{P}(E)$ which satisfies

$$\int_E J(\gamma, \bar\eta)\bar\eta(d\gamma) \leq \int_E J(\gamma, \bar\eta)\eta(d\gamma) \qquad \forall \eta \in \mathcal{P}(E).$$

We set is the sequel $m(t) := e_t \sharp \bar\eta$ *(*$t \in [0, T]$*).*

4. (The limit problem) We consider the value function u, defined, for any $(t, x) \in [0, T] \times \mathbb{R}^d$, by

$$u(t,x) := \inf\left\{\int_t^T F(\gamma(s), m(s))ds \text{ where } \gamma \in W^{1,\infty}([t,T], \mathbb{R}^d),\ \gamma(t) = x,\ \|\dot\gamma\|_{L^\infty([t,T])} \leq 1\right\}.$$

Show that

$$u(0, x_0) = \int_E J(\gamma, \bar\eta)\bar\eta(d\gamma).$$

5. (The MFG system) We assume that there exists a map $v : [0, T] \times \mathbb{R}^d \to \mathbb{R}$, of class C^1, solution to

$$\begin{cases} -\partial_t v(t,x) + |Dv(t,x)| = F(x, m(t)) & \text{in } (0,T)\times\mathbb{R}^d, \\ v(T,x) = 0 \quad \text{in } \mathbb{R}^d. \end{cases}$$

   (a) Show that $v \leq u$ in $[0, T] \times \mathbb{R}^d$.

   (b) Let us assume in addition that $Dv(t, x) \neq 0$ and that the map $(t, x) \to Dv(t, x)/|Dv(t, x)|$ is Lipschitz continuous. For $(t, x) \in (0, T] \times \mathbb{R}^d$, let $\bar\gamma$ be the solution to

$$\begin{cases} \dot{\bar\gamma}(s) = -\frac{Dv(s,\bar\gamma(s))}{|Dv(s,\bar\gamma(s))|} & \forall s \in [t, T], \\ \bar\gamma(t) = x. \end{cases}$$

   Show that $\bar\gamma$ is optimal in the problem defining $u(t, x)$ and that $v(t, x) = u(t, x)$.

   (c) Show that, under the same assumptions as above, m is a solution, in the sens of distributions, of

$$\begin{cases} \partial_t m - \mathrm{div}(m\frac{Dv}{|Dv|}) = 0 & \text{in } (0,T)\times\mathbb{R}^d, \\ m(0) = \delta_{x_0} \quad \text{in } \mathbb{R}^d. \end{cases}$$

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Throughout the text, $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P})$ is a filtered probability space satisfying the usual conditions. Let $(B_t)_{t\in[0,T]}$ be a standard one-dimensional Brownian motion (with $B_0 = 0$) adapted to the filtration $(\mathcal{F}_t)_{t\in[0,T]}$ and X_0 be a real-valued initial condition X_0 which is $\mathcal{F}_0-$mesurable and such that $\mathbb{E}[|X_0|] < \infty$. Let $\mathcal{A}$ be the set of controls, i.e., the set of $(\mathcal{F}_t)_{t\in[0,T]}-$progressively measurable processes $\alpha = (\alpha_t)_{t\in[0,T]}$ (with values in $\mathbb{R}$) such that

$$\mathbb{E}\left[\int_0^T |\alpha_t|^2 dt\right] < \infty.$$

We denote by $\mathcal{P}_1$ the set of Borel probability measures on $\mathbb{R}$ with a finite first order moment, endowed with the usual $\mathbf{d}_1$ distance. Finally, let $F : \mathbb{R} \times \mathcal{P}_1 \to \mathbb{R}$ be a locally bounded, continuous map with a derivative F_x with respect to the first variable which is bounded and globally Lipschitz continuous: there exists $L > 0$ with

$$|F_x(x,m) - F_x(y,\bar{m})| \le L(|x-y| + \mathbf{d}_1(m,\bar{m})) \qquad \forall (x,m),(y,\bar{m}) \in \mathbb{R}\times\mathcal{P}_1.$$

For $(\alpha, m) \in \mathcal{A} \times C^0([0,T], \mathcal{P}_1)$, we set

$$J(\alpha,m) := \mathbb{E}\left[\int_0^T (\frac{|\alpha_s|^2}{2} ds + F(X_s^\alpha, m_s))\ ds\right]$$

where $X_s^\alpha := X_0 + \displaystyle\int_0^s \alpha_r dr + B_s$ for $s \in [0,T]$.

Definition: We say that $(\alpha^*, m^*) \in \mathcal{A} \times C^0([0,T], \mathcal{P}_1)$ is an MFG equilibrium if

(i) α^* is a minimizer of the problem

$$\inf_{\alpha\in\mathcal{A}} J(\alpha, m^*),$$

(ii) the law $\mathbb{P}_{X_t^{\alpha^*}}$ of $X_t^{\alpha^*}$ is $m^*(t)$ for any $t \in [0,T]$.

First part. The goal of this first part is to show the existence of a unique MFG equilibrium if $T > 0$ is small enough. For this we will admit that, for any $m \in C^0([0,T], \mathcal{P}_1)$, there exists a least a minimum in $\mathcal{A}$ to $\alpha \to J(\alpha, m)$.

1. (optimality condition) Let $m \in C^0([0,T], \mathcal{P}_1)$ and $\alpha^* \in \mathcal{A}$ be optimal for $J(\cdot, m)$. Prove that, for any $\beta \in \mathcal{A}$,

$$\mathbb{E}\left[\int_0^T (\alpha_s^*\beta_s + F_x(X_s^{\alpha^*}, m_s)(\int_0^s \beta_r dr))\ ds\right] = 0.$$

(Hint: compare $J(\alpha^*, m)$ and $J(\alpha^* + h\beta, m)$ for $h \in \mathbb{R}$ small.)

2. Let m and α^* be as in the previous question. Show that

$$\mathbb{E}\left[\int_0^T (\alpha_s^* + \int_s^T F_x(X_r^{\alpha^*}, m_r)dr)\beta_s \, ds\right] = 0 \qquad \forall \beta \in \mathcal{A},$$

and conclude that

$$\alpha_s^* = -\mathbb{E}\left[\int_s^T F_x(X_r^{\alpha^*}, m_r)dr|\mathcal{F}_s\right] \ a.s., \text{ for a.e. } s \in [0,T].$$

3. Prove that there exists constants $T_0 > 0$ and $C_0 > 0$ such that, if $T \in (0, T_0)$, for any $m, \bar{m} \in C^0([0,T], \mathcal{P}_1)$ and any minimizer α^* of $J(\cdot, m)$ and $\bar{\alpha}^*$ of $J(\cdot, \bar{m})$,

$$\mathbb{E}\left[\int_0^T |\alpha_s^* - \bar{\alpha}_s^*|ds\right] \leq C_0 T \sup_{s\in[0,T]} \mathbf{d}_1(m_s, \bar{m}_s).$$

4. Let $\alpha, \bar{\alpha} \in \mathcal{A}$. Show that

$$\sup_{s\in[0,T]} \mathbf{d}_1(\mathbb{P}_{X_s^\alpha}, \mathbb{P}_{X_s^{\bar{\alpha}}}) \leq \mathbb{E}\left[\int_0^T |\alpha_s - \bar{\alpha}_s|ds\right].$$

5. Infer from the two previous questions the existence of $T_1 > 0$ such that, if $T \in (0, T_1)$, there exists a unique MFG equilibrium.
(Hint: Define the MFG equilibrium as the fixed point of a contraction on $C^0([0,T], \mathcal{P}_1)$, which is a complete metric space when endowed with the distance $\delta(m, \bar{m}) := \sup_{t\in[0,T]} \mathbf{d}_1(m_s, \bar{m}_s)$ for $m, \bar{m} \in C^0([0,T], \mathcal{P}_1)$).

Second part. Throughout this part (except in the last question), we assume that

$$F(x,m) = x\int_{\mathbb{R}} ym(dy) \qquad \forall (x,m) \in \mathbb{R}\times\mathcal{P}_1. \tag{1}$$

6. Using Question 2, check that, if $m \in C^0([0,T], \mathcal{P}_1)$, then $\alpha^* \in \mathcal{A}$ is optimal for $J(\cdot, m)$, if and only if,

$$\alpha_s^* = -\int_s^T (\int_{\mathbb{R}} ym_r(dy))dr \qquad a.s., \text{ for a.e. } s \in [0,T].$$

7. Assume that (α^*, m^*) is an MFG equilibrium for F defined by (1). Set $e_t = \mathbb{E}\left[X_t^{\alpha^*}\right]$ for $t \in [0,T]$. Show that $t \to e_t$ is of class C^2 and that

$$\frac{d^2}{dt^2}e_t = e_t, \qquad \forall t \in [0,T],$$

with $e_0 = \mathbb{E}[X_0]$ and $\frac{d}{dt}e_t = 0$ at $t = T$.

8. Still assuming that F is given by (1), show that there exists a unique MFG equilibrium (α^*, m^*) and give α^* explicitely.

9. What happens if

$$F(x,m) = -x \int_{\mathbb{R}} y m(dy) \qquad \forall (x,m) \in \mathbb{R} \times \mathcal{P}_1 \ ?$$

Third part. The goal of this part is to extend ideas from the first part to a simple MFG with a common Brownian noise. We assume that the probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P})$ carries another standard Brownian motion W (with $W_0 = 0$) which is $(\mathcal{F}_t)_{t\in[0,T]}-$adapted and independent of B and of X_0. The coupling function between players is given by a smooth map $\Phi : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ such that Φ and its first order derivatives are bounded and Lipschitz continuous in both arguments.

We denote by Λ the set of $(\mathcal{F}_t)_{t\in[0,T]}-$progressively measurable processes $\lambda = (\lambda_t)_{t\in[0,T]}$ (with values in $\mathbb{R}$) such that

$$\|\lambda\|_\Lambda := \mathbb{E}\left[\sup_{t\in[0,T]} |\lambda_t|\right] < \infty.$$

Given $(\alpha, \lambda) \in \mathcal{A} \times \Lambda$, we set

$$J(\alpha, \lambda) := \mathbb{E}\left[\int_0^T \left(\frac{|\alpha_s|^2}{2} ds + \Phi(X_s^\alpha, \lambda_s)\right) ds\right],$$

where $X_s^\alpha := X_0 + \int_0^s \alpha_r dr + B_s + W_s$ for $s \in [0,T]$,

Definition: A pair $(\alpha^*, \lambda^*) \in \mathcal{A} \times \Lambda$ is an MFG equilibrium with common noise W if

(i) α^* is a minimizer of the problem

$$\inf_{\alpha\in\mathcal{A}} J(\alpha, \lambda^*)$$

(ii) for all $s \in [0,T]$, λ_s^* is the conditional expectation of $X_s^{\alpha^*}$ given W: $\lambda_s^* = \mathbb{E}\left[X_s^{\alpha^*} | W\right]$.

10. Admit that, for any $\lambda \in \Lambda$, there exists a least a minimum in $\mathcal{A}$ to $\alpha \to J(\alpha, \lambda)$. Following and adapting ideas from the first part, show that there exists $T_1 > 0$ such that, if $T \in (0, T_1)$, there exists a unique MFG equilibrium with common noise W.