

A short course on Mean field games and Mean field control problems

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Chapter 1

Introduction

Mean field game theory is devoted to the analysis of differential games with infinitely many players. For such large population dynamic games, it is unrealistic for a player to collect detailed state information about all other players. Fortunately this impossible task is useless: mean field game theory explains that one just needs to implement strategies based on the distribution of the other players. Such a strong simplification is well documented in the (static) game community since the seminal works of Aumann [4]. However, for differential games, this idea has been considered only very recently: the starting point is a series of papers by Lasry and Lions [44, 45, 46], who introduced the terminology in around 2005. The term mean field comes for an analogy with the mean field models in mathematical physics, which analyse the behavior of many identical particles (see for instance Sznitman's notes [59]). Here the particules are replaced by agents or players, whence the name of mean field games. Lions' ideas have been largely developed during his lectures at the Collège de France, available online. Related ideas have been developed independently, and at about the same time, by Caines, Huang and Malhamé [37, 38, 39, 40], under the name of Nash certainty equivalence principle. This analysis led also to a renewal of the interest of mean field control, in which a global planer aims at controlling a system made of a large number of particles.

This text aims at a short (and very incomplete) presentation of mean field games and mean field control problems, as self-contained as possible. Besides the papers already quoted, the main and central reference on these topics is the monograph by Carmona and Delarue [14].

Chapter 2

An appetizer: Nonatomic games

Before starting the analysis of differential games with a large number of players, it is not uninteresting to have a look at this question for classical (one-shot) games.

The general framework is the following: let N be a (large) number of players. We assume that the players are symmetric. In particular, the set of strategies Q is the same for all players. We denote by $F_i^N = F_i^N(x_1, \dots, x_N)$ the payoff (= the cost) of player $i \in \{1, \dots, N\}$. Our symmetry assumption means that

$$F_{\sigma(i)}^N(x_{\sigma(1)}, \dots, x_{\sigma(N)}) = F_i(x_1, \dots, x_N)$$

for all permutation σ on $\{1, \dots, N\}$. We consider Nash equilibria for this game and want to analyze their behavior as $N \rightarrow +\infty$.

For this we first recall the notion of Nash equilibria for one shot games. In order to proceed with the analysis of large population games, we describe next the limit of maps of many variable. Then we explain the limit, as the number of players tends to infinity, of Nash equilibria in pure, and then in mixed, strategies. We finally discuss the uniqueness of the solution of the limit equation and present some examples.

2.1 Nash equilibria in classical differential games

In this section, we introduce the notion of Nash equilibria in one-shot games.

Let $S_1, \dots, S_N$ be compact metric spaces, $J_1, \dots, J_N$ be continuous real valued functions on $\prod_{i=1}^N S_i$.

Definition 2.1.1. A Nash equilibrium in pure strategies is a N -tuple $(\bar{s}_1, \dots, \bar{s}_N) \in \prod_{i=1}^N S_i$ such that, for any $i = 1, \dots, N$,

$$J_i(\bar{s}_1, \dots, \bar{s}_N) \leq J_i(s_i, (\bar{s}_j)_{j \neq i}) \quad \forall s_i \in S_i .$$

Unfortunately Nash equilibria in pure strategies seldom exist and we have to introduce the notion of mixed strategies. For this we denote by $\mathcal{P}(S_i)$ the compact metric space of all Borel probability measures defined on S_i .

Definition 2.1.2. A Nash equilibrium in mixed strategies is a N -tuple $(\bar{\pi}_1, \dots, \bar{\pi}_N) \in \prod_{i=1}^N \mathcal{P}(S_i)$ such that, for any $i = 1, \dots, N$,

$$(2.1) \quad J_i(\bar{\pi}_1, \dots, \bar{\pi}_N) \leq J_i((\bar{\pi}_j)_{j \neq i}, \pi_i) \quad \forall \pi_i \in \mathcal{P}(S_i) .$$

where by abuse of notation

$$J_i(\pi_1, \dots, \pi_N) = \int_{S_1 \times \dots \times S_N} J_i(s_1, \dots, s_N) d\pi_1(s_1) \dots d\pi_N(s_N) .$$

Remark 2.1.3. Note that condition (2.1) is equivalent to

$$J_i(\bar{\pi}_1, \dots, \bar{\pi}_N) \leq J_i((\bar{\pi}_j)_{j \neq i}, s_i) \quad \forall s_i \in S_i .$$

This later characterization is very convenient and used throughout the notes.

Theorem 2.1.4 (Nash (1950), Glicksberg (1952)). *Under the above assumptions, there exists at least one equilibrium point in mixed strategies.*

Proof. It is a straightforward application of Fan's fixed point Theorem [29]: let X be a non-empty, compact and convex subset of a locally convex topological vector space. Let $\phi : X \rightarrow 2^X$ be an upper semicontinuous set-valued map such that $\phi(x)$ is non-empty, compact and convex for all $x \in X$. Then ϕ has a fixed point: $\exists \bar{x} \in X$ with $\bar{x} \in \phi(\bar{x})$.

Let us recall that the upper semicontinuity of set-valued function $\phi : X \rightarrow 2^X$ means that, for every open set $W \subset X$, the set $\{x \in X, \phi(x) \cap W\}$ is open in X .

Let us set $X = \prod_{j=1}^N \mathcal{P}(S_i)$ and let us consider the best response map $\mathcal{R}_i : X \rightarrow \mathcal{P}(S_i)$ of player i defined by

$$\mathcal{R}_i((\pi_j)_{j=1,\dots,N}) = \left\{ \pi \in \mathcal{P}(S_i), J_i((\pi_j)_{j \neq i}, \pi) = \min_{\pi' \in \mathcal{P}(S_i)} J_i((\pi_j)_{j \neq i}, \pi') \right\}.$$

Then the map $\phi((\pi_j)_{j=1,\dots,N}) = \prod_{i=1}^N \mathcal{R}_i((\pi_j)_{j=1,\dots,N})$ is upper semicontinuous with non-empty, compact and convex values. Therefore it has a fixed point, which is a Nash equilibrium. $\square$

We now consider the case where the game is symmetric. Namely, we assume that, for all $i \in \{1, \dots, N\}$, $S_i = S$ and $J_i(s_1, \dots, s_N) = J_{\theta(s_i)}(s_{\theta(1)}, \dots, s_{\theta(N)})$ for all and all permutation θ on $\{1, \dots, N\}$.

Theorem 2.1.5 (Symmetric games). *If the game is symmetric, then there is an equilibrium of the form $(\bar{\pi}, \dots, \bar{\pi})$, where $\bar{\pi} \in \mathcal{P}(S)$ is a mixed strategy.*

Proof. Let $X = \mathcal{P}(S)$ and $\mathcal{R} : X \rightarrow 2^X$ be the set-valued map defined by

$$\mathcal{R}(\pi) = \left\{ \sigma \in X, J_i(\sigma, \pi, \dots, \pi) = \min_{\sigma' \in X} J_i(\sigma', \pi, \dots, \pi) \right\}.$$

Then $\mathcal{R}$ is upper semicontinuous with nonempty convex compact values. By Fan's fixed point Theorem, it has a fixed point $\bar{\pi}$ and, from the symmetry of the game, the N -tuple $(\bar{\pi}, \dots, \bar{\pi})$ is a Nash equilibrium. $\square$

2.2 Symmetric functions of many variables

Let Q be a compact metric space and $u_N : Q^N \rightarrow \mathbb{R}$ be a symmetric function:

$$u_N(x_1, \dots, x_N) = u_N(x_{\sigma(1)}, \dots, x_{\sigma(N)}) \quad \text{for any permutation } \sigma \text{ on } \{1, \dots, n\}.$$

Our aim is to define a limit for the u_N .

For this let us introduce the set $\mathcal{P}(Q)$ of Borel probability measures on Q . This set is endowed with the topology of weak-* convergence: a sequence (m_N) of $\mathcal{P}(Q)$ converges to $m \in \mathcal{P}(Q)$ if

$$\lim_N \int_Q \varphi(x) dm_N(x) = \int_Q \varphi(x) dm(x) \quad \forall \varphi \in \mathcal{C}^0(Q).$$

Let us recall that $\mathcal{P}(Q)$ is a compact metric space for this topology, which can be metrized by the distance (often called the Kantorowich-Rubinstein distance)

$$\mathbf{d}_1(\mu, \nu) = \sup \left\{ \int_Q f d(\mu - \nu) \text{ where } f : Q \rightarrow \mathbb{R} \text{ is 1-Lipschitz continuous} \right\}.$$

Other formulations for this distance will be given later (section 3.1).

In order to show that the (u_N) have a limit (at least up to a subsequence), we assume the following:

1. (Uniform bound) there is some $C > 0$ with

$$(2.2) \quad \|u_N\|_{L^\infty(Q)} \leq C$$

2. (Uniform continuity) there is a modulus of continuity ω independent of n such that

$$(2.3) \quad |u_N(X) - u_N(Y)| \leq \omega(\mathbf{d}_1(m_X^N, m_Y^N)) \quad \forall X, Y \in Q^N, \forall N \in \mathbb{N},$$

where $m_X^N = \frac{1}{N} \sum_{i=1}^N \delta_{x_i}$ and $m_Y^N = \frac{1}{N} \sum_{i=1}^N \delta_{y_i}$ if $X = (x_1, \dots, x_N)$ and $Y = (y_1, \dots, y_N)$.

Theorem 2.2.1. *If the u_N are symmetric and satisfy (2.2) and (2.3), then there is a subsequence (u_{N_k}) of (u_N) and a continuous map $U : \mathcal{P}(Q) \rightarrow \mathbb{R}$ such that*

$$\lim_{k \rightarrow +\infty} \sup_{X \in Q^{N_k}} |u_{N_k}(X) - U(m_X^{N_k})| = 0.$$

Proof of Theorem 2.2.1. Without loss of generality we can assume that the modulus ω is concave. Let us define the sequence of maps $U^N : \mathcal{P}(Q) \rightarrow \mathbb{R}$ by

$$U^N(m) = \inf_{X \in Q^N} \{u_N(X) + \omega(\mathbf{d}_1(m_X^N, m))\} \quad \forall m \in \mathcal{P}(Q).$$

Then, by condition (2.3), $U^N(m_X^N) = u_N(X)$ for any $X \in Q^N$. Let us show that the U^N have ω for modulus of continuity on $\mathcal{P}(Q)$: indeed, if $m_1, m_2 \in \mathcal{P}(Q)$ and if $X \in Q^N$ is ϵ -optimal in the definition of $U^N(m_2)$, then

$$\begin{aligned} U^N(m_1) &\leq u_N(X) + \omega(\mathbf{d}_1(m_X^N, m_1)) \\ &\leq U^N(m_2) + \epsilon + \omega(\mathbf{d}_1(m_X^N, m_2) + \mathbf{d}_1(m_1, m_2)) - \omega(\mathbf{d}_1(m_X^N, m_2)) \\ &\leq U^N(m_2) + \omega(\mathbf{d}_1(m_1, m_2)) + \epsilon \end{aligned}$$

because ω is concave. Hence the U^N are equicontinuous on the compact set $\mathcal{P}(Q)$ and uniformly bounded. We complete the proof thanks to Ascoli Theorem. $\square$

Remark 2.2.2. Some uniform continuity condition is needed: for instance if Q is a compact subset of $\mathbb{R}^d$ and $u_N(X) = \max_i |x_i|$, then u_N “converges” to $U(m) = \sup_{x \in \text{spt}(m)} |x|$ which is not continuous. Of course the convergence is not uniform.

Remark 2.2.3. If Q is a compact subset of some finite dimensional space $\mathbb{R}^d$, a typical condition which ensures (2.3) is the existence of a constant $C > 0$, independent of N , such that

$$\sup_{i=1, \dots, N} \|D_{x_i} u_N\|_\infty \leq \frac{C}{N} \quad \forall N.$$

2.3 Limits of Nash equilibria in pure strategies

Let Q be a compact metric space and $\mathcal{P}(Q)$ be the set of Borel probability measures on Q .

We consider a one-shot game with a large number N of players. Our main assumption is that the payoffs $F_1^N, \dots, F_N^N$ of the players are symmetric. In particular, under suitable bounds and uniform continuity, we know from Theorem 2.2.1 that the F_i^N have a limit, which has the form $F(x, m)$ (the dependence on x is here to keep track of the fact of the dependence in i of the function F_i^N). So the payoffs of the players are very close to payoffs of the form $F(x_1, \frac{1}{N-1} \sum_{j \geq 2} \delta_{x_j}), \dots, F(x_N, \frac{1}{N-1} \sum_{j \leq N-1} \delta_{x_j})$.

In order to keep the presentation as simple as possible, we suppose that the payoffs have already this form. That is, we suppose that there is a continuous map $F : Q \times \mathcal{P}(Q) \rightarrow \mathbb{R}$ such that, for any $i \in \{1, \dots, N\}$

$$F_i^N(x_1, \dots, x_N) = F\left(x_i, \frac{1}{N-1} \sum_{j \neq i} \delta_{x_j}\right) \quad \forall (x_1, \dots, x_N) \in Q^N.$$

Let us recall that a Nash equilibrium for the game $(F_1^N, \dots, F_N^N)$ is an element $(\bar{x}_1^N, \dots, \bar{x}_N^N) \in Q^N$ such that

$$F_i^N(\bar{x}_1^N, \dots, \bar{x}_{i-1}^N, y_i, \bar{x}_{i+1}^N, \dots, \bar{x}_N^N) \geq F_i^N(\bar{x}_1^N, \dots, \bar{x}_N^N) \quad \forall y_i \in Q.$$

We set

$$\bar{X}^N = (\bar{x}_1^N, \dots, \bar{x}_N^N) \quad \text{and} \quad m_{\bar{X}^N}^N = \frac{1}{N} \sum_{i=1}^N \delta_{\bar{x}_i^N} .$$

Theorem 2.3.1. *Assume that, for any N , $\bar{X}^N = (\bar{x}_1^N, \dots, \bar{x}_N^N)$ is a Nash equilibrium in pure strategies for the game $F_1^N, \dots, F_N^N$. Then up to a subsequence, the sequence of measures $(m_{\bar{X}^N}^N)$ converges to a measure $\bar{m} \in \mathcal{P}(Q)$ such that*

$$(2.4) \quad \int_Q F(y, \bar{m}) d\bar{m}(y) = \inf_{m \in \mathcal{P}(Q)} \int_Q F(y, m) dm(y) .$$

Remark 2.3.2. The “mean field equation” (2.4) is equivalent to saying that the support of $\bar{m}$ is contained in the set of minima of $F(y, \bar{m})$.

Indeed, if $\text{Spt} \bar{m} \subset \arg - \min_{y \in Q} F(y, \bar{m})$, then clearly $\bar{m}$ satisfies (2.4). Conversely, if (2.4) holds, then choosing $m = \delta_x$ shows that $\int_Q F(y, \bar{m}) d\bar{m}(y) \leq F(x, \bar{m})$ for any $x \in Q$. Therefore $\int_Q F(y, \bar{m}) d\bar{m}(y) \leq \min_{x \in Q} F(x, \bar{m})$, which implies that $\bar{m}$ is supported in $\arg - \min_{y \in Q} F(y, \bar{m})$.

Remark 2.3.3. The result is not completely satisfying because it requires the existence of Nash equilibria in the N -player game, which does not always hold. However there always exists Nash equilibria in mixed strategies, i.e., when the player are allowed to randomize their behavior by playing strategies in $\mathcal{P}(Q)$ instead of Q . We discuss this point below.

Proof. Without loss of generality we can assume that the sequence $(m_{\bar{X}^N}^N)$ converges to some $\bar{m}$. Let us check that $\bar{m}$ satisfies (2.4).

For this we note that, by definition, the measure $\delta_{\bar{x}_i^N}$ is a minimum of the problem

$$\inf_{m \in \mathcal{P}(Q)} \int_Q F(y, \frac{1}{N-1} \sum_{j \neq i} \delta_{\bar{x}_j^N}) dm(y) .$$

Since

$$\mathbf{d}_1 \left(\frac{1}{N-1} \sum_{j \neq i} \delta_{\bar{x}_j^N}, m_{\bar{X}^N}^N \right) \leq \frac{2}{N} ,$$

and since F is uniformly continuous, the measure $\delta_{\bar{x}_i^N}$ is also ϵ -optimal for the problem

$$\inf_{m \in \mathcal{P}(Q)} \int_Q F(y, m_{\bar{X}^N}^N) dm(y)$$

as soon as N is sufficiently large. By linearity, so is $m_{\bar{X}^N}^N$:

$$\int_Q F(y, m_{\bar{X}^N}^N) dm_{\bar{X}^N}^N(y) \leq \inf_{m \in \mathcal{P}(Q)} \int_Q F(y, m_{\bar{X}^N}^N) dm(y) + \epsilon .$$

Letting $N \rightarrow +\infty$ gives the result. □

2.4 Limit of Nash equilibria in mixed strategies

We now assume that the players play the same game $F_1^N, \dots, F_N^N$ as before, but there are allowed to play in *mixed strategies*, i.e., they minimize over elements of $\mathcal{P}(Q)$ instead of minimizing over elements of Q (which are now viewed as *pure strategies*). If the players play the mixed strategies $\pi_1, \dots, \pi_N \in \mathcal{P}(Q)$, then the outcome of Player i (still denoted, by abuse of notation, F_N^i) is

$$(2.5) \quad F_N^i(\pi_1, \dots, \pi_N) = \int_{Q^N} F \left(x_i, \frac{1}{N-1} \sum_{j \neq i} \delta_{x_j} \right) d\pi_1(x_1) \dots d\pi_N(x_N) .$$

Let us recall that the notion of Nash equilibria in mixed strategies is defined in Definition 2.1.2 and that we explained in Theorem 2.1.5 that symmetric Nash equilibria do exist.

Theorem 2.4.1. *We assume that F is Lipschitz continuous. Let, for any N , $(\bar{\pi}^N, \dots, \bar{\pi}^N)$ be a symmetric Nash equilibrium in mixed strategies for the game $F_1^N, \dots, F_N^N$. Then, up to a subsequence, $(\bar{\pi}^N)$ converges to a measure $\bar{m}$ satisfying (2.4).*

Remark 2.4.2. In particular the above Theorem proves the existence of a solution to the “mean field equation” (2.4).

Proof. Let $\bar{m}$ be a limit, up to subsequences, of the $(\bar{\pi}^N)$. Since the map $x_j \rightarrow F(y, \frac{1}{N-1} \sum_{j \neq i} \delta_{x_j})$ is $\text{Lip}(F)/(N-1)$ -Lipschitz continuous, we have, by definition of the distance $\mathbf{d}_1$,

$$(2.6) \quad \left| \int_{Q^{N-1}} F(y, \frac{1}{N-1} \sum_{j \neq i} \delta_{x_j}) \prod_{j \neq i} d\bar{\pi}^N(x_j) - \int_{Q^{N-1}} F(y, \frac{1}{N-1} \sum_{j \neq i} \delta_{x_j}) \prod_{j \neq i} d\bar{m}(x_j) \right| \leq \text{Lip}(F) \mathbf{d}_1(\bar{\pi}^N, \bar{m}) \quad \forall y \in Q.$$

A direct application of the law of large numbers (see Theorem 3.1.9 below) gives

$$(2.7) \quad \lim_{N \rightarrow +\infty} \int_{Q^{N-1}} F(y, \frac{1}{N-1} \sum_{j \neq i} \delta_{x_j}) \prod_{j \neq i} d\bar{m}(x_j) = F(y, \bar{m}),$$

where the convergence is uniform with respect to $y \in Q$ thanks to the (Lipschitz) continuity of F . Since $(\bar{\pi}_1, \dots, \bar{\pi}_N)$ is a Nash equilibrium, inequality (2.6) implies that, for any $\epsilon > 0$ and if we choose N large enough,

$$\int_{Q^N} F(y, \frac{1}{N-1} \sum_{j \neq i} \delta_{x_j}) \prod_{j \neq i} d\bar{m}(x_j) d\bar{m}(x_i) \leq \int_{Q^N} F(y, \frac{1}{N-1} \sum_{j \neq i} \delta_{x_j}) \prod_{j \neq i} d\bar{m}(x_j) dm(x_i) + \epsilon,$$

for any $m \in \mathcal{P}(Q)$. Letting $N \rightarrow +\infty$ on both sides of the inequality gives, in view of (2.7),

$$\int_Q F(x_i, \bar{m}) d\bar{m}(x_i) \leq \int_Q F(x_i, \bar{m}) dm(x_i) + \epsilon \quad \forall m \in \mathcal{P}(Q),$$

which gives the result, since ϵ is arbitrary. $\square$

We can also investigate the converse statement: suppose that a measure $\bar{m}$ satisfying the equilibrium condition (2.4) is given. In what extent can it be used in N -player games?

Theorem 2.4.3. *Let F be as in Theorem 2.4.1. For any $\epsilon > 0$, there exists $N_0 \in \mathbb{N}^*$ such that, if $N \geq N_0$, the symmetric mixed strategy $\bar{m}, \dots, \bar{m}$ is ϵ -optimal in the N -player game with costs (F_i^N) defined by (2.5). Namely,*

$$F_i^N(\bar{m}, \dots, \bar{m}) \leq F_i^N(x_i, (\bar{m})_{j \neq i}) \quad \forall x_i \in Q.$$

Proof. Indeed, as explained in the proof of Theorem 2.4.1, we have

$$\lim_{N \rightarrow +\infty} F_i^N(x_i, (\bar{m})_{j \neq i}) = F(x_i, \bar{m})$$

and this limit holds uniformly with respect to $x_i \in Q$. So we can find N_0 such that

$$(2.8) \quad \sup_{x_i \in Q} |F_i^N(x_i, (\bar{m})_{j \neq i}) - F(x_i, \bar{m})| \leq \epsilon/2 \quad \forall N \geq N_0.$$

Then, for any $x_i \in Q$, we have

$$F_i^N(x_i, (\bar{m})_{j \neq i}) \geq F(x_i, \bar{m}) - \epsilon/2 \geq \int_Q F(y_i, \bar{m}) d\bar{m}(y_i) - \epsilon/2$$

where the last inequality comes from the equilibrium condition (2.4) on $\bar{m}$. Using again (2.8) we finally get

$$F_i^N(x_i, (\bar{m})_{j \neq i}) \geq \int_Q F(y_i, \bar{m}) d\bar{m}(y_i) - \epsilon/2 \geq F_i^N(\bar{m}, \dots, \bar{m}) - \epsilon.$$

$\square$

2.5 A uniqueness result

One obtains the full convergence of the measure $m_{\bar{x}^N}^N$ (or $\bar{\pi}^N$) if there is a unique measure $\bar{m}$ satisfying the condition (2.4). This is the case under the following (very strong) assumption:

Proposition 2.5.1. *Assume that F satisfies*

$$(2.9) \quad \int_Q (F(y, m_1) - F(y, m_2)) d(m_1 - m_2)(y) > 0 \quad \forall m_1 \neq m_2 .$$

Then there is at most one measure satisfying (2.4).

Remark 2.5.2. Requiring at the same time the continuity of F and the above monotonicity condition seems rather restrictive for applications.

Condition (2.9) is more easily fulfilled for mapping defined on strict subsets of $\mathcal{P}(Q)$. For instance, if Q is a compact subset of $\mathbb{R}^d$ of positive measure and $\mathcal{P}_{ac}(Q)$ is the set of absolutely continuous measures on Q (absolutely continuous with respect to the Lebesgue measure), then

$$F(y, m) = \begin{cases} G(m(y)) & \text{if } m \in \mathcal{P}_{ac}(Q) \\ +\infty & \text{otherwise} \end{cases}$$

satisfies (2.9) as soon as $G : \mathbb{R} \rightarrow \mathbb{R}$ is continuous and increasing.

If we assume that Q is the closure of some smooth open bounded subset Ω of $\mathbb{R}^d$, another example is given by

$$F(y, m) = \begin{cases} u_m(y) & \text{if } m \in \mathcal{P}_{ac}(Q) \cap L^2(Q) \\ +\infty & \text{otherwise} \end{cases}$$

where u_m is the solution in $H^1(Q)$ of

$$\begin{cases} -\Delta u_m = m & \text{in } \Omega \\ u_m = 0 & \text{on } \partial\Omega \end{cases}$$

Note that in this case the map $y \rightarrow F(y, m)$ is continuous.

Proof of Proposition 2.5.1. Let $\bar{m}_1, \bar{m}_2$ satisfying (2.4). Then

$$\int_Q F(y, \bar{m}_1) d\bar{m}_1(y) \leq \int_Q F(y, \bar{m}_1) d\bar{m}_2(y)$$

and

$$\int_Q F(y, \bar{m}_2) d\bar{m}_2(y) \leq \int_Q F(y, \bar{m}_2) d\bar{m}_1(y) .$$

Therefore

$$\int_Q (F(y, \bar{m}_1) - F(y, \bar{m}_2)) d(\bar{m}_1 - \bar{m}_2)(y) \leq 0 ,$$

which implies that $\bar{m}_1 = \bar{m}_2$ thanks to assumption (2.9). $\square$

2.6 Example: potential games

We now consider a class of nonatomic games for which the equilibria can be reached by minimizing a functional. To fix the idea, we now assume that $Q \subset \mathbb{R}^d$. The heuristic idea is that, if $F(x, m)$ can somehow be represented as the derivative of some mapping $\Phi(x, m)$ with respect to the m -variable, and if the problem

$$\inf_{m \in \mathcal{P}(Q)} \int_Q \Phi(x, m) dx$$

has a minimum $\bar{m}$, then

$$\int_Q \Phi'(x, \bar{m})(m - \bar{m}) \geq 0 \quad \forall m \in \mathcal{P}(Q) .$$

So

$$\int_Q F(x, \bar{m}) dm \geq \int_Q F(x, \bar{m}) d\bar{m} \quad \forall m \in \mathcal{P}(Q) ,$$

which shows that $\bar{m}$ is an equilibrium.

For instance let us assume that

$$F(x, m) = \begin{cases} V(x)m(x) + G(m(x)) & \text{if } m \in \mathcal{P}_{ac}(Q) \\ +\infty & \text{otherwise} \end{cases}$$

where $V : Q \rightarrow \mathbb{R}$ is continuous and $G : (0, +\infty) \rightarrow \mathbb{R}$ is continuous, strictly increasing, with $G(0) = 0$ and $G(s) \geq cs$ for some $c > 0$. Then let

$$\Phi(x, m) = V(x)m(x) + H(m(x)) \quad \text{if } m \text{ is a.c.}$$

where H is a primitive of G with $H(0) = 0$. Note that G is strictly convex with $G(s) \geq \frac{c}{2}s^2 - ds$. Hence the problem

$$\inf_{m \in \mathcal{P}_{ac}(Q)} \int_Q V(x)m(x) + H(m(x)) \, dx$$

has a unique solution $\bar{m} \in L^2(Q)$. Then we have, for any $m \in \mathcal{P}_{ac}(Q)$,

$$\int_Q (V(x) + G(\bar{m}(x)))m(x)dx \geq \int_Q (V(x) + G(\bar{m}(x)))\bar{m}(x)dx ,$$

so that $\bar{m}$ satisfies (a slightly modified version of) the mean field equation (2.4). In particular, we have $V(x) + G(m(x)) = \min_y V(y) + G(\bar{m}(y))$ for any $x \in \text{Spt}(\bar{m})$. Let us set $\lambda = \min_y V(y) + G(\bar{m}(y))$. Then

$$\bar{m}(x) = G^{-1}((\lambda - V(x))_+)$$

For instance, if we plug formally $Q = \mathbb{R}^d$, $V(x) = \frac{|x|^2}{2}$ and $G(s) = \log(s)$ into the above equality, we get $m(x) = e^{-|x|^2/2} / (2\pi)^{d/2}$.

2.7 Comments

There is a huge literature on games with a continuum of players, starting from the seminal work by Aumann [4]. Schmeidler [56], and then Mas-Colell [49], introduced a notion of non-cooperative equilibrium in games with a continuum of agents and established several existence results in a much more general framework where the agents have *types*, i.e., personal characteristics; in that set-up, the equilibria are known under the name of Cournot-Nash equilibria. Blanchet and Carlier [8] investigated classes of problems in which such equilibrium is unique and can be fully characterized.

Theorem 2.3.1 is borrowed from [48]. The variational approach described in section 2.6 presents strong similarities with the potential games of Monderer and Shapley [50].

Chapter 3

Preliminaries

In this section we give a crash course on the space of probability measures, mean field limits and optimal control. As mean field control problems consist in a combination of these three topics, it is important to have a hint on all these domains separately.

3.1 On the space of measures

3.1.1 The Wasserstein distances

Most results of this part are given without proof: we refer to the classical monographs by Villani [61, 62], Ambrosio-Gigli-Savaré [3] and Santambrogio [55] for a general introduction.

We fix $d \in \mathbb{N}$, $d \geq 1$ and will work in $\mathbb{R}^d$. We denote by $\mathcal{P}(\mathbb{R}^d)$ the set of Borel probability measures on $\mathbb{R}^d$. Let us recall that two Borel probability measures m and m' on $\mathbb{R}^d$ are equal if and only if

$$\int_{\mathbb{R}^d} \phi(x) m(dx) = \int_{\mathbb{R}^d} \phi(x) m'(dx) \quad \forall \phi \in C_b^0(\mathbb{R}^d),$$

where $C_b^0(\mathbb{R}^d)$ is the set of continuous and bounded maps on $\mathbb{R}^d$. We say that a sequence (m_n) of $\mathcal{P}(\mathbb{R}^d)$ narrowly converges to a measure $m \in \mathcal{P}(\mathbb{R}^d)$ if, for any test function $\phi \in C_b^0(\mathbb{R}^d)$, we have

$$\lim_n \int_{\mathbb{R}^d} \phi(x) m_n(dx) = \int_{\mathbb{R}^d} \phi(x) m(dx).$$

Let us recall *Prokhorov compactness criterium*: a subset $\mathcal{K}$ of $\mathcal{P}(\mathbb{R}^d)$ is relatively compact for the sequential narrow convergence if and only if it is tight: for any $\epsilon > 0$ there exists a compact subset K of $\mathbb{R}^d$ such that

$$\sup_{\mu \in \mathcal{K}} m(\mathbb{R}^d \setminus K) \leq \epsilon.$$

Fix $p \geq 1$, we denote by $\mathcal{P}_p(\mathbb{R}^d)$ the set measures $m \in \mathcal{P}(\mathbb{R}^d)$ such that

$$\int_{\mathcal{X}} |x|^p m(dx) < +\infty.$$

We endow $\mathcal{P}_p(\mathcal{X})$ with the **Wasserstein distance**: for any $m_1, m_2 \in \mathcal{P}_p(\mathbb{R}^d)$,

$$\mathbf{d}_p(m_1, m_2) = \inf_{\Pi(m_1, m_2)} \left[\int_{\mathbb{R}^d \times \mathbb{R}^d} |x_1 - x_2|^p \pi(dx_1, dx_2) \right]^{1/p},$$

where $\Pi(m_1, m_2)$ is the set of Borel probability measures π on $\mathbb{R}^d \times \mathbb{R}^d$ with first and second marginals are m_1 and m_2 respectively, i.e., $\pi(A \times \mathbb{R}^d) = m_1(A)$ and $\pi(\mathbb{R}^d \times A) = m_2(A)$ for any Borel set $A \subset \mathbb{R}^d$. The set $\Pi(m_1, m_2)$ is the set of couplings between m_1 and m_2 . We will mostly work with $p = 1$ and

$p = 2$. Note that, by Cauchy-Schwartz inequality, $\mathbf{d}_1 \leq \mathbf{d}_2$. Note finally that the set $\Pi(m_1, m_2)$ being compact for the narrow convergence, there exists at least a minimizer for $\mathbf{d}_p(m_1, m_2)$ (often called an optimal transport plan).

Theorem 3.1.1. $\mathbf{d}_p$ is a distance on $\mathcal{P}_p(\mathbb{R}^d)$ and, for any sequence (m_n) of $\mathcal{P}_p(\mathbb{R}^d)$ and $m \in \mathcal{P}_p(\mathbb{R}^d)$,

$$\lim \mathbf{d}_p(m_n, m) = 0 \quad \Leftrightarrow \quad \begin{cases} m_n \rightarrow m \text{ narrowly} \\ \int_{\mathbb{R}^d} |x|^p m_n(dx) \rightarrow \int_{\mathbb{R}^d} |x|^p m(dx) \end{cases}$$

An example of explicit computation of the distance is the case of empirical measures. Fix $N \in \mathbb{N}$, $N \geq 1$ (N is seen as the number of particles). Given $\mathbf{x} = (x^1, \dots, x^N) \in (\mathbb{R}^d)^N$, we set $m_{\mathbf{x}}^N = \frac{1}{N} \sum_{i=1}^N \delta_{x^i}$. Note that $m_{\mathbf{x}}^N$ is a probability measure on $\mathbb{R}^d$. In fact, $m_{\mathbf{x}}^N$ belongs to $\mathcal{P}_p(\mathbb{R}^d)$ since

$$\int_{\mathbb{R}^d} |x|^p m_{\mathbf{x}}^N(dx) = \frac{1}{N} \sum_{i=1}^N |x^i|^p < \infty.$$

Proposition 3.1.2. Given $\mathbf{x} = (x^1, \dots, x^N), \mathbf{y} = (y^1, \dots, y^N) \in (\mathbb{R}^d)^N$,

$$\mathbf{d}_p(m_{\mathbf{x}}^N, m_{\mathbf{y}}^N) = \inf_{\sigma \in \mathcal{S}(N)} \left(\frac{1}{N} \sum_{i=1}^N |x^i - y^{\sigma(i)}|^p \right)^{1/p},$$

where $\mathcal{S}(N)$ is the set of permutations over $\{1, \dots, N\}$.

Proof. Let us fix $\sigma \in \mathcal{S}(N)$ and define the coupling $\pi \in \Pi(m_{\mathbf{x}}^N, m_{\mathbf{y}}^N)$ by

$$\pi = \frac{1}{N} \sum_{i=1}^N \delta_{(x^i, y^{\sigma(i)})}.$$

Then

$$\mathbf{d}_p^p(m_{\mathbf{x}}^N, m_{\mathbf{y}}^N) \leq \int_{\mathbb{R}^d \times \mathbb{R}^d} |x - y|^p \pi(dx, dy) = \frac{1}{N} \sum_{i=1}^N |x^i - y^{\sigma(i)}|^p.$$

Thus

$$\mathbf{d}_p^p(m_{\mathbf{x}}^N, m_{\mathbf{y}}^N) \leq \inf_{\sigma \in \mathcal{S}(N)} \frac{1}{N} \sum_{i=1}^N |x^i - y^{\sigma(i)}|^p.$$

Conversely, let $\pi \in \Pi(m_{\mathbf{x}}^N, m_{\mathbf{y}}^N)$. As π has for marginals $m_{\mathbf{x}}^N$ and $m_{\mathbf{y}}^N$ respectively, its support is contained in the set $\{(x^i, y^j), i, j \in \{1, \dots, N\}\}$. Thus

$$\pi = \frac{1}{N} \sum_{i=1}^N \sum_{j=1}^N \pi_{i,j} \delta_{(x^i, y^j)}, \quad \text{with } \pi_{i,j} = N\pi(\{(x^i, y^j)\})$$

and

$$\int_{\mathbb{R}^d \times \mathbb{R}^d} |x - y|^p \pi(dx, dy) = \frac{1}{N} \sum_{i,j} \pi_{i,j} |x^i - y^j|^p.$$

Note that the matrix $(\pi_{i,j})$ is bi-stochastic: its entries are non-negative and $\sum_{j'} \pi_{i,j'} = \sum_{i'} \pi_{i',j} = 1$ for any i, j . Let us denote by M_N the set of such matrices. There is a one-to-one correspondance between this set and $\Pi(m_{\mathbf{x}}^N, m_{\mathbf{y}}^N)$, so that

$$\mathbf{d}_p^p(m_{\mathbf{x}}^N, m_{\mathbf{y}}^N) = \inf_{(\pi_{i,j}) \in M_N} \frac{1}{N} \sum_{i,j} \pi_{i,j} |x^i - y^j|^p.$$

The set M_N is a compact convex subset of $\mathbb{R}^{2N}$ and the criterium is linear: the infimum in the optimization problem above is achieved at an extreme point $(\bar{\pi}_{i,j})$ of M_N . Birkhoff Theorem says extreme points of M_N are permutations and thus $\bar{\pi}_{i,j} \in \{0, 1\}$ for any i, j . This completes the proof. $\square$

We will repetitively use the following remark (see also Theorem 3.1.4 below): if $m_1, m_2 \in \mathcal{P}_p(\mathbb{R}^d)$ and $\phi : \mathbb{R}^d \rightarrow \mathbb{R}$ is Lipschitz continuous with Lipschitz constant L , then

$$\int_{\mathbb{R}^d} \phi(x)(m_1 - m_2)(dx) \leq L \mathbf{d}_p(m_1, m_2).$$

Indeed, for any coupling $\pi \in \Pi(m_1, m_2)$,

$$\begin{aligned} \int_{\mathbb{R}^d} \phi(x)(m_1 - m_2)(dx) &= \int_{\mathbb{R}^d \times \mathbb{R}^d} (\phi(x_1) - \phi(x_2))\pi(dx_1, dx_2) \leq L \int_{\mathbb{R}^d \times \mathbb{R}^d} |x_1 - x_2|\pi(dx_1, dx_2) \\ &\leq L \left[\int_{\mathbb{R}^d \times \mathbb{R}^d} |x_1 - x_2|^p \pi(dx_1, dx_2) \right]^{1/p}. \end{aligned}$$

We conclude by taking the infimum over π .

We will also repeatedly use the following compactness criterium:

Lemma 3.1.3. *Let $r > p$ and $\mathcal{K} \subset \mathcal{P}_p(\mathbb{R}^d)$ be such that*

$$(3.1) \quad \sup_{\mu \in \mathcal{K}} \int_{\mathbb{R}^d} |x|^r \mu(dx) < +\infty.$$

Then $\mathcal{K}$ is relatively compact for the $\mathbf{d}_p$ distance.

Note that bounded subsets of $\mathcal{P}_p(\mathbb{R}^d)$ are not relatively compact for the $\mathbf{d}_p$ distance. For instance, if $p = 1$, the sequence of measures $\mu_n = \frac{n-1}{n}\delta_0 + \frac{1}{n}\delta_n$ converges narrowly to δ_0 , but $\mathbf{d}_1(\mu_n, \delta_0) = 1$ for any $n \geq 1$.

Proof of Lemma 3.1.3. By (3.1), the set $\mathcal{K}$ is tight. By Prokhorov Theorem, it is therefore relatively compact for the narrow convergence. Let (m_n) be a sequence in $\mathcal{K}$ converging to a probability measure m for the narrow convergence and let us check that (m_n) converges to m for $\mathbf{d}_p$: this will show that $\mathcal{K}$ is relatively compact for the $\mathbf{d}_p$ distance. For this it is enough to check that

$$\lim_n \int |x|^p m_n = \int |x|^p m.$$

Note that, by narrow convergence, for any $M > 0$,

$$\lim_n \int (|x|^p \wedge M) m_n = \int (|x|^p \wedge M) m.$$

Thus

$$\liminf_n \int |x|^p m_n \geq \sup_M \lim_n \int (|x|^p \wedge M) m_n = \sup_M \int (|x|^p \wedge M) m = \int |x|^p m.$$

On the other hand, if $C := \sup_{\mu \in \mathcal{K}} \int_{\mathbb{R}^d} |x|^r \mu(dx)$, then, for any $M > 0$,

$$\left| \int (|x|^p \wedge M) m_n - \int |x|^p m_n \right| = \int_{\{|x| > M^{1/p}\}} (|x|^p - M) m_n \leq \int_{\{|x| > M^{1/p}\}} |x|^p m_n \leq \int \frac{|x|^r}{M^{\frac{r-p}{p}}} m_n \leq \frac{C}{M^{\frac{r-p}{p}}}.$$

Thus

$$\limsup_n \int |x|^p m_n \leq \limsup_n \int (|x|^p \wedge M) m_n + \frac{C}{M^{\frac{r-p}{p}}} = \int (|x|^p \wedge M) m + \frac{C}{M^{\frac{r-p}{p}}}.$$

As this holds for any $M > 0$, letting $M \rightarrow \infty$ we find:

$$\limsup_n \int |x|^p m_n \leq \int |x|^p m.$$

Whence the result. $\square$

3.1.2 Duality

We only sketch here—again without proof—this important aspect of optimal transport.

Theorem 3.1.4. *Fix $p \geq 1$, $m^1, m^2 \in \mathcal{P}_p(\mathbb{R}^d)$. Then*

$$\mathbf{d}_p(m^1, m^2) = \sup_{\phi, \psi} \int_{\mathbb{R}^d} \phi(x) m^1(dx) + \int_{\mathbb{R}^d} \psi(x) m^2(dx)$$

where the supremum is taken over all continuous, bounded maps $\phi, \psi : \mathbb{R}^d \rightarrow \mathbb{R}$ such that

$$\phi(x) + \psi(y) \leq |x - y|^p \quad \forall x, y \in \mathbb{R}^d.$$

In particular, if $p = 1$, then

$$\mathbf{d}_1(m^1, m^2) = \sup_{\phi \text{ 1-Lipschitz}} \int_{\mathbb{R}^d} \phi(x) (m^1 - m^2)(dx).$$

Heuristic explanation of Theorem 3.1.4. Let $\mathcal{M}(\mathbb{R}^{2d})$ be the set of Radon measures on $\mathbb{R}^{2d}$.

$$\begin{aligned} \mathbf{d}_p^p(m^1, m^2) &= \inf_{\pi \in \Pi(m^1, m^2)} \int_{\mathbb{R}^{2d}} |x - y|^p \pi(dx, dy) \\ &= \inf_{\pi \in \mathcal{M}(\mathbb{R}^{2d})} \sup_{\phi, \psi \in C_b^0(\mathbb{R}^d)} \int_{\mathbb{R}^{2d}} (|x - y|^p - \phi(x) - \psi(y)) \pi(dx, dy) + \int_{\mathbb{R}^d} \phi(x) m^1(dx) + \int_{\mathbb{R}^d} \psi(x) m^2(dx) \\ &\text{“=”} \sup_{\phi, \psi \in C_b^0(\mathbb{R}^d)} \inf_{\pi \in \mathcal{M}(\mathbb{R}^{2d})} \int_{\mathbb{R}^{2d}} (|x - y|^p - \phi(x) - \psi(y)) \pi(dx, dy) + \int_{\mathbb{R}^d} \phi(x) m^1(dx) + \int_{\mathbb{R}^d} \psi(x) m^2(dx) \\ &= \sup_{\phi, \psi \in C_b^0(\mathbb{R}^d), |x - y|^p - \phi(x) - \psi(y) \geq 0} \int_{\mathbb{R}^d} \phi(x) m^1(dx) + \int_{\mathbb{R}^d} \psi(x) m^2(dx). \end{aligned}$$

The justification of the “=” requires a careful application of the min-max theorem.

If $p = 1$, one can replace $\phi(x)$ by $\sup_y |x - y| - \psi(y)$ without changing the cost, so that we can assume that ϕ is 1-Lipschitz continuous. Then we can replace $\psi(y)$ by

$$\sup_x |x - y| - \phi(x) = -\psi(y) + \sup_x (|x - y| - \phi(x) + \phi(y)) = -\phi(y)$$

so that we can also assume that the supremum is taken over the pair $(\phi, -\phi)$ with ϕ 1-Lipschitz continuous. $\square$

3.1.3 Characterization through random variables.

We refer here to the monograph by Carmona-Delarue [15].

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be an atomless probability space¹. In such a space, for any Polish space E and any Borel probability measure m on E , there exists a random variable on E with law m : namely there exists a measurable map $X : \Omega \rightarrow E$ with law m : $\mathcal{L}(X) = X\#\mathbb{P} = m$.

The following characterization of the Wasserstein distance will be useful:

Proposition 3.1.5. *For any $m_1, m_2 \in \mathcal{P}_p(\mathbb{R}^d)$,*

$$(3.2) \quad \mathbf{d}_p^p(m_1, m_2) = \inf_{X_1, X_2} \mathbb{E}[|X_1 - X_2|^p],$$

where the infimum is taken over random variables X_1 and X_2 (with values in $\mathbb{R}^d$) such that the law of X_i is m_i .

¹This means that, for any $A \in \mathcal{F}$ with $\mathbb{P}(A) > 0$, there exists $B \in \mathcal{F}$ such that $B \subset A$ and $0 < \mathbb{P}(B) < \mathbb{P}(A)$.

Proof. If X_1 and X_2 have law m_1 and m_2 respectively, then the law π of (X_1, X_2) belongs to $\Pi(m_1, m_2)$. Hence

$$\mathbb{E}[|X_1 - X_2|^p] = \int_{\mathbb{R}^d \times \mathbb{R}^d} |x_1 - x_2|^p \pi(dx_1, dx_2) \geq \mathbf{d}_p^p(m_1, m_2).$$

Conversely, if $\pi \in \Pi(m_1, m_2)$, then there exists a random variable X on $\mathbb{R}^d \times \mathbb{R}^d$ with law π . We write $X = (X_1, X_2)$ with X_1 and X_2 random variables on $\mathbb{R}^d$. We note that, as the first marginal of π is m_1 , the law of X_1 is m_1 . The same argument shows that X_2 has law m_2 . Hence

$$\int_{\mathbb{R}^d \times \mathbb{R}^d} |x_1 - x_2|^p \pi(dx_1, dx_2) = \mathbb{E}[|X_1 - X_2|^p] \geq \inf_{X'_1, X'_2} \mathbb{E}[|X'_1 - X'_2|^p],$$

where the infimum is taken over random variables X'_1 and X'_2 with law m_1 and m_2 respectively. Taking the infimum over π concludes the proof. $\square$

3.1.4 Symmetric functions of many variables

Let Q be a compact subset of $\mathbb{R}^d$ and $u_N : Q^N \rightarrow \mathbb{R}$ be a symmetric function:

$$(3.3) \quad u_N(x_1, \dots, x_N) = u_N(x_{\sigma(1)}, \dots, x_{\sigma(n)}) \quad \text{for any permutation } \sigma \text{ on } \{1, \dots, n\}.$$

Our aim is to define a limit for the u_N . For this, we assume in addition to the symmetry condition the following:

1. (Uniform bound) there is some $C > 0$ with

$$(3.4) \quad \|u_N\|_{L^\infty(Q)} \leq C$$

2. (Uniform continuity) there is a modulus of continuity² ω independent of N such that

$$(3.5) \quad |u_N(X) - u_N(Y)| \leq \omega\left(\frac{1}{N} \sum_{i=1}^N |x_i - y_i|\right) \quad \forall X, Y \in Q^N, \forall N \in \mathbb{N}.$$

where $X = (x_1, \dots, x_N)$ and $Y = (y_1, \dots, y_N)$.

Theorem 3.1.6. *If the u_N are symmetric (i.e., satisfy condition (3.3)) and satisfy (3.4) and (3.5), then there is a subsequence (u_{N_k}) of (u_N) and a continuous map $U : \mathcal{P}(Q) \rightarrow \mathbb{R}$ such that*

$$\lim_{k \rightarrow +\infty} \sup_{X \in Q^{N_k}} |u_{N_k}(X) - U(m_X^{N_k})| = 0.$$

Proof of Theorem 3.1.6. Without loss of generality we can assume that the modulus ω is concave. Using Proposition 3.1.2 and the symmetry of u_N , condition (3.5) can be rewritten as

$$(3.6) \quad |u_N(X) - u_N(Y)| \leq \omega\left(\inf_{\sigma \in \mathcal{S}(N)} \frac{1}{N} \sum_{i=1}^N |x_i - y_{\sigma(i)}|\right) = \omega(\mathbf{d}_1(m_X^N, m_Y^N)).$$

Let us define the sequence of maps $U^N : \mathcal{P}(Q) \rightarrow \mathbb{R}$ by

$$U^N(m) = \inf_{X \in Q^N} \{u_N(X) + \omega(\mathbf{d}_1(m_X^N, m))\} \quad \forall m \in \mathcal{P}(Q).$$

Then, by condition (3.5), $U^N(m_X^N) = u_N(X)$ for any $X \in Q^N$. Let us show that the U^N have ω for modulus of continuity on $\mathcal{P}(Q)$: indeed, if $m_1, m_2 \in \mathcal{P}(Q)$ and if $X \in Q^N$ is ϵ -optimal in the definition of $U^N(m_2)$, then

$$\begin{aligned} U^N(m_1) &\leq u_N(X) + \omega(\mathbf{d}_1(m_X^N, m_1)) \\ &\leq U^N(m_2) - \omega(\mathbf{d}_1(m_X^N, m_2)) + \epsilon + \omega(\mathbf{d}_1(m_X^N, m_2) + \mathbf{d}_1(m_1, m_2)) \\ &\leq U^N(m_2) + \omega(\mathbf{d}_1(m_1, m_2)) + \epsilon \end{aligned}$$

²This means that the $\omega : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is continuous, nondecreasing and vanishes at 0.

because ω is concave. We now claim that

$$U^N(m_Y^N) = u_N(Y) \quad \forall Y \in Q^N.$$

Indeed,

$$U^N(m_Y^N) = u_N(Y) + \inf_{X \in Q^N} \{u_N(X) - u_N(Y) + \omega(\mathbf{d}_1(m_X^N, m_Y^N))\}.$$

By (3.6),

$$u_N(X) - u_N(Y) + \omega(\mathbf{d}_1(m_X^N, m_Y^N)) \geq -\omega(\mathbf{d}_1(m_X^N, m_Y^N)) - \omega(\mathbf{d}_1(m_X^N, m_Y^N)) \geq 0,$$

so that it is optimal in the definition of $U^N(m_Y^N)$ to take $X = Y$. Hence $U^N(m_Y^N) = u_N(Y)$.

We now check that the U^N are uniformly bounded. As $\mathcal{P}(Q)$ is compact, it has a finite diameter D . Fix $q \in Q$ and note that $u_N(q, \dots, q) = U^N(\delta_q)$. By (3.4), the $(u_N(q, \dots, q))$ are bounded in N , so that the $U^N(\delta_q)$ are also bounded in N . As the U^N are uniformly continuous on $\mathcal{P}(Q)$ which has a finite diameter, this implies that the U^N are uniformly bounded.

Hence the U^N are equicontinuous on the compact set $\mathcal{P}(Q)$ and uniformly bounded. By Ascoli-Arzelà Theorem, there exists a subsequence (U^{N_k}) and a continuous map $U : \mathcal{P}(Q) \rightarrow \mathbb{R}$ such that (U^{N_k}) converges uniformly to U . Then

$$\begin{aligned} \lim_{k \rightarrow +\infty} \sup_{X \in Q^{N_k}} |u_{N_k}(X) - U(m_X^{N_k})| &= \lim_{k \rightarrow +\infty} \sup_{X \in Q^{N_k}} |U^{N_k}(X) - U(m_X^{N_k})| \\ &\leq \lim_{k \rightarrow +\infty} \sup_{m \in \mathcal{P}(Q)} |U^{N_k}(m) - U(m)| = 0 \end{aligned}$$

□

Remark 3.1.7. Some uniform continuity condition is needed: for instance if $u_N(X) = \max_i |x_i|$, then u_N “converges” to $U(m) = \sup_{x \in \text{spt}(m)} |x|$ which is not continuous. Of course the convergence is not uniform.

Remark 3.1.8. A typical condition which ensures (3.5) is the existence of a constant $C > 0$, independent of N , such that

$$\sup_{i=1, \dots, N} \|D_{x_i} u_N\|_\infty \leq \frac{C}{N} \quad \forall N.$$

3.1.5 The Glivenko-Cantelli law of large numbers

We consider (X_n) a sequence of i.i.d. random variables on $\mathbb{R}^d$ on a fixed (atomless) probability space $(\Omega, \mathcal{F}, \mathbb{P})$, with $\mathbb{E}[|X_1|] < +\infty$. We denote by m the law of X_1 and note that $m \in \mathcal{P}_1(\mathcal{X})$. The law of large numbers states that, for any continuous map $\phi : \mathcal{X} \rightarrow \mathbb{R}$ with at most a linear growth, we have, a.s. and in L^1 ,

$$\lim_{N \rightarrow +\infty} \frac{1}{N} \sum_{n=1}^N \phi(X_n) = \mathbb{E}[\phi(X_1)].$$

Our aim is to show that a slightly stronger convergence holds: let

$$m_X^N := \frac{1}{N} \sum_{n=1}^N \delta_{X_n}$$

Note that m_X^N is a random measure, in the sense that m_X^N is a.s. a measure and, for any Borel set $A \subset \mathcal{X}$, $m_X^N(A)$ is a random variable. The following result is known as the Glivenko-Cantelli Theorem:

Theorem 3.1.9. *A.s. and in L^1 ,*

$$\lim_{N \rightarrow +\infty} \mathbf{d}_1(m_X^N, m) = 0.$$

Sketch of proof. Let $\phi : \mathbb{R}^d \rightarrow \mathbb{R}$ be continuous with at most a linear growth (i.e., $|\phi(x)| \leq ad(x, x_0) + b$ for some constants a, b). Then, by the law of large numbers,

$$\lim_{N \rightarrow \infty} \int_{\mathbb{R}^d} \phi(x) m_X^N(dx) = \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N \phi(X_n) = \mathbb{E}[\phi(X_1)] \quad a.s..$$

By a separability argument, it is not difficult to check that the set of zero probability in the above convergence can be chosen independent of $\phi \in C_b^0(\mathcal{X})$. So (m_X^N) converges weakly to m a.s. Choosing now $\phi(x) = |x|$, we also have

$$\int_{\mathcal{X}} |x| m_X^N(dx) \rightarrow \int_{\mathcal{X}} |x| m(dx) \quad a.s..$$

Therefore, by Theorem 3.1.1, (m_X^N) converges a.s. in $\mathbf{d}_1$ to m . It remains to show that this convergence also holds in expectation. For this we note that

$$\mathbf{d}_1(m_X^N, m) = \mathbf{d}_1(m_X^N, \delta_0) + \mathbf{d}_1(\delta_0, m) \leq \frac{1}{N} \sum_{i=1}^N |X_i| + \mathbf{d}_1(\delta_0, m).$$

As the right-hand side converges in L^1 to $2\mathbb{E}[|X_1|]$ a.s., $\mathbf{d}_1(m_X^N, m)$ is uniformly integrable which implies its convergence in expectation to 0. $\square$

3.1.6 Further reading

Classical references on the distances over the space of probability measures are the monographs by Ambrosio, Gigli and Savaré [3], by Rachev and Rüschendorf [53], Santambrogio [55], and Villani [61], [62]. See also Chap. V in the monograph by Carmona and Delarue [15].

It is also useful to quantify the convergence speed in the law of large numbers. Such results can be found in [53] or, in a sharper form, in [15].

3.2 Dynamics of large particle systems

In this part we discuss the dynamic of systems made of a large number of particles in interaction. As in the largest part of the text we assume that particles interact through a continuous function of their empirical density.

3.2.1 Deterministic particle systems

We first investigate the case of systems of ordinary differential equations. Let $b : \mathbb{R}^d \times \mathcal{P}_1(\mathbb{R}^d) \rightarrow \mathbb{R}^d$ be a C^1 and globally Lipschitz continuous vector field, with Lipschitz constant L . We denote by N the number of particles, and fix $\mathbf{x} = (x_0^1, \dots, x_0^N) \in (\mathbb{R}^d)^N$ the initial configuration. We investigate the system

$$(3.7) \quad \begin{cases} \dot{X}^i(t) = b(X^i(t), m_{\mathbf{X}(t)}^N), & t \geq 0, i = 1, \dots, N \\ X^i(0) = x_0^i, & i = 1, \dots, N \end{cases}$$

Here $\mathbf{X}(t) = (X^1(t), \dots, X^N(t))$ and $m_{\mathbf{X}(t)}^N = \frac{1}{N} \sum_{i=1}^N \delta_{X^i(t)}$ is the empirical measure at time t .

Proposition 3.2.1. *Let $\mathbf{x} = (x_0^1, \dots, x_0^N), \mathbf{y} = (y_0^1, \dots, y_0^N) \in (\mathbb{R}^d)^N$ be two initial configurations, and $\mathbf{X}(t) = (X^1(t), \dots, X^N(t)), \mathbf{Y}(t) = (Y^1(t), \dots, Y^N(t))$ be the associated solutions to (3.7). Then*

$$(3.8) \quad \mathbf{d}_1(m_{\mathbf{X}(t)}^N, m_{\mathbf{Y}(t)}^N) \leq \mathbf{d}_1(m_{\mathbf{x}_0}^N, m_{\mathbf{y}_0}^N) e^{2Lt}, \quad t \geq 0.$$

Moreover, $m_{\mathbf{X}}^N$ is a weak solution in $C^0(\mathbb{R}_+, \mathcal{P}_1(\mathbb{R}^d))$ to the continuity equation

$$(3.9) \quad \begin{cases} \partial_t m + \operatorname{div}(mb(x, m)) = 0 & \text{in } (0, \infty) \times \mathbb{R}^d, \\ m(0) = m_{\mathbf{x}_0}^N & \text{in } \mathbb{R}^d. \end{cases}$$

Proof. Recalling Proposition 3.1.2, we can assume without loss of generality that the labelling of the particles is arranged in such a way that

$$\mathbf{d}_1(m_{\mathbf{x}_0}^N, m_{\mathbf{y}_0}^N) = \frac{1}{N} \sum_{i=1}^N |x_0^i - y_0^i|.$$

In view of the equation satisfied by $\mathbf{X}$ and $\mathbf{Y}$, we have

$$\begin{aligned} \frac{1}{N} \sum_{i=1}^N |X^i(t) - Y^i(t)| &\leq \frac{1}{N} \sum_{i=1}^N |x_0^i - y_0^i| + \frac{1}{N} \int_0^t \sum_{i=1}^N |b(X^i(s), m_{\mathbf{X}(s)}^N) - b(Y^i(s), m_{\mathbf{Y}(s)}^N)| ds \\ &\leq \mathbf{d}_1(m_{\mathbf{x}_0}^N, m_{\mathbf{y}_0}^N) + \frac{1}{N} \int_0^t \sum_{i=1}^N L \left(|X^i(s) - Y^i(s)| + \mathbf{d}_1(m_{\mathbf{X}(s)}^N, m_{\mathbf{Y}(s)}^N) \right) ds \\ &\leq \mathbf{d}_1(m_{\mathbf{x}_0}^N, m_{\mathbf{y}_0}^N) + 2L \int_0^t \frac{1}{N} \sum_{i=1}^N |X^i(s) - Y^i(s)| ds. \end{aligned}$$

By Gronwall Lemma and Proposition 3.1.2, we obtain

$$\mathbf{d}_1(m_{\mathbf{X}(t)}^N, m_{\mathbf{Y}(t)}^N) \leq \frac{1}{N} \sum_{i=1}^N |X^i(t) - Y^i(t)| \leq \mathbf{d}_1(m_{\mathbf{x}_0}^N, m_{\mathbf{y}_0}^N) e^{2Lt}.$$

For the rest of the proof, we assume for simplicity that b is bounded (and leave the general case as an exercise). We note that, for $0 \leq s \leq t$,

$$\mathbf{d}_1(m_{\mathbf{X}(s)}^N, m_{\mathbf{X}(t)}^N) \leq \frac{1}{N} \sum_{i=1}^N |X^i(t) - X^i(s)| \leq \frac{1}{N} \sum_{i=1}^N \int_s^t |b(X^i(r), m_{\mathbf{X}(r)}^N)| dr \leq \|b\|_\infty (t - s).$$

Thus $m_{\mathbf{X}}^N \in C^0(\mathbb{R}_+, \mathcal{P}_1(\mathbb{R}^d))$. Finally, for any test function $\phi \in \mathcal{C}_c^\infty([0, \infty) \times \mathbb{R}^d)$,

$$\begin{aligned} \frac{d}{dt} \int_{\mathbb{R}^d} \phi(t, x) m_{\mathbf{X}(t)}^N(dx) &= \frac{d}{dt} \frac{1}{N} \sum_{i=1}^N \phi(t, X^i(t)) \\ &= \frac{1}{N} \sum_{i=1}^N \partial_t \phi(t, X^i(t)) + D\phi(t, X^i(t)) \cdot b(X^i(t), m_{\mathbf{X}(t)}^N) \\ &= \int_{\mathbb{R}^d} \left(\partial_t \phi(t, x) + D\phi(t, x) \cdot b(x, m_{\mathbf{X}(t)}^N) \right) m_{\mathbf{X}(t)}^N(dx). \end{aligned}$$

Integrating in time over $[0, \infty)$ gives (as ϕ has a compact support)

$$-\int_{\mathbb{R}^d} \phi(0, x) m_{\mathbf{x}_0}^N(dx) = \int_0^\infty \int_{\mathbb{R}^d} \left(\partial_t \phi(t, x) + D\phi(t, x) \cdot b(x, m_{\mathbf{X}(t)}^N) \right) m_{\mathbf{X}(t)}^N(dx) dt.$$

Thus $m_{\mathbf{X}}^N$ is a weak solution to (3.9). □

Given an initial condition $m_0 \in \mathcal{P}_1(\mathbb{R}^d)$, we now consider the weak solution to:

$$(3.10) \quad \begin{cases} \partial_t m + \operatorname{div}(mb(x, m)) = 0 & \text{in } (0, \infty) \times \mathbb{R}^d, \\ m(0) = m_0 & \text{in } \mathbb{R}^d. \end{cases}$$

Theorem 3.2.2. *For any $m_0 \in \mathcal{P}_1(\mathbb{R}^d)$, there exists a unique solution to (3.10) in $C^0(\mathbb{R}_+, \mathcal{P}_1(\mathbb{R}^d))$. In addition, if $m_{0,1}, m_{0,2} \in \mathcal{P}_1(\mathbb{R}^d)$ and if $m^1, m^2 \in C^0(\mathbb{R}_+, \mathcal{P}_1(\mathbb{R}^d))$ are the associated solutions, then*

$$(3.11) \quad \mathbf{d}_1(m_1(t), m_2(t)) \leq \mathbf{d}_1(m_{0,1}, m_{0,2}) e^{2Lt}, \quad \forall t \geq 0.$$

Proof. Existence of a solution and Estimate (3.11) are an easy consequence of Proposition 3.2.1: indeed, Let (Z^n) be a sequence of independent random variables on $\mathbb{R}^d$ of law m_0 . Set $\mathbf{Z}^N = (Z^1, \dots, Z^N)$. By the law of large numbers, $m_{\mathbf{Z}^N}^N$ converges to m_0 in $\mathbf{d}_1$. We fix a realization for which this holds. We denote by $\mathbf{X}^N = (X^{N,1}, \dots, X^{N,N})$ the solution to (3.9) associated with the initial condition $\mathbf{Z}^N$. Note that, for any integer $k < l$, $m_{\mathbf{Z}^{N^k}}^{N^k}$ can be considered as system of N^l particles, by looking at each particle in the N^k empirical measure as N^{l-k} identical particles. We can thus use inequality (3.11) in Proposition 3.2.1 to assert that

$$\sup_{t \geq 0} \mathbf{d}_1(m_{\mathbf{X}^{N^k}(t)}^{N^k}, m_{\mathbf{X}^{N^l}(t)}^{N^l}) e^{-Lt} \leq \mathbf{d}_1(m_{\mathbf{Z}^{N^k}}^{N^k}, m_{\mathbf{Z}^{N^l}}^{N^l}).$$

Therefore $(m_{\mathbf{X}^{N^k}}^{N^k})$ is a Cauchy sequence in $C^0(\mathbb{R}_+, \mathcal{P}_1)$ for the norm $d(m, m') = \sup_t e^{-Lt} \mathbf{d}_1(m(t), m'(t))$. As this space is complete, $(m_{\mathbf{X}^{N^k}}^{N^k})$ has a limit which can be easily seen to solve (3.10). Then (3.11) is a simple consequence of (3.11) and of the construction above.

Let us now prove the uniqueness of the solution to (3.10). Let m^1, m^2 be two solutions to (3.10). Set $\mu = m^1 - m^2$. Then μ solves in the sense of distribution

$$\begin{cases} \partial_t \mu + \operatorname{div}(b(x, m^1(t)) \mu) + \operatorname{div}(m_2(b(x, m^1(t)) - b(x, m^2(t))) = 0 & \text{in } (0, \infty) \times \mathbb{R}^d, \\ \mu(0, dx) = 0 & \text{in } \mathbb{R}^d. \end{cases}$$

Fix $T_0 > 0$, $T \in (0, T_0]$ and $\psi_T \in C_c^\infty(\mathbb{R}^d)$. By the method of characteristics, there is a C^1 solution to

$$\begin{cases} \partial_t \psi + b(x, m^1(t)) \cdot D\psi = 0 & \text{in } (0, \infty) \times \mathbb{R}^d, \\ \psi(T, x) = \psi & \text{in } \mathbb{R}^d. \end{cases}$$

Note that ψ is continuous on $[0, T] \times \mathbb{R}^d$, has a compact support, with $\|D\psi\|_\infty \leq C\|D\psi_T\|_\infty$, the constant C depending on the C^1 regularity of b and on T_0 . We use ψ as a test function in the equation satisfied by μ : we obtain

$$\begin{aligned} \int_{\mathbb{R}^d} \psi_T(x) \mu(T, dx) &= \int_0^T \int_{\mathbb{R}^d} (\partial_t \psi(t, x) + D\psi(t, x) \cdot b(x, m^1(t))) m^1(t, dx) dt \\ &\quad + \int_0^T \int_{\mathbb{R}^d} D\psi(t, x) \cdot (b(x, m^2(t)) - b(x, m^1(t))) m^2(t, x) dt. \end{aligned}$$

Using the equation for ψ , the regularity of b and the fact that $m^2(t)$ is a probability measure, we obtain

$$\begin{aligned} \int_{\mathbb{R}^d} \psi_T(x) \mu(T, dx) &\leq L\|D\psi\|_\infty \int_0^T \mathbf{d}_1(m^2(t), m^1(t)) dt \\ &\leq C\|D\psi_T\|_\infty L \int_0^T \mathbf{d}_1(m^2(t), m^1(t)) dt. \end{aligned}$$

We take the supremum over 1-Lipschitz continuous maps to obtain, thanks to Theorem 3.1.4,

$$\mathbf{d}_1(m^2(T), m^1(T)) \leq C\|D\psi_T\|_\infty L \int_0^T \mathbf{d}_1(m^2(t), m^1(t)) dt \quad \forall t \in [0, T_0].$$

We conclude that $m^1 = m^2$ in $[0, T_0]$ thanks to Gronwall Lemma. As T_0 is arbitrary, the proof is complete. $\square$

3.2.2 Stochastic particle systems

We complete this preliminary part by the analysis of large particle systems. We consider a system of N -particles (where $N \in \mathbb{N}^*$ is a large number) and we want to understand the behavior of the system as the number N tends to infinity. We work with the following system: for $i = 1, \dots, N$,

$$(3.12) \quad \begin{cases} dX_t^i = b(X_t^i, m_{X_t}^N) dt + dB_t^i, & m_{X_t}^N := \frac{1}{N} \sum_{j=1}^N \delta_{X_t^j} \\ X_0^i = Z^i \end{cases}$$

where the (B^i) are independent Brownian motions, the Z^i are i.i.d. random variables in $\mathbb{R}^d$ which are also independent of the (B^i) . The map $b : \mathbb{R}^d \times \mathcal{P}_1(\mathbb{R}^d) \rightarrow \mathbb{R}^d$ is assumed to be globally Lipschitz continuous. Note that, under these assumptions, the solution (X^i) to (3.12) exists and is unique, since this is an ordinary system of SDEs with Lipschitz continuous drift. A key point is that, because the (Z^i) have the same law and the equations satisfied by the X^i are symmetric, the X^i have the same law (they are actually “exchangeable”).

We want to understand the limit of the (X^i) as $N \rightarrow +\infty$. The heuristic idea is that, as N is large, the (X^i) become more and more independent, so that they become almost i.i.d. The law of large numbers then implies that

$$b(X_t^i, m_{X_t^i}^N) \approx \tilde{\mathbb{E}} [b(X_t^i, \mathcal{L}(X_t^i))] = \int_{\mathbb{R}^d} b(X_t^i, y) \mathbb{P}_{X_t^i}(dy),$$

where $\mathcal{L}(X)$ denotes the law of a random variable X . Therefore we expect the X^i to be close to the solution $\bar{X}^i$ to the McKean-Vlasov equation

$$(3.13) \quad \begin{cases} d\bar{X}_t^i = b(\bar{X}_t^i, \mathcal{L}(\bar{X}_t^i))dt + dB_t^i, \\ \bar{X}_0^i = Z^i \end{cases}$$

This is exactly what we are going to show. For doing so, we proceed in 2 steps: first, we prove the existence and the uniqueness of a solution to the McKean-Vlasov equation (3.13) and, second, we establish the convergence.

3.2.3 The well-posedness of the McKean-Vlasov equation.

Theorem 3.2.3. *Let us assume that $b : \mathbb{R}^d \times \mathcal{P}_1(\mathbb{R}^d) \rightarrow \mathbb{R}^d$ is globally Lipschitz continuous and let $Z \in L^2(\Omega)$. Then the McKean-Vlasov equation*

$$\begin{cases} dX_t = b(X_t, \mathcal{L}(X_t))dt + dB_t \\ X_0 = Z \end{cases}$$

has a unique solution, i.e., a progressively measurable process such that $\mathbb{E} \left[\int_0^T |X_s|^2 ds \right] < +\infty$ for any $T > 0$.

Remark 3.2.4. By Itô’s formula, the law m_t of a solution X_t solves in the sense of distributions the McKean-Vlasov equation

$$\begin{cases} \partial_t m_t - \frac{1}{2} \Delta m_t + \operatorname{div}(m_t b(x, m_t)) = 0 & \text{in } (0, T) \times \mathbb{R}^d \\ m_0 = \mathcal{L}(Z) & \text{in } \mathbb{R}^d. \end{cases}$$

One can show (and we will admit) that this equation has a unique solution, which proves the uniqueness in law of the process X .

Proof. Let $\alpha > 0$ to be chosen later and E be the set of progressively measurable processes (X_t) such that

$$\|X\|_E := \mathbb{E} \left[\int_0^\infty e^{-\alpha t} |X_t| dt \right] < +\infty.$$

Then $(E, \|\cdot\|_E)$ is a Banach space. On E we define the map Φ by

$$\Phi(X)_t = Z + \int_0^t b(X_s, \mathcal{L}(X_s)) ds + B_t, \quad t \geq 0.$$

Let us check that the map Φ is well defined from E to E . Note first that $\Phi(X)$ is indeed progressively measurable. By the L -Lipschitz continuity of b (for some $L > 0$),

$$\begin{aligned} |\Phi(X)_t| &\leq |Z| + \int_0^t |b(X_s, \mathcal{L}(X_s))| ds + |B_t| \\ &\leq |Z| + t|b(0, \delta_0)| + L \int_0^t (|X_s| + \mathbf{d}_1(\mathcal{L}(X_s), \delta_0)) ds + |B_t|, \end{aligned}$$

where one can easily check that $\mathbf{d}_1(\mathcal{L}(X_s), \delta_0) = \mathbb{E} \left[\int_0^t |X_s| ds \right]$. So

$$\begin{aligned} \mathbb{E} \left[\int_0^{+\infty} e^{-\alpha t} |\Phi(X)_t| dt \right] &\leq \alpha^{-1} \mathbb{E}[|Z|] + \alpha^{-2} |b(0, \delta_0)| + 2L \mathbb{E} \left[\int_0^{+\infty} e^{-\alpha t} \int_0^t |X_s| ds dt \right] + \int_0^{+\infty} e^{-\alpha t} \mathbb{E}[|B_t|] dt \\ &= \alpha^{-1} \mathbb{E}[|Z|] + \alpha^{-2} |b(0, \delta_0)| + \frac{2L}{\alpha} \mathbb{E} \left[\int_0^{+\infty} e^{-\alpha s} |X_s| ds \right] + C_d \int_0^{+\infty} t^{1/2} e^{-\alpha t} dt, \end{aligned}$$

where C_d depends only on dimension. This proves that $\Phi(X)$ belongs to E .

Let us finally check that Φ is a contraction. We have, if $X, Y \in E$,

$$\begin{aligned} |\Phi(X)_t - \Phi(Y)_t| &\leq \int_0^t |b(X_s, \mathcal{L}(X_s)) - b(Y_s, \mathcal{L}(Y_s))| dt \\ &\leq Lip(b) \left(\int_0^t \mathbf{d}_1(\mathbb{P}_{X_s}, \mathbb{P}_{Y_s}) dt + \int_0^t |X_s - Y_s| dt \right). \end{aligned}$$

Recall that $\mathbf{d}_1(\mathbb{P}_{X_s}, \mathbb{P}_{Y_s}) \leq \mathbb{E}[|X_s - Y_s|]$. So multiplying by $e^{-\alpha t}$ and taking expectation, we obtain:

$$\begin{aligned} \|\Phi(X) - \Phi(Y)\|_E &= \mathbb{E} \left[\int_0^{+\infty} e^{-\alpha t} |\Phi(X)_t - \Phi(Y)_t| dt \right] \\ &\leq 2Lip(b) \int_0^{+\infty} e^{-\alpha t} \int_0^t \mathbb{E}[|X_s - Y_s|] ds dt \\ &\leq \frac{2Lip(b)}{\alpha} \|X - Y\|_E. \end{aligned}$$

If we choose $\alpha > 2Lip(b)$, then Φ is a contraction in the Banach space E and therefore has a unique fixed point. It is easy to check that this fixed point is the unique solution to our problem. $\square$

3.2.4 The mean field limit

Let (X^i) be the solution to the particle system (3.12) and $(\bar{X}^i)$ be the solution to (3.13). Let us note that, as the (B^i) and the (Z^i) are independent with the same law, the $(\bar{X}^i)$ are i.i.d. for any $t \geq 0$.

Theorem 3.2.5. *We have, for any $T > 0$,*

$$\lim_{N \rightarrow +\infty} \sup_{i=1, \dots, N} \mathbb{E} \left[\sup_{t \in [0, T]} |X_t^i - \bar{X}_t^i| \right] = 0.$$

Remark: a similar result holds when there is a non constant volatility term σ in front of the Brownian motion. The proof is then slightly more intricate.

Proof. We consider

$$X_t^i - \bar{X}_t^i = \int_0^t (b(X_s^i, m_{X_s^i}^N) - b(\bar{X}_s^i, \mathcal{L}(\bar{X}_s^i))) ds.$$

By the uniqueness in law of the solution to the McKean-Vlasov equation we can denote by $m(t) := \mathcal{L}(\bar{X}_t^i)$

(it is independent of i). Then, setting $m_{\bar{X}_t^i}^N := \frac{1}{N} \sum_{j=1}^N \delta_{\bar{X}_t^j}$ and using the triangle inequality, we have

$$\begin{aligned} |X_t^i - \bar{X}_t^i| &\leq \int_0^t |b(X_s^i, m_{X_s^i}^N) - b(\bar{X}_s^i, m_{\bar{X}_s^i}^N)| ds + \int_0^t |b(\bar{X}_s^i, m_{\bar{X}_s^i}^N) - b(\bar{X}_s^i, m(s))| ds \\ &\leq Lip(b) \int_0^t (|X_s^i - \bar{X}_s^i| + \mathbf{d}_1(m_{X_s^i}^N, m_{\bar{X}_s^i}^N)) ds + Lip(b) \int_0^t \mathbf{d}_1(m_{\bar{X}_s^i}^N, m(s)) ds \\ (3.14) \quad &\leq Lip(b) \int_0^t (|X_s^i - \bar{X}_s^i| + \frac{1}{N} \sum_{j=1}^N |X_s^j - \bar{X}_s^j|) ds + Lip(b) \int_0^t \mathbf{d}_1(m_{\bar{X}_s^i}^N, m(s)) ds, \end{aligned}$$

since

$$\mathbf{d}_1(m_{X_t}^N, m_{\bar{X}_s}^N) \leq \frac{1}{N} \sum_{j=1}^N |X_s^j - \bar{X}_s^j|.$$

Summing over $i = 1, \dots, N$, we get

$$\frac{1}{N} \sum_{i=1}^N |X_t^i - \bar{X}_t^i| \leq 2\text{Lip}(b) \int_0^t \frac{1}{N} \sum_{j=1}^N |X_s^j - \bar{X}_s^j| ds + \text{Lip}(b) \int_0^t \mathbf{d}_1(m_{X_s}^N, m(s)) ds.$$

Using Gronwall Lemma, we find, for any $T > 0$, and for some constant C_T depending on $\text{Lip}(b)$,

$$(3.15) \quad \sup_{t \in [0, T]} \frac{1}{N} \sum_{i=1}^N |X_t^i - \bar{X}_t^i| \leq C_T \int_0^T \mathbf{d}_1(m_{X_s}^N, m(s)) ds,$$

where C_T depends on T and $\text{Lip}(b)$ (but not on N). Then we can come back to (3.14), use first Gronwall Lemma and then (3.15) to get, for any $T > 0$, and for some (new) constant C_T depending on $\text{Lip}(b)$ and which might change from line to line,

$$\begin{aligned} \sup_{t \in [0, T]} |X_t^i - \bar{X}_t^i| &\leq C_T \int_0^T \left(\frac{1}{N} \sum_{j=1}^N |X_s^j - \bar{X}_s^j| + \mathbf{d}_1(m_{X_s}^N, m(s)) \right) ds \\ &\leq C_T \int_0^T \mathbf{d}_1(m_{X_s}^N, m(s)) ds. \end{aligned}$$

We now take expectation to obtain

$$\mathbb{E} \left[\sup_{t \in [0, T]} |X_t^i - \bar{X}_t^i| \right] \leq C_T \int_0^T \mathbb{E} \left[\mathbf{d}_1(m_{X_s}^N, m(s)) \right] ds.$$

One can finally check exactly as in the proof of Theorem 3.1.9 that the right-hand side tends to 0. $\square$

3.2.5 Further reading

An introductive exposition of the topic can be found in the notes of Villani [60]. Classical references for the mean field limit of particle systems are the monographs or textbooks by Sznitman [59], Spohn [58] and Golse [34].

3.3 Optimal control

3.3.1 The controlled problem

Let us consider a stochastic controlled problem where the state (X_s) of the system is governed by the stochastic differential equation (SDE) with values in $\mathbb{R}^d$:

$$(3.16) \quad X_s = x + \int_t^s b(r, X_r, \alpha_r) dr + \int_t^s \sigma(r, X_r, \alpha_r) dB_r.$$

In the above equation, $B = (B_s)_{s \geq 0}$ is a N -dimensional Brownian motion (starting at 0) adapted to a fixed filtration $(\mathcal{F}_t)_{t \geq 0}$, $b : [0, T] \times \mathbb{R}^d \times A \rightarrow \mathbb{R}^d$ and $\sigma : [0, T] \times \mathbb{R}^d \times A \rightarrow \mathbb{R}^{d \times N}$ satisfy some regularity conditions given below and the process $\alpha = (\alpha_s)$ is progressively measurable with values in some set A . We denote by $\mathcal{A}$ the set of such processes. The elements of $\mathcal{A}$ are called the control processes.

The controller controls the process X through the control α in order to reach some goal: here we consider optimal control problems, in which the controller aims at minimizing some cost J . We will mostly focus on the finite horizon problem, where J takes the form:

$$J(t, x, \alpha) = \mathbb{E} \left[\int_t^T L(s, X_s, \alpha_s) ds + g(X_T) \right].$$

Here $T > 0$ is the finite horizon of the problem, $L : [0, T] \times \mathbb{R}^d \times A \rightarrow \mathbb{R}$ and $g : \mathbb{R}^d \rightarrow \mathbb{R}$ are given continuous maps (again we are more precise in the next section on the assumptions on L and g). The controller minimizes J by using controls in $\mathcal{A}$.

We introduce the value function the map $u : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}$ defined by

$$u(t, x) = \inf_{\alpha \in \mathcal{A}} J(t, x, \alpha).$$

3.3.2 Dynamic programming and the verification Theorem

The main interest of the value function is that it indicates how the controller should choose her control in order to play in an optimal way. We explain the key ideas in a very informal way. A rigorous treatment of the question is described in the references indicated below.

Let us start with the **dynamic programming principle**, which states the following identity: for any $t_1 \leq t_2$,

$$(3.17) \quad u(t_1, x) = \inf_{\alpha \in \mathcal{A}} \mathbb{E} \left[\int_{t_1}^{t_2} L(s, X_s, \alpha_s) ds + u(t_2, X_{t_2}) \right].$$

The interpretation is that, to play optimally at time t_2 , the controller can forget the past trajectory and only remember the position X_{t_2} , thus reducing the problem to the computation of $u(t_2, X_{t_2})$.

Fix now $t \in [0, T)$. Choosing $t_1 = t$, $t_2 = t + h$ (for $h > 0$ small) and assuming that u is smooth enough, we obtain by Itô's formula and (3.17) that

$$\begin{aligned} u(t, x) = \inf_{\alpha \in \mathcal{A}} \mathbb{E} \left[\int_t^{t+h} L(s, X_s, \alpha_s) ds + u(t, x) + \int_t^{t+h} (\partial_t u(s, X_s) + Du(s, X_s) \cdot b(s, X_s, \alpha_s) \right. \\ \left. + \frac{1}{2} \text{Tr}(\sigma \sigma^*(s, X_s, \alpha_s) D^2 u(s, X_s))) ds \right]. \end{aligned}$$

Simplifying by $u(t, x)$, dividing by h and letting $h \rightarrow 0^+$ gives (informally) the Hamilton-Jacobi equation

$$0 = \inf_{a \in A} \left[L(t, x, a) + \partial_t u(t, x) + Du(t, x) \cdot b(t, x, a) + \frac{1}{2} \text{Tr}(\sigma \sigma^*(t, x, a) D^2 u(t, x)) \right].$$

The proof of this fact is delicate in general, even more since the solution u is seldom enough differentiable to fully justify this computation. Let us explain the argument in the case where u is assumed to be twice differentiable in x and once in t with bounded derivatives. Fix a (time independent) control a . Then, by dynamic programming principle and $h > 0$ small, we have

$$u(t, x) \leq \mathbb{E} \left[\int_t^{t+h} L(s, X_s, a) ds + u(t+h, X_{t+h}) \right].$$

We now use Itô's formula to expand the term in u in the right-hand side:

$$u(t, x) \leq \mathbb{E} \left[\int_t^{t+h} (L(s, X_s, a) + \partial_t u(s, X_s) + Du(s, X_s) \cdot b(s, X_s, a) + \frac{1}{2} \text{Tr}(\sigma \sigma^*(s, X_s, a) D^2 u(s, X_s))) ds + u(t, x) \right].$$

We simplify by $u(t, x)$ and divide by h to get:

$$0 \leq \mathbb{E} \left[\frac{1}{h} \int_t^{t+h} (L(s, X_s, a) + \partial_t u(s, X_s) + Du(s, X_s) \cdot b(s, X_s, a) + \frac{1}{2} \text{Tr}(\sigma \sigma^*(s, X_s, a) D^2 u(s, X_s))) ds \right].$$

As $h \rightarrow 0$ and for $s \in [t, t+h]$, we have by continuity of the trajectories of the SDE that $X_s \rightarrow X_t = x$. So we can let $h \rightarrow 0^+$ and obtain

$$\begin{aligned} 0 &\leq \mathbb{E} \left[(L(s, x, a) + \partial_t u(s, x) + Du(s, x) \cdot b(s, x, a) + \frac{1}{2} \text{Tr}(\sigma \sigma^*(s, x, a) D^2 u(s, x))) \right] \\ &= L(s, x, a) + \partial_t u(s, x) + Du(s, x) \cdot b(s, x, a) + \frac{1}{2} \text{Tr}(\sigma \sigma^*(s, x, a) D^2 u(s, x)). \end{aligned}$$

This can be rearranged into

$$-\partial_t u(s, x) - Du(s, x) \cdot b(s, x, a) - L(s, x, a) - \frac{1}{2} \text{Tr}(\sigma \sigma^*(s, x, a) D^2 u(s, x)) \leq 0.$$

Taking the sup over the controls $a \in A$ gives the first inequality. The reverse inequality is slightly more technical to obtain and we omit the argument.

Let us introduce the Hamiltonian H of our problem: for $p \in \mathbb{R}^d$ and $M \in \mathbb{R}^{d \times d}$,

$$H(t, x, p, M) := \sup_{a \in A} \left[-L(t, x, a) - p \cdot b(t, x, a) - \frac{1}{2} \text{Tr}(\sigma \sigma^*(t, x, a) M) \right].$$

Then the **Hamilton-Jacobi equation** can be rewritten as a terminal value problem:

$$\begin{cases} -\partial_t u(t, x) + H(t, x, Du(t, x), D^2 u(t, x)) = 0 & \text{in } (0, T) \times \mathbb{R}^d, \\ u(T, x) = g(x) & \text{in } \mathbb{R}^d. \end{cases}$$

The first equation is backward in time (the map H being nonincreasing with respect to $D^2 u$). The terminal condition comes just from the definition of u for $t = T$.

Let us now introduce $\alpha^*(t, x) \in A$ as a maximum point in the definition of H when $p = Du(t, x)$ and $M = D^2(t, x)$. Namely

$$(3.18) \quad \begin{aligned} H(t, x, Du(t, x), D^2 u(t, x)) &= -L(t, x, \alpha^*(t, x)) - Du(t, x) \cdot b(t, x, \alpha^*(t, x)) \\ &\quad - \frac{1}{2} \text{Tr}(\sigma \sigma^*(t, x, \alpha^*(t, x)) D^2 u(t, x)). \end{aligned}$$

We assume that α^* is sufficiently smooth to justify the computation below. We are going to show that α^* is the **optimal feedback**, namely the optimal control to play at time t in the state x . Indeed, one has the following “**Verification Theorem**”:

$$u(t, x) = J(t, x, \alpha^*).$$

Proof. Let us denote by X^* the solution to

$$X_s^* = x + \int_t^s b(r, X_r^*, \alpha^*(r, X_r^*)) dr + \int_t^s \sigma(r, X_r^*, \alpha^*(r, X_r^*)) dB_r$$

and set, to simplify the expression, $\alpha_s^* := \alpha^*(s, X_s^*)$ (note now that (α_s^*) is a control, namely it belongs to $\mathcal{A}$. Strictly speaking, (α_t^*) is the optimal control, $\alpha^* = \alpha^*(t, x)$ being the optimal feedback). By Itô's formula, we have

$$\begin{aligned} g(X_T^*) = u(T, X_T^*) &= u(t, x) + \int_t^T (\partial_t u(s, X_s^*) + Du(s, X_s^*) \cdot b(s, X_s^*, \alpha_s^*)) \\ &\quad + \frac{1}{2} \text{Tr}(\sigma \sigma^*(s, X_s^*, \alpha_s^*) D^2 u(s, X_s^*)) ds + \int_t^T \sigma^*(s, X_s^*, \alpha_s^*) Du(s, X_s^*) \cdot dB_s. \end{aligned}$$

Taking expectation, using first the optimality of α^* in (3.18) and then the Hamilton-Jacobi equation satisfied by u , we obtain

$$\begin{aligned} \mathbb{E}[g(X_T^*)] &= u(t, x) + \mathbb{E} \left[\int_t^T (\partial_t u(s, X_s^*) - H(s, X_s^*, Du(s, X_s^*), D^2 u(s, X_s^*)) - L(s, X_s^*, \alpha_s^*)) ds \right] \\ &= u(t, x) - \mathbb{E} \left[\int_t^T L(s, X_s^*, \alpha_s^*) ds \right]. \end{aligned}$$

Rearranging, we find

$$u(t, x) = \mathbb{E} \left[\int_t^T L(s, X_s^*, \alpha_s^*) ds + g(X_T^*) \right],$$

which shows the optimality of α^* . □

The above arguments, although largely heuristic, can be partially justified. Surprisingly, the dynamic programming principle is the hardest step to prove, and only holds under strong restrictions on the probability space (but these restrictions are merely theoretical, and one can most often assume that they are met in practice). In general, the value function is smooth only under very strong assumptions on the system. However, under milder conditions, it is at least continuous and then it satisfies the Hamilton-Jacobi equation in the viscosity sense. Besides, the Hamilton-Jacobi has a unique (viscosity) solution so that it characterizes the value function. If the diffusion is strongly non degenerate (i.e., $M = d$ and σ is invertible with a smooth and bounded inverse) and if the Hamiltonian is smooth, then the value function is smooth as well. In this setting the verification Theorem makes perfectly sense.

3.3.3 Estimates on SDEs

In the previous parts, we were intentionally very fuzzy about the assumptions and the results. A complete rigorous treatment of the problem is far beyond the aim of these notes. However, we need to clarify a bit the definition of our problem. For this, we assume the maps b and σ to be continuous and Lipschitz continuous in x independently of t and a : There is a constant $K > 0$ such that

$$|b(t, x, a) - b(t, y, a)| + |\sigma(t, x, a) - \sigma(t, y, a)| \leq K|x - y| \quad \forall x, y \in \mathbb{R}^d, t \in [0, +\infty), a \in A.$$

Under these assumptions, for any bounded control $\alpha \in \mathcal{A}$, there exists a unique solution to (3.16). By a solution we mean a progressively measurable process X such that, for any $T > 0$,

$$\mathbb{E} \left[\int_t^T |X_s|^2 ds \right] < +\infty$$

and (3.16) holds $\mathbb{P}$ -a.s. More precisely, we have:

Lemma 3.3.1. *Let α be a bounded control in $\mathcal{A}$. Then there exists a unique solution X to (3.16) and this solution satisfies, for any $T > 0$ and $p \in [2, +\infty)$,*

$$\mathbb{E} \left[\sup_{t \in [0, T]} |X_t|^p \right] \leq C(1 + |x|^p + \|b(\cdot, 0, \alpha)\|_\infty^p + \|\sigma(\cdot, 0, \alpha)\|_\infty^p),$$

where $C = C(T, p, d)$ depends only on T , p and d .

Remark 3.3.2. In view of the above result, the cost J is well-defined provided, for instance, that the maps $L : [0, T] \times \mathbb{R}^d \times A \rightarrow \mathbb{R}$ and $g : \mathbb{R}^d \rightarrow \mathbb{R}$ are continuous with at most a polynomial growth.

Proof. The existence can be proved by a fixed point argument, exactly as for the McKean-Vlasov equation (see below). Let us show the bound. We set $M := \|b(\cdot, 0, \alpha)\|_\infty + \|\sigma(\cdot, 0, \alpha)\|_\infty$. We have, by Hölder's inequality

$$|X_s|^p \leq 3^{p-1} \left(|x|^p + \int_t^s |b(r, X_r, \alpha_r)|^p dr + \left| \int_t^s \sigma(r, X_r, \alpha_r) dB_r \right|^p \right).$$

Thus

$$\mathbb{E} \left[\sup_{t \leq r \leq s} |X_r|^p \right] \leq 3^{p-1} \left(|x|^p + \int_t^s \mathbb{E} [|b(r, X_r, \alpha_r)|^p] dr + \mathbb{E} \left[\sup_{t \leq r \leq s} \left| \int_t^r \sigma(u, X_u, \alpha_u) dB_u \right|^p \right] \right).$$

Note that

$$|b(s, X_s, \alpha_s)| \leq |b(s, 0, \alpha_s)| + L|X_s| \leq M + L|X_s|$$

and, in the same way,

$$(3.19) \quad |\sigma(s, X_s, \alpha_s)| \leq M + L|X_s|.$$

So we have

$$\int_t^s \mathbb{E} [|b(r, X_r, \alpha_r)|^p] dr \leq 2^{p-1} (M^p(s-t) + L^p \int_t^s \mathbb{E} [|X_r|^p] dr).$$

By the Burkholder-Davis-Gundy inequality (see Theorem IV.4.1 in [54]), we have

$$\mathbb{E} \left[\sup_{t \leq r \leq s} \left| \int_t^r \sigma(u, X_u, \alpha_u) dB_u \right|^p \right] \leq C_p \mathbb{E} \left[\left(\int_t^s \text{Tr}(\sigma \sigma^*(r, X_r, \alpha_r)) dr \right)^{p/2} \right],$$

where the constant C_p depends on p only. Combining Hölder's inequality (since $p/2 \geq 1$) with (3.19), we then obtain

$$\mathbb{E} \left[\sup_{t \leq r \leq s} \left| \int_t^r \sigma(u, X_u, \alpha_u) dB_u \right|^p \right] \leq C_p (s-t)^{p/2-1} 2^{p-1} \left(M^p (s-t) + L^p \int_t^s \mathbb{E} [|X_r|^p] dr \right).$$

Putting together the different estimates we get therefore, for $s \in [t, T]$,

$$\begin{aligned} \mathbb{E} \left[\sup_{t \leq r \leq s} |X_r|^p \right] &\leq C(p, T, d) \left(1 + |x|^p + M^p + \int_t^s \mathbb{E} [|X_r|^p] dr \right) \\ &\leq C(p, T, d) \left(1 + |x|^p + M^p + \int_t^s \mathbb{E} \left[\sup_{t \leq u \leq r} |X_u|^p \right] dr \right), \end{aligned}$$

where the constant $C(p, T, d)$ depends only on p , T and d . We can then conclude by Gronwall's Lemma. $\square$

3.3.4 Further reading

Classical references on stochastic optimal control problems are the monographs by Fleming and Rischel [30], Fleming and Soner [31], Yong and Zhou [63]. Several introductory courses on optimal control can be found online: for instance

- B. Bouchard, “Introduction to stochastic control of mixed diffusion processes, viscosity solutions and applications in finance and insurance”,
<https://www.ceremade.dauphine.fr/~bouchard/bouchard.htm>
- N. Touzi, “Deterministic and Stochastic Control, Application to Finance”,
<http://www.cmap.polytechnique.fr/~touzi/#Lecture>

Chapter 4

Mean field control problems

We study here mean field control problems, in which a global planner controls a large number of devices (for instance drones, or robots, or electric devices). Having in mind the behavior of large particle systems, we formalize the problem as an optimal control of a McKean-Vlasov equation. Namely, in a general situation, we aim at minimizing a cost of the form

$$J(\alpha) = \int_0^T \int_{\mathbb{R}^d} L(x, \alpha(t, x), m(t)) dx dt + \mathcal{G}(m(T))$$

where $\alpha : [0, T] \times \mathbb{R}^d \rightarrow A$ is a distributed control taking values in some space A and m solves a controlled McKean-Vlasov equation of the form

$$\partial_t m - \Delta m + \operatorname{div}(mb(x, \alpha, m)) = 0, \quad m(0) = m_0.$$

Here $b : \mathbb{R}^d \times A \times \mathcal{P}(\mathbb{R}^d) \rightarrow \mathbb{R}^d$ is a vector field. The diffusion term could also be more complex, involving a control. Our aim is to show—at least for a simplified problem—the existence of an optimal solution and to characterize it.

4.1 Statement of the problem and main result

The (toy model) of optimal control problem we are interested in is the following: we want to minimize the cost

$$\tilde{J}(\alpha) = \mathbb{E} \left[\int_0^T \left(\frac{1}{2} |\alpha_s|^2 + \mathcal{F}(\mathcal{L}(X_s)) \right) ds \right],$$

where (α_s) is a process with values in $\mathbb{R}^d$ defined on a filtrated probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ and where

$$X_t = \bar{X}_0 + \int_0^t \alpha_s ds + \sqrt{2} B_t, \quad t \in [0, T],$$

(B_t) being a d -dimensional Brownian motion on the probability space. The initial condition $\bar{X}_0$ is random and belongs to $\mathcal{F}_0$. The coupling function $\mathcal{F} : \mathcal{P}_1 \rightarrow \mathbb{R}$ will be assumed to be (at least) continuous and bounded. It is important to note that the above problem is not a standard optimal control problem as soon as $\mathcal{F}$ is nonlinear. For instance the standard dynamic programming principle (in $\mathbb{R}^d$) does not apply for this problem: we will see below that a dynamic programming principle actually holds, but in the space of measures.

In order to study this problem, it will be convenient to rewrite it in terms of the law of $m_s = \mathcal{L}(X_s)$, assuming that the process α is in feedback form. This yields to the problem of minimizing

$$(4.1) \quad J(m, \alpha) = \int_0^T \left(\int_{\mathbb{R}^d} \frac{1}{2} |\alpha(s, x)|^2 m(s, dx) + \mathcal{F}(m(s)) \right) ds$$

where $m \in C^0([0, T], \mathcal{P}_2)$, $\alpha \in L^0([0, T] \times \mathbb{R}^d, \mathbb{R}^d)$ (the set of Borel measurable maps from $[0, T] \times \mathbb{R}^d$ to $\mathbb{R}^d$) such that $\int \int |\alpha|^2 m(t, dx) dt < +\infty$ and where the pair (m, α) solves in the sense of distribution

$$(4.2) \quad \partial_t m - \Delta m + \operatorname{div}(\alpha m) = 0 \text{ in } (0, T) \times \mathbb{R}^d, \quad m(0) = m_0 \text{ in } \mathbb{R}^d.$$

The coupling function $\mathcal{F}$ is as above.

At a first glance it is not obvious that both problems are equivalent. We shall not discuss this equivalence here (and we won't use it). It is proved under suitable conditions on the control in [42] and relies on a mimicking argument on SDEs and the so-called superposition principle for McKean-Vlasov equations.

In order to state the optimality conditions, we suppose that the map $\mathcal{F}$ has some derivative on $\mathcal{P}_1$. For this we assume the existence of a continuous and bounded map $F : \mathbb{R}^d \times \mathcal{P}_1 \rightarrow \mathbb{R}$ such that

$$(4.3) \quad \mathcal{F}(m') - \mathcal{F}(m) = \int_0^1 \int_{\mathbb{R}^d} F(x, (1-t)m + tm')(m - m')(dx) \quad \forall m, m' \in \mathcal{P}_1(\mathbb{R}^d).$$

The above equality means that F is “the (flat) derivative” of $\mathcal{F}$. We will discuss the notion of differentiability later in the text. For instance, if $F(x, m) = \int_{\mathbb{R}^d} \phi(x, y)m(dy)$, where ϕ is continuous, bounded and symmetric, it is not difficult to show that one can take:

$$\mathcal{F}(m) = \frac{1}{2} \int_{\mathbb{R}^d} \phi(x, y)m(dy)m(dx).$$

We now state the main result of this section.

Theorem 4.1.1. *If $\mathcal{F} : \mathcal{P}_1 \rightarrow \mathbb{R}$ is continuous and bounded, then for any $m_0 \in \mathcal{P}_2$ there exists at least one minimizer (m, α) of $J(m_0, \cdot, \cdot)$ under the constraint (4.2).*

If in addition there exists a map $F : \mathbb{R}^d \times \mathcal{P}_1 \rightarrow \mathbb{R}$ which is globally Lipschitz continuous and bounded and such that (4.3) holds, then for any minimizer (m, α) , there exists a map $u : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}$, of class $C^{1,2}$, such that $\alpha(t, x) = -Du(t, x)$ and the pair (u, m) solves the system

$$(4.4) \quad \begin{cases} -\partial_t u - \Delta u + \frac{1}{2}|Du|^2 = F(x, m(t)) \text{ in } (0, T) \times \mathbb{R}^d, \\ \partial_t m - \Delta m - \operatorname{div}(Dum) = 0 \text{ in } (0, T) \times \mathbb{R}^d, \\ m(0) = m_0, u(T, x) = 0 \text{ in } \mathbb{R}^d. \end{cases}$$

Finally, if $\mathcal{F}$ is convex on $\mathcal{P}_1$, then the condition above is also sufficient: if (u, m) is a classical solution to (4.4), then the pair (m, α) with $\alpha = -Du$ is a solution to the minimization problem.

The theorem is stated in a very simple framework: the existence of a minimizer hold under much more general assumptions on the Hamiltonian (here it is just $\frac{1}{2}|Du|^2 - \mathcal{F}$) and on the Fokker-Planck equation (the equation for m). In terms of stochastic processes, the results are even more general: the existence of (weak) minimizers is for instance discussed in [42] which also discusses the link with the optimal control of large particle systems.

The proof of Theorem 4.1.1 relies on standard arguments in convex analysis. We prove the existence part in Section 4.3, the optimality condition is formally explained in Section 4.4 and rigorously proved in Section 4.5 while Section 4.6 is dedicated to the sufficient part.

4.2 An elementary example

We illustrate Theorem 4.1.1 by an elementary example (which, however, does not exactly fit the assumptions). We assume that $\mathcal{F}(m) = \frac{1}{2}|M(m)|^2$, where $M(m) = \int_{\mathbb{R}^d} xm(dx)$ is the mean of m . Then we can take $F(x, m) = M(m) \cdot x$ (Exercise). Note that $\mathcal{F}$ and F are not bounded and therefore the above result has to be adapted to that set-up (this is however easy). The system of optimality conditions becomes

$$\begin{cases} -\partial_t u - \Delta u + \frac{1}{2}|Du|^2 = M(m(t)) \cdot x \text{ in } (0, T) \times \mathbb{R}^d, \\ \partial_t m - \Delta m - \operatorname{div}(Dum) = 0 \text{ in } (0, T) \times \mathbb{R}^d, \\ m(0) = m_0, u(T, x) = 0 \text{ in } \mathbb{R}^d. \end{cases}$$

One looks for a solution of the Hamilton-Jacobi equation in the form $u(t, x) = a(t) \cdot x + b(t)$ where $a : [0, T] \rightarrow \mathbb{R}^d$ and $b : [0, T] \rightarrow \mathbb{R}$ are assumed to be smooth and have to satisfy

$$-(a'(t) \cdot x + b'(t)) + \frac{1}{2}|a(t)|^2 = M(m(t)) \cdot x.$$

This implies that

$$a'(t) = -M(m(t)), \quad b'(t) = -\frac{1}{2}|a(t)|^2, \quad a(T) = 0, \quad b(T) = 0.$$

On the other hand (at least formally), letting $M_i(m)$ begin the i -th component of $M(m)$, we have

$$\frac{d}{dt}M_i(m(t)) = \int_{\mathbb{R}^d} x_i (\Delta m + \operatorname{div}(Dum)) = - \int_{\mathbb{R}^d} \partial_{x_i} um = -a_i(t).$$

The third inequality is obtained by integration by parts. This computation is licit since m has finite second order moments (the rigorous proof requires an approximation of the map $x \rightarrow x_i$ by smooth functions with a compact support). So we obtain

$$\frac{d^2}{dt^2}M(m(t)) = M(m(t)), \quad M(m(0)) = M(m_0), \quad \frac{d}{dt}M(m(T)) = -a(T) = 0.$$

This system has a unique solution given by

$$M(m(t)) = \frac{M(m_0)}{e^T + e^{-T}}(e^{t-T} + e^{T-t})$$

and we have

$$a(t) = -\frac{M(m_0)}{e^T + e^{-T}}(e^{t-T} - e^{T-t}) = -\frac{M(m_0) \sinh(t-T)}{\cosh(T)}.$$

One can then easily derive from this $u(t, x)$. Then the optimal solution (m, α) to the mean field control problem is given by the solution to

$$\partial_t m - \Delta m + \operatorname{div}(ma(t)) = 0, \quad m(0) = m_0.$$

Link with the optimal control of a particle system Let us check, in this elementary example, that (m, α) can be obtained as the limit of optimal trajectories associated with a control problem with many particles. Let N be the number of particles. We consider the optimal control problem consisting in minimizing over (α^i) the quantity

$$J^N(t_0, (x_0^i), (\alpha^i)) = \mathbb{E} \left[\int_{t_0}^T \frac{1}{2N} \sum_{i=1}^N |\alpha_s^i|^2 + \mathcal{F}(m_{\mathbf{X}_s}^N) ds \right],$$

where $t_0 \in [0, T]$ is the initial time, $\mathbf{x} = (x_0^i) \in (\mathbb{R}^d)^N$ is the initial condition, $\mathbf{X}_t = (X_t^1, \dots, X_t^N)$ satisfies

$$dX_t^i = x_0^i + \int_{t_0}^T \alpha_t^i dt + \sqrt{2}(B_t^i - B_{t_0}^i),$$

the (B^i) being independent d -dimensional Brownian motions defined on a stochastic basic $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$. The controls α^i are adapted to the filtration and take values in $\mathbb{R}^d$.

We denote by $v^N = v^N(t_0, \mathbf{x})$ the value function associated with this optimal control problem:

$$v^N(t_0, \mathbf{x}) = \inf_{(\alpha^i)} J^N(t_0, (x_0^i), (\alpha^i)).$$

It is a (a priori viscosity) solution to the following equation

$$\begin{cases} -\partial_t v^N - \sum_{i=1}^N \Delta_{x^i} v^N + \frac{N}{2} \sum_{i=1}^N |D_{x^i} v^N|^2 = \mathcal{F}(m_{\mathbf{x}}^N) & \text{in } (0, T) \times (\mathbb{R}^d)^N \\ v^N(T, \mathbf{x}) = 0 & \text{in } (\mathbb{R}^d)^N \end{cases}$$

The form of the Hamiltonian comes from the computation

$$H^N((x^i)(p^i)) = \sup_{(a^i) \in (\mathbb{R}^d)^N} - \sum_{i=1}^N a^i p^i - \frac{1}{N} \sum_{i=1}^N |a^i|^2 - \mathcal{F}(m_{\mathbf{x}}^N) = \frac{2}{N} \sum_{i=1}^N |a^i|^2 - \mathcal{F}(m_{\mathbf{x}}^N)$$

for any $(p^i) \in (\mathbb{R}^d)^N$, with optimal control $a^i = -Np^i$.

In view of the particular form of $\mathcal{F}$, one is led to look for a solution in the form of

$$v^N(t, \mathbf{x}) = c(t) \left| \frac{1}{N} \sum_{i=1}^N x^i \right|^2 + d(t)$$

where $c, d : [0, T] \rightarrow \mathbb{R}$ are of class C^1 . Note that

$$D_{x^i} v^N(t, \mathbf{x}) = \frac{2c(t)}{N} \left(\frac{1}{N} \sum_{j=1}^N x^j \right), \quad D_{x^i x^i}^2 v^N(t, \mathbf{x}) = \frac{2c(t)}{N^2} I_d.$$

Thus the expression above satisfies the Hamilton-Jacobi equation if and only if

$$-(c'(t) \left| \frac{1}{N} \sum_{i=1}^N x^i \right|^2 + d'(t)) - \frac{2c(t)d}{N} + 2c^2(t) \left| \frac{1}{N} \sum_{i=1}^N x^i \right|^2 = \frac{1}{2} \left| \frac{1}{N} \sum_{i=1}^N x^i \right|^2,$$

i.e., if and only if

$$-c'(t) + 2c^2(t) = \frac{1}{2}, \quad -d'(t) - \frac{2c(t)d}{N} = 0,$$

with boundary condition $c(T) = d(T) = 0$ (because $v^N(T) = 0$). This system has a unique solution, given by $c(t) = -\tanh(t-T)/2$ since $\tanh' = 1 - \tanh^2$. Recalling the form of the optimal control in the Hamiltonian, the optimal solution to the control of the N -particle problem is given by

$$dX_t^{i,N} = -2c(t) \left(\frac{1}{N} \sum_{j=1}^N X_t^{j,N} \right) dt + \sqrt{2} dB_t^i.$$

If one starts at initial time $t_0 = 0$ with a chaotic initial condition $(\bar{Z}^i)$ which are i.i.d. with law m_0 and independent of the (B^i) , we know from Section 3.2 that $X^{1,N}$ (for instance) converges to the solution X of the McKean-Vlasov equation

$$dX_t = -2c(t) \mathbb{E}[X_t] dt + \sqrt{2} dB_t^1, \quad X_0 = Z^1, \quad i \in \{1, \dots, N\}.$$

An elementary computation shows that $\mathbb{E}[X_t] = \mathbb{E}[Z^1] \exp\{-2 \int_0^t c(s) ds\} = M(m_0) \cosh(t-T)/\cosh(T)$ (since $(\ln(\cosh))' = \tanh$). As

$$-2c(t) \mathbb{E}[X_t] = M(m_0) \frac{\sinh(t-T)}{\cosh(T)} = a(t),$$

the law of X_t solves the same Fokker-Planck equation as m . Thus $\mathcal{L}(X_t) = m(t)$ for any t .

4.3 Proof of the existence of a minimizer

In this part we show the existence part of Theorem 4.1.1. We first note that the infimum is finite because, as $\mathcal{F}$ is bounded below, the functional J is bounded below and, on the other hand, if we choose $\alpha = 0$ and let m be the solution of the heat equation starting from m_0 , the criterium is finite for the pair $(m, 0)$.

Let (m^n, α^n) be a minimizing sequence for $J(m_0, \cdot, \cdot)$ under the constraint (4.2). We set $E^n = \alpha^n m^n$ and see E^n as a vector measure on $[0, T] \times \mathbb{R}^d$ with values in $\mathbb{R}^d$.

Lemma 4.3.1. *There exists a constant C , independent of n , such that*

$$\sup_{t \in [0, T]} \int_{\mathbb{R}^d} |x|^2 m^n(t, dx) + \sup_{t \neq t'} \frac{\mathbf{d}_1(m^n(t), m^n(t'))}{|t' - t|^{1/2}} + |E^n|([0, T] \times \mathbb{R}^d) \leq C.$$

Sketch of proof. Throughout the proof, C denotes a constant independent of n and which might change from line to line. For simplicity, we prove the result under the additional condition that α^n is a smooth, bounded and globally Lipschitz vector field. The extension to the general setting can be obtained by smoothing argument. Under this extra assumption, the solution to (4.2) is given by $m^n(t) = \mathcal{L}(X_t^n)$ where X_t^n solves

$$X_t^n = \bar{X}_0 + \int_0^t \alpha^n(s, X_s^n) ds + \sqrt{2} B_t,$$

where $\bar{X}_0$ is a random variable defined on a filtrated probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ with $\bar{X}_0 \in \mathcal{F}_0$ and $\mathcal{L}(\bar{X}_0) = m_0$ and B is a Brownian motion with $B_0 = 0$ ($\bar{X}_0$ and B being independent). Then by the triangle inequality and then by Cauchy-Schwarz,

$$\begin{aligned} \int_{\mathbb{R}^d} |x|^2 m^n(t, dx) &= \mathbb{E}[|X_t^n|^2] \leq 3 \left(\mathbb{E}[|\bar{X}_0|^2] + \mathbb{E} \left[\left| \int_0^t \alpha^n(s, X_s^n) ds \right|^2 \right] + 2\mathbb{E}[|B_t|^2] \right) \\ &\leq C + 3T \left(\int_0^T \int_{\mathbb{R}^d} |\alpha^n(s, x)|^2 m^n(s, dx) ds \right). \end{aligned}$$

The right-hand side of the above expression is bounded independently of n because $J(m_0, m^n, \alpha^n)$ is uniformly bounded as well as $\mathcal{F}$. On the other hand we have, for any $0 \leq t < t' \leq T$,

$$\begin{aligned} \mathbf{d}_1(m^n(t), m^n(t')) &\leq \mathbb{E}[|X_t^n - X_{t'}^n|] \leq \mathbb{E} \left[\int_t^{t'} |\alpha^n(s, X_s^n)| ds \right] + \sqrt{2} \mathbb{E}[|B_t - B_{t'}|] \\ &\leq (t' - t)^{1/2} \left(\int_t^{t'} \int_{\mathbb{R}^d} |\alpha^n(s, x)|^2 m^n(s, dx) ds \right)^{1/2} + C(t' - t)^{1/2} \\ &\leq C(t' - t)^{1/2} \end{aligned}$$

Finally, again by Cauchy-Schwarz,

$$|E^n|([0, T] \times \mathbb{R}^d) = \int_0^T \int_{\mathbb{R}^d} |\alpha^n(t, x)| m^n(t, dx) dt \leq T^{1/2} \left(\int_0^T \int_{\mathbb{R}^d} |\alpha^n(s, x)|^2 m^n(s, dx) ds \right)^{1/2} \leq C.$$

□

Corollary 4.3.2. *The sequence m^n is relatively compact in $C^0([0, T], \mathcal{P}_1)$.*

Proof. We know by Lemma 4.3.1 that the sequence m^n is uniformly Holder continuous on $\mathcal{P}_1$ and takes values in the compact subset of $\mathcal{P}_1$ consisting in measures $m \in \mathcal{P}_1$ such that $M_2(m) \leq C^{1/2}$ (see Lemma 3.1.3 for the compactness). We can therefore conclude by Ascoli-Arzelà theorem. □

As (E^n) has bounded variations it converges—up to a subsequence denoted in the same way—weakly- $*$ in $C_c^0([0, T] \times \mathbb{R}^d)$ to a vector measure E . We note that the pair (m, E) satisfies, in the sense of distributions, the Fokker-Planck equation

$$(4.5) \quad \partial_t m - \Delta m - \operatorname{div}(E) = 0 \text{ in } (0, T) \times \mathbb{R}^d, \quad m(0) = m_0$$

(since $m^n(0) = m_0$ and (m^n) converges uniformly to m in $\mathcal{P}_1$). The main difficulty is to write E in the form of $E = \alpha m$. This is the aim of the next lemma.

Lemma 4.3.3. *The measure E is absolutely continuous with respect to m and its Radon-Nikodym derivative α satisfies*

$$\int_0^T \int_{\mathbb{R}^d} |\alpha(t, x)|^2 m(t, dx) dt \leq \liminf \int_0^T \int_{\mathbb{R}^d} |\alpha^n(t, x)|^2 m^n(t, dx) dt$$

In particular, the pair (m, α) satisfies the Fokker-Planck equation (4.2).

Proof. We first claim that, for any $f \in C_c^0([0, T] \times \mathbb{R}^d; \mathbb{R}^d)$ one has

$$(4.6) \quad \int_0^T \int_{\mathbb{R}^d} f(t, x) \cdot E(dt, dx) - \frac{1}{2} \int_0^T \int_{\mathbb{R}^d} |f(t, x)|^2 m(t, dx) dt \leq \liminf_n \frac{1}{2} \int_0^T \int_{\mathbb{R}^d} |\alpha(t, x)|^2 m^n(t, dx) dt \leq C.$$

Indeed, for any $f \in C_c^0([0, T] \times \mathbb{R}^d; \mathbb{R}^d)$ we have

$$\begin{aligned} & \int_0^T \int_{\mathbb{R}^d} f(t, x) \cdot E^n(dt, dx) - \frac{1}{2} \int_0^T \int_{\mathbb{R}^d} |f(t, x)|^2 m^n(t, dx) dt \\ &= \int_0^T \int_{\mathbb{R}^d} f(t, x) \cdot \alpha^n(t, x) m^n(t, dx) dt - \frac{1}{2} \int_0^T \int_{\mathbb{R}^d} |f(t, x)|^2 m^n(t, dx) dt \\ &\leq \frac{1}{2} \int_0^T \int_{\mathbb{R}^d} |\alpha(t, x)|^2 m^n(t, dx) dt \leq C, \end{aligned}$$

where the first inequality comes from the fact that, for any $f, \alpha \in \mathbb{R}^d$, $f \cdot \alpha - \frac{1}{2}|f|^2 \leq \frac{1}{2}|\alpha|^2$ and the second one by Lemma 4.3.1. So we obtain the claim by letting $n \rightarrow +\infty$ since (m^n) converges to m in $C^0([0, T], \mathcal{P}_1)$ and (E^n) converges weakly- $*$ to E .

We now check that $E \ll m$. Let $E = (E_1, \dots, E_d)$. Fix $\lambda \in \mathbb{R}$ and K be a compact subset of $[0, T] \times \mathbb{R}^d$. Considering the sequence of continuous maps $f^k(t, x) = \lambda((1 - kd_K(t, x))_+, 0, \dots, 0)$ (where $(s)_+ = \max\{s, 0\}$ and $d_K(t, x) = \inf_{(s, y) \in E} |(s, y) - (t, x)|$ is the distance to K). Note that $f^k \in C_c^0([0, T] \times \mathbb{R}^d; \mathbb{R}^d)$. We have, by (4.6) and monotone convergence,

$$\begin{aligned} C &\geq \lim_k \lambda \int_0^T \int_{\mathbb{R}^d} f^k(t, x) \cdot E_1(dt, dx) - \lambda^2 \frac{1}{2} \int_0^T \int_{\mathbb{R}^d} |f^k(t, x)|^2 m(t, dx) dt \\ &= \lambda \int_0^T \int_{\mathbb{R}^d} \mathbf{1}_K(t, x) \cdot E_1(dt, dx) - \lambda^2 \frac{1}{2} \int_0^T \int_{\mathbb{R}^d} \mathbf{1}_K(t, x) m(t, dx) dt = \lambda E_1(K) - \lambda^2 m(K). \end{aligned}$$

Assume $m(K) = 0$. Then $\lambda E_1(K) \leq C$ for any $\lambda \in \mathbb{R}$, which implies that $E_1(K) = 0$. This shows that E_1 is absolutely continuous with respect to m . We can argue in the same way for $E_2, \dots, E_d$, and obtain therefore that E is absolutely continuous with respect to m . We denote by α its Radon-Nikodym derivative: $E = \alpha m$ with $\alpha \in L_m^1$.

Coming back to (4.6) we have therefore, for any $f \in C_c^0([0, T] \times \mathbb{R}^d; \mathbb{R}^d)$,

$$\int_0^T \int_{\mathbb{R}^d} (f(t, x) \cdot \alpha(t, x) - \frac{1}{2}|f(t, x)|^2) m(t, dx) dt \leq \liminf_n \frac{1}{2} \int_0^T \int_{\mathbb{R}^d} |\alpha(t, x)|^2 m^n(t, dx) dt.$$

We conclude by the next lemma. □

Lemma 4.3.4. *We have*

$$\sup_{f \in C_c^0([0, T] \times \mathbb{R}^d; \mathbb{R}^d)} \int_0^T \int_{\mathbb{R}^d} (f(t, x) \cdot \alpha(t, x) - \frac{1}{2}|f(t, x)|^2) m(t, dx) dt = \int_0^T \int_{\mathbb{R}^d} \frac{1}{2} |\alpha(t, x)|^2 m(t, dx) dt.$$

Proof. Let us set

$$S := \sup_{f \in C_c^0([0, T] \times \mathbb{R}^d; \mathbb{R}^d)} \int_0^T \int_{\mathbb{R}^d} (f(t, x) \cdot \alpha(t, x) - \frac{1}{2}|f(t, x)|^2) m(t, dx) dt.$$

Let $M > 0$ large and let us set $\alpha_M := \alpha \mathbf{1}_{\{|\alpha| \leq M\}}$. A corollary of Lusin's theorem (see Evans-Gariepy, Corollary 1 p.16) asserts that, for any $\epsilon > 0$ there exists a continuous map $\bar{\alpha}_M$ such that $\{(t, x) \in [0, T] \times \mathbb{R}^d, \alpha_M(t, x) \neq \bar{\alpha}_M(t, x)\}$ has a measure less than ϵ for m . Since α_M is bounded by M , we can assume without loss of generality that $\bar{\alpha}_M$ is bounded by M as well.

For any $R \in \mathbb{N}$ let us consider a continuous map ϕ_R such that $\phi_R(t, x) = 1$ if $|(t, x)| \leq R$, $\phi_R(t, x) = 0$ if $|(t, x)| \geq R + 1$ and $0 \leq \phi_R \leq 1$. Then $f_{R,M} := \phi_R \bar{\alpha}_M$ belongs to $C_c^0([0, T] \times \mathbb{R}^d; \mathbb{R}^d)$ and we have

$$\begin{aligned} S &\geq \int_0^T \int_{\mathbb{R}^d} (f_{R,M}(t, x) \cdot \alpha(t, x) - \frac{1}{2} |f_{R,M}(t, x)|^2) m(t, dx) dt \\ &= \int \int_{\{\alpha_M = \bar{\alpha}_M, |\alpha| \leq M\}} |\alpha(t, x)|^2 (\phi_R(t, x) - \frac{1}{2} \phi_R^2(t, x)) m(t, dx) dt \\ &\quad + \int \int_{\{\alpha_M \neq \bar{\alpha}_M\} \cup \{|\alpha| > M\}} (f_{R,M}(t, x) \cdot \alpha(t, x) - \frac{1}{2} |f_{R,M}(t, x)|^2) m(t, dx) dt \\ &\geq \int \int_{\{|\alpha| \leq M\}} |\alpha(t, x)|^2 (\phi_R(t, x) - \frac{1}{2} \phi_R^2(t, x)) m(t, dx) dt \\ &\quad - \int \int_{\{\alpha_M \neq \bar{\alpha}_M, |\alpha| \leq M\}} |\alpha(t, x)|^2 (\phi_R(t, x) - \frac{1}{2} \phi_R^2(t, x)) m(t, dx) dt \\ &\quad + \int \int_{\{\alpha_M \neq \bar{\alpha}_M\} \cup \{|\alpha| > M\}} (f_{R,M}(t, x) \cdot \alpha(t, x) - \frac{1}{2} |f_{R,M}(t, x)|^2) m(t, dx) dt. \end{aligned}$$

The second integral in the right-hand side is bounded below by $-M^2 m(\{\alpha_M \neq \bar{\alpha}_M\})$ which is above $-M^2 \epsilon$. Note that, by the definition of α_M and $\bar{\alpha}_M$, the set $\{|\alpha| > M\}$ is contained in the set $\{\alpha_M \neq \bar{\alpha}_M\}$. Thus the second term in the third integral in the right-hand side can be treated as the second above. As for the first one, we have, for any $M' > M$,

$$\begin{aligned} \int \int_{\{\alpha_M \neq \bar{\alpha}_M\}} f_{R,M}(t, x) \cdot \alpha(t, x) m(t, dx) dt &\geq -M \int \int_{\{\alpha_M \neq \bar{\alpha}_M\}} |\alpha(t, x)| m(t, dx) dt \\ &\geq -MM' m(\{\alpha_M \neq \bar{\alpha}_M, |\alpha| \leq M'\}) - M \int \int_{\{|\alpha| > M'\}} |\alpha| m \geq -MM' \epsilon - M \int \int_{\{|\alpha| > M'\}} |\alpha| m. \end{aligned}$$

Therefore

$$S \geq \int \int_{\{|\alpha| \leq M\}} |\alpha(t, x)|^2 (\phi_R(t, x) - \frac{1}{2} \phi_R^2(t, x)) m(t, dx) dt - MM' \epsilon - M \int \int_{\{|\alpha| > M'\}} |\alpha| m - 2M^2 \epsilon.$$

We first let $\epsilon \rightarrow 0$ and then $M' \rightarrow +\infty$ and find

$$S \geq \int \int_{\{|\alpha| \leq M\}} |\alpha(t, x)|^2 (\phi_R(t, x) - \frac{1}{2} \phi_R^2(t, x)) m(t, dx) dt.$$

By monotone convergence, the term in the right-hand side tends to $(1/2) \int \int |\alpha|^2 m$ as $R \rightarrow +\infty$ and then $M \rightarrow +\infty$. This proves the claim. $\square$

We are now ready to prove the existence of a solution to our minimization problem:

Proof of Theorem 4.1.1: existence of a solution. As (m^n, α^n) is a minimizing sequence, we have

$$\begin{aligned} \inf_{(m', \alpha')} J(m', \alpha') &= \lim_n \int_0^T \left(\int_{\mathbb{R}^d} \frac{1}{2} |\alpha^n(s, x)|^2 m^n(s, dx) + \mathcal{F}(m^n(s)) \right) ds \\ &\geq \int_0^T \left(\int_{\mathbb{R}^d} \frac{1}{2} |\alpha(s, x)|^2 m(s, dx) + \mathcal{F}(m(s)) \right) ds, \end{aligned}$$

where we used Lema 4.3.3 for the first term in the right-hand side and Corollary 4.3.2 for the second one. As the pair (m, α) satisfies the Fokker-Planck equation (4.2), we conclude that (m, α) is a minimizer of the problem. $\square$

4.4 Optimality conditions: case of classical solutions

The aim of this part is to show the following result:

Proposition 4.4.1. *Assume that (m, α) is a smooth minimizer of the functional J defined in (4.1), with $\bar{m} > 0$. Then there exists a smooth map $u : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}$ such that $\alpha = -Du$ and (m, u) solves (4.4).*

Note that Theorem 4.1.1 does not require the regularity of the minimizer. The proof of the optimality conditions without this extra condition is slightly more involved and postponed to Subsection 4.5. The main interest of the approach developed here is to find quickly the optimality condition, without bothering about technical aspects.

Proof. Fix a smooth minimizer $(\bar{m}, \bar{\alpha})$ with $\bar{m} > 0$. Let (m, α) satisfy be smooth and satisfy the constraint (4.2). For any $\lambda \in (0, 1]$, we set

$$(m_\lambda, \alpha_\lambda) := ((1 - \lambda)\bar{m} + \lambda m, \frac{(1 - \lambda)\bar{\alpha}\bar{m} + \lambda\alpha m}{m_\lambda}).$$

Then the pair $(m_\lambda, \alpha_\lambda)$ satisfies the constraint (4.2). By optimality of $(\bar{m}, \bar{\alpha})$ we have

$$\begin{aligned} 0 \leq J(m_\lambda, \alpha_\lambda) - J(\bar{m}, \bar{\alpha}) &= \int_0^T \int_{\mathbb{R}^d} \left(\frac{1}{2} |\alpha_\lambda|^2 m_\lambda - \frac{1}{2} |\bar{\alpha}|^2 \bar{m} \right) dx dt + \int_0^T (\mathcal{F}(m_\lambda(t)) - \mathcal{F}(\bar{m}(t))) dt \\ &= \int_0^T \int_{\mathbb{R}^d} \left(\frac{|(1 - \lambda)\bar{\alpha}\bar{m} + \lambda\alpha m|^2}{2((1 - \lambda)\bar{m} + \lambda m)} - |\bar{\alpha}|^2 \bar{m} \right) dx dt \\ &\quad + \int_0^T \int_0^1 \int_{\mathbb{R}^d} F(x, (1 - s)\bar{m}(t) + s m_\lambda(t)) (m_\lambda - \bar{m})(t, x) dx ds dt \end{aligned}$$

Dividing by λ and letting $\lambda \rightarrow 0^+$ gives

$$\begin{aligned} 0 &\leq \int_0^T \int_{\mathbb{R}^d} \bar{\alpha} \cdot (\alpha m - \bar{\alpha} \bar{m}) - \frac{|\bar{\alpha}|^2}{2} (m - \bar{m}) dx dt + \int_0^T \int_{\mathbb{R}^d} F(x, \bar{m}(t)) (m - \bar{m})(t, x) dx ds dt \\ (4.7) \quad &= \int_0^T \int_{\mathbb{R}^d} (\bar{\alpha} \cdot (\alpha - \bar{\alpha}) m + \frac{|\bar{\alpha}|^2}{2} (m - \bar{m}) + F(x, \bar{m}(t)) (m - \bar{m})(t, x)) dx dt. \end{aligned}$$

The goal is now to get rid of the term in $(m - \bar{m})$ in the expression above. For this we consider the classical solution u to the backward equation

$$(4.8) \quad \begin{cases} -\partial_t u - \Delta u - \bar{\alpha} \cdot Du = F(x, \bar{m}(t)) + \frac{1}{2} |\bar{\alpha}(t, x)|^2 & \text{in } (0, T) \times \mathbb{R}^d, \\ u(T, x) = 0 & \text{in } \mathbb{R}^d. \end{cases}$$

Note that u solves the dual equation of the equation satisfied by $\bar{m}$ with a right-hand side given by the term multiplied by $(m - \bar{m})$ in (4.7). Plugging this relation into (4.7), we have (since $u(T) = m(0) - \bar{m}(0) = 0$)

$$\begin{aligned} 0 &\leq \int_0^T \int_{\mathbb{R}^d} (\bar{\alpha} \cdot (\alpha - \bar{\alpha}) m + (-\partial_t u - \Delta u - \bar{\alpha} \cdot Du)(m - \bar{m})) dx dt \\ &= \int_0^T \int_{\mathbb{R}^d} (\bar{\alpha} \cdot (\alpha - \bar{\alpha}) m + u(\partial_t(m - \bar{m}) - \Delta(m - \bar{m})) - \bar{\alpha} \cdot Du(m - \bar{m})) dx dt \\ &= \int_0^T \int_{\mathbb{R}^d} (\bar{\alpha} \cdot (\alpha - \bar{\alpha}) m + u(-\operatorname{div}(\alpha m) + \operatorname{div}(\bar{\alpha} \bar{m})) - \bar{\alpha} \cdot Du(m - \bar{m})) dx dt \\ &= \int_0^T \int_{\mathbb{R}^d} ((\bar{\alpha} + Du) \cdot (\alpha - \bar{\alpha}) m) dx dt. \end{aligned}$$

Now we choose $\alpha = \bar{\alpha} + h\beta$ for $h > 0$ and β smooth and let m_h be the associated solution to the Fokker-Planck equation (4.2). As $h \rightarrow 0^+$, m_h converges to $\bar{m}$. Then the previous inequality becomes, after dividing by h and letting $h \rightarrow 0^+$,

$$0 \leq \int_0^T \int_{\mathbb{R}^d} (\bar{\alpha} + Du) \cdot \beta \bar{m} dx dt.$$

As β is arbitrary and $\bar{m} > 0$, we obtain that $D\bar{u} = -\bar{\alpha}$. This proves that $\bar{u}$ actually solves

$$\begin{cases} -\partial_t \bar{u} - \Delta \bar{u} + \frac{1}{2}|D\bar{u}(t, x)|^2 = F(x, \bar{m}(t)) & \text{in } (0, T) \times \mathbb{R}^d, \\ \bar{u}(T, x) = 0 & \text{in } \mathbb{R}^d. \end{cases}$$

and $\bar{m}$ solves

$$\partial_t \bar{m} - \Delta \bar{m} - \operatorname{div}(\bar{m} D\bar{u}) = 0 \text{ in } (0, T) \times \mathbb{R}^d, \quad m(0, x) = m_0(x).$$

So the pair $(\bar{u}, \bar{m})$ is a solution to (4.4). $\square$

4.5 Optimality conditions: case of general solutions

Let $\mathcal{E}$ be the set of pairs (m, E) , where $m \in C^0([0, T], \mathcal{P}_1)$ and E is a vector measure on $[0, T] \times \mathbb{R}^d$ with values in $\mathbb{R}^d$. We introduce the map Φ , defined on $\mathcal{E}$ by:

$$\Phi(m, E) = \begin{cases} \int_0^T \int_{\mathbb{R}^d} |\alpha(t, x)|^2 m(t, dx) dt & \text{if } E \ll m, \alpha = \frac{dE}{dm}, \\ +\infty & \text{otherwise} \end{cases}$$

By Lemma 4.3.4, Φ is convex and lower semi-continuous.

Lemma 4.5.1. *Let $(\bar{m}, \bar{\alpha})$ be a minimum of J (under constraint (4.2)) and set $\bar{E} = \bar{\alpha} \bar{m}$. Then $(\bar{m}, \bar{E})$ is a minimum of the linearized functional $\tilde{J}$ under constraint (4.5), where $\tilde{J}$ is defined by*

$$\tilde{J}(m, E) = \Phi(m, E) + \int_0^T \int_{\mathbb{R}^d} F(x, \bar{m}(t)) m(dt, dx),$$

where we recall that F is defined in (4.3).

Proof. Let $(m, E) \in \mathcal{E}$. For any $\lambda \in (0, 1)$, we set $(m_\lambda, E_\lambda) = (1 - \lambda)(\bar{m}, \bar{E}) + \lambda(m, E)$. Note that (m_λ, E_λ) belongs to $\mathcal{E}$ and satisfies the linear constraint (4.5). As $(\bar{m}, \bar{E})$ is optimal for J and Φ is convex, we get

$$J(\bar{m}, \bar{E}) \leq J(m_\lambda, E_\lambda) \leq (1 - \lambda)\Phi(\bar{m}, \bar{E}) + \lambda\Phi(m, E) + \int_0^T \int_{\mathbb{R}^d} \mathcal{F}(x, m_\lambda(t)) dt$$

Thus

$$\lambda\Phi(\bar{m}, \bar{E}) \leq \lambda \int_0^T \int_{\mathbb{R}^d} (\mathcal{F}(x, m_\lambda(t)) - \mathcal{F}(x, \bar{m}(t))) dt.$$

Using the definition of F in (4.3), dividing this inequality by $\lambda > 0$ and letting $\lambda \rightarrow 0^+$, we get

$$\Phi(\bar{m}, \bar{E}) \leq \int_0^T \int_{\mathbb{R}^d} F(x, \bar{m}(t)) (m(t, dx) - \bar{m}(t, dx)) dt.$$

This proves that $(\bar{m}, \bar{E})$ is a minimum of the linearized functional $\tilde{J}$ under constraint (4.5). $\square$

Proof of Theorem 4.1.1: optimality conditions. We now consider the solution $\bar{u}$ to the Hamilton-Jacobi equation

$$\begin{cases} -\partial_t \bar{u} - \Delta \bar{u} + \frac{1}{2}|D\bar{u}|^2 = F(x, \bar{m}(t)) & \text{in } (0, T) \times \mathbb{R}^d, \\ \bar{u}(T, x) = 0 & \text{in } \mathbb{R}^d. \end{cases}$$

As $\bar{m} \in C^{1/2}([0, T], \mathcal{P}_1(\mathbb{R}^d))$ and F is globally Lipschitz continuous, the right-hand side is Holder continuous and bounded. The solution u is therefore a classical one, with $\partial_t u$, Du and D^2u bounded and Holder continuous.

Let (m, E) satisfy constraint (4.5) and be such that $\tilde{J}(m, E) < \infty$. Then $E \ll m$ and we denote by α the Radon-Nikodym derivative of E with respect to m . As (m, E) belongs to $\mathcal{E}$ and satisfies (4.5) and as $\bar{u}$ is such that $\partial_t \bar{u}$, $D\bar{u}$ and $D^2\bar{u}$ are bounded and continuous, we get

$$\int_{\mathbb{R}^d} \bar{u}(0, x) m_0(dx) + \int_0^T \int_{\mathbb{R}^d} (\partial_t \bar{u}(t, x) + \Delta \bar{u}(t, x)) m(dt, dx) + \int_0^T \int_{\mathbb{R}^d} Da(t, x) \cdot E(dt, dx) = 0,$$

which, in view of the equation satisfied by u , can be rewritten as

$$\int_{\mathbb{R}^d} \bar{u}(0, x) m_0(dx) + \int_0^T \int_{\mathbb{R}^d} \left(\frac{1}{2} |D\bar{u}(t, x)|^2 - F(x, \bar{m}(t)) + D\bar{u}(t, x) \cdot \alpha(t, x) \right) m(dt, dx) = 0.$$

As $\frac{1}{2} |D\bar{u}(t, x)|^2 + D\bar{u}(t, x) \cdot \alpha(t, x) \geq -\frac{1}{2} |\alpha(t, x)|^2$, we obtain

$$(4.9) \quad \int_{\mathbb{R}^d} \bar{u}(0, x) m_0(dx) \leq \int_0^T \left(\frac{1}{2} |\alpha(t, x)|^2 + F(x, \bar{m}(t)) \right) m(dt, dx) = \tilde{J}(\bar{m}, E),$$

with a strict inequality if equality $\frac{1}{2} |D\bar{u}(t, x)|^2 + \alpha(t, x) \cdot D\bar{u}(t, x) = -\frac{1}{2} |\alpha(t, x)|^2$ does not hold m -a.e.

On the other hand, if we denote by $\tilde{m}$ the unique solution to

$$(4.10) \quad -\partial_t \tilde{m} - \Delta \tilde{m} - \operatorname{div}(\tilde{m} D\bar{u}) = 0, \quad \tilde{m}(0) = m_0,$$

and set $\tilde{E} = -D\bar{u}\tilde{m}$, then the argument above shows that

$$\int_{\mathbb{R}^d} \bar{u}(0, x) m_0(dx) = \tilde{J}(\tilde{m}, \tilde{E}).$$

This proves that $(\tilde{m}, \tilde{E})$ is another minimum of $\tilde{J}$ under constraint (4.5), so that

$$\int_{\mathbb{R}^d} \bar{u}(0, x) m_0(dx) = \tilde{J}(\tilde{m}, \tilde{E}) = \tilde{J}(\bar{m}, \bar{E}).$$

Recalling the case of equality in (4.9), we infer that $\frac{1}{2} |D\bar{u}(t, x)|^2 + \bar{\alpha}(t, x) \cdot D\bar{u}(t, x) = -\frac{1}{2} |\bar{\alpha}(t, x)|^2$ holds $\bar{m}$ -a.e., i.e., $\bar{\alpha} = -D\bar{u}$ $\bar{m}$ -a.e. Thus $\bar{m}$ is another solution of (4.10), which implies that $\bar{m} = \tilde{m}$. As $D\bar{u}$ and $D^2\bar{u}$ are Holder, classical arguments show that $\bar{m}$ has a density which is positive and in $C_{loc}^{1,2}$. Hence $\bar{\alpha} = -D\bar{u}$ a.e. $\square$

4.6 (Sketch of) proof of the sufficient conditions

Assume that the pair (u, m) solves system (4.4), in the sense that u is a classical solution of the first equation (with u , $\partial_t u$, Du and D^2u continuous and bounded) while $m \in C^0([0, T], \mathcal{P}_2)$ solves the second one in the sense of distribution. We also suppose that $\mathcal{F}$ is convex on $\mathcal{P}_1$. Fix $\alpha' : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ a globally Lipschitz continuous map. Let $\bar{X}_0$ be a random variable of law m_0 independent of a d -dimensional Brownian motion B and let X be the unique solution to

$$X_t^{\alpha'} = \bar{X}_0 + \int_0^t \alpha'(s, X_s^{\alpha'}) ds + \sqrt{2} dB_t, \quad t \in [0, T].$$

We set $m'(t) = \mathcal{L}(X_t^{\alpha'})$. Note that, if $\alpha' = \alpha := -Du$, then $m' = m$.

By Itô's formula, we have

$$u(T, X_T^{\alpha'}) = u(0, \bar{X}_0) + \int_0^T (\partial_t u + Du \cdot \alpha'(t, X_t^{\alpha'}) + \Delta u) dt + \sqrt{2} \int_0^T Du \cdot dB_t,$$

where u and its derivatives are computed at $(t, X_t^{\alpha'})$. Recalling the equation satisfied by u one has, taking expectation:

$$0 = \mathbb{E} \left[u(0, \bar{X}_0) + \int_0^T (Du \cdot \alpha'(t, X_t^{\alpha'}) + \frac{1}{2} |Du|^2 - F(X_t^{\alpha'}, m(t))) dt \right].$$

Note that $Du(t, x) \cdot \alpha' + \frac{1}{2} |Du(t, x)|^2 \geq -\frac{1}{2} |\alpha'|^2$ for any $\alpha' \in \mathbb{R}^d$, with an equality for $\alpha' = -Du(t, x) = \alpha(t, x)$. Thus

$$\begin{aligned} \mathbb{E} [u(0, \bar{X}_0)] &\leq \mathbb{E} \left[\int_0^T \frac{1}{2} |\alpha'(t, X_t^{\alpha'})|^2 + F(X_t^{\alpha'}, m(t)) dt \right] \\ &= \int_0^T \int_{\mathbb{R}^d} \left(\frac{1}{2} |\alpha'(t, x)|^2 + F(x, m(t)) \right) m'(t, dx) dx dt, \end{aligned}$$

with an equality if $\alpha' = \alpha$. Since F is convex, we have

$$\mathcal{F}(m(t)) + \int_{\mathbb{R}^d} F(x, m(t))(m'(t, dx) - m(t, dx)) \leq \mathcal{F}(m'(t)).$$

Thus

$$\mathbb{E} [u(0, \bar{X}_0)] \leq \int_0^T \left(\int_{\mathbb{R}^d} \frac{1}{2} |\alpha'(t, x)|^2 + \mathcal{F}(m'(t)) \right) dt + \int_0^T \left(\int_{\mathbb{R}^d} F(x, m(t)) m(t, dx) - \mathcal{F}(m(t)) \right) dt$$

with an equality for $\alpha' = \alpha$ (and thus $m' = m$). This shows the optimality of (m, α) among the Lipschitz competitors. The optimality among general competitors can be achieved by standard regularization and we omit the proof.

Chapter 5

The second order MFG systems

This chapter is devoted to the most classical mean field game system. We first explain—in a very general framework—the derivation and the meaning of the system and illustrate the concept through an example. Then we prove the existence of a solution in a very particular setting. Finally we also discuss potential MFG, for which one can obtain the equilibrium as the minimum of a functional.

5.1 Description of the system

5.1.1 Heuristic derivation of the MFG system

We describe here the simplest, standard class of mean field games. In this control problem with infinitely many agents, each small agent controls her own dynamics:

$$(5.1) \quad X_s = x + \int_t^s b(r, X_r, \alpha_r, m(r))dr + \int_t^s \sigma(r, X_r, \alpha_r, m(r))dB_r,$$

where X lives in $\mathbb{R}^d$, α is the control (taking its values in a fixed set A) and B is a given M –dimensional Brownian motion. The difference with Subsection 3.3.1 is the dependence of the coefficients with respect to the distribution $(m(t))$ of the all the players. This (time dependent) distribution $(m(t))$ belongs to the set $\mathcal{P}(\mathbb{R}^d)$ of Borel probability measures on $\mathbb{R}^d$ and is, at this stage, supposed given: we think at $(m(t))$ as the anticipation made by the agents on their future time dependent distribution. The coefficients $b : [0, T] \times \mathbb{R}^d \times A \times \mathcal{P}(\mathbb{R}^d) \rightarrow \mathbb{R}^d$ and $\sigma : [0, T] \times \mathbb{R}^d \times A \times \mathcal{P}(\mathbb{R}^d) \rightarrow \mathbb{R}^{d \times M}$ are assumed to be smooth enough for the solution (X_t) to exist.

The cost of a small player is given by

$$J(t, x, \alpha) = \mathbb{E} \left[\int_t^T L(s, X_s, \alpha_s, m(s))ds + g(X_T, m(T)) \right].$$

Here $T > 0$ is the finite horizon of the problem, $L : [0, T] \times \mathbb{R}^d \times A \times \mathcal{P}(\mathbb{R}^d) \rightarrow \mathbb{R}$ and $g : \mathbb{R}^d \times \mathcal{P}(\mathbb{R}^d) \rightarrow \mathbb{R}$ are given continuous maps.

If we define the value function u as

$$u(t, x) = \inf_{\alpha} J(t, x, \alpha),$$

then, at least in a formal way, u solves the Hamilton-Jacobi equation

$$\begin{cases} -\partial_t u(t, x) + H(t, x, Du(t, x), D^2 u(t, x), m(t)) = 0 & \text{in } (0, T) \times \mathbb{R}^d \\ u(T, x) = g(x, m(T)) & \text{in } \mathbb{R}^d. \end{cases}$$

where the Hamiltonian $H : [0, T] \times \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^{d \times d} \times \mathcal{P}(\mathbb{R}^d) \rightarrow \mathbb{R}$ is defined by

$$H(t, x, p, M, m) := \sup_{a \in A} \left[-L(t, x, a, m) - p \cdot b(t, x, a, m) - \frac{1}{2} \text{Tr}(\sigma \sigma^*(t, x, a, m) M) \right].$$

Let us now introduce $\alpha^*(t, x) \in A$ as a maximum point in the definition of H when $p = Du(t, x)$ and $M = D^2(t, x)$. Namely

$$(5.2) \quad \begin{aligned} H(t, x, Du(t, x), D^2u(t, x), m(t)) &= -L(t, x, \alpha^*(t, x), m(t)) - Du(t, x) \cdot b(t, x, \alpha^*(t, x), m(t)) \\ &\quad - \frac{1}{2} \text{Tr}(\sigma \sigma^*(t, x, \alpha^*(t, x)) D^2u(t, x), m(t)). \end{aligned}$$

Recall from Subsection 3.3.1 that α^* is the optimal feedback for the problem. Let us strongly emphasize that u and α^* depend on the time-dependent family of measures $(m(t))$.

We now discuss the evolution of the population density. For this we make two important assumptions: First we assume that all the agents control the same system (5.1) (although not necessarily starting from the same initial position) and minimize the same cost J . As a consequence, the dynamics at optimum of each player is given by

$$dX_s^* = b(s, X_s^*, \alpha^*(s, X_s^*), m(s))ds + \sigma(s, X_s^*, \alpha^*(s, X_s^*), m(s))dB_s.$$

Second we assume that the initial position of the agents and the noise driving their dynamics are independent: in particular, there is no “common noise” impacting all the players. The initial distribution of the agents at time $t = 0$ is denoted by $\bar{m}_0 \in \mathcal{P}(\mathbb{R}^d)$. From the analysis of the mean field limit (in the simple case where the coefficients do not depend on the other agents) the actual distribution $(\tilde{m}(s))$ of all agents at time s is simply given by the law of (X_s^*) with $\mathcal{L}(X_0^*) = \bar{m}_0$.

Let us now write the equation satisfied by $(\tilde{m}(s))$. By Itô's formula, we have, for any smooth map $\phi : [0, T) \times \mathbb{R}^d \rightarrow \mathbb{R}$ with a compact support:

$$\begin{aligned} 0 &= \mathbb{E}[\phi(T, X_T^*)] = \mathbb{E}[\phi(0, X_0^*)] + \int_0^T \mathbb{E} \left[\partial_t \phi(s, X_s^*) + b(s, X_s^*, \alpha^*(s, X_s^*), m(s)) \cdot D\phi(s, X_s^*) \right. \\ &\quad \left. + \frac{1}{2} \text{Tr}(\sigma \sigma^*(s, X_s^*, \alpha^*(s, X_s^*), m(s)) D^2 \phi(s, X_s^*)) \right] ds \\ &= \int_{\mathbb{R}^d} \phi(0, x) \bar{m}_0(dx) + \int_0^T \int_{\mathbb{R}^d} \left[\partial_t \phi(s, x) + b(s, x, \alpha^*(s, x), m(s)) \cdot D\phi(s, x) \right. \\ &\quad \left. + \frac{1}{2} \text{Tr}(\sigma \sigma^*(s, x, \alpha^*(s, x), m(s)) D^2 \phi(s, x)) \right] \tilde{m}(t, dx) ds \\ &= \int_{\mathbb{R}^d} \phi(0, x) \bar{m}_0(dx) + \int_0^T \int_{\mathbb{R}^d} \left[\partial_t \phi(s, x) + \sum_{i=1}^d b_i(s, x, \alpha^*(s, x), m(s)) \partial_{x_i} \phi(s, x) \right. \\ &\quad \left. + \frac{1}{2} \sum_{i,j=1}^d a_{ij}(s, x, \alpha^*(s, x), m(s)) \partial_{x_i x_j}^2 \phi(s, x) \right] \tilde{m}(t, dx) ds \end{aligned}$$

where $a = (a_{ij}) = \sigma \sigma^*$. After two integration by parts (and assuming that the boundary terms vanish), we obtain that $(\tilde{m}(t))$ satisfies, in the sense of distributions,

$$\begin{cases} \partial_t \tilde{m} - \frac{1}{2} \sum_{i,j=1}^d \partial_{x_i x_j}^2 (\tilde{m}(t, x) a_{ij}(t, x, \alpha^*(t, x), m(t))) + \sum_{i=1}^d \partial_{x_i} (\tilde{m}(t, x) b_i(s, x, \alpha^*(s, x), m(s))) = 0 & \text{in } (0, T) \times \mathbb{R}^d, \\ \tilde{m}(0) = \bar{m}_0 & \text{in } \mathbb{R}^d, \end{cases}$$

or, in a more compact way,

$$\begin{cases} \partial_t \tilde{m} - \frac{1}{2} \sum_{i,j} D_{ij}^2 (\tilde{m}(t, x) a_{ij}(t, x, \alpha^*(t, x), m(t))) + \text{div}(\tilde{m}(t, x) b(s, x, \alpha^*(s, x), m(s))) = 0 & \text{in } (0, T) \times \mathbb{R}^d, \\ \tilde{m}(0) = \bar{m}_0 & \text{in } \mathbb{R}^d, \end{cases}$$

where, for a vector field $\phi = (\phi_i)$, $\text{div}(\phi)(x) = \sum_{i=1}^d \partial_{x_i} \phi_i(x)$.

At equilibrium, one expects the anticipation $(m(t))$ made by the agents to be correct: $\tilde{m}(t) = m(t)$. Collecting the above equation leads to the MFG system:

$$\begin{cases} -\partial_t u(t, x) + H(t, x, Du(t, x), D^2 u(t, x), m(t)) = 0 & \text{in } (0, T) \times \mathbb{R}^d, \\ \partial_t m - \frac{1}{2} \sum_{ij} D_{ij}^2(m(t, x) a_{ij}(t, x, \alpha^*(t, x), m(t))) + \operatorname{div}(m(t, x) b(s, x, \alpha^*(s, x), m(s))) = 0 & \text{in } (0, T) \times \mathbb{R}^d, \\ m(0) = \tilde{m}_0, u(T, x) = g(x, m(T)) & \text{in } \mathbb{R}^d, \end{cases}$$

where α^* is given by (5.2) and $a = \sigma \sigma^*$.

In order to simplify a little this system, let us assume that $M = d$ and $\sigma = \sqrt{2}I_d$. We set (warning! abuse of notation!)

$$H(t, x, p, m) := \sup_{a \in A} [-L(t, x, a, m) - p \cdot b(t, x, a, m)]$$

and note that, by the Envelope Theorem (see Lemma 5.1.1 below), one can expect to get (under suitable assumptions):

$$D_p H(t, x, Du(t, x), m(t)) = -b(t, x, \alpha^*(t, x), m(t)).$$

In this case the MFG system becomes

$$\begin{cases} -\partial_t u(t, x) - \Delta u(t, x) + H(t, x, Du(t, x), m(t)) = 0 & \text{in } (0, T) \times \mathbb{R}^d, \\ \partial_t m - \Delta m(t, x) - \operatorname{div}(m(t, x) D_p H(t, x, Du(t, x), m(t))) = 0 & \text{in } (0, T) \times \mathbb{R}^d, \\ m(0) = \tilde{m}_0, u(T, x) = g(x, m(T)) & \text{in } \mathbb{R}^d, \end{cases}$$

Here is the Envelope Theorem:

Lemma 5.1.1. *Let A be a compact metric space, $\mathcal{O}$ be an open subset of $\mathbb{R}^d$ and $f : A \times \mathcal{O} \rightarrow \mathbb{R}$ be continuous and with $D_x f$ continuous on $A \times \mathcal{O}$. Then the marginal map*

$$V(x) = \inf_{a \in A} f(a, x)$$

is differentiable at each point $x \in \mathcal{O}$ at which infimum in $V(x)$ is reached at a unique point $a_x \in A$. Then

$$DV(x) = D_x f(a_x, x).$$

Proof. Let $x \in \mathcal{O}$ be such that the infimum in $V(x)$ is a unique point $a_x \in A$. Then an easy compactness argument shows that, if a_y is a minimum point of $V(y)$ for $y \in \mathcal{O}$ and $y \rightarrow x$, then $a_y \rightarrow a_x$.

Fix $y \in \mathcal{O}$. Note first that, as $a_x \in A$,

$$V(y) \leq f(a_x, y) = f(a_x, x) + D_x f(a_x, z_y) \cdot (y - x) = V(x) + D_x f(a_x, z_y) \cdot (y - x),$$

for some $z_y \in [x, y]$.

Conversely,

$$V(x) \leq f(a_y, x) = f(a_y, y) + D_x f(a_y, z'_y) \cdot (x - y) = V(y) + D_x f(a_y, z'_y) \cdot (x - y),$$

for some $z'_y \in [x, y]$.

By continuity of $D_x f$ and convergence of a_y , we infer that

$$\begin{aligned} \lim_{y \rightarrow x} \frac{|V(y) - V(x) - D_x f(a_x, x) \cdot (y - x)|}{|y - x|} \\ \leq \liminf_{y \rightarrow x} |D_x f(a_x, z_y) - D_x f(a_x, x)| + |D_x f(a_y, z'_y) - D_x f(a_x, x)| = 0. \end{aligned}$$

□

5.1.2 A simple example of optimal trading

We discuss here a toy example of optimal trading. Let us warn the reader that this example does not fit the framework described above. However, it can be solved explicitly and illustrates quite well the mean field interactions.

A continuum of investors decide to buy or sell a given tradable instrument. The decision is a signed quantity Q_0 to buy (in such a case Q_0 is negative: the investor has a negative inventory at the initial time $t = 0$) or to sell (when Q_0 is positive). All investors have to buy or sell before a given terminal time T , and we assume that they have the same risk aversion parameters ϕ and A .

Each investor will control its trading speed α_t through time, in order to fulfill its goal. The price S_t of the tradable instrument is submitted to two kinds of moves: an exogenous innovation supported by a standard Wiener process B_t (with its natural probability space and the associated filtration $\mathcal{F}_t$), and the *permanent market impact* generated linearly from the buying or selling pressure $\theta\mu_t$ where μ_t is the net sum of the trading speed of all investors and $\theta > 0$ is a fixed parameter.

$$(5.3) \quad dS_t = \theta\mu_t dt + \sigma dB_t.$$

In principle one should expect the term (μ_t) to be stochastic and adapted to the filtration generate by B , which, in this setting, should be interpreted as a common noise. Because of the specific structure of the problem it is not the case here: so in the rest of the example we consider (μ_t) as deterministic.

The state of each investor is described by two variables: its inventory Q_t and its wealth X_t (starting with $X_0 = 0$ for all investors). The evolution of Q reads

$$(5.4) \quad dQ_t = \alpha_t dt,$$

since for a seller, $Q_0 > 0$ (the associated control α will be mostly negative) and the wealth suffers from linear trading costs (or *temporary*, or *immediate market impact*, parametrized by κ):

$$(5.5) \quad dX_t = -\alpha_t(S_t + \kappa \cdot \alpha_t) dt.$$

Meaning the wealth of a seller will be positive (and the faster you sell—i.e. α is largely negative—, the smaller the sell price).

The cost function of each investor is made of the wealth at T , plus the value of the inventory penalized by a terminal market impact, and minus a running cost quadratic in the inventory:

$$(5.6) \quad V_t := \sup_{\alpha} \mathbb{E} \left[X_T + Q_T(S_T - A \cdot Q_T) - \phi \int_t^T (Q_s)^2 ds \right].$$

The Hamilton-Jacobi-Bellman satisfied by the valued function $V = V(t, s, q, x; \mu)$ associated to (5.6) is

$$(5.7) \quad 0 = \partial_t V - \phi q^2 + \frac{1}{2} \sigma^2 \partial_{ss}^2 V + \theta \mu \partial_s V + \sup_{\alpha} \{ \alpha \partial_q V - \alpha(s + \kappa \alpha) \partial_x V \}$$

with terminal condition

$$V(T, x, s, q; \mu) = x + q(s - Aq).$$

To find V , we will use the following ansatz:

$$(5.8) \quad V = x + qs + v(t, q; \mu).$$

Thus the HJB on v is

$$-\theta \mu q = \partial_t v - \phi q^2 + \sup_{\alpha} \{ \alpha \partial_q v - \kappa \alpha^2 \},$$

with terminal condition

$$v(T, q; \mu) = -Aq^2$$

and the associated optimal feedback is

$$(5.9) \quad \alpha(t, q) = \frac{\partial_q v(t, q)}{2\kappa}.$$

Further simplification In view of the equation, we look for $v(t, q)$ as a quadratic function of q :

$$v(t, q) = h_0(t) + q h_1(t) - q^2 \frac{h_2(t)}{2}.$$

Then the equation for v can be split in three parts

$$(5.10) \quad \begin{aligned} 0 &= -2\kappa h_2'(t) - 4\kappa\phi + (h_2(t))^2, \\ 2\kappa\theta\mu(t) &= -2\kappa h_1'(t) + h_1(t)h_2(t), \\ -(h_1(t))^2 &= 4\kappa h_0'(t), \end{aligned}$$

One also has to add the terminal condition: as $V_T = x + q(s - Aq)$, $v(T, q) = -Aq^2$. This implies that

$$(5.11) \quad h_0(T) = h_1(T) = 0, \quad h_2(T) = 2A.$$

Defining the mean field. In our model, the net trading flow μ is the average of the optimal feedback in the population. If the initial inventories of the investors is distributed as the random variable $\tilde{Q}_0$, by (5.9) the inventories at time t of these investors is the random variable given by

$$\tilde{Q}_t = \tilde{Q}_0 + \int_0^t \frac{\partial_q v(s, \tilde{Q}_s)}{2\kappa} ds = \tilde{Q}_0 + \frac{1}{2\kappa} \int_0^t (-h_2(s)\tilde{Q}_s + h_1(s)) ds.$$

Calling $E(t)$ the expectation of $\tilde{Q}_t$, we find that E solves the ODE

$$E'(t) = -\frac{h_2(t)}{2\kappa} E(t) + \frac{h_1(t)}{2\kappa}, \quad E(0) = E_0 := \mathbb{E}[\tilde{Q}_0]$$

while

$$\mu(t) = \mathbb{E} \left[\frac{\partial_q v(s, \tilde{Q}_s)}{2\kappa} \right] = \frac{1}{2\kappa} \mathbb{E} [-h_2(t)\tilde{Q}_t + h_1(t)] = E'(t).$$

The full system To summarize, we find the system:

$$\begin{aligned} 0 &= -2\kappa h_2'(t) - 4\kappa\phi + (h_2(t))^2, \\ 2\kappa\theta E'(t) &= -2\kappa h_1'(t) + h_1(t)h_2(t), \\ -(h_1(t))^2 &= 4\kappa h_0'(t), \\ E'(t) &= -\frac{h_2(t)}{2\kappa} E(t) + \frac{h_1(t)}{2\kappa}, \\ h_0(T) &= h_1(T) = 0, \quad h_2(T) = 2A, \quad E(0) = E_0. \end{aligned}$$

Note that this is a forward-backward system and thus the standard theory of ODEs does not directly apply.

An example of resolution The (backward) equation for h_2 can be solved separately. Once h_1 is known, one can also find h_0 . Let us now show that one can reduce the system of equations satisfied (h_1, E) to a single equation. We have

$$h_1(t) = 2\kappa E'(t) + h_2(t)E(t),$$

so that

$$\begin{aligned} h_1'(t) &= 2\kappa E''(t) + h_2(t)E'(t) + h_2'(t)E(t) \\ &= -\theta E'(t) + \frac{h_2(t)}{2\kappa} h_1(t) = -\theta E'(t) + \frac{h_2(t)}{2\kappa} (2\kappa E'(t) + h_2(t)E(t)). \end{aligned}$$

Thus

$$\begin{aligned} 0 &= -2\kappa E''(t) - \theta E'(t) + (-h_2'(t) + \frac{h_2^2(t)}{2\kappa})E(t) \\ &= -2\kappa E''(t) - \theta E'(t) + 2\phi E(t). \end{aligned}$$

with boundary conditions

$$E(0) = E_0, \quad 0 = h_1(T) = 2\kappa E'(T) + h_2(T)E(T) = 2\kappa E'(T) + 2AE(T).$$

To summarize we find the problem

$$(5.12) \quad \begin{cases} -2\kappa E''(t) - \theta E'(t) + 2\phi E(t) = 0 & \text{in } (0, T), \\ E(0) = E_0, \quad 2\kappa E'(T) + 2AE(T) = 0. \end{cases}$$

Let us consider the Hilbert space $H := \{x \in H^1([0, T]), x(0) = 0\}$. Then by the equation satisfied by $x(t) = E(t) - E_0$, we have, for any $y \in H$,

$$\begin{aligned} 0 &= \int_0^T y(t)(-2\kappa x''(t) - \theta x'(t) + 2\phi(x(t) + E_0))dt \\ &= \left[-2y(t)x'(t)\right]_0^T + \int_0^T (2\kappa x'(t)y'(t)x''(t) - \theta x'(t)y(t) + 2\phi x(t)y(t))dt + E_0 \int_0^T y(t)dt \\ &= \frac{2A}{\kappa}(x(T) + E_0)y(T) + \int_0^T (2\kappa x'(t)y'(t) - \theta x'(t)y(t) + 2\phi x(t)y(t))dt + E_0 \int_0^T y(t)dt \end{aligned}$$

Let us consider on H the bilinear form

$$\mathcal{A}(x, y) := \frac{2A}{\kappa}x(T)y(T) + \int_0^T (2\kappa x'(t)y'(t)x''(t) - \theta x'(t)y(t) + 2\phi x(t)y(t))dt.$$

One easily checks that it is bilinear continuous on $H \times H$. Moreover,

$$\begin{aligned} \mathcal{A}(x, x) &= \frac{2A}{\kappa}(x(T))^2 + \int_0^T (2\kappa(x'(t))^2 - \theta x'(t)x(t) + 2\phi(x(t))^2)dt \\ &= \left(\frac{2A}{\kappa} - \theta\right)(x(T))^2 + \int_0^T (2\kappa(x'(t))^2 + 2\phi(x(t))^2)dt \end{aligned}$$

So $\mathcal{A}$ is coercive (at least if) if $\theta \leq \frac{2A}{\kappa}$. From Lax-Milgram Theorem we deduce that, if ξ is the linear form

$$\xi(y) := -\frac{2AE_0}{\kappa}y(T) - E_0 \int_0^T y(t)dt \quad \forall y \in H,$$

there exists a unique $\bar{x} \in H$ such that

$$\mathcal{A}(\bar{x}, y) = \xi(y) \quad \forall y \in H.$$

One can check by standard arguments that this is equivalent to saying that $\bar{x}$ solves system (5.12).

5.2 Analysis of a second order quadratic MFG

We discuss the existence of a solution of the MFG system in the simple setting of second order equations with a quadratic Hamiltonian:

$$(5.13) \quad \begin{cases} (i) & -\partial_t u - \Delta u + \frac{1}{2}|Du|^2 = F(x, m(t)) & \text{in } \mathbb{R}^d \times (0, T) \\ (ii) & \partial_t m - \Delta m - \operatorname{div}(m Du) = 0 & \text{in } \mathbb{R}^d \times (0, T) \\ (iii) & m(0) = m_0, \quad u(x, T) = G(x, m(T)) & \text{in } \mathbb{R}^d \end{cases}$$

Our aim is to prove the existence of classical solutions for this system and give some interpretation in terms of game with finitely many players.

For this our main assumption is that F and G are regularizing on the set of probability measures on $\mathbb{R}^d$. To make our life simple, we assume that all measures considered in this section have a finite first order moment: let $\mathcal{P}_1$ be the set of such Borel probability measures m on $\mathbb{R}^d$ such that $\int_{\mathbb{R}^d} |x| dm(x) < +\infty$. The set $\mathcal{P}_1$ can be endowed with following (Kantorovitch-Rubinstein) introduced in Subsection 3.1.

Here are our main assumptions on F , G and m_0 : we suppose that there is some constant C_0 such that

1. (Bounds on F and G) F and G are uniformly bounded by C_0 over $\mathbb{R}^d \times \mathcal{P}_1$.
2. (Uniform regularity of G) $G(\cdot, m)$ is bounded in $C^{2+\alpha}(\mathbb{R}^d)$ (for some $\alpha \in (0, 1)$), uniformly with respect to the measure m .
3. (Lipschitz continuity of F and G)

$$|F(x_1, m_1) - F(x_2, m_2)| \leq C_0 [|x_1 - x_2| + \mathbf{d}_1(m_1, m_2)] \quad \forall (x_1, m_1), (x_2, m_2) \in \mathbb{R}^d \times \mathcal{P}_1 .$$

and

$$|G(x_1, m_1) - G(x_2, m_2)| \leq C_0 [|x_1 - x_2| + \mathbf{d}_1(m_1, m_2)] \quad \forall (x_1, m_1), (x_2, m_2) \in \mathbb{R}^d \times \mathcal{P}_1 .$$

4. The probability measure m_0 is absolutely continuous with respect to the Lebesgue measure, has a $C^{2+\alpha}$ density (still denoted m_0) which satisfies $\int_{\mathbb{R}^d} |x|^2 m_0(x) dx < +\infty$.

A pair (u, m) is a classical solution to (5.13) if $u, m : \mathbb{R}^d \times [0, T] \rightarrow \mathbb{R}$ are continuous, of class $\mathcal{C}^2$ in space and $\mathcal{C}^1$ in time and (u, m) satisfies (5.13) in the classical sense. The main result of this section is the following:

Theorem 5.2.1. *Under the above assumptions, there is at least one classical solution to (5.13).*

The proof is relatively easy and relies on basic estimates for the heat equation as well as some remarks on the Fokker-Planck equation (5.13-(ii)).

5.2.1 On the Fokker-Planck equation

Let $b : \mathbb{R}^d \times [0, T] \rightarrow \mathbb{R}$ be a given vector field. Our aim is to analyse the Fokker-Planck equation

$$(5.14) \quad \begin{cases} \partial_t m - \Delta m + \operatorname{div}(m b) = 0 & \text{in } \mathbb{R}^d \times (0, T) \\ m(0) = m_0 \end{cases}$$

as an evolution equation in the space of probability measures. We assume here that the vector field $b : \mathbb{R}^d \times [0, T] \rightarrow \mathbb{R}^d$ is continuous, uniformly Lipschitz continuous in space, and bounded.

Definition 5.2.2 (Weak solution to (5.14)). *We say that m is a weak solution to (5.14) if $m \in L^1([0, T], \mathcal{P}_1)$ is such that, for any test function $\varphi \in \mathcal{C}_c^\infty(\mathbb{R}^d \times [0, T])$, we have*

$$\int_{\mathbb{R}^d} \phi(x, 0) dm_0(x) + \int_0^T \int_{\mathbb{R}^d} (\partial_t \varphi(x, t) + \Delta \varphi(x, t) + \langle D\varphi(x, t), b(x, t) \rangle) m(t, dx) = 0 .$$

In order to analyse some particular solutions of (5.14), it is convenient to introduce the following stochastic differential equation (SDE)

$$(5.15) \quad \begin{cases} dX_t = b(X_t, t)dt + \sqrt{2}dB_t, & t \in [0, T] \\ X_0 = Z_0 \end{cases}$$

where (B_t) is a standard d -dimensional Brownian motion over some probability space $(\Omega, \mathcal{A}, \mathbb{P})$ and where the initial condition $Z_0 \in L^1(\Omega)$ is random and independent of (B_t) . Under the above assumptions on b , there is a unique solution to (5.15). This solution is closely related to equation (5.14):

Lemma 5.2.3. *If $\mathcal{L}(Z_0) = m_0$, then $m(t) := \mathcal{L}(X_t)$ a weak solution of (5.14).*

Proof : This is a straightforward consequence of Itô's formula, which says that, if $\varphi : \mathbb{R}^d \times [0, T] \rightarrow \mathbb{R}$ is bounded, of class $\mathcal{C}^2$ in space and $\mathcal{C}^1$ in time, then

$$\begin{aligned} \varphi(X_t, t) &= \varphi(Z_0, 0) \\ &+ \int_0^t [\varphi_t(X_s, s) + \langle D\varphi(X_s, s), b(X_s, s) \rangle + \Delta \varphi(X_s, s)] ds + \int_0^t \langle D\varphi(X_s, s), dB_s \rangle . \end{aligned}$$

Taking the expectation on both sides of the equality, we have, since

$$\mathbb{E} \left[\int_0^t \langle D\varphi(X_s, s), dB_s \rangle \right] = 0$$

because $t \rightarrow \int_0^t \langle D\varphi(X_s, s), dB_s \rangle$ is a martingale,

$$\mathbb{E} [\varphi(X_t, t)] = \mathbb{E} \left[\varphi(Z_0, 0) + \int_0^t [\varphi_t(X_s, s) + \langle D\varphi(X_s, s), b(X_s, s) \rangle + \Delta\varphi(X_s, s)] ds \right].$$

So by definition of $m(t)$, we get

$$\begin{aligned} \int_{\mathbb{R}^d} \varphi(x, t) m(t, dx) &= \int_{\mathbb{R}^d} \varphi(x, 0) m_0(dx) \\ &+ \int_0^t \int_{\mathbb{R}^d} [\varphi_t(x, s) + \langle D\varphi(x, s), b(x, s) \rangle + \Delta\varphi(x, s)] m(s, dx) ds, \end{aligned}$$

i.e., m is a weak solution to (5.14). $\square$

This above interpretation of the continuity equation allows to get very easily some estimates on the map $t \rightarrow m(t)$ in $\mathcal{P}_2$.

Lemma 5.2.4. *Let m be defined as above. There is a constant $c_0 = c_0(T)$, such that*

$$\mathbf{d}_1(m(t), m(s)) \leq c_0(1 + \|b\|_\infty) |t - s|^{\frac{1}{2}} \quad \forall s, t \in [0, T].$$

Proof : Recalling the definition of $\mathbf{d}_1$ we note that the law γ of the pair (X_t, X_s) belongs to $\Pi(m(t), m(s))$, so that

$$\mathbf{d}_1(m(t), m(s)) \leq \int_{\mathbb{R}^{2d}} |x - y| d\gamma(x, y) = \mathbb{E} [|X_t - X_s|].$$

Therefore, if for instance $s < t$,

$$\begin{aligned} \mathbb{E} [|X_t - X_s|] &\leq \mathbb{E} \left[\int_s^t |b(X_\tau, \tau)| d\tau + \sqrt{2} |B_t - B_s| \right] \\ &\leq \|b\|_\infty (t - s) + \sqrt{2(t - s)} \end{aligned}$$

$\square$

Moreover we also obtain some integral estimates:

Lemma 5.2.5. *There is a constant $c_0 = c_0(T)$ such that*

$$\int_{\mathbb{R}^d} |x|^2 m(t, dx) \leq c_0 \left(\int_{\mathbb{R}^d} |x|^2 m_0(dx) + 1 + \|b\|_\infty^2 \right) \quad \forall t \in [0, T].$$

Proof : Indeed:

$$\begin{aligned} \int_{\mathbb{R}^d} |x|^2 m(t, dx) &= \mathbb{E} [|X_t|^2] \leq 2\mathbb{E} \left[|X_0|^2 + \left| \int_0^t b(X_\tau, \tau) d\tau \right|^2 + 2|B_t|^2 \right] \\ &\leq 2 \left[\int_{\mathbb{R}^d} |x|^2 m_0(dx) + t^2 \|b\|_\infty^2 + 2t \right] \end{aligned}$$

$\square$

5.2.2 Proof of the existence Theorem

Before starting the proof of Theorem 5.2.1, let us recall some basic existence/uniqueness result for the heat equation

$$(5.16) \quad \begin{cases} w_t - \Delta w + \langle a(x, t), Dw \rangle + b(x, t)w = f(x, t) & \text{in } \mathbb{R}^d \times [0, T] \\ w(x, 0) = w_0(x) & \text{in } \mathbb{R}^d \end{cases}$$

For this it will be convenient to denote by $\mathcal{C}^{s+\alpha}$ (for an integer $s \geq 0$ and $\alpha \in (0, 1)$) the set of maps $z : \mathbb{R}^d \times [0, T] \rightarrow \mathbb{R}$ such that the derivatives $\partial_t^k D_x^l z$ exist for any pair (k, l) with $2k + l \leq s$ and such that these derivatives are bounded and α -Hölder continuous in space and $(\alpha/2)$ -Hölder continuous in time. If we assume that, for some $\alpha \in (0, 1)$, $a : \mathbb{R}^d \times [0, T] \rightarrow \mathbb{R}$, $b, f : \mathbb{R}^d \times [0, T] \rightarrow \mathbb{R}$ and $w_0 : \mathbb{R}^d \rightarrow \mathbb{R}$ belong to $\mathcal{C}^\alpha$, then the above heat equation has a unique weak solution. Furthermore this solution belongs to $\mathcal{C}^{2+\alpha}$ (Theorem 5.1 p. 320 of [43]).

We will also need the following interior estimate (Theorem 11.1 p. 211 of [43]): if $a = b = 0$ and f is continuous and bounded, any classical, bounded solution w of (5.16) satisfies, for any compact set $K \subset \mathbb{R}^d \times (0, T)$,

$$(5.17) \quad \sup_{(x,t),(y,s) \in K} \frac{|Dw(x,t) - Dw(y,s)|}{|x-y|^\beta + |t-s|^{\beta/2}} \leq C(K, \|w\|_\infty) \|f\|_\infty,$$

where $\beta \in (0, 1)$ depends only on the dimension d while $C(K, \|w\|_\infty)$ depends on the compact set K , on $\|w\|_\infty$ and on d .

Let C_1 be a large constant to be chosen below and $\mathcal{C}$ be the set of maps $\mu \in \mathcal{C}^0([0, T], \mathcal{P}_1)$ such that

$$(5.18) \quad \sup_{s \neq t} \frac{\mathbf{d}_1(\mu(s), \mu(t))}{|t-s|^{\frac{1}{2}}} \leq C_1$$

and

$$\sup_{t \in [0, T]} \int_{\mathbb{R}^d} |x|^2 m(t, dx) \leq C_1.$$

Then $\mathcal{C}$ is a convex closed subset of $\mathcal{C}^0([0, T], \mathcal{P}_1)$. It is actually compact, because the set of probability measures m for which $\int_{\mathbb{R}^d} |x|^2 dm(x) \leq C_1$ is finite, is compact in $\mathcal{P}_1$ (see Lemma 3.1.3).

To any $\mu \in \mathcal{C}$ we associate $m = \Psi(\mu) \in \mathcal{C}$ in the following way: Let u be the unique solution to

$$(5.19) \quad \begin{cases} -\partial_t u - \Delta u + \frac{1}{2}|Du|^2 = F(x, \mu(t)) & \text{in } \mathbb{R}^d \times (0, T) \\ u(x, T) = G(x, \mu(T)) & \text{in } \mathbb{R}^d \end{cases}$$

Then we define $m = \Psi(\mu)$ as the solution of the Fokker-Planck equation

$$(5.20) \quad \begin{cases} \partial_t m - \Delta m - \operatorname{div}(m Du) = 0 & \text{in } \mathbb{R}^d \times (0, T) \\ m(0) = m_0 & \text{in } \mathbb{R}^d \end{cases}$$

Let us check that Ψ is well-defined and continuous. To see that a solution to (5.19) exists and is unique, we use the Hopf-Cole transform: setting $w = e^{u/2}$ we easily check that u is a solution of (5.19) if and only if w is a solution of the linear (backward) equation

$$\begin{cases} -\partial_t w - \Delta w = wF(x, \mu(t)) & \text{in } \mathbb{R}^d \times (0, T) \\ w(x, T) = e^{G(x, \mu(T))/2} & \text{in } \mathbb{R}^d \end{cases}$$

Note that the maps $(x, t) \rightarrow F(x, \mu(t))$ and $x \rightarrow e^{G(x, \mu(T))/2}$ belong to $\mathcal{C}^{1/2}$ and $\mathcal{C}^{2+\alpha}$ respectively, because μ satisfies (5.18) and from our assumptions on F and G . Therefore the above equation is uniquely solvable and the solution belongs to $\mathcal{C}^{2+\alpha}$ with $\alpha = \frac{1}{2}$, which in turn implies the unique solvability of (5.19) with a solution u which belongs to $\mathcal{C}^{2+\alpha}$. Recall that the maps $x \rightarrow F(x, \mu(t))$ and $x \rightarrow G(x, \mu(T))$

are bounded by C_0 , so that a straightforward application of the comparison principle implies that u is bounded by $(1 + T)C_0$. In the same way, since moreover the maps $x \rightarrow F(x, \mu(t))$ and $x \rightarrow G(x, \mu(T))$ are C_0 -Lipschitz continuous (again by our assumptions on F and G), u is also C_0 -Lipschitz continuous. Hence Du is bounded by C_0 .

Next we turn to the Fokker-Planck equation (5.20), that we write into the form

$$\partial_t m - \Delta m - \langle Dm, Du(x, t) \rangle - m \Delta u(x, t) = 0 .$$

Since $u \in \mathcal{C}^{2+\alpha}$, the maps $(x, t) \rightarrow Du(x, t)$ and $(x, t) \rightarrow \Delta u(x, t)$ belong to $\mathcal{C}^\alpha$, so that this equation is uniquely solvable and the solution m belongs to $\mathcal{C}^{2+\alpha}$. Moreover, in view of the discussion of subsection 5.2.1, we have the following estimates on m :

$$\mathbf{d}_1(m(t), m(s)) \leq c_0(1 + C_0)|t - s|^{\frac{1}{2}} \quad \forall s, t \in [0, T]$$

and

$$\int_{\mathbb{R}^d} |x|^2 m(t, dx) \leq c_0(1 + C_0^2) \quad \forall t \in [0, T] ,$$

where c_0 depends only on T . So if we choose $C_1 = \max\{c_0(1 + C_0), c_0(1 + C_0^2)\}$, m belongs to $\mathcal{C}$.

We have just proved that the mapping $\Psi : \mu \rightarrow m = \Psi(\mu)$ is well-defined. Let us check that it is continuous. Let $\mu_n \in \mathcal{C}$ converge to some μ . Let (u_n, m_n) and (u, m) be the corresponding solutions. Note that $(x, t) \rightarrow F(x, \mu_n(t))$ and $x \rightarrow G(x, \mu_n(T))$ locally uniformly converge to $(x, t) \rightarrow F(x, \mu(t))$ and $x \rightarrow G(x, \mu(T))$. Then one gets the local uniform convergence of (u_n) to u by standard arguments (of viscosity solutions for instance). Since the (Du_n) are uniformly bounded, the (u_n) solve an equation of the form

$$\partial_t u_n - \Delta u_n = f_n$$

where $f_n = \frac{1}{2}|Du_n|^2 - F(x, m_n)$ is uniformly bounded in x and n . Then the interior regularity result (5.17) implies that (Du_n) is locally uniformly Hölder continuous and therefore locally uniformly converges to Du . This easily implies that any converging subsequence of the relatively compact sequence m_n is a weak solution of (5.20). But m is the unique weak solution of (5.20), which proves that (m_n) converges to m .

We conclude by Schauder fixed point Theorem that the continuous map $\mu \rightarrow m = \Psi(\mu)$ has a fixed point in $\mathcal{C}$. Then this fixed point is a solution of our system (5.13). $\square$

5.2.3 Uniqueness

Let us assume that, besides the assumptions given at the beginning of the section, the following conditions hold:

$$(5.21) \quad \int_{\mathbb{R}^d} (F(x, m_1) - F(x, m_2)) d(m_1 - m_2)(x) \geq 0 \quad \forall m_1, m_2 \in \mathcal{P}_1, \quad m_1 \neq m_2$$

and

$$(5.22) \quad \int_{\mathbb{R}^d} (G(x, m_1) - G(x, m_2)) d(m_1 - m_2)(x) \geq 0 \quad \forall m_1, m_2 \in \mathcal{P}_1 .$$

Theorem 5.2.6. *Under the above conditions, there is a unique classical solution to the mean field equation (5.13).*

Proof. Before starting the proof, let us notice that we can use as a test function for m any map w which is of class $\mathcal{C}^2$: indeed, the result follows easily by regularizing and truncating w by cut-off functions of the form $\phi_\epsilon(t)\psi_R(x)$ where

$$\phi_\epsilon(t) = \begin{cases} 1 & \text{if } t \leq T - \epsilon \\ 1 + (T - \epsilon - t)/\epsilon & \text{if } T - \epsilon \leq t \leq T \\ 0 & \text{if } t \geq T \end{cases} \quad \text{and } \psi_R(x) = \begin{cases} 1 & \text{if } |x| \leq R \\ R + 1 - |x| & \text{if } R \leq |x| \leq R + 1 \\ 0 & \text{if } |x| \geq R + 1 \end{cases}$$

Let now consider (u_1, m_1) and (u_2, m_2) two classical solutions of (5.13). We set $\bar{u} = u_1 - u_2$ and $\bar{m} = m_1 - m_2$. Then

$$(5.23) \quad -\partial_t \bar{u} - \Delta \bar{u} + \frac{1}{2} (|Du_1|^2 - |Du_2|^2) - (F(x, m_1) - F(x, m_2)) = 0$$

while

$$\partial_t \bar{m} - \Delta \bar{m} - \operatorname{div} (m_1 Du_1 - m_2 Du_2) = 0.$$

Let us use $\bar{u}$ as a test function in the second equation. Since $\bar{u}$ is $\mathcal{C}^2$ we have (recall the remark at the beginning of the proof)

$$-\int_{\mathbb{R}^d} (\bar{m} \bar{u})(T) + \int_{\mathbb{R}^d} m_0 \bar{u}(0) + \int_0^T \int_{\mathbb{R}^d} (\partial_t \bar{u} + \Delta \bar{u}) d\bar{m} - \int_0^T \int_{\mathbb{R}^d} \langle D\bar{u}, m_1 Du_1 - m_2 Du_2 \rangle = 0$$

Let us multiply equality (5.23) by $\bar{m}$, integrate over $\mathbb{R}^d \times (0, T)$ and add to the previous equality. We get, after simplification and using that $\bar{m}(0) = 0$,

$$\begin{aligned} & -\int_{\mathbb{R}^d} (m_1(T) - m_2(T)) (G(m_1(T)) - G(m_2(T))) \\ & + \int_0^T \int_{\mathbb{R}^d} \left(\frac{\bar{m}}{2} (|Du_1|^2 - |Du_2|^2) - \bar{m} (F(x, m_1) - F(x, m_2)) - \langle D\bar{u}, m_1 Du_1 - m_2 Du_2 \rangle \right) = 0. \end{aligned}$$

Let us recall that

$$\int_{\mathbb{R}^d} (m_1(T) - m_2(T)) (G(m_1(T)) - G(m_2(T))) \geq 0,$$

and also note that

$$\frac{\bar{m}}{2} (|Du_1|^2 - |Du_2|^2) - \langle D\bar{u}, m_1 Du_1 - m_2 Du_2 \rangle = -\frac{m_1 + m_2}{2} |Du_1 - Du_2|^2.$$

So we find

$$\int_0^T \int_{\mathbb{R}^d} \frac{m_1 + m_2}{2} |Du_1 - Du_2|^2 \leq 0,$$

which shows that $Du_1 = Du_2$ m_1 - and m_2 -a.s. Thus m_1 and m_2 solve the same Kolmogorov equation, which, by uniqueness, shows that $m_1 = m_2$. As u_1 and u_2 solve the same Hamilton-Jacobi equation, we infer that $u_1 = u_2$ as well. $\square$

5.2.4 Potential MFGs

In this part we consider a specific class of MFGs where the equilibria come from a minimization problem. This part is largely redundant with Chapter 4. Namely we assume that there exist continuous maps $\mathcal{F}, \mathcal{G} : \mathcal{P}_2(\mathbb{R}^d) \rightarrow \mathbb{R}$ such that

$$\mathcal{F}(m') - \mathcal{F}(m) = \int_0^1 \int_{\mathbb{R}^d} F(x, (1-t)m + tm')(m - m')(dx) \quad \forall m, m' \in \mathcal{P}_2(\mathbb{R}^d)$$

and the symmetric equality holds for $\mathcal{G}$ and G . The above equality means that F is “the derivative” of $\mathcal{F}$. We will discuss this idea in details in the next chapter.

For instance, if $F(x, m) = \int_{\mathbb{R}^d} \phi(x, y) m(dy)$, where ϕ is continuous, bounded and symmetric, it is not difficult to show that one can take:

$$\mathcal{F}(m) = \frac{1}{2} \int_{\mathbb{R}^d} \phi(x, y) m(dy) m(dx).$$

We consider the problem

$$\inf_{m, v} \mathcal{J}(m, v) \text{ where } \mathcal{J}(m, v) := \frac{1}{2} \int_0^T \int_{\mathbb{R}^d} |v(t, x)|^2 m(t, x) dx dt + \int_0^T \mathcal{F}(m(t)) dt + \mathcal{G}(m(T)),$$

the minimum being taken over the set of smooth maps $(m, v) : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R} \times \mathbb{R}^d$ such that

$$(5.24) \quad \partial_t m - \Delta m + \operatorname{div}(mv) = 0 \text{ in } (0, T) \times \mathbb{R}^d, \quad m(0, x) = m_0(x).$$

Our aim is to show the following result:

Proposition 5.2.7. *Assume that $(\bar{m}, \bar{v})$ is a smooth minimizer of the above problem with $m > 0$. Then there exists a map $\bar{u} = \bar{u}(t, x)$ such that the pair $(\bar{u}, \bar{m})$ is a solution to the MFG system (5.13).*

Sketch of proof. Let (m, v) satisfy the constraint (5.24). Then, for any $\lambda > 0$,

$$(m_\lambda, v_\lambda) := ((1 - \lambda)\bar{m} + \lambda m, \frac{(1 - \lambda)\bar{m}\bar{v} + \lambda m v}{(1 - \lambda)\bar{m} + \lambda m})$$

satisfies the constraint (5.24). By optimality of $(\bar{m}, \bar{v})$ we have

$$\begin{aligned} 0 \leq \mathcal{J}(m_\lambda, v_\lambda) - \mathcal{J}(\bar{m}, \bar{v}) &= \int_0^T \int_{\mathbb{R}^d} \left(\frac{|(1 - \lambda)\bar{m}\bar{v} + \lambda m v|^2}{2((1 - \lambda)\bar{m} + \lambda m)} - |\bar{v}|^2 \bar{m} \right) dx dt \\ &\quad + \int_0^T (\mathcal{F}(m_\lambda(t)) - \mathcal{F}(\bar{m}(t))) dt + \mathcal{G}(m_\lambda(T)) - \mathcal{G}(\bar{m}(T)) \\ &= \int_0^T \int_{\mathbb{R}^d} \left(\frac{|(1 - \lambda)\bar{m}\bar{v} + \lambda m v|^2}{2((1 - \lambda)\bar{m} + \lambda m)} - |\bar{v}|^2 \bar{m} \right) dx dt \\ &\quad + \int_0^T \int_0^1 \int_{\mathbb{R}^d} F(x, (1 - s)\bar{m}(t) + s m_\lambda(t)) (m_\lambda - \bar{m})(t, x) dx ds dt \\ &\quad + \int_0^1 \int_{\mathbb{R}^d} G(x, (1 - s)\bar{m}(T) + s m_\lambda(T)) (m_\lambda - \bar{m})(T, x) dx ds \end{aligned}$$

Note that m_λ tends to $\bar{m}$ as $\lambda \rightarrow 0^+$. So, dividing by λ and letting $\lambda \rightarrow 0^+$ gives

$$(5.25) \quad \begin{aligned} 0 \leq & \int_0^T \int_{\mathbb{R}^d} ((v - \bar{v}) \cdot \bar{v} \bar{m} + \frac{1}{2} |\bar{v}|^2 (m - \bar{m})) dx dt \\ & + \int_0^T \int_{\mathbb{R}^d} F(x, \bar{m}(t)) (m - \bar{m})(t, x) dx ds dt + \int_0^1 \int_{\mathbb{R}^d} G(x, \bar{m}(T)) (m - \bar{m})(T, x) dx ds. \end{aligned}$$

Let u be the solution to the backward equation

$$(5.26) \quad \begin{cases} -\partial_t u - \Delta u - v \cdot Du = F(x, \bar{m}(t)) + \frac{1}{2} |\bar{v}(t, x)|^2 & \text{in } (0, T) \times \mathbb{R}^d, \\ u(T, x) = G(x, \bar{m}(T)) & \text{in } \mathbb{R}^d. \end{cases}$$

Plugging this relation into (5.25), we have

$$\begin{aligned} 0 \leq & \int_0^T \int_{\mathbb{R}^d} ((v - \bar{v}) \cdot \bar{v} \bar{m} + (-\partial_t u - \Delta u - v \cdot Du)(m - \bar{m})) dx dt \\ & + \int_0^1 \int_{\mathbb{R}^d} G(x, \bar{m}(T)) (m - \bar{m})(T, x) dx ds \\ & = \int_0^T \int_{\mathbb{R}^d} ((v - \bar{v}) \cdot \bar{v} \bar{m} + u(\partial_t(m - \bar{m}) - \Delta(m - \bar{m})) - v \cdot Du(m - \bar{m})) dx dt \\ & = \int_0^T \int_{\mathbb{R}^d} ((v - \bar{v}) \cdot \bar{v} \bar{m} + Du \cdot (mv - \bar{m}\bar{v}) - v \cdot Du(m - \bar{m})) dx dt \\ & = \int_0^T \int_{\mathbb{R}^d} (v - \bar{v}) \cdot \bar{v} \bar{m} + \bar{m} Du \cdot (v - \bar{v}) dx dt. \end{aligned}$$

Now we choose $v = \bar{v} + hz$ for $h > 0$ and z smooth and let u_h be the associated solution to the HJ equation (5.26). As $h \rightarrow 0^+$, u_h converges to $\bar{u}$ solution to

$$\begin{cases} -\partial_t \bar{u} - \Delta \bar{u} - \bar{v} \cdot D\bar{u} = F(x, \bar{m}(t)) + \frac{1}{2} |\bar{v}(t, x)|^2 & \text{in } (0, T) \times \mathbb{R}^d, \\ \bar{u}(T, x) = G(x, \bar{m}(T)) & \text{in } \mathbb{R}^d. \end{cases}$$

Then

$$0 = \int_0^T \int_{\mathbb{R}^d} z \cdot \bar{v} \bar{m} + \bar{m} D\bar{u} \cdot z dx dt.$$

As z is arbitrary, (meaning that, for any $h > 0$, there exists a unique classical solution to (5.26) associated with the drift $v = \bar{v} + hz$) and $\bar{m} > 0$, we obtain that $D\bar{u} = -\bar{v}$. This proves that $\bar{u}$ actually solves

$$\begin{cases} -\partial \bar{u} - \Delta \bar{u} + \frac{1}{2} |D\bar{u}(t, x)|^2 = F(x, \bar{m}(t)) & \text{in } (0, T) \times \mathbb{R}^d, \\ \bar{u}(T, x) = G(x, \bar{m}(T)) & \text{in } \mathbb{R}^d. \end{cases}$$

and $\bar{m}$ solves

$$\partial_t \bar{m} - \Delta \bar{m} - \operatorname{div}(m D\bar{u}) = 0 \text{ in } (0, T) \times \mathbb{R}^d, \quad m(0, x) = m_0(x).$$

So the pair $(\bar{u}, \bar{m})$ is a solution to (5.13). $\square$

5.2.5 Application to games with finitely many players

Before starting the discussion of games with a large number of players, let us fix a solution (u, m) of the mean field equation (5.13) and investigate the optimal strategy of a generic player who considers the density m “of the other players” as given. He faces the following minimization problem

$$\inf_{\alpha} \mathcal{J}(\alpha) \quad \text{where} \quad \mathcal{J}(\alpha) = \mathbb{E} \left[\int_0^T \frac{1}{2} |\alpha_s|^2 + F(X_s, m(s)) ds + G(X_T, m(T)) \right].$$

In the above formula, $X_t = X_0 + \int_0^t \alpha_s ds + \sqrt{2} B_s$, X_0 is a fixed random initial condition with law m_0 and the control α is adapted to some filtration $(\mathcal{F}_t)$. We assume that (B_t) is an d -dimensional Brownian motion adapted to $(\mathcal{F}_t)$ and that X_0 and (B_t) are independent. We claim that the feedback strategy $\bar{\alpha}(x, t) := -Du(x, t)$ is optimal for this optimal stochastic control problem.

Lemma 5.2.8. *Let $(\bar{X}_t)$ be the solution of the stochastic differential equation*

$$\begin{cases} d\bar{X}_t = \bar{\alpha}(\bar{X}_t, t) dt + \sqrt{2} dB_t \\ \bar{X}_0 = X_0 \end{cases}$$

and $\tilde{\alpha}(t) = \bar{\alpha}(\bar{X}_t, t)$. Then

$$\inf_{\alpha} \mathcal{J}(\alpha) = \mathcal{J}(\tilde{\alpha}) = \int_{\mathbb{R}^N} u(x, 0) dm_0(x).$$

Proof : This kind of result is known as a verification Theorem: one has a good candidate for an optimal control, and one checks, using the equation satisfied by the value function u , that this is indeed the minimum. Let α be an adapted control. We have, from Itô's formula,

$$\begin{aligned} \mathbb{E}[G(X_T, m(T))] &= \mathbb{E}[u(X_T, T)] \\ &= \mathbb{E} \left[u(X_0, 0) + \int_0^T (\partial_t u(X_s, s) + \langle \alpha_s, Du(X_s, s) \rangle + \Delta u(X_s, s)) ds \right] \\ &= \mathbb{E} \left[u(X_0, 0) + \int_0^T \left(\frac{1}{2} |Du(X_s, s)|^2 + \langle \alpha_s, Du(X_s, s) \rangle - F(X_s, m(s)) \right) ds \right] \\ &\geq \mathbb{E} \left[u(X_0, 0) + \int_0^T \left(-\frac{1}{2} |\alpha_s|^2 - F(X_s, m(s)) \right) ds \right] \end{aligned}$$

This shows that $\mathbb{E}[u(X_0, 0)] \leq \mathcal{J}(\alpha)$ for any adapted control α . If we replace α by $\tilde{\alpha}$ in the above computations, then, since the process (X_t) becomes $(\bar{X}_t)$, the above inequalities are all equalities. So $\mathbb{E}[u(X_0, 0)] = \mathcal{J}(\tilde{\alpha})$ and the result is proved. $\square$

We now consider a differential game with N players which consists in a kind of discrete version of the mean field game. In this game player i ($i = 1, \dots, N$) is controlling through his control α^i a dynamics of the form

$$(5.27) \quad dX_t^i = \alpha_t^i dt + \sqrt{2} dB_t^i$$

where (B_t^i) is a d -dimensional brownian motion. The initial condition X_0^i for this system is also random and has for law m_0 . We assume that the all X_0^i and all the brownian motions (B_t^i) ($i = 1, \dots, N$) are independent. However player i can choose his control α^i adapted to the filtration $(\mathcal{F}_t = \sigma(X_0^j, B_s^j, s \leq t, j = 1, \dots, N))$. His payoff is then given by

$$\begin{aligned} & \mathcal{J}_i^N(\alpha^1, \dots, \alpha^N) \\ &= \mathbb{E} \left[\int_0^T \frac{1}{2} |\alpha_s^i|^2 + F \left(X_s^i, \frac{1}{N-1} \sum_{j \neq i} \delta_{X_s^j} \right) ds + G \left(X_T^i, \frac{1}{N-1} \sum_{j \neq i} \delta_{X_T^j} \right) \right] \end{aligned}$$

Our aim is to explain that the strategy given by the mean field game is suitable for this problem. More precisely, let (u, m) be one classical solution to (5.13) and let us set $\bar{\alpha}(x, t) = -Du(x, t)$. With the closed loop strategy $\bar{\alpha}$ one can associate the open-loop control $\tilde{\alpha}^i$ obtained by solving the SDE

$$(5.28) \quad d\bar{X}_t^i = \bar{\alpha}(\bar{X}_t^i, t) dt + \sqrt{2} dB_t^i$$

with random initial condition X_0^i and setting $\tilde{\alpha}_t^i = \bar{\alpha}(\bar{X}_t^i, t)$. Note that this control is just adapted to the filtration $(\mathcal{F}_t^i = \sigma(X_0^i, B_s^i, s \leq t))$, and not to the full filtration $(\mathcal{F}_t)$ defined above.

Theorem 5.2.9. *For any $\epsilon > 0$, there is some N_0 such that, if $N \geq N_0$, then the symmetric strategy $(\tilde{\alpha}^1, \dots, \tilde{\alpha}^N)$ is an ϵ -Nash equilibrium in the game $\mathcal{J}_1^N, \dots, \mathcal{J}_N^N$: Namely*

$$\mathcal{J}_i^N(\tilde{\alpha}^1, \dots, \tilde{\alpha}^N) \leq \mathcal{J}_i^N((\tilde{\alpha}^j)_{j \neq i}, \alpha) + \epsilon$$

for any control α adapted to the filtration $(\mathcal{F}_t)$ and any $i \in \{1, \dots, N\}$.

Remark 5.2.10. This result is very close to one-shot games and its proof is mainly based on the stability property of the mean field equation. In some sense it is rather “cheap”: the difficult question is in what extent Nash equilibria for differential games in *feedback strategies* give rise to a mean field equation. This question is for a large extent open.

Proof : Fix $\epsilon > 0$. Since the problem is symmetrical, it is enough to show that

$$(5.29) \quad \mathcal{J}_1^N(\tilde{\alpha}^1, \dots, \tilde{\alpha}^N) \leq \mathcal{J}_1^N((\tilde{\alpha}^j)_{j \neq 1}, \alpha) + \epsilon$$

for any control α , as soon as N is large enough. Let us denote by $\bar{X}_t^j$ the solution of the stochastic differential equation (5.28) with initial condition X_0^j . We note that the $(\bar{X}_t^j)$ are independent and identically distributed with law $m(t)$ (the law comes from Lemma 5.2.3). Therefore, using (as in subsection 2.4) the law of large numbers 3.1.9, there is some N_0 such that, if $N \geq N_0$,

$$(5.30) \quad \mathbb{E} \left[\sup_{|y| \leq 1/\sqrt{\epsilon}} \left| F \left(y, \frac{1}{N-1} \sum_{j \geq 2} \delta_{\bar{X}_s^j} \right) - F(y, m(t)) \right| \right] \leq \epsilon$$

for any $t \in [0, T]$ and

$$(5.31) \quad \mathbb{E} \left[\sup_{|y| \leq 1/\sqrt{\epsilon}} \left| G \left(y, \frac{1}{N-1} \sum_{j \geq 2} \delta_{\bar{X}_T^j} \right) - G(y, m(T)) \right| \right] \leq \epsilon.$$

For the first inequality, one can indeed choose N_0 independent of t because, F being C_0 -Lipschitz continuous with respect to m , we have

$$\begin{aligned} & \mathbb{E} \left[\sup_{|y| \leq \sqrt{\epsilon}} \left| F \left(y, \frac{1}{N-1} \sum_{j \geq 2} \delta_{\bar{X}_t^j} \right) - F \left(y, \frac{1}{N-1} \sum_{j \geq 2} \delta_{\bar{X}_s^j} \right) \right| \right] \\ & \leq \mathbb{E} \left[C_0 \mathbf{d}_1 \left(\frac{1}{N-1} \sum_{j \geq 2} \delta_{\bar{X}_t^j}, \frac{1}{N-1} \sum_{j \geq 2} \delta_{\bar{X}_s^j} \right) \right] \\ & \leq \frac{1}{N-1} \sum_{j \geq 2} \mathbb{E} \left[\left| \bar{X}_t^j - \bar{X}_s^j \right| \right] \leq c_0 (1 + \|\bar{\alpha}\|_\infty) (t-s)^{1/2}, \end{aligned}$$

where the last inequality easily comes from computations similar to that for Lemma 5.2.4.

Let now α be a control adapted to the filtration $(\mathcal{F}_t)$ and X_t be the solution to

$$dX_t = \alpha_t dt + \sqrt{2} dB_t^1$$

with random initial condition X_0^1 . Let us set $K = 2(T\|F\|_\infty + \|G\|_\infty) + \mathbb{E}[\int_0^T \frac{1}{2} |\bar{\alpha}_s^1|^2 ds]$. Note that, if $\mathbb{E}[\int_0^T \frac{1}{2} |\alpha_s|^2 ds] \geq K$, then (5.29) holds.

Let us now assume that $\mathbb{E}[\int_0^T \frac{1}{2} |\alpha_s|^2 ds] \leq K$. We first estimate $\mathbb{E}[\sup_{t \in [0, T]} |X_t|^2]$:

$$\mathbb{E} \left[\sup_{t \in [0, T]} |X_t|^2 \right] \leq 2\mathbb{E} \left[|X_0^1|^2 + \int_0^T \frac{1}{2} |\alpha_s|^2 ds + 2 \sup_{t \in [0, T]} |B_t|^2 \right] \leq 2\mathbb{E}[|X_0^1|^2] + 2K + 4T$$

where the last estimates comes from the Burkholder-Davis-Gundy inequality. Denoting by K_1 the right-hand side of the above inequality we obtain therefore that

$$\mathbb{P} \left[\sup_{t \in [0, T]} |X_t| \geq 1/\sqrt{\epsilon} \right] \leq K_1 \epsilon.$$

Let us now fix $N \geq N_0$ and estimate $\mathcal{J}_1^N((\tilde{\alpha}^j)_{j \neq 1}, \alpha)$ by separating the expectation for the F and G terms according to the fact that $\sup_{t \in [0, T]} |X_t| \geq 1/\sqrt{\epsilon}$ or not. Taking into account (5.30) and (5.31) we have

$$\begin{aligned} \mathcal{J}_1^N((\tilde{\alpha}^j)_{j \neq 2}, \alpha) & \geq \mathbb{E} \left[\int_0^T \frac{1}{2} |\alpha_s|^2 ds + F(X_s^i, m(t)) + G(X_T^i, m(T)) \right] - (1+T)\epsilon \\ & \quad - 2\mathbb{P} \left[\sup_{t \in [0, T]} |X_t| \geq 1/\sqrt{\epsilon} \right] (T\|F\|_\infty + \|G\|_\infty) \\ & \geq \mathcal{J}_1^N((\tilde{\alpha}^j)_{j \neq 1}) - C\epsilon \end{aligned}$$

for some constant C independent of N and α , where the last inequality comes from the optimality of $\bar{\alpha}$ in Lemma 5.2.8. $\square$

5.3 Comments

Existence: Existence of solutions for the MFG system can be achieved either by Banach fixed point Theorem (as in the papers by Caines, Huang and Malhamé [40], under a smallness assumption on the coefficients or on the time interval) or by Schauder arguments (we used this argument in Theorem 5.2.1), as in Lasry and Lions [45, 46]). For MFG systems with local coupling functions, i.e., when $F = F(x, m(t, x))$, one can also use variational methods (see Subsection 5.2.4) similar to the techniques for optimal transport problems (see [46]).

Uniqueness: Concerning the uniqueness of the solution, one can distinguish two kinds of regimes. Of course the Banach fixed point argument provides directly uniqueness of the solution of the MFG system.

However, as explained above, it mostly concerns local in time results. For the large time uniqueness, one can rely on the monotonicity conditions (5.21) and (5.22). These conditions first appear in [45, 46].

Nash equilibria for the N -player games: the use of the MFG system to obtain ϵ -Nash equilibria (Theorem 5.2.9) has been initiated—in a slightly different framework—in a series of papers due to Caines, Huang and Malhamé: see in particular [37] (for linear dynamics) and [40] (for nonlinear dynamics). In these papers, the dependence with respect of the empirical measure of dynamics and payoff occurs through an average, so that the CTL implies that the error term is a order $N^{-1/2}$ (instead of $N^{-1/(d+4)}$ as in Theorem 5.2.9). The genuinely non linear version of the result given above is a variation on a result by Carmona and Delarue [14].

The reverse statement, namely in what extend the MFG system pops up as the limit of Nash equilibria, is much more difficult and investigated in [10].

Extensions: it is difficult to discuss all the extensions of the MFG systems since the number of papers on this subject has grown exponentially in the last years. We give here only a brief overview.

The ergodic MFG system has been introduced by Lasry and Lions in [44] as the limit, when the number of players tends to infinity, of Nash equilibria in ergodic differential games. As explained in Lions [48], this system also pops up as the limit, as the horizon tends to infinity, of the finite horizon MFG system.

The natural issue of boundary conditions has not been thoroughly investigated up to now. For the PDE approach, the authors have mostly worked with periodic data (as we did above), which completely eliminates this question. In the “probabilistic literature” (as in the work by Caines, Huang and Malhamé), the natural set-up is the full space. Beside these two extreme cases, little has been written (see however Cirant [19], for Neumann boundary condition in ergodic multi-population MFG systems).

The interesting MFG systems with several populations were introduced in the early paper by Caines, Huang and Malhamé [40] and revisited by Cirant [19] (for Neumann boundary conditions) and by Kolokoltsov, Li and Yang [41] (for very general diffusions, possibly with jumps).

A very general MFG model (the so-called extended MFG) for a single population is described in Gomes, Patrizi and Voskanyan [35] and Gomes and Voskanyan[36]. There the velocity of the population is a nonlocal function of the (repartition of) actions of the players. See also Chapter I.4.6 of the monograph by Carmona and Delarue [15]. The example in Subsection 5.1.2 is also of this form and borrowed from the paper [13].

Chapter 6

Differentiability of maps on the Wasserstein space

In this chapter, we discuss different notions of derivatives in the space of probability measures and explain how they are related. We also explain how to work with them (optimality conditions, Itô's formulas). This part is, to a large extent, borrowed from [15].

6.1 The various notions of derivatives

6.1.1 The flat derivative

Definition 6.1.1. Let $U : \mathcal{P}_2 \rightarrow \mathbb{R}$. We say that U is of class C^1 if there exists a jointly continuous and bounded map $\frac{\delta U}{\delta m} : \mathcal{P}_2 \times \mathbb{R}^d \rightarrow \mathbb{R}$ such that

$$U(m') - U(m) = \int_0^1 \int_{\mathbb{R}^d} \frac{\delta U}{\delta m}((1-h)m + hm', y)(m' - m)(dy)dh \quad \forall m, m' \in \mathcal{P}_2.$$

Moreover we adopt the normalization convention

$$(6.1) \quad \int_{\mathbb{R}^d} \frac{\delta U}{\delta m}(m, y)m(dy) = 0 \quad \forall m \in \mathcal{P}_2.$$

Note that, if U is of class C^1 , then the following equality holds for any $m \in \mathcal{P}_2$ and $y \in \mathbb{R}^d$

$$\frac{\delta U}{\delta m}(m, y) = \lim_{h \rightarrow 0^+} \frac{1}{h} (U((1-h)m + h\delta_y) - U(m)).$$

Here is a kind of converse.

Proposition 6.1.2. Let $U : \mathcal{P}_2 \rightarrow \mathbb{R}$ and assume that the limit

$$V(m, y) := \lim_{h \rightarrow 0^+} \frac{1}{h} (U((1-h)m + h\delta_y) - U(m))$$

exists and is jointly continuous and bounded on $\mathcal{P}_2 \times \mathbb{R}^d$. Then U is C^1 and $\frac{\delta U}{\delta m}(m, y) = V(m, y)$.

Proof. Although the result can be expected, the proof is a little involved and can be found in [11]. $\square$

Let us recall that, if $\phi : \mathbb{R}^d \rightarrow \mathbb{R}^d$ is a Borel measurable map and m is a Borel probability measure on $\mathbb{R}^d$, the image of m by ϕ is the Borel probability measure $\phi\#m$ defined by

$$\int_{\mathbb{R}^d} f(x)\phi\#m(dx) = \int_{\mathbb{R}^d} f(\phi(y))m(dy) \quad \forall f \in C_b^0(\mathbb{R}^d).$$

Proposition 6.1.3. *Let U be C^1 and be such that $D_y \frac{\delta U}{\delta m}$ exists and is jointly continuous and bounded on $\mathcal{P}_2 \times \mathbb{R}^d$. Then, for any Borel measurable map $\phi : \mathbb{R}^d \rightarrow \mathbb{R}^d$ with at most a linear growth, the map $s \rightarrow U((id_{\mathbb{R}^d} + s\phi)\sharp m)$ is differentiable at 0 and*

$$\frac{d}{ds}\bigg|_{s=0} U((id_{\mathbb{R}^d} + s\phi)\sharp m) = \int_{\mathbb{R}^d} D_y \frac{\delta U}{\delta m}(m, y) \cdot \phi(y) m(dy).$$

Proof. Indeed

$$\begin{aligned} U((id_{\mathbb{R}^d} + s\phi)\sharp m) - U(m) &= \int_0^1 \int_{\mathbb{R}^d} \frac{\delta U}{\delta m}(m_{h,s}, y) ((id_{\mathbb{R}^d} + s\phi)\sharp m - m)(dy) dh \\ &= \int_0^1 \int_{\mathbb{R}^d} \left(\frac{\delta U}{\delta m}(m_{h,s}, y + s\phi(y)) - \frac{\delta U}{\delta m}(m_{h,s}, y) \right) m(dy) dh \\ &= s \int_0^1 \int_0^1 \int_{\mathbb{R}^d} D_y \frac{\delta U}{\delta m}(m_{h,s}, y + s\tau\phi(y)) \cdot \phi(y) m(dy) dh d\tau, \end{aligned}$$

where

$$m_{h,s} = (1-h)m + h(id_{\mathbb{R}^d} + s\phi)\sharp m.$$

Dividing by s and letting $s \rightarrow 0^+$ gives the result. $\square$

If $m, m' \in \mathcal{P}_2$, let us denote by $\Pi^{opt}(m, m')$ the set of optimal transport plans between m and m' .

Proposition 6.1.4. *Under the assumptions of the previous Proposition, let $m, m' \in \mathcal{P}_2$ and $\pi \in \Pi^{opt}(m, m')$. Then*

$$\left| U(m') - U(m) - \int_{\mathbb{R}^{2d}} D_y \frac{\delta U}{\delta m}(m, x) \cdot (y - x) \pi(dx, dy) \right| \leq \epsilon_m(m') \mathbf{d}_2(m, m'),$$

where $\epsilon_m(m')$ tends to 0 as m' tends to m in $\mathcal{P}_2$.

Remark 6.1.5. The same proof shows that, if π is a transport plan between m and m' (not necessarily optimal), then

$$\left| U(m') - U(m) - \int_{\mathbb{R}^{2d}} D_y \frac{\delta U}{\delta m}(m, x) \cdot (y - x) \pi(dx, dy) \right| \leq \epsilon_m(m') \left(\int_{\mathbb{R}^{2d}} |x - y|^2 \pi(dx, dy) \right)^{1/2}.$$

Proof. Let $\phi_t(x, y) = (1-t)x + ty$ and $m_t = \phi_t\sharp\pi$. Then $m_0 = m$ and $m_1 = m'$ and, for any $t \in (0, 1)$ and any s small we have

$$\begin{aligned} U(\phi_{t+s}\sharp\pi) - U(\phi_t\sharp\pi) &= \int_0^1 \int_{\mathbb{R}^d} \frac{\delta U}{\delta m}(m_{s,h}, y) (\phi_{t+s}\sharp\pi - \phi_t\sharp\pi)(dy) dh \\ &= \int_0^1 \int_{\mathbb{R}^{2d}} \frac{\delta U}{\delta m}(m_{s,h}, (1-t-s)x + (t+s)y) - \frac{\delta U}{\delta m}(m_{s,h}, (1-t)x + ty) \pi(dx, dy) dh \\ &= s \int_0^1 \int_0^1 \int_{\mathbb{R}^{2d}} D_y \frac{\delta U}{\delta m}(m_{s,h}, (1-t-\tau s)x + (t+\tau s)y) \cdot (y - x) \pi(dx, dy) dh d\tau, \end{aligned}$$

where $m_{s,h} = (1-h)\phi_{t+s}\sharp\pi + h\phi_t\sharp\pi$. So, dividing by s and letting $s \rightarrow 0$ we find:

$$\frac{d}{dt} U(\phi_t\sharp\pi) = \int_{\mathbb{R}^{2d}} D_y \frac{\delta U}{\delta m}(\phi_t\sharp\pi, (1-t)x + ty) \cdot (y - x) \pi(dx, dy).$$

As $D_y \frac{\delta U}{\delta m}$ is continuous and bounded by C , for any $\epsilon, R > 0$, there exists $r > 0$ such that, if $\mathbf{d}_2(m, m') \leq r$ and $|x|, |y| \leq R$, then

$$|D_y \frac{\delta U}{\delta m}(\phi_t\sharp\pi, (1-t)x + ty) - D_y \frac{\delta U}{\delta m}(m, x)| \leq \epsilon + 2C\mathbf{1}_{|y-x| \geq r}.$$

So

$$\begin{aligned} & \left| \int_{\mathbb{R}^{2d}} D_y \frac{\delta U}{\delta m}(\phi_t \# \pi, (1-t)x + ty) \cdot (y-x) \pi(dx, dy) - \int_{\mathbb{R}^{2d}} D_y \frac{\delta U}{\delta m}(m, x) \cdot (y-x) \pi(dx, dy) \right| \\ & \leq \delta_R + \int_{(B_R)^2} (\epsilon + 2C \mathbf{1}_{|x-y| \geq r}) |y-x| \pi(dx, dy) \leq \delta_R + \epsilon \mathbf{d}_2(m, m') + \frac{2C}{r} \mathbf{d}_2^2(m, m'). \end{aligned}$$

where

$$\begin{aligned} \delta_R &:= \int_{\mathbb{R}^{2d} \setminus (B_R)^2} \left(|D_y \frac{\delta U}{\delta m}(\phi_t \# \pi, (1-t)x + ty) \cdot (y-x)| + |D_y \frac{\delta U}{\delta m}(m, x) \cdot (y-x)| \right) \pi(dx, dy) \\ &\leq 2C \int_{\mathbb{R}^{2d} \setminus (B_R)^2} |y-x| \pi(dx, dy) \leq 2C \mathbf{d}_2(m, m') (\pi(\mathbb{R}^{2d} \setminus (B_R)^2))^{1/2} = \mathbf{d}_2(m, m') o_R(1). \end{aligned}$$

This proves the result. $\square$

6.1.2 W -differentiability

Next we turn to a more geometric definition of derivative in the space of measure. For this, let us introduce the notation $L_m^2(\mathbb{R}^d)$ as the set of Borel measurable vector fields $\phi : \mathbb{R}^d \rightarrow \mathbb{R}^d$ such that $\int_{\mathbb{R}^d} |\phi|^2 dm$ is finite. We now explain the notion of tangent space to a measure m in $\mathcal{P}_2$.

Definition 6.1.6 (Tangent space). *The tangent space $\text{Tan}_m(\mathcal{P}_2)$ of $\mathcal{P}_2$ at $m \in \mathcal{P}_2$ is the closure in $L_m^2(\mathbb{R}^d)$ of $\{D\phi, \phi \in C_c^\infty(\mathbb{R}^d)\}$.*

Following [2] we define the super and the subdifferential of a map defined on $\mathcal{P}_2$:

Definition 6.1.7. *Let $U : \mathcal{P}_2 \rightarrow \mathbb{R}$, $m \in \mathcal{P}_2$ and $\xi \in L_m^2(\mathbb{R}^d, \mathbb{R}^d)$. We say that ξ belongs to the superdifferential $\partial^+ U(m)$ to U at m if, for any $m' \in \mathcal{P}_2$ and any transport plan π from m to m' ,*

$$U(m') \leq U(m) + \int_{\mathbb{R}^d \times \mathbb{R}^d} \xi(x) \cdot (y-x) \pi(dx, dy) + o \left(\left(\int_{\mathbb{R}^{2d}} |x-y|^2 \pi(dx, dy) \right)^{1/2} \right).$$

We say that ξ belongs to the subdifferential $\partial^- U(m)$ to U at m if $-\xi$ belongs to $D^+(-U)(m)$. Finally, we say that the map U is W -differentiable at m if $\partial^+ U(m) \cap \partial^- U(m)$ is not empty.

Proposition 6.1.8. *If U is W -differentiable at m , then $\partial^+ U(m)$ and $\partial^- U(m)$ are equal and reduce to a singleton, denoted $\{D_m U(m, \cdot)\}$.*

Remark 6.1.9. On can actually check that $D_m U(m, \cdot)$ belongs to $\text{Tan}_m(\mathcal{P}_2)$. Note that we have seen in Remark 6.1.5 that if U is C^1 with $D_y \delta U / \delta m$ continuous and bounded, then U is W -differentiable. In this case it is obvious that $D_m U(m)$ belongs to $\text{Tan}_m(\mathcal{P}_2)$ by definition.

Proof. Let $\xi_1 \in D^+ U(m)$ and $\xi_2 \in D^- U(m)$. We have, for any $m' \in \mathcal{P}_2$ and any transport plan π from m to m' ,

$$\begin{aligned} & \int_{\mathbb{R}^d \times \mathbb{R}^d} \xi_2(x) \cdot (y-x) \pi(dx, dy) + o \left(\left(\int_{\mathbb{R}^{2d}} |x-y|^2 \pi(dx, dy) \right)^{1/2} \right) \\ & \leq U(m') - U(m) \leq \int_{\mathbb{R}^d \times \mathbb{R}^d} \xi_1(x) \cdot (y-x) \pi(dx, dy) + o \left(\left(\int_{\mathbb{R}^{2d}} |x-y|^2 \pi(dx, dy) \right)^{1/2} \right). \end{aligned}$$

In particular, if we choose $m' = (1+h\phi) \# m$ and $\pi = (Id, Id+h\phi) \# m$ for some $\phi \in L_m^2(\mathbb{R}^d, \mathbb{R}^d)$ and $h > 0$ small, we obtain

$$h \int_{\mathbb{R}^d} \xi_2(x) \cdot \phi(x) m(dx) + o(h) \leq U(m') - U(m) \leq h \int_{\mathbb{R}^d} \xi_1(x) \cdot \phi(x) m(dx) + o(h),$$

from which we easily infer that $\xi_1 = \xi_2$ in $L_m^2(\mathbb{R}^d)$. $\square$

6.1.3 Link with the L -derivative

Another possibility for the notion of derivative is to look at the set of Borel probability measures as the law of random variables with values in $\mathbb{R}^d$ and to use the fact that this set has a Hilbertian structure. This approach has been developed by Lions in his courses at Collège de France, whence the name “Lions-derivative”, or “L-derivative”.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ an atomless probability space. Given a map $U : \mathcal{P}_2 \rightarrow \mathbb{R}$, we consider its extension $\tilde{U}$ to the set of random variables $L^2(\Omega, \mathbb{R}^d)$:

$$\tilde{U}(X) = U(\mathcal{L}(X)) \quad \forall X \in L^2(\Omega, \mathbb{R}^d).$$

(recall that $\mathcal{L}(X)$ is the law of X , i.e., $\mathcal{L}(X) := X\#\mathbb{P}$. Note that $\mathcal{L}(X)$ belongs to $\mathcal{P}_2$ because $X \in L^2(\Omega)$). The important point is that $L^2(\Omega, \mathcal{F}, \mathbb{P})$ is a Hilbert space, in which the notion of Frechet differentiability makes sense.

For instance, if U is a map of the form

$$(6.2) \quad U(m) = \int_{\mathbb{R}^d} \phi(x) m(dx) \quad \forall m \in \mathcal{P}_2,$$

where $\phi \in C_c^0(\mathbb{R}^d)$ is given, then

$$\tilde{U}(X) = \mathbb{E}[\phi(X)] \quad \forall X \in L^2(\Omega, \mathbb{R}^d).$$

Definition 6.1.10. *The map $U : \mathcal{P}_2 \rightarrow \mathbb{R}$ is L -differentiable at $m \in \mathcal{P}_2$ if there exists $X \in L^2(\Omega, \mathbb{R}^d)$ such that $\mathcal{L}(X) = m$ and the extension $\tilde{U}$ of U is Frechet differentiable at X .*

The following result says that the notion of L -differentiability coincides with that of W -differentiability and is independent of the probability space and of the representative X . The first statement in that direction goes back to Lions course at Collège de France; the version given here is can be found in [33] (see also [1], from which the sketch of proof of Lemma 6.1.13 is largely inspired).

Theorem 6.1.11. *The map U is W -differentiable at $m \in \mathcal{P}_2$ if and only if U is L -differentiable at some (or thus any) $X \in L^2(\Omega, \mathbb{R}^d)$ with $\mathcal{L}(X) = m$. In this case*

$$\nabla \tilde{U}(X) = D_m U(m, X).$$

The result can be considered as a structure theorem for the L-derivative. For instance, if U is as in (6.2) for some map $\phi \in C_c^1(\mathbb{R}^d)$, then it is almost obvious that

$$\nabla \tilde{U}(X) = D\phi(X)$$

and thus

$$D_m U(m, x) = D\phi(x).$$

The proof of Theorem 6.1.11 is difficult and we only sketch it briefly. Complete proofs can be found in [33] or [1]. The first step is the fact that, if X and X' have the same law, then so do $\nabla U(X)$ and $\nabla U(X')$:

Lemma 6.1.12. *Let $U : \mathcal{P}_2 \rightarrow \mathbb{R}$ and $\tilde{U}$ be its extension. Let X, X' be two random variables in $L^2(\Omega, \mathbb{R}^d)$ with $\mathcal{L}(X) = \mathcal{L}(X')$. If $\tilde{U}$ is Frechet differentiable at X , then $\tilde{U}$ is differentiable at X' and $(X, \nabla U(X))$ has the same law as $(X', \nabla U(X'))$.*

(Sketch of) proof. The idea behind this fact is that, if X and X' have the same law, then one can “almost” find a bi-measurable and measure-preserving transformation $\tau : \Omega \rightarrow \Omega$ such that $X' = X \circ \tau$. Admitting this statement for a while, we have, for any $H' \in L^2$ small,

$$\begin{aligned} \tilde{U}(X' + H') &= \tilde{U}((X' + H') \circ \tau) = \tilde{U}(X + H' \circ \tau) = \tilde{U}(X) + \mathbb{E} \left[\nabla \tilde{U}(X) \cdot H' \circ \tau \right] + o(\|H' \circ \tau\|_2) \\ &= \tilde{U}(X') + \mathbb{E} \left[\nabla \tilde{U}(X) \circ \tau^{-1} \cdot H' \right] + o(\|H'\|_2). \end{aligned}$$

This shows that $\tilde{U}$ is differentiable at X' with differential given by $\nabla\tilde{U}(X) \circ \tau^{-1}$. Thus $(X', \nabla\tilde{U}(X')) = (X, \nabla\tilde{U}(X)) \circ \tau^{-1}$, which shows that $(X, \nabla U(X))$ and $(X', \nabla U(X'))$ have the same law.

In fact the existence of τ does not hold in general. However, one can show that, for any $\epsilon > 0$, there exists $\tau : \Omega \rightarrow \Omega$ bi-measurable and measure preserving and such that $\|X' - X \circ \tau\|_\infty \leq \epsilon$. A (slightly technical) adaptation of the proof above then gives the result (see [15] for the details). $\square$

Next we show that $\nabla\tilde{U}(X)$ is a function of X :

Lemma 6.1.13. *Assume that $\tilde{U}$ is differentiable at $X \in L^2(\Omega, \mathbb{R}^d)$. Then there exists a Borel measurable map $g : \mathbb{R}^d \rightarrow \mathbb{R}^d$ such that $\nabla\tilde{U}(X) = g(X)$ a.s..*

(Sketch of) proof. To prove the claim, we just need to check that $\nabla\tilde{U}(X)$ is $\sigma(X)$ -measurable (see Theorem 20.1 in [7]), which can be recasted into the fact that $\nabla\tilde{U}(X) = \mathbb{E}[\nabla\tilde{U}(X)|X]$. Let $\mu = \mathcal{L}(X, \nabla\tilde{U}(X))$ and let $\mu(dx, dy) = (\delta_x \otimes \nu_x(dy))\mathbb{P}_X(dx)$ be its disintegration with respect to its first marginal $\mathbb{P}_X$. Let λ be the restriction of the Lebesgue measure to $Q_1 := [0, 1]^d$. Then, as λ has an L^1 density, the optimal transport from λ to ν_x is unique and given by the gradient of a convex map $\psi_x(\cdot)$ (Brenier's Theorem, see [62]). So we can find¹ a measurable map $\psi : \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ such that, for $\mathbb{P}_X$ -a.e. $x \in \mathbb{R}^d$, $\psi_x(\cdot)\# \lambda = \nu_x$. Let Z be a random variable with law λ and independent of $(X, \nabla\tilde{U}(X))$.

Note that $\mu = \mathcal{L}(X, \nabla\tilde{U}(X)) = \mathcal{L}(X, \psi_X(Z))$ because, for any $f \in C_b^0(\mathbb{R}^d \times \mathbb{R}^d)$,

$$\begin{aligned} \mathbb{E}[f(X, \psi_X(Z))] &= \int_{\mathbb{R}^d} \int_{Q_1} f(x, \psi_x(z)) \lambda(dz) \mathbb{P}_X(dx) = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f(x, y) (\psi_x \# \lambda)(dy) \mathbb{P}_X(dx) \\ &= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f(x, y) \nu_x(dy) \mathbb{P}_X(dx) = \int_{\mathbb{R}^{2d}} f(x, y) \mu(dx, dy). \end{aligned}$$

So, for any ϵ ,

$$\tilde{U}(X + \epsilon \nabla\tilde{U}(X)) = \tilde{U}(X + \epsilon \psi_X(Z)),$$

from which we infer, taking the derivative with respect to ϵ at $\epsilon = 0$:

$$\mathbb{E}\left[\left|\nabla\tilde{U}(X)\right|^2\right] = \mathbb{E}\left[\nabla\tilde{U}(X) \cdot \psi_X(Z)\right].$$

Note that, as Z is independent of $(X, \nabla\tilde{U}(X))$, we have

$$\mathbb{E}\left[\nabla\tilde{U}(X) \cdot \psi_X(Z)\right] = \mathbb{E}\left[\nabla\tilde{U}(X) \cdot \mathbb{E}[\psi_x(Z)]_{x=X}\right],$$

where, for $\mathbb{P}_X$ -a.e. x ,

$$\mathbb{E}[\psi_x(Z)] = \int_{Q_1} \psi_x(z) \lambda(dz) = \int_{Q_1} y (\psi_x \# \lambda)(dy) = \int_{\mathbb{R}^d} y \nu_x(dy) = \mathbb{E}[\nabla\tilde{U}(X)|X = x].$$

So, by the tower property of the conditional expectation, we have

$$\mathbb{E}\left[\left|\nabla\tilde{U}(X)\right|^2\right] = \mathbb{E}\left[\nabla\tilde{U}(X) \cdot \mathbb{E}[\nabla\tilde{U}(X)|X]\right] = \mathbb{E}\left[\left|\mathbb{E}[\nabla\tilde{U}(X)|X]\right|^2\right].$$

Using again standard properties of the conditional expectation we infer the equality $\nabla\tilde{U}(X) = \mathbb{E}[\nabla\tilde{U}(X)|X]$, which shows the result. $\square$

Proof of Theorem 6.1.11. Let us first assume that U is W -differentiable at some $m \in \mathcal{P}_2$. Then there exists $\xi := D_m U(m, \cdot) \in L_m^2(\mathbb{R}^d)$ such that, for any $m' \in \mathcal{P}_2$ and any transport plan π between m and m' we have

$$\left|U(m') - U(m) - \int_{\mathbb{R}^d \times \mathbb{R}^d} \xi(x) \cdot (y - x) \pi(dx, dy)\right| \leq o\left(\left(\int_{\mathbb{R}^{2d}} |x - y|^2 \pi(dx, dy)\right)^{1/2}\right).$$

¹Warning: here the proof is sloppy and the possibility of a measurable selection should be justified.

Therefore, for any $X \in L^2$ such that $\mathcal{L}(X) = m$, for any $H \in L^2$, if we denote by m' the law of $X + H$ and by π the law of $(X, X + H)$, we have

$$\begin{aligned} \left| \tilde{U}(X + H) - \tilde{U}(X) - \mathbb{E}[\xi(X) \cdot H] \right| &= \left| U(m') - \tilde{U}(m) - \int_{\mathbb{R}^{2d}} \xi(x) \cdot (y - x) \pi(x, y) \right| \\ &\leq o \left(\left(\int_{\mathbb{R}^{2d}} |x - y|^2 \pi(dx, dy) \right)^{1/2} \right) \\ &= o \left(\mathbb{E}[|X - Y|^2]^{1/2} \right). \end{aligned}$$

This shows that U is L-differentiable.

Conversely, let us assume that U is L-differentiable at m . We know from Lemma 6.1.13 that, for any $X \in L^2$ such that $\mathcal{L}(X) = m$, $\tilde{U}$ is differentiable at X and $\nabla \tilde{U}(X) = \xi(X)$ for some Borel measurable map $\xi : \mathbb{R}^d \rightarrow \mathbb{R}^d$. In view of Lemma 6.1.12, the map ξ does not depend on the choice of X . So, for any $\epsilon > 0$, there exists $r > 0$ such that, for any X with $\mathcal{L}(X) = m$ and any $H \in L^2$ with $\|H\| \leq r$, one has

$$\left| \tilde{U}(X + H) - \tilde{U}(X) - \mathbb{E}[\xi(X) \cdot H] \right| \leq \epsilon.$$

Let now $m' \in \mathcal{P}_2$ and π be a transport plan between m and m' such that $\int_{\mathbb{R}^{2d}} |x - y|^2 \pi(dx, dy) \leq r^2$. Let (X, Y) with law π . We set $H = Y - X$ and note that $\|H\|_2 \leq r$. So we have

$$\left| U(m') - \tilde{U}(m) - \int_{\mathbb{R}^{2d}} \xi(x) \cdot (y - x) \pi(x, y) \right| = \left| \tilde{U}(X + H) - \tilde{U}(X) - \mathbb{E}[\xi(X) \cdot H] \right| \leq \epsilon.$$

This proves the W-differentiability of U . $\square$

6.1.4 Higher order derivatives

We say that U is partially C^2 if U is C^1 and if $D_y \delta U / \delta m$ and $D_{yy}^2 \delta U / \delta m$ exist and are continuous and bounded on $\mathcal{P}_2 \times \mathbb{R}^d$.

We say that U is C^2 if $\frac{\delta U}{\delta m}$ is C^1 in m with a continuous and bounded derivative: namely $\frac{\delta^2 U}{\delta m^2} = \frac{\delta}{\delta m} \left(\frac{\delta U}{\delta m} \right) : \mathcal{P}_2 \times \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}$ is continuous in all variables and bounded. We say that U is twice L-differentiable if the map $D_m U$ is L-differentiable with respect to m with a second order derivative $D_{mm}^2 U = D_{mm}^2 U(m, y, y')$ which is continuous and bounded on $\mathcal{P}_2 \times \mathbb{R}^d \times \mathbb{R}^d$ with values in $\mathbb{R}^{d \times d}$. One can check that this second order derivative enjoys standard properties of derivatives, such as the symmetry:

$$D_{mm}^2 U(m, y, y') = D_{mm}^2 U(m, y', y).$$

See [12].

6.1.5 To go further

For a general description of the notion of derivatives and the historical background, we refer to [15], Chap. V. The notion of flat derivative is very natural and has been introduced in several contexts and under various assumptions. We follow here [12].

The initial definition of sub and super differential in the space $\mathcal{P}_2$, introduced in [3], is the following: ξ belongs to $\partial^+ U(m)$ if $\xi \in \text{Tan}_m(\mathcal{P}_2)$ and

$$U(m') \leq U(m) + \inf_{\pi \in \Pi^{opt}(m, m')} \int_{\mathbb{R}^d \times \mathbb{R}^d} \xi(x) \cdot (y - x) \pi(dx, dy) + o(\mathbf{d}_2(m, m')).$$

It is proved in [33] that this definition coincides with the one introduced in Definition 6.1.7.

The notion of L-derivative and the structure of this derivative has been first discussed by Lions in his course at Collège de France. The proof, without an extra continuity condition, of this structure property is due to Gangbo and Tudorascu [33] (see also [1] for simpler arguments, revisited here in a loose way).

6.2 Calculus on the Wasserstein space

In this part we collect several useful tools in relation with the calculus in $\mathcal{P}_2$: optimality conditions, projection over empirical measures and various Itô's formulas.

6.2.1 Optimality conditions

This part is borrowed from [11].

Proposition 6.2.1 (First order conditions). *Assume that the map $U : \mathcal{P}_2 \rightarrow \mathbb{R}$ has a maximum at some measure $\bar{m} \in \mathcal{P}_2$. If U has a linear derivative, then*

$$(6.3) \quad \frac{\delta U}{\delta m}(\bar{m}, y) \leq 0 \quad \forall y \in \mathbb{R}^d \quad \text{and} \quad \frac{\delta U}{\delta m}(\bar{m}, y) = 0 \text{ for } \bar{m}\text{-a.e. } y \in \mathbb{R}^d.$$

In particular, if U is partially C^2 , then

$$(6.4) \quad D_m U(m, y) = 0 \text{ and } D_{yy}^2 D_m U(m, y) \geq 0 \quad \text{for } \bar{m}\text{-a.e. } y \in \mathbb{R}^d$$

A trivial example: Assume that $U(m) = \int_{\mathbb{R}^d} \phi(y) dm(y)$ where $\phi : \mathbb{R}^d \rightarrow \mathbb{R}$ is a smooth map such that $\phi(0) = 0$, $\phi < 0$ outside 0 and $D^2 \phi(0) < 0$. Then U has a maximum at $\hat{m} = \delta_0$ (the Dirac mass at 0). As

$$\frac{\delta U}{\delta m}(m, y) = \phi(y) - \int_{\mathbb{R}^d} \phi(y') dm(y'),$$

we have

$$\frac{\delta U}{\delta m}(\hat{m}, y) = \phi(y), \quad D_m U(\hat{m}, y) = D\phi(y), \quad D_{ym} U(\hat{m}, y) = D^2 \phi(y).$$

In particular, $\frac{\delta U}{\delta m}(\hat{m}, y)$ and $D_m U(\hat{m}, y)$ vanish only on the support of $\hat{m} = \delta_0$ and $D_{ym} U(\delta_0, y)$ is negative on the support of $\hat{m}$.

Proof. For any $y_0 \in \mathbb{R}^d$ and $s \in (0, 1]$, we have, by then optimality of $\bar{m}$,

$$U((1-s)\bar{m} + s\delta_{y_0}) - U(\bar{m}) = \int_0^s \int_{\mathbb{R}^d} \frac{\delta U}{\delta m}((1-t)\bar{m} + t\delta_{y_0}, y)(\delta_{y_0} - \bar{m})(dy) \leq 0.$$

We divide by s and let $s \rightarrow 0^+$. By the continuity of $\delta U / \delta m$ we obtain

$$\int_{\mathbb{R}^d} \frac{\delta U}{\delta m}(\bar{m}, y)(\delta_{y_0} - \bar{m})(dy) \leq 0,$$

which implies that, for any $y_0 \in \mathbb{R}^d$ and because of the convention in (6.1),

$$(6.5) \quad \frac{\delta U}{\delta m}(\bar{m}, y_0) \leq 0.$$

If we integrate this inequality in y_0 against $\bar{m}$, we find an equality (again because of the convention in (6.1)). So

$$(6.6) \quad \frac{\delta U}{\delta m}(\bar{m}, y_0) = 0 \quad \text{for } \bar{m}\text{-a.e. } y_0 \in \mathbb{R}^d.$$

Then (6.5) and (6.6) imply that the map $y \rightarrow \frac{\delta U}{\delta m}(\bar{m}, y)$ has a maximum at $\bar{m}$ -a.e. $y_0 \in \mathbb{R}^d$, and thus, using the standard optimality conditions for a map defined on $\mathbb{R}^d$, we obtain (6.4). $\square$

Proposition 6.2.2 (Second order condition). *Assume that the map $U : \mathcal{P}_2 \rightarrow \mathbb{R}$ has a maximum at some measure $\bar{m} \in \mathcal{P}_2$ and that U is fully C^2 . Then, for any bounded Borel map $\phi : \mathbb{R}^d \rightarrow \mathbb{R}^d$, we have*

$$(6.7) \quad \int_{\mathbb{R}^d \times \mathbb{R}^d} D_{mm}^2 U(\bar{m}, y, y') \phi(y) \cdot \phi(y') \bar{m}(dy) \bar{m}(dy') \leq 0.$$

Proof. We fix a Borel measurable, bounded vector field $\phi : \mathbb{R}^d \rightarrow \mathbb{R}^d$ and note that, for any $s \in \mathbb{R}$ and by optimality of $\bar{m}$,

$$U((id + s\phi)\# \bar{m}) \leq U(\bar{m}).$$

Therefore

$$\frac{d}{ds} \Big|_{s=0} U((id + s\phi)\# \bar{m}) = 0 \text{ and } \frac{d^2}{ds^2} \Big|_{s=0} U((id + s\phi)\# \bar{m}) \leq 0.$$

As

$$\frac{d}{ds} U((id + s\phi)\# \bar{m}) = \int_{\mathbb{R}^d} D_m U((id + s\phi)\# \bar{m}, y) \cdot \phi(y) \bar{m}(dy),$$

we have

$$\frac{d^2}{ds^2} \Big|_{s=0} U((id + s\phi)\# \bar{m}) = \int_{\mathbb{R}^{2d}} D_{mm}^2 U(y, y') \phi(y) \cdot \phi(y') \bar{m}(dy) \bar{m}(dy'),$$

where U and its derivatives are computed at $\bar{m}$. This gives (6.7). $\square$

6.2.2 Projection over finite dimensional spaces

It is often very convenient to translate results known on a map defined on the Wasserstein space to its restriction to the set of empirical measures. The computations here come from [12].

Given $U : \mathcal{P}_2 \rightarrow \mathbb{R}$ and $N \in \mathbb{N}$ be a large integer, we set

$$U^N(x_1, \dots, x_N) = U(m_{\mathbf{x}}^N), \text{ where } \mathbf{x} = (x_1, \dots, x_N), \quad m_{\mathbf{x}}^N = \frac{1}{N} \sum_{n=1}^N \delta_{x_n}.$$

Note that $U^N : (\mathbb{R}^d)^N \rightarrow \mathbb{R}$.

Proposition 6.2.3. *Assume that $U : \mathcal{P}_2 \rightarrow \mathbb{R}$ is L -continuous differentiable with bounded derivative. Then U^N is of class C^1 on $(\mathbb{R}^d)^N$ and*

$$D_{x_i} U^N(\mathbf{x}) = \frac{1}{N} D_m U(m_{\mathbf{x}}^N, x_i) \quad \forall i \in \{1, \dots, N\}.$$

If in addition U is fully C^2 , then

$$D_{x_i x_j} U^N(\mathbf{x}) = \frac{1}{N^2} D_{mm}^2 U(m, x_i, x_j) \quad \forall i \neq j \in \{1, \dots, N\}$$

and

$$D_{x_i x_i} U^N(\mathbf{x}) = \frac{1}{N^2} D_{mm}^2 U(m, x_i, x_i) + \frac{1}{N} D_y D_m U(m_{\mathbf{x}}^N, x_i) \quad \forall i \in \{1, \dots, N\}.$$

Proof. We first assume that the (x_i) are all distinct. Then, for any direction $\mathbf{v} = (v_1, \dots, v_N) \in (\mathbb{R}^d)^N$, we can find a smooth map $\phi : \mathbb{R}^d \rightarrow \mathbb{R}^d$ such that $\phi(x_i) = v_i$ for any i . Note that

$$(Id + h\phi)\# m_{\mathbf{x}}^N = m_{\mathbf{x} + h\mathbf{v}}^N$$

for any $h \in \mathbb{R}$, so that, by Proposition 6.1.3 we have, at $h = 0$,

$$\frac{d}{dh} U^N(\mathbf{x} + h\mathbf{v}) = \frac{d}{dh} U((Id + h\phi)\# m_{\mathbf{x}}^N) = \int_{\mathbb{R}^d} D_m U(m_{\mathbf{x}}^N, y) \cdot \phi(y) m_{\mathbf{x}}^N(dy).$$

So

$$\frac{d}{dh} U^N(\mathbf{x} + h\mathbf{v}) = \frac{1}{N} \sum_{i=1}^N D_m U(m_{\mathbf{x}}^N, x_i) \cdot \phi(x_i) = \frac{1}{N} \sum_{i=1}^N D_m U(m_{\mathbf{x}}^N, x_i) \cdot v_i.$$

This shows that U^N has continuous directional derivatives in the dense open set $\{\mathbf{x} \in (\mathbb{R}^d)^N, x_i \neq x_j \text{ if } i \neq j\}$. Therefore U^N is C^1 and the above formula holds. The second order differentiability can be proved by the same arguments. $\square$

6.2.3 Itô's formula

Given a flow $(m(t))$ of probability measures satisfying a differential equation (typically a continuity equation, a Fokker-Planck equation, etc...) we consider the derivative of a map U along this flow.

First order Itô's formula

We start with a simple continuity equation:

$$\partial_t m + \operatorname{div}(mb) = 0.$$

Here we assume that the continuous drift $b : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ is such that $b(t, \cdot)$ is uniformly (in t) Lipschitz continuous on $\mathbb{R}^d$. Then the Cauchy-Lipschitz Theorem states that there exists a unique solution $(X_t^{t_0, x_0})$ to

$$x'(t) = b(t, x(t)), \quad t \in [0, T], \quad x(t_0) = x_0.$$

We denote by $X_t^{x_0}$ the solution to this equation.

Given an initial measure $m_0 \in \mathcal{P}_2$, the flow of measures $m(t) := X_t^\# m_0$ is a solution in the sense of distribution of

$$\partial_t m + \operatorname{div}(mb) = 0 \quad \text{in } (0, T) \times \mathbb{R}^d, \quad m(0) = m_0 \text{ in } \mathbb{R}^d.$$

We explained in Theorem 3.2.2 that $(m(t))$ is the unique solution to this equation.

Proposition 6.2.4. *Let $m_0 \in \mathcal{P}_2$ and $m(t) = X_t^{t_0, \cdot \#} m_0$. If U is $L - C^1$, then $t \rightarrow U(m(t))$ is of class C^1 and*

$$\frac{d}{dt} U(m(t)) = \int_{\mathbb{R}^d} D_m U(m(t), y) \cdot b(t, y) m(t)(dy).$$

Proof. It is only a small variant of the proof of Proposition 6.1.3 and we omit it. $\square$

Itô's formula for the law of SDEs (or the Fokker-Planck equation)

When U is partially C^2 , Itô's formula holds for the law of a diffusion of the form

$$dX_t = b_t dt + \sigma_t dB_t.$$

The following result is [15, Theorem 5.98].

Theorem 6.2.5. *Assume that where (b_t) and (σ_t) are progressively measurable with values in $\mathbb{R}^d$ and $\mathbb{R}^{n \times d}$ respectively and satisfy, for some $T > 0$,*

$$\mathbb{E} \left[\int_0^T (|b_s|^2 + |\sigma_s|^4) ds \right] < +\infty.$$

Assume in addition that U is partially C^2 with

$$(6.8) \quad \sup_{\mu \in \mathcal{P}_2} \int_{\mathbb{R}^d} |D_y D_m U(\mu, y)|^2 \mu(dy) < +\infty.$$

Then, for any $t \in [0, T]$,

$$(6.9) \quad U(\mathcal{L}(X_t)) = \mathcal{L}(X_0) + \int_0^t \mathbb{E} [D_m U(\mathcal{L}(X_s), X_s) \cdot b_s] ds + \frac{1}{2} \int_0^t \mathbb{E} [\operatorname{Tr} (a_s D_y D_m U(\mathcal{L}(X_s), X_s))] ds,$$

where $a_s = \sigma_s \sigma_s^*$.

Remark: if $b_s = b(s, X_s)$ and $\sigma_s = \sigma(s, X_s)$, the above expression becomes, if we set $m(t) =: \mathcal{L}(X_t)$,

$$\begin{aligned} U(m(t)) &= U(m_0) + \int_0^t \int_{\mathbb{R}^d} D_m U(m(s), y) \cdot b(s, y) m(s, dy) ds \\ &\quad + \frac{1}{2} \int_0^t \int_{\mathbb{R}^d} \operatorname{Tr} (a(s, y) D_y D_m U(m(s), y)) m(s, dy) ds. \end{aligned}$$

Sketch of proof in a particular case. For simplicity, we do the proof only in the case $a = \sqrt{2}Id$ and $b = b(s, x)$ is a smooth and globally Lipschitz continuous vector field. In this case it is known that the law $m(t)$ of X_t has smooth density which satisfies

$$\partial_t m - \Delta m + \operatorname{div}(mb) = 0.$$

Therefore

$$\begin{aligned} \frac{d}{dt}U(m(t)) &= \int_{\mathbb{R}^d} \frac{\delta U}{\delta m}(m(t), y) \partial_t m(t, y) dy \\ &= \int_{\mathbb{R}^d} \frac{\delta U}{\delta m}(m(t), y) (\Delta m(t, y) + \operatorname{div}(m(t, y)b(t, y))) dy \\ &= \int_{\mathbb{R}^d} D_y D_m U(m(t), y) m(t, y) dy - \int_{\mathbb{R}^d} D_m U(m(t), y) \cdot b(t, y) m(t, y) dy. \end{aligned}$$

□

An alternative proof is to use the law of large numbers as described in Section 3.1.5. Let (X_t^i) be i.i.d. copies of the process X and let m_t^N be the associated empirical measure:

$$m_t^N := \frac{1}{N} \sum_{i=1}^N \delta_{X_t^i}.$$

We know that m_t^N converges to the law of X_t in $\mathcal{P}_2$ (Theorem 3.1.9). On the other hand, if we consider the finite dimensional projection U^N of U :

$$U^N(\mathbf{x}) := U(m_{\mathbf{x}}^N), \forall \mathbf{x} = (x_1, \dots, x_N) \in (\mathbb{R}^d)^N,$$

then we can use Itô's calculus and the expressions of the derivatives of U^N in Proposition 6.2.3 to obtain the result: see [15], Chap. 5 for the details.

Itô's formula for the conditional law

A similar Itô's formula holds for the *conditional law* of an Itô's process of the form,

$$dX_t = b_t dt + \sigma_t^0 dB_t + \sigma_t^1 dW_t,$$

where B and W are independent d -dimensional Brownian motions living on different probability spaces $(\Omega^0, \mathcal{F}^0, \mathbb{P}^0)$ and $(\Omega^1, \mathcal{F}^1, \mathbb{P}^1)$ and where b , σ^0 and σ^1 are progressively measurable with respect to the filtration generated by W and B , with

$$\mathbb{E} \left[\int_0^T (|b_s|^2 + |\sigma_s^0|^4 + |\sigma_s^1|^4) ds \right] < +\infty.$$

We assume that U is globally C^2 with

$$\sup_{\mu \in \mathcal{P}_2} \int_{\mathbb{R}^d} |D_m U(\mu, x)|^2 \mu(dx) + \int_{\mathbb{R}^d} |D_x D_m U(\mu, x)|^2 \mu(dx) + \int_{\mathbb{R}^d \times \mathbb{R}^d} |D_{\mu\mu}^2 U(\mu, x, y)|^2 \mu(dx) \mu(dy) < +\infty.$$

Then, letting $\mu_t(\omega^1) = [X_t|W](\omega^1)$, we have, $\mathbb{P}^1$ -a.s.,

(6.10)

$$\begin{aligned} U(\mu_t) &= U(\mu_0) + \int_0^t \mathbb{E}^0 [D_m U(\mu_s, X_s) \cdot b_s] ds + \int_0^t \mathbb{E}^0 [(\sigma_s^1)^* D_m U(\mu_s, X_s)] \cdot dW_s \\ &\quad + \frac{1}{2} \int_0^t \mathbb{E}^0 [\operatorname{Tr}(a_s D_y D_m U(\mu_s, X_s))] ds + \frac{1}{2} \int_0^t \mathbb{E}^0 \tilde{\mathbb{E}}^0 \left[\operatorname{Tr} \left(D_{\mu\mu}^2 U(\mu_s, X_s, \tilde{X}_s) \sigma_s^1 (\tilde{\sigma}_s^1)^* \right) \right] ds \end{aligned}$$

where $\tilde{X}$ and $\tilde{\sigma}^1$ are independent copies of X and σ^1 is defined on the space $(\tilde{\Omega}^0 \times \tilde{\Omega}^1, \tilde{\mathbb{P}}^0 \otimes \tilde{\mathbb{P}}^1)$, while $a_s := (\sigma_s^0 (\sigma_s^0)^* + \sigma_s^1 (\sigma_s^1)^*)$. See [15, Theorem 11.13].

Chapter 7

The master equation

In this chapter we consider a mean field game problem and investigate in what extend the optimal feedback for the players depends on the time, the current position of the player *and the current distribution* of the other players. Compared to the standard MFG system (where the action of a player depends on time and position only), having such a feedback would yield a more robust strategy: it would allow to take into account errors in the computation of the mean field.

This question can be answered by the “master equation”, which subsume in a single function the two unknowns of the MFG system. The price to pay, however, is the fact that this equation is in infinite dimension: Actually it lives in the space of measures. Let us also point out that the master equation allows to handle MFG problems with common noise; it is also the good object to understand the limit, as the number of players tends to infinity, of Nash equilibria in differential games. Unfortunately, these two last points largely exceed the scope of these notes and won’t be discussed.

7.1 The first order Master equation

To fix the ideas (and the notations), we consider here the MFG system of Section 5.2. We consider the second order MFG system with a quadratic Hamiltonian:

$$(7.1) \quad \begin{cases} (i) & -\partial_t u - \Delta u + \frac{1}{2}|Du|^2 = F(x, m) & \text{in } (0, T) \times \mathbb{R}^d \\ (ii) & \partial_t m - \Delta m - \operatorname{div}(m Du) = 0 & \text{in } (0, T) \times \mathbb{R}^d \\ (iii) & m(0) = m_0, u(x, T) = G(x, m(T)) & \text{in } \mathbb{R}^d \end{cases}$$

Let us recall that this system corresponds to a problem in which each agent has a dynamic of the form

$$dX_t = \alpha_t dt + \sqrt{2}dB_t$$

(where (α_t) is the control with values in $\mathbb{R}^d$ and B a Brownian motion) and a cost of the form

$$J(\alpha) = \mathbb{E} \left[\int_0^T \frac{1}{2} |\alpha_t|^2 + F(X_t, m(t)) dt + G(X_T, m(T)) \right].$$

As we explain below, the master equation associated with the above MFG problem is the following equation, where $U : [0, T] \times \mathbb{R}^d \times \mathcal{P}_2 \rightarrow \mathbb{R}$ is the unknown:

$$(7.2) \quad \begin{cases} -\partial_t U(t, x, m) - \Delta_x U(t, x, m) + \frac{1}{2}|D_x U(t, x, m)|^2 - \int_{\mathbb{R}^d} \operatorname{div}_y D_m U(t, x, m, y) m(dy) \\ \quad + \int_{\mathbb{R}^d} D_m U(t, x, m, y) \cdot D_x U(t, y, m) m(dy) = F(x, m) \\ \text{in } (0, T) \times \mathbb{R}^d \times \mathcal{P}_2 \\ U(T, x, m) = G(x, m) \quad \text{in } \mathbb{R}^d \times \mathcal{P}_2 \end{cases}$$

Note that this equation consists in two parts: the part involving only local terms (local derivatives), which are exactly the ones which appear in the Hamilton-Jacobi equation of the MFG system; and the part with nonlocal terms (the two integrals) which involve derivatives with respect to the measure. This later part corresponds to the Kolmogorov equation in the MFG system.

By a (classical) solution we mean a map U such that the function and all its derivatives involved in the equation exist, are Lipschitz continuous and bounded. Note for later use that these condition implies that $D_x U$ is Lipschitz continuous in (x, m) and bounded.

Proposition 7.1.1. *Assume that U is a classical solution to the master equation (7.2). Let $m_0 \in \mathcal{P}_2$.*

(i) *there exists a (weak) solution to the McKean-Vlasov equation*

$$\partial_t m - \Delta m - \operatorname{div}(m D_x U(t, x, m(t))) = 0 \text{ in } (0, T) \times \mathbb{R}^d, \quad m(0) = m_0,$$

(ii) *if we set $u(t, x) = U(t, x, m(t))$, then the pair (u, m) solves the MFG system (7.1).*

Proof. Existence of a solution to the McKean-Vlasov equation can be achieved through the SDE

$$dX_t = X_0 - \int_0^t D_x U(t, X_t, [X_t]) dt + \sqrt{2} dB_t,$$

where $[X_0] = m_0$. As the map $D_x U = D_x U(t, x, m)$ is bounded and Lipschitz continuous in (x, m) , this solution exists and is unique. It is then easy to check that $m(t) = [X_t]$ is a weak solution to the equation. This proves (i).

To prove (ii), we use Itô's formula in Proposition ??: For a fixed $x \in \mathbb{R}^d$, we have

$$\begin{aligned} U(T, x, [X_T]) &= U(t, x, [X_t]) + \int_t^T \mathbb{E} \left[\partial_t U(s, x, [X_s]) - D_m U(s, x, [X_s], X_s) \cdot D_x U(t, X_s, [X_t]) \right. \\ &\quad \left. + \operatorname{Tr}(D_y D_m U(s, x, [X_s], X_s)) \right] ds \\ &= U(t, x, m(t)) + \int_t^T \partial_t U(s, x, m(s)) \\ &\quad - \int_0^t \int_{\mathbb{R}^d} D_m U(s, x, m(s), y) \cdot D_x U(t, y, m(s)) m(s, dy) ds \\ &\quad + \int_0^t \int_{\mathbb{R}^d} \operatorname{div}_y D_m U(s, x, m(s), y) m(s, dy) ds. \end{aligned}$$

So using the equation satisfied by U , we obtain:

$$G(x, [X_T]) = U(t, x, m(t)) + \int_t^T -\Delta U(s, x, m(s)) + \frac{1}{2} |D_x U(s, x, m(s))|^2 - F(x, m(s)) ds.$$

Recalling the definition of u and m , we conclude that the pair (u, m) solves the MFG system. $\square$

Let us now state an existence result for the solution to the master equation:

Theorem 7.1.2. *Assume that F and G are monotone and have smooth derivatives in space and measure (up to order 3 in space and to order 2 in measure). Then there exists a solution to the master equation.*

Rough ideas of proof. In view of the discussion of Proposition 7.1.1, it is natural to state, for any $(t_0, x, m_0) \in [0, T] \times \mathbb{R}^d \times \mathcal{P}_2$,

$$U(t_0, x, m_0) = u(t_0, x)$$

where (u, m) is the solution to the MFG system (7.1) on the time interval $[t_0, T]$ and with the initial condition $m(t_0) = m_0$. Note that, if (u, m) is such a solution, then, calling $(\tilde{u}, \tilde{m})$ the restriction of (u, m)

to the time interval $[t_0 + h, T]$ (for $h > 0$ small) gives a solution of MFG (7.1) on $[t_0 + h, T]$ with initial condition $\tilde{m}(t_0 + h) = m(t_0 + h)$. This implies that $U(t_0 + h, x, m(t_0 + h)) = \tilde{u}(t_0 + h, x) = u(t_0 + h, x)$. Assuming that U is smooth enough and taking the derivative of this equality at $h = 0$ yields:

$$\begin{aligned}\partial_t u(t_0, x) &= \partial_t U(t_0, x, m_0) + \int_{\mathbb{R}^d} \frac{\delta U}{\delta m}(t_0, x, m_0, y) \partial_t m(t_0, y) dy \\ &= \partial_t U(t_0, x, m_0) + \int_{\mathbb{R}^d} \frac{\delta U}{\delta m}(t_0, x, m_0, y) (\Delta m(t_0, y) + \operatorname{div}(m(t_0, y) Du(t_0, y))) dy \\ &= \partial_t U(t_0, x, m_0) + \int_{\mathbb{R}^d} \operatorname{div}_y D_m U(t_0, x, m_0, y) m_0(y) dy - \int_{\mathbb{R}^d} D_m U(t_0, x, m_0, y) \cdot Du(t_0, y) m_0(y) dy,\end{aligned}$$

where we used the equation of m in the second line and integrations by part in the last one. Note that $D_x U = Du$ and $D_{xx}^2 U = D^2 u$. So, by the equation satisfied by u and the above equality we obtain:

$$\begin{aligned}\partial_t u(t_0, x) &= -\Delta U(t_0, x, m_0) + \frac{1}{2} |D_x U(t_0, x, m_0)|^2 - F(x, m_0) \\ &= \partial_t U(t_0, x, m_0) + \int_{\mathbb{R}^d} \operatorname{div}_y D_m U(t_0, x, m_0, y) m_0(y) dy - \int_{\mathbb{R}^d} D_m U(t_0, x, m_0, y) \cdot Du(t_0, y) m_0(y) dy.\end{aligned}$$

This shows that U is a solution to the master equation (7.2).

The (relatively) difficult part of the proof, that we will not present here, is the fact that the map U is smooth enough to justify the above computation. The interested reader can find the argument (for a probability approach) in [18] and [15] and in [10] for a PDE approach. $\square$

7.2 An elementary problem with common noise

In many applications, the system is perturbed by randomness which affect all agents simultaneously. We call this a “common noise”. This is the case for instance in the simple example of optimal trading of Subsection 5.1.2: Indeed, in this example, the fluctuation of the price of the tradable instrument affects all the traders. It turns out that, by the specific form of the cost, this common noise plays no role in the analysis. However, this is more an exception than a rule. Another very interesting example is the case of a crowd of small agents facing a “major agent”: if the dynamics of the major agent is random, this randomness might affect all the small agents.

We consider here a problem in which the agents face a common noise which, in this elementary example, is a random variable Z on which the coupling cost F depends: $F = F(x, m, Z)$. The exact value of Z is revealed to the agents at time $T/2$ (to fix the ideas).

As before the agents directly control their drift: Let us recall that this system corresponds to a problem in which each agent has a dynamic of the form

$$dX_t = \alpha_t dt + \sqrt{2} dB_t$$

(where (α_t) is the control with values in $\mathbb{R}^d$ and B a Brownian motion). In contrast to the previous discussions, the control α_t is now adapted to the filtration generated by B and to the noise Z when $t \geq T/2$. The cost is now of the form

$$J(\alpha) = \mathbb{E} \left[\int_0^T \frac{1}{2} |\alpha_t|^2 + F(X_t, m(t), W) dt + G(X_T, m(T)) \right].$$

As all the agents will probably choose their optimal control in function of the realization of Z (of course after time $T/2$), one expect the distribution of players to be random after $T/2$ and to depend on the noise Z .

On the time interval $[T/2, T]$, the agents have to solve a classical control problem (which depends on Z and on $(m(t))$):

$$u(t, x) := \inf_{\alpha} \mathbb{E} \left[\int_t^T \frac{1}{2} |\alpha_t|^2 + F(X_t, m(t), W) dt + G(X_T, m(T)) \mid Z \right]$$

which depends on the realization of Z and solves the HJ equation (with random coefficients):

$$\begin{cases} -\partial_t u - \Delta u + \frac{1}{2}|Du|^2 = F(x, m(t), Z) & \text{in } (T/2, T) \times \mathbb{R}^d \\ u(T, x, Z) = G(x, m(T)) & \text{in } \mathbb{R}^d. \end{cases}$$

On the other hand, on the time interval $[0, T/2)$, the agent has no information on Z and, by dynamic programming, one has

$$u(t, x) := \inf_{\alpha} \mathbb{E} \left[\int_t^{T/2} \frac{1}{2} |\alpha_t|^2 + \bar{F}(X_t, m(t)) dt + u(T/2^+, X_{T/2}) \right],$$

where $\bar{F}(x, m) = \mathbb{E}[F(x, m, Z)]$ (recall that $m(t)$ is deterministic on $[0, T/2]$). Thus, on the time interval $[0, T/2]$, u solves

$$\begin{cases} -\partial_t u - \Delta u + \frac{1}{2}|Du|^2 = \bar{F}(x, m(t)) & \text{in } (0, T/2) \times \mathbb{R}^d \\ u(T/2^-, x) = \mathbb{E}[u(T/2^+, x)] & \text{in } \mathbb{R}^d \end{cases}$$

As for the associated Kolmogorov equation, on the time interval $[0, T/2]$ (where the optimal feedback $-Du$ is purely deterministic) we have as usual:

$$\partial_t m - \Delta m - \operatorname{div}(m Du(t, x)) = 0 \text{ in } (0, T/2) \times \mathbb{R}^d, \quad m(0) = m_0.$$

while on the time interval $[T/2, T]$, m becomes random (as the control $-Du$) and solves

$$\partial_t m - \Delta m - \operatorname{div}(m Du(t, x, Z)) = 0 \text{ in } (T/2) \times \mathbb{R}^d, \quad m(T/2^-) = m(T/2^+).$$

Note the relation: $m(T/2^-) = m(T/2^+)$, which means that the dynamics of the crowd is continuous in time.

Let us point out some remarkable features of the problem. First the pairs (u, m) are no longer deterministic, and are adapted to the filtration generated by the common noise (here this filtration is trivial up to time $T/2$ and is the σ -algebra generated by Z after $T/2$). Second the map u is discontinuous: this is due to the shock of information at time $T/2$.

7.3 Comment

Most formal properties of the Master equation have been introduced and presented by Lions in [48]. Lions discusses also the existence and uniqueness of the solution in several framework. The most general reference on this subject is the monograph of Carmona and Delarue [15] (see also the interesting but mostly formal approach in the monograph by Bensoussan, Frehse and Yam [5]). One of the main interests of the master equation is that it allows to understand in a rigorous way the limit of Nash equilibria in N -player differential games as $N \rightarrow +\infty$: see [10] and [15].

Chapter 8

The Hamilton-Jacobi equation

In this chapter we come back to the mean field control problem and discuss the associated Hamilton-Jacobi equation. We will present a verification principle and some ideas on the notion viscosity solutions. We will also explain, at least in simple settings, that the mean field control indeed describes the limit of optimal control problems of a large number of devices.

8.1 The value function and its associated HJB equation

8.1.1 Dynamic programming

In this part we consider the value function $\mathcal{U}$ associated with our optimal control problem: for any $(t_0, m_0) \in [0, T] \times \mathcal{P}_2$,

$$\mathcal{U}(t_0, m_0) = \inf_{(m, \alpha)} \int_{t_0}^T \int_{\mathbb{R}^d} \frac{1}{2} |\alpha(t, x)|^2 m(t, dx) dt + \int_{t_0}^T \mathcal{F}(m(t)) dt,$$

where the infimum is taken over the pairs (m, α) with $m \in \mathcal{C}^0([t_0, T], \mathcal{P}_1)$ and $\alpha \in L_m^2$ such that

$$\partial_t m - \Delta m + \operatorname{div}(m\alpha) = 0, \quad m(t_0) = m_0.$$

Throughout this part, we assume that $\mathcal{F} : \mathcal{P}_1 \rightarrow \mathbb{R}$ is continuous and bounded, and possesses a Lipschitz continuous and bounded derivative $F : \mathbb{R}^d \times \mathcal{P}_1 \rightarrow \mathbb{R}$, in the sense that the following holds

$$(8.1) \quad \mathcal{F}(m') - \mathcal{F}(m) = \int_0^1 \int_{\mathbb{R}^d} F(x, (1-t)m + tm')(m - m')(dx) \quad \forall m, m' \in \mathcal{P}_1(\mathbb{R}^d).$$

Proposition 8.1.1. *The map $m \rightarrow \mathcal{U}(t, m)$ is Lipschitz continuous, uniformly in $t \in [0, T]$.*

Proof. Let $m_0^1, m_0^2 \in \mathcal{P}_2$ be two initial conditions and let (m^1, α^1) be optimal for $\mathcal{U}(t_0, m_0^1)$. By Theorem 4.1.1, we know that $\alpha^1 = -Du$, where (u, m^1) solves the system (4.4). Recall that u is bounded in $C^{1,2}$, with a bound independent of m_0^1 . Thus α^1 is Lipschitz in space, uniformly in time and in the initial measure m_0^1 . Let now m^2 be the solution to

$$\partial_t m^2 - \Delta m^2 - \operatorname{div}(m^2 Du) = 0, \quad m^2(0) = m_0^2.$$

Then $(m^1(t))$ and $(m^2(t))$ are respectively the law of the processes (X_t^1) and (X_t^2) solution to (for $k = 1, 2$)

$$dX_t^k = \alpha^1(t, X_t^k) dt + \sqrt{2} dB_t, \quad X_{t_0}^k = Z^k,$$

where Z^k have law m_0^k and are independent of the Brownian motion B . We also assume that $\mathbf{d}_2(m_0^1, m_0^2) = \|Z^1 - Z^2\|_2$. As α^1 is Lipschitz in space, uniformly in time, there is a constant $C > 0$ such that

$$\mathbb{E} \left[\sup_{t \in [t_0, T]} |X_t^1 - X_t^2|^2 \right] \leq C \mathbb{E} [|Z^1 - Z^2|^2] \leq C \mathbf{d}_2^2(m_0^1, m_0^2).$$

Hence

$$\sup_{t \in [t_0, T]} \mathbf{d}_2(m^1(t), m^2(t)) \leq C \mathbf{d}_2^2(m_0^1, m_0^2).$$

Thus

$$\begin{aligned} J(m^2, \alpha^1) &\leq \int_{t_0}^T \int_{\mathbb{R}^d} \frac{1}{2} |\alpha^1(t, x)|^2 m^2(t, dx) dt + \int_{t_0}^T \mathcal{F}(m^2(t)) dt \\ &\leq J(m^1, \alpha^1) + \int_{t_0}^T (\text{Lip}(\frac{1}{2} |\alpha^1(t, \cdot)|^2) + \text{Lip}(\mathcal{F})) \mathbf{d}_2(m^1(t), m^2(t)) dt \\ &\leq \mathcal{U}(t_0, m_0^1) + C \mathbf{d}_2^2(m_0^1, m_0^2). \end{aligned}$$

We can therefore conclude that $\mathcal{U}(t_0, m_0^2) \leq \mathcal{U}(t_0, m_0^1)$. $\square$

Theorem 8.1.2 (Dynamic programming). *For any $0 \leq t_0 \leq t_1 \leq T$, we have*

$$\mathcal{U}(t_0, m_0) = \inf_{(m, \alpha)} \int_{t_0}^{t_1} \int_{\mathbb{R}^d} \frac{1}{2} |\alpha(t, x)|^2 m(t, dx) dt + \int_{t_0}^{t_1} \mathcal{F}(m(t)) dt + \mathcal{U}(t_1, m(t_1)),$$

where the infimum is taken over the pairs (m, α) with $m \in \mathcal{C}^0([t_1, t_1], \mathcal{P}_1)$ and $\alpha \in L_m^2$ such that

$$\partial_t m - \Delta m + \text{div}(m\alpha) = 0, \quad m(t_0) = m_0.$$

Proof. The proof is completely standard, exactly as for deterministic control problems. $\square$

8.1.2 Verification Theorem

From the dynamic programming, one can derive that the value function satisfies, at least formally, assuming the value function to be smooth, that it satisfies an Hamilton-Jacobi (HJ equation). Our aim is to show the converse: assuming that there exists a classical solution to the HJ equation, show that it is the value function and show the convergence of the optimal control of large particle systems.

Theorem 8.1.3. *Assume that $\mathcal{V} : [0, T] \times \mathcal{P}_2 \rightarrow \mathbb{R}$ is a classical solution to*

$$(8.2) \quad \begin{cases} -\partial_t \mathcal{V}(t, m) + \frac{1}{2} \int_{\mathbb{R}^d} |D_m \mathcal{V}(t, m, y)|^2 m(dy) - \int_{\mathbb{R}^d} \text{div}_y D_m \mathcal{V}(t, m, y) m(dy) = \mathcal{F}(m) & \text{in } (0, T) \times \mathcal{P}_2, \\ \mathcal{V}(T, m) = 0 & \text{in } \mathcal{P}_2. \end{cases}$$

Then $\mathcal{V} \leq \mathcal{U}$.

If, in addition, $(m, x) \rightarrow D_m \mathcal{V}(t, m, x)$ is Lipschitz continuous in (m, y) uniformly in t , then $\mathcal{V} = \mathcal{U}$ and, for any initial condition $(t_0, m_0) \in [0, T] \times \mathcal{P}_2$, the map $(t, x, m) \rightarrow -D_m \mathcal{V}(t, m(t), x)$ is an optimal feedback: namely, if $\bar{m}$ is the unique solution to the McKean-Vlasov equation

$$\begin{cases} \partial_t \bar{m} - \Delta \bar{m} - \text{div}(D_m \mathcal{V}(t, \bar{m}(t), y) \bar{m}) = 0 & \text{in } (t_0, T) \times \mathbb{R}^d, \\ \bar{m}(t_0, dx) = m_0, \end{cases}$$

and if we set $\bar{\alpha}(t, x) := -D_m \mathcal{V}(t, m(t), x)$, then the pair $(\bar{m}, \bar{\alpha})$ is optimal for $\mathcal{U}(t_0, m_0)$.

Proof. Fix (m, α) with $m \in \mathcal{C}^0([t_0, T], \mathcal{P}_1)$ and $\alpha \in L_m^2$ such that

$$\partial_t m - \Delta m + \text{div}(m\alpha) = 0, \quad m(t_0) = m_0.$$

According to Proposition 6.2.5 and using the equation satisfied by $\mathcal{V}$, we have

$$\begin{aligned} 0 = \mathcal{V}(T, m(T)) &= \mathcal{V}(t_0, m_0) + \int_{t_0}^T \int_{\mathbb{R}^d} (\partial_t \mathcal{V}(t, m(t)) + D_m \mathcal{V}(t, m(t), y) \cdot \alpha(t, y) + \text{div}_y D_m \mathcal{V}(t, m(t), y)) m(t, dy) dy \\ &= \mathcal{V}(t_0, m_0) + \int_{t_0}^T \int_{\mathbb{R}^d} (\frac{1}{2} \int_{\mathbb{R}^d} |D_m \mathcal{V}(t, m(t), y)|^2 m(dy) + D_m \mathcal{V}(t, m(t), y) \cdot \alpha(t, y) - \mathcal{F}(m(t))) m(t, dy) dy \end{aligned}$$

As

$$\frac{1}{2}|D_m \mathcal{V}(t, m(t), y)|^2 m(dy) + D_m \mathcal{V}(t, m(t), y) \cdot \alpha(t, y) \geq -\frac{1}{2}|\alpha(t, x)|^2,$$

with an equality is $\alpha(t, x) = -D_m \mathcal{V}(t, m(t), y)$, we obtain

$$0 \geq \mathcal{V}(t_0, m_0) - \int_{t_0}^T \int_{\mathbb{R}^d} \frac{1}{2} |\alpha(t, x)|^2 m(t, dx) dt - \int_{t_0}^T \mathcal{F}(m(t)) dt,$$

with an equality only if $\alpha(t, x) = -D_m \mathcal{V}(t, m(t), y)$ m -a.e.. Thus

$$\mathcal{V}(t_0, m_0) \leq \int_{t_0}^T \int_{\mathbb{R}^d} \frac{1}{2} |\alpha(t, x)|^2 m(t, dx) dt + \int_{t_0}^T \mathcal{F}(m(t)) dt.$$

As α is arbitrary, we obtain that $\mathcal{V}(t_0, m_0) \leq \mathcal{U}(t_0, m_0)$. If in addition, we choose $\alpha = \bar{\alpha}$, then the same computation as above proves that

$$\mathcal{V}(t_0, m_0) = \int_{t_0}^T \int_{\mathbb{R}^d} \frac{1}{2} |\bar{\alpha}(t, x)|^2 \bar{m}(t, dx) dt + \int_{t_0}^T \mathcal{F}(\bar{m}(t)) dt \geq \mathcal{U}(t_0, m_0),$$

so that there is an equality everywhere: $\mathcal{V}(t_0, m_0) = \mathcal{U}(t_0, m_0)$ and the pair $(\bar{m}, \bar{\alpha})$ is optimal for $\mathcal{U}(t_0, m_0)$.

□

8.1.3 Mean field limit in the smooth case

We consider the optimal control of N particle systems, in which, given an initial configuration $(t_0, \mathbf{x}^N = (t_0, x^{N,1}, \dots, x^{N,N}) \in [0, T] \times (\mathbb{R}^d)^N$,

$$\mathcal{V}^N(t_0, \mathbf{x}^N) = \inf_{(\alpha^{N,i})} \mathbb{E} \left[\int_{t_0}^T \left(\frac{1}{2} \left(\sum_{i=1}^N |\alpha_s^{N,i}|^2 \right) + \mathcal{F}(m_{\mathbf{x}_s^N}^N) \right) ds \right],$$

where $\mathbf{X}_t^N = (X_t^{N,1}, \dots, X_t^{N,N})$ with

$$X_t^{N,i} = x_0^{N,i} + \int_{t_0}^t \alpha_s^{N,i} ds + \sqrt{2}(B_t^i - B_{t_0}^i).$$

Recall that $\mathcal{V}^N$ is a classical solution to the HJ equation

$$(8.3) \quad \begin{cases} -\partial_t \mathcal{V}^N(t, \mathbf{x}) - \sum_{i=1}^N \Delta_{x^i} \mathcal{V}^N(t, \mathbf{x}) + \frac{N}{2} \sum_{i=1}^N |\Delta_{x^i} \mathcal{V}^N(t, \mathbf{x})|^2 = \mathcal{F}(m_{\mathbf{x}}^N) & \text{in } (0, T) \times (\mathbb{R}^d)^N, \\ \mathcal{V}^N(T, \mathbf{x}) = 0 & \text{in } (\mathbb{R}^d)^N. \end{cases}$$

Theorem 8.1.4. *Assume that $\mathcal{U}$ is smooth and thus a classical solution to (8.2). Then*

$$|\mathcal{U}(t_0, m_{\mathbf{x}^N}^N) - \mathcal{V}^N(t_0, \mathbf{x}^N)| \leq CN^{-1}.$$

Proof. Set $\mathcal{U}^N(t, \mathbf{x}^N) = \mathcal{U}(t, m_{\mathbf{x}^N}^N)$. Then recalling Proposition 6.2.3, we have

$$\partial_t \mathcal{U}^N(t, \mathbf{x}^N) = \partial_t \mathcal{U}(t, m_{\mathbf{x}^N}^N), \quad D_{x^i} \mathcal{U}^N(t, \mathbf{x}^N) = \frac{1}{N} D_m \mathcal{U}(t, m_{\mathbf{x}^N}^N, x^i),$$

$$D_{x^i x^i}^2 \mathcal{U}^N(t, \mathbf{x}^N) = \frac{1}{N^2} D_{mm}^2 \mathcal{U}(t, m_{\mathbf{x}^N}^N, x^i, x^i) + \frac{1}{N} D_y D_n \mathcal{U}(t, m_{\mathbf{x}^N}^N, x^i).$$

Thus

$$\begin{aligned}
& -\partial_t \mathcal{U}^N(t, \mathbf{x}) - \sum_{i=1}^N \Delta_{x^i} \mathcal{U}^N(t, \mathbf{x}) + \frac{1}{2} \sum_{i=1}^N |\Delta_{x^i} \mathcal{U}^N(t, \mathbf{x})|^2 \\
& = -\partial_t \mathcal{U}^N(t, \mathbf{x}) - \frac{1}{N} \sum_{i=1}^N \operatorname{div}_y D_n \mathcal{U}(t, m_{\mathbf{x}^N}^N, x^i) - \frac{1}{N^2} \sum_{i=1}^N \operatorname{Tr} D_{mm}^2 \mathcal{U}(t, m_{\mathbf{x}^N}^N, x^i, x^i) \\
& \quad + \frac{1}{2N} \sum_{i=1}^N |D_m \mathcal{U}(t, m_{\mathbf{x}^N}^N, x^i)|^2 \\
& = -\partial_t \mathcal{V}(t, m) + \frac{1}{2} \int_{\mathbb{R}^d} |D_m \mathcal{V}(t, m, y)|^2 m(dy) - \int_{\mathbb{R}^d} \operatorname{div}_y D_m \mathcal{V}(t, m, y) m(dy) + O(1/N) \\
& = \mathcal{F}(m_{\mathbf{x}}^N) + O(1/N),
\end{aligned}$$

where

$$O(1/N) = -\frac{1}{N^2} \sum_{i=1}^N \operatorname{Tr} D_{mm}^2 \mathcal{U}(t, m_{\mathbf{x}^N}^N, x^i, x^i)$$

and where we have used the equation satisfied by $\mathcal{U}$. Hence $\mathcal{U}^N$ solves (8.3) with an error term $O(1/N)$. We obtain the result by comparison. $\square$

8.2 Problems with no diffusion

We study here problems without diffusion. The limit problem takes the form

$$(8.4) \quad \mathcal{U}(t_0, m_0) = \inf_{(m, \alpha)} J(t_0, m_0, m, \alpha) \quad \text{where } J(t_0, m_0, m, \alpha) = \int_{t_0}^T \int_{\mathbb{R}^d} \frac{1}{2} |\alpha|^2 m + \int_{t_0}^T \mathcal{F}(m(t))$$

and where the pair (m, α) solves

$$\partial_t m + \operatorname{div}(m\alpha) = 0, \quad m(t_0) = m_0.$$

As for the N -particle system, the value function is given by

$$(8.5) \quad \mathcal{V}^N(t_0, \mathbf{x}_0) = \inf_{(\alpha^i)} J^N(t_0, \mathbf{x}_0, (\alpha^i)) \quad \text{where } J^N(t_0, \mathbf{x}_0, (\alpha^i)) = \frac{1}{2N} \sum_{i=1}^N \int_{t_0}^T |\alpha^i(t)|^2 dt + \int_{t_0}^T \mathcal{F}(m_{\mathbf{x}(t)}^N) dt,$$

and where

$$\dot{x}^i(t) = \alpha^i(t), \quad x^i(t_0) = x_0^i, \quad i = 1, \dots, N$$

and we use the notation $\mathbf{x}_0 = (x_0^1, \dots, x_0^N)$, $\mathbf{x}(t) = (x^1(t), \dots, x^N(t))$ and $m_{\mathbf{x}(t)}^N = \frac{1}{N} \sum_{i=1}^N \delta_{x^i(t)}$.

Our goal is to characterize $\mathcal{U}$ as a viscosity solution to a Hamilton-Jacobi (HJ) equation and to show that $\mathcal{V}^N$ converges in a suitable sense to $\mathcal{U}$. On these topics, one can refer to Lions's lectures and to the papers by Gangbo-Mayorga-Swiecz [32] and by Cavagnari-Lisini-Orrieri-Savaré [16].

8.2.1 Mean field limit: a direct proof

Theorem 8.2.1. *Assume that $\mathcal{F} : \mathcal{P}_1 \rightarrow \mathbb{R}$ is Lipschitz continuous and bounded. Then, for any $(t_0, m_0) \in [0, T] \times \mathcal{P}_2$, and for any sequence $(\mathbf{x}_0^N)$ with $\mathbf{x}_0^N \in (\mathbb{R}^d)^N$ such that $m_{\mathbf{x}_0^N}^N \rightarrow m_0$ in $\mathcal{P}_2$,*

$$\lim_{N \rightarrow \infty} \mathcal{V}^N(t_0, \mathbf{x}_0^N) = \mathcal{U}(t_0, m_0).$$

The proof requires a few preliminary remarks.

Lemma 8.2.2. *If $\mathcal{F} : \mathcal{P}_1 \rightarrow \mathbb{R}$ is Lipschitz continuous and bounded, then the $\mathcal{V}^N$ are bounded and there is a constant C such that*

$$|\mathcal{V}^N(t, \mathbf{x}) - \mathcal{V}^N(s, \mathbf{y})| \leq C(|t - s| + \mathbf{d}_1(m_{\mathbf{x}}^N, m_{\mathbf{y}}^N)).$$

Proof. □

Another remark is the fact that the discrete problem can be embedded into the continuous one.

Lemma 8.2.3. *Let $\alpha := (\alpha^i)$ be a family of controls and $\mathbf{x} := (x^i)$ be the associated solutions. Let $w^N(t) := \frac{1}{N} \sum_{i=1}^N \alpha_t^i \delta_{x_t^i}$. Then the pair $(m_{\mathbf{x}}^N, w^N)$ is admissible for the initial condition $m_{\mathbf{x}_0}^N$ and*

$$J^N(t_0, x_0^1, \dots, x_0^N, \alpha^1, \dots, \alpha^N) = J(t_0, m_{\mathbf{x}_0}^N, w^N).$$

Remark 8.2.4. The fact that the discrete problem is embedded into the continuous implies that

$$\mathcal{V}^N(t_0, \mathbf{x}) \geq \mathcal{U}(t_0, m_{\mathbf{x}}^N).$$

Let us warn the reader that there is no equality in general, since in the continuous problem the mass can split.

Proof. By definition, $w^N(t)$ is a.c. with respect to $m^N(t)$ and its density is given by

$$\frac{dw^N(t)}{dm^N(t)}(x) = \sum_{i=1}^N \alpha_t^i \mathbf{1}_{x=x_t^i}.$$

Moreover $(m_{\mathbf{x}}^N, w^N)$ is admissible (for the initial condition $m_{\mathbf{x}_0}^N$) and

$$\begin{aligned} J(m^N, w^N) &= \int_{t_0}^T \int_{\mathbb{R}^d} \frac{1}{2} \left| \frac{dw^N}{dm^N}(t, x) \right|^2 m^N(t, dx) dt + \int_{t_0}^T \mathcal{F}(m^N(t)) dt \\ &= \frac{1}{2N} \sum_{i=1}^N |\alpha_t^i|^2 dt + \int_{t_0}^T \mathcal{F}(m^N(t)) dt = J^N(t_0, \mathbf{x}_0, (\alpha^i)). \end{aligned}$$

Note that the second equality holds because $\alpha^j = \alpha^i$ a.e. on $x^j = x^i$. □

We now state a tightness property:

Lemma 8.2.5. *Assume that $(\bar{\mathbf{x}}_0^N)$ is a sequence of initial conditions with*

$$m_{\bar{\mathbf{x}}_0}^N \rightarrow \bar{m}_0 \quad \text{in } \mathcal{P}_2.$$

Let $(\bar{\alpha}^i)$ be a minimum for $J^N(t_0, \bar{\mathbf{x}}_0^N, \cdot)$ with associated trajectory $\bar{\mathbf{x}}_t = (\bar{x}_t^1, \dots, \bar{x}_t^N)$ and let $\bar{m}^N(t) := m_{\bar{\mathbf{x}}_t}^N$ be the associated empirical measure. Then the sequence $(\bar{m}^N)$ is tight in $C^0([0, T], \mathcal{P}_{2-\epsilon}(\mathbb{R}^d))$ (for any $\epsilon \in (0, 2)$) and, for any cluster point $m \in C^0([0, T], \mathcal{P}_2(\mathbb{R}^d))$, there exists α such that (m, α) is admissible and

$$J(m, \alpha) \leq \limsup_N J^N(\mathbf{x}_0^{N, \epsilon_N}, \bar{\alpha}^N).$$

Proof. We already know by Lemma 8.2.3 that, if we set $\bar{w}^N(t) := \frac{1}{N} \sum_{i=1}^N \bar{\alpha}_t^i \delta_{\bar{x}_t^i}$, then

$$J^N(x_0^1, \dots, x_0^N, \bar{\alpha}^1, \dots, \bar{\alpha}^N) = J(\bar{m}^N, \bar{w}^N).$$

It is easily to check that $J^N(\mathbf{x}_0^{N, \epsilon_N}, \bar{\alpha}^N)$ is bounded. Then arguing exactly as in the proof of the existence result (existence part of Theorem 4.1.1), the claim follows. □

Proof of Theorem 8.2.1. Because of the uniform continuity of $\mathcal{V}^N$ given in Lemma 8.2.2 we only need to build a sequence $(\mathbf{x}_0^N)$ for which the claimed convergence holds.

Let $(\bar{m}, \bar{\alpha})$ be a minimum for J with initial condition (t_0, m_0) (it exists thanks to a small variant of Theorem 4.1.1). By Ambrosio's superposition principle [3, Theorem 8.2.1], there exists a measure η on $C^0([0, T], \mathbb{R}^d)$ such that $\bar{m}(t) = e_t \# \eta$ and, for η -a.e. curve γ , γ is a solution of the ODE

$$\frac{d}{dt}\gamma(t) = \bar{\alpha}(t, \gamma(t)), \quad \text{a.e. } t \in [0, T].$$

Let $(\gamma^k)_{k \in \mathbb{N}}$ be an i.i.d. family of curves in $C^0([0, T], \mathbb{R}^d)$ with law η . Note that, for any $t \in [0, T]$, $(\gamma^k(t))_{k \in \mathbb{N}}$ is an i.i.d. sequence in $\mathbb{R}^d$ with law $\bar{m}(t)$. Moreover, the quantity $e^k := \int_{t_0}^T \frac{1}{2} |\alpha(t, \gamma^k(t))|^2 dt$ is an i.i.d. sequence of real numbers with law

$$\left(\gamma \rightarrow \int_{t_0}^T \frac{1}{2} |\alpha(t, \gamma(t))|^2 dt \right) \# \eta,$$

which is in L^1 since, by Fubini,

$$\begin{aligned} \mathbb{E}[e^k] &= \int_{C^0} \int_0^T \frac{1}{2} |\alpha(t, \gamma(t))|^2 dt \eta(d\gamma) \\ &= \int_{t_0}^T \int_{\mathbb{R}^d} \frac{1}{2} |\bar{\alpha}(t, x)|^2 \bar{m}(t, dx) dt = J(t_0, m_0, \bar{m}, \bar{\alpha}) < +\infty. \end{aligned}$$

Let us set $x_0^k = \gamma^k(0)$, $\alpha^k(t) := \alpha(t, \gamma^k(t))$ and

$$\bar{m}_{\mathbf{x}_0}^N = \frac{1}{N} \sum_{k=1}^N \delta_{x_0^k}$$

Note that, a.s., $(\alpha^k)_{k=1, \dots, N}$ is an admissible control associated with the initial position $\mathbf{x}_0^N := (x_0^k)_{k=1, \dots, N}$. By the law of large numbers, a.s. $(\bar{m}_{\mathbf{x}(t)}^N)$ converges weakly to $\bar{m}(t)$ while

$$J^N(t_0, (x_0^k)_{k=1, \dots, N}, (\alpha^k)_{k=1, \dots, N}) = \frac{1}{N} \sum_{k=1}^N e^k + \int_{t_0}^T \mathcal{F}(m_{\mathbf{x}(t)}^N) dt$$

converges to $J(t_0, m_0, \bar{m}, \bar{w})$. Let us fix from now on such an event for which these convergences hold. Then

$$\limsup_N \inf_{(\alpha^{k,i})} J^N((x_0^k)_{k=1, \dots, N}, (\alpha^{k,i})_{k=1, \dots, N}) \leq \limsup_N J^N((x_0^k)_{k=1, \dots, N}, (\alpha^k)_{k=1, \dots, N}) = J(\bar{m}, \bar{w}).$$

Let $(\bar{\alpha}^{N,i})$ be a minimizer for $J^N((x_0^i), \cdot)$, with associated trajectory $\bar{\mathbf{x}}_t = (\bar{x}_t^1, \dots, \bar{x}_t^N)$. Let us denote by $\bar{m}^N(t) := m_{\bar{\mathbf{x}}_t}^N$ the associated empirical measure. Note that $\bar{m}^N(0) := \bar{m}_{\mathbf{x}_0}^N$. Then Lemma 8.2.5 states that the sequence $(\bar{m}^N)$ is tight in $C^0([0, T], \mathcal{P}_{2-\epsilon}(\mathbb{R}^d))$ (for any $\epsilon \in (0, 2)$) and, for any cluster point $m \in C^0([0, T], \mathcal{P}_2(\mathbb{R}^d))$, there exists α such that (m, α) is admissible with

$$J(t_0, m_0, m, \alpha) \leq \liminf_N J^N(t_0, (x_0^i), (\bar{\alpha}^{N,i})) \leq J(t_0, m_0, \bar{m}, \bar{\alpha}).$$

As $\bar{m}^N(0) = \bar{m}_{\mathbf{x}_0}^N$ tends to m_0 , we conclude that $m(0) = m_0$ and therefore that (m, w) is a minimum of J . $\square$

8.2.2 Some variational principle

Hamilton-Jacobi in infinite dimensional spaces requires to construction of minima of functionals, which is not too obvious because of the lack of local compactness. We discuss here a few of these variational principle.

Let us start with Ekeland variational principle.

Theorem 8.2.6 (Ekeland [26]). *Let (X, d) be a complete metric space and $\phi : X \rightarrow \mathbb{R} \cup \{+\infty\}$ be a proper lower semi-continuous maps which is bounded from below. For any $\epsilon > 0$ and any $x_0 \in X$ there is some $\bar{x} \in X$ such that*

$$\begin{cases} i) & f(\bar{x}) + \epsilon d(\bar{x}, x_0) \leq f(x_0) \\ ii) & f(\bar{x}) < f(x) + \epsilon d(x, \bar{x}) \quad \forall x \in X \setminus \{\bar{x}\} \end{cases}$$

As an immediate consequence we have:

Corollary 8.2.7. *Under the same assumptions of Theorem 8.2.6, let $\lambda, \epsilon > 0$ and $x_0 \in X$ be such that $f(x_0) \leq \inf_X f + \lambda\epsilon$. Then there is some $\bar{x} \in X$ such that*

$$\begin{cases} i) & f(\bar{x}) \leq f(x_0) \\ ii) & d(\bar{x}, x_0) \leq \lambda \\ iii) & f(\bar{x}) \leq f(x) + \epsilon d(x, \bar{x}) \quad \forall x \in X \end{cases}$$

Proof of Theorem 8.2.6: It is enough to do the proof for $\epsilon = 1$. Let us set

$$F(x) = \{y \in X, f(y) + d(x, y) \leq f(x)\} \quad \text{and} \quad v(x) = \inf_{y \in F(x)} f(y) \quad \forall x \in X.$$

Note that $F(x)$ is closed (because f is l.s.c.), $F(x) \neq \emptyset$, because $x \in F(x)$, that

$$y \in F(x) \quad \Rightarrow \quad F(y) \subset F(x)$$

and that

$$\text{diam}(F(x)) \leq f(x) - v(x) \quad \forall x \in X.$$

We define by induction a sequence (x_n) starting at x_0 and such that

$$x_{n+1} \in F(x_n) \quad \text{and} \quad f(x_{n+1}) \leq v(x_n) + 2^{-n} \quad \forall n \in \mathbb{N}.$$

Then $(F(x_n))$ is a nonincreasing family of closed sets. Its diameter converges to 0 because

$$\text{diam}(F(x_{n+1})) \leq f(x_{n+1}) - v(x_{n+1}) \leq v(x_n) - v(x_{n+1}) + 2^{-n} \leq 2^{-n}.$$

Since X is complete, this implies that there is some $\bar{x} \in \bigcap_n F(x_n)$.

We claim that $F(\bar{x}) = \{\bar{x}\}$. Indeed, if $y \in F(\bar{x})$, then $y \in F(x_n)$ for any n so that $d(\bar{x}, y) \leq \text{diam}(F(x_n)) \rightarrow 0$. So $F(\bar{x}) = \{\bar{x}\}$, which implies that $f(\bar{x}) < f(y) + d(\bar{x}, y)$ for any $y \in X \setminus \{\bar{x}\}$. Finally $\bar{x} \in F(x_0)$, so that $f(\bar{x}) + d(\bar{x}, x_0) \leq f(x_0)$. $\square$

As an application, we have the following generic differentiability of a map defined in a Hilbert space (Ekeland-Lebourg [27]). For this we assume that $(X, \|\cdot\|)$ be a real Banach space. The closed unit ball around the origin in X is denoted by B_X . We say that X is dentable if for every $\epsilon > 0$ every nonempty bounded set $D \subset X$ has a slice with norm diameter at most ϵ , i.e., there are $\xi \in X^*$ and $\alpha > 0$ so that $\|x_1 - x_2\| \leq \epsilon$ whenever $x_i \in D$ and $\langle \xi, x_i \rangle > \sup\langle \xi, D \rangle - \alpha$, $i = 1, 2$. There exist plenty of conditions which are equivalent to the dentability of X : for instance “the Radon-Nikodym property”, which states that “for every nonempty closed bounded set $D \subset X$ there exist $x \in D$ and $0 \neq \xi \in X^*$ such that $\langle \xi, x \rangle = \sup\langle \xi, D \rangle$ ”. In particular Hilbert spaces are dentable.

A map $f : X \rightarrow [0, +\infty]$ is said to attain a strong minimum at x^* if x^* is a minimum of f and $x_n \rightarrow x^*$ whenever $x_n \in X$ for any n and $f(x_n) \rightarrow f(x^*)$.

Theorem 8.2.8. *For a real Banach space $(X, \|\cdot\|)$ the following assertions are equivalent.*

- (i) X is dentable.
- (ii) For every coercive lower semicontinuous function $f : X \rightarrow [0, +\infty]$ and for every $\epsilon > 0$ there are $x \in X$ and $\xi \in X^*$, with $\|\xi\| < \epsilon$, such that the function $f - \xi$ attains a strong minimum at x .
- (iii) For every coercive continuous convex function $f : X \rightarrow [0, +\infty)$ there are $x \in X$ and $\xi \in X^*$ such that the function $f - \xi$ attains a strong minimum at x .

Ekeland and Lebourg [27] proved that (ii) and (iii) hold in a Banach space which has a “bump function” (a nonzero, nonnegative function which is zero outside of a bounded set and which is Fréchet differentiable at any point where it is positive); the version above is given in [28]. See also the monograph by Phelps [52].

Let us finally mention the Borwein-Preiss variational principle [9] which is another extension frequently used.

Definition 8.2.9. *Let (X, d) be a metric space. We say that a continuous function $\rho : X \times X \rightarrow [0, \infty)$ is a gauge-type function if*

- (i) $\rho(x, x) = 0$ for any $x \in X$,
- (ii) for any $\epsilon > 0$, there exists $\delta > 0$ such that for all $y, z \in X$ we have

$$\rho(y, z) \leq \delta \text{ implies that } d(y, z) < \epsilon.$$

Theorem 8.2.10. *Let (X, d) be a complete metric space and let $f : X \rightarrow \mathbb{R} \cup \{+\infty\}$ be a lsc function bounded from below. Suppose that ρ is a gauge-type function and (δ_i) is a sequence of positive numbers, and suppose that $\epsilon > 0$ and $z \in X$ satisfy*

$$f(z) \leq \inf_X f + \epsilon.$$

Then there exist y and a sequence $\{x_i\} \subset X$ such that

- (i) $\rho(z, y) \leq \epsilon/\delta_0$, $\rho(x_i, y) \leq \epsilon/(2^i \delta_0)$,
- (ii) $f(y) + \sum_{i=0}^{\infty} \delta_i \rho(y, x_i) \leq f(z)$, and
- (iii) $f(x) + \sum_{i=0}^{\infty} \delta_i \rho(x, x_i) > f(y) + \sum_{i=0}^{\infty} \delta_i \rho(y, x_i)$, for all $x \in X \setminus \{y\}$.

The interest of the result is that, if the map ρ is sufficiently “smooth” and if the sequence (δ_i) decays fast enough, then the map $x \rightarrow \sum_{i=0}^{\infty} \delta_i \rho(x, x_i)$ is also “smooth” and close to its minimum at $x = y$. See an application the Hamilton-Jacobi equation in spaces of probability measures in [22] for instance.

8.2.3 Viscosity solutions in terms of L-derivatives

We study here the viscosity solution to the HJ equation

$$(8.6) \quad \begin{cases} -\partial_t \mathcal{U}(t, m) + \frac{1}{2} \int_{\mathbb{R}^d} |D_m \mathcal{U}(t, m, x)|^2 dm = \mathcal{F}(m) & \text{in } (0, T) \times \mathcal{P}_2, \\ \mathcal{U}(T, m) = 0 & \text{in } \mathcal{P}_2 \end{cases}$$

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be an atomless probability space. Following the discussion in Subsection 6.1.3 we define the map $\hat{\mathcal{U}} : [0, T] \times L^2(\Omega, \mathbb{R}^d) \rightarrow \mathbb{R}$ by

$$\hat{\mathcal{U}}(t, X) = \mathcal{U}(t, \mathcal{L}(X)).$$

If $\mathcal{U}$ solves (8.6), then $\hat{\mathcal{U}}$ formally solves

$$(8.7) \quad \begin{cases} -\partial_t \hat{\mathcal{U}}(t, X) + \frac{1}{2} \mathbb{E} \left[|\nabla_X \hat{\mathcal{U}}(t, X)|^2 \right] = \mathcal{F}(\mathcal{L}(X)) & \text{in } (0, T) \times L^2, \\ \hat{\mathcal{U}}(T, X) = 0 & \text{in } L^2 \end{cases}$$

This is very elementary example of a HJ equation in infinite dimension (the Hilbert L^2). As no unbounded operator is involved here, the only difficulty is the fact that the space L^2 is not locally compact, which can be overcome by using Ekeland-Lebourg’s variational principle.

Definition 8.2.11. *We say that U is a subsolution of the HJ equation (8.7) if, for any test function $\phi \in \mathcal{C}^1([0, T] \times L^2)$ such that the map $U - \phi$ has a local maximum at $(t_0, X_0) \in [0, T] \times L^2$ one has*

$$-\partial_t \phi(t_0, X_0) + \frac{1}{2} \mathbb{E} [|\nabla_X \phi(t_0, X_0)|^2] \leq \mathcal{F}(\mathcal{L}(X_0)).$$

In a symmetric way U is a supersolution of the HJ equation (8.7) if, for any test function $\phi \in \mathcal{C}^1([0, T] \times L^2)$ such that the map $U - \phi$ has a local minimum at $(X_0, t_0) \in L^2 \times (0, T]$ one has

$$-\partial_t \phi(t_0, X_0) + \frac{1}{2} \mathbb{E} [|\nabla_X \phi(t_0, X_0)|^2] \geq \mathcal{F}(\mathcal{L}(X_0)).$$

Finally U is a solution of (8.7) if U is a sub and a super solution (8.7).

We will say that a map $\mathcal{U} : [0, T] \times \mathcal{P}_2 \rightarrow \mathbb{R}$ is a viscosity solution to (8.6) if $\hat{\mathcal{U}}$ is a viscosity solution to (8.7). Note that the notion is not completely satisfactory as the notion is not intrinsic. It has however the advantage of (relative) simplicity.

The main point is the following comparison Theorem:

Theorem 8.2.12. *Let $U_1, U_2 : [0, T] \times L^2 \rightarrow \mathbb{R}$ be two uniformly continuous, bounded maps such that u_1 is a subsolution of (8.7) and u_2 is a supersolution of (8.7) with $U_1 \leq U_2$ at $t = T$. Then $U_1 \leq U_2$.*

Proof. The proof is a direct application of [23]. Let $\epsilon, \sigma, \delta, \alpha \in (0, 1)$ to be choose later and look at the map

$$\Phi(t, X, s, Y) = U_1(t, X) - U_2(s, Y) - \frac{1}{2\epsilon} \|(t, X) - (s, Y)\|_2^2 - \frac{\alpha}{2} (\|X\|_2^2 + \|Y\|_2^2) + \sigma t$$

From Theorem 8.2.8 there are $\xi_s, \xi_t \in \mathbb{R}$, $\xi_X, \xi_Y \in L^2$ such that $|\xi_s|, |\xi_t|, \|\xi_X\|_2, \|\xi_Y\|_2 \leq \delta$ and the map $(t, X, s, Y) \rightarrow \Phi(t, X, s, Y) - \langle \xi_X, X \rangle - \langle \xi_Y, Y \rangle - \xi_t t - \xi_s s$ has a maximum at a point $(\bar{t}, \bar{X}, \bar{s}, \bar{Y})$.

Since the U_i are bounded by some M and

$$\Phi(\bar{X}, \bar{t}, \bar{Y}, \bar{s}) - \langle \xi_X, X \rangle - \langle \xi_Y, Y \rangle - \xi_t t - \xi_s s \geq \Phi(0, 0, 0, 0)$$

we have

$$\alpha(\|\bar{X}\|_2^2 + \|\bar{Y}\|_2^2) + \frac{1}{\epsilon} \|(\bar{X}, \bar{t}) - (\bar{Y}, \bar{s})\|_2^2 \leq C(1 + \delta(\|\bar{X}\|_2^2 + \|\bar{Y}\|_2^2)^{\frac{1}{2}})$$

where C only depends on M . So

$$(8.8) \quad (\|\bar{X}\|_2^2 + \|\bar{Y}\|_2^2)^{\frac{1}{2}} \leq C \left(\frac{\delta}{\alpha} + \frac{1}{\sqrt{\alpha}} \right)$$

and

$$(8.9) \quad \|(\bar{t}, \bar{X}) - (\bar{s}, \bar{Y})\|_2 \leq C\sqrt{\epsilon} \left(1 + \sqrt{\delta} \left(\frac{\delta}{\alpha} + \frac{1}{\sqrt{\alpha}} \right)^{\frac{1}{2}} \right).$$

Let us now assume for a while that $\bar{t} < T$ and $\bar{s} < T$. Since the map $(t, X) \rightarrow \Phi(t, X, \bar{s}, \bar{Y}) - \langle \xi_X, X \rangle - \langle \xi_Y, \bar{Y} \rangle - \xi_t t - \xi_s \bar{s} + \sigma t$ has a maximum at $(\bar{t}, \bar{X})$ and U_1 is a subsolution, we have

$$-\xi_t + \sigma - \frac{\bar{t} - \bar{s}}{\epsilon} + \frac{1}{2} \left\| \xi_X + \alpha \bar{X} + \frac{\bar{X} - \bar{Y}}{\epsilon} \right\|_2^2 \leq 0.$$

In the same way, since the map $(s, Y) \rightarrow \Phi(\bar{t}, \bar{X}, s, Y) - \langle \xi_X, \bar{X} \rangle - \langle \xi_Y, Y \rangle - \xi_t \bar{t} - \xi_s s$ has a maximum at $(\bar{s}, \bar{Y})$ and U_2 is a supersolution, we have

$$\xi_s - \frac{\bar{t} - \bar{s}}{\epsilon} + \frac{1}{2} \left\| -\xi_Y - \alpha \bar{Y} + \frac{\bar{X} - \bar{Y}}{\epsilon} \right\|_2^2 \geq 0.$$

Computing the difference in the two inequalities gives

$$\sigma - \xi_t - \xi_s + \frac{1}{2} \left\| \xi_X + \alpha \bar{X} + \frac{\bar{X} - \bar{Y}}{\epsilon} \right\|_2^2 - \frac{1}{2} \left\| -\xi_Y - \alpha \bar{Y} + \frac{\bar{X} - \bar{Y}}{\epsilon} \right\|_2^2 \leq 0.$$

Hence

$$\sigma - 2\delta - 2(\|\xi_X\|_2 + \|\xi_Y\|_2 + \alpha(\|\bar{X}\|_2 + \|\bar{Y}\|_2)) \left\| \frac{\bar{X} - \bar{Y}}{\epsilon} \right\|_2 - (\|\xi_Y\|_2 + \alpha\|\bar{Y}\|_2)^2 \leq 0.$$

Using estimates (8.8) and (8.9) and the fact that $\epsilon \leq 1$ we get

$$\sigma - 2\delta - C(\delta + \sqrt{\alpha}) \left(1 + \sqrt{\delta} \left(\frac{\delta}{\alpha} + \frac{1}{\sqrt{\alpha}} \right)^{\frac{1}{2}} \right) - (\delta + \sqrt{\alpha})^2 \leq 0.$$

If we let first $\delta \rightarrow 0$, and then $\alpha \rightarrow 0$ we get a contradiction because $\sigma > 0$. So we can choose δ and α small enough (independently of ϵ) such that either $\bar{t} = 0$ or $\bar{s} = 0$.

Let us assume to fix the ideas that $\bar{t} = T$. Let ω be a modulus of continuity of U_1 and U_2 . Then for any (t, X) ,

$$\Phi(t, X, t, X) - \langle (\xi_X + \xi_Y), X \rangle - (\xi_t + \xi_s)t \leq \Phi(\bar{X}, 0, \bar{Y}, \bar{s}) - \langle \xi_X, \bar{X} \rangle - \langle \xi_Y, \bar{Y} \rangle - \xi_s \bar{s}$$

where, from (8.8)

$$\begin{aligned} \Phi(\bar{X}, 0, \bar{Y}, \bar{s}) &\leq U_1[\bar{X}, 0] - U_2[\bar{Y}, \bar{s}] \leq U_2[\bar{X}, 0] - U_2[\bar{Y}, \bar{s}] \leq \omega(\|(\bar{X}, 0) - (\bar{Y}, \bar{s})\|_2) \\ &\leq \omega\left(C\sqrt{\epsilon}\left(1 + \sqrt{\delta}\left(\frac{\delta}{\alpha} + \frac{1}{\sqrt{\alpha}}\right)^{\frac{1}{2}}\right)\right) \end{aligned}$$

while, from (8.9),

$$-\langle \xi_X, \bar{X} \rangle - \langle \xi_Y, \bar{Y} \rangle - \xi_s \bar{s} \leq C\delta \left(1 + \frac{\delta}{\alpha} + \frac{1}{\sqrt{\alpha}}\right).$$

So, letting $\epsilon \rightarrow 0$, and then $\delta \rightarrow 0$ and $\alpha \rightarrow 0$, we get

$$U_1[X, t] - U_2[X, t] \leq 0.$$

□

We now make the link with our optimal control problem with value function $\mathcal{U}$ defined in (8.4).

Proposition 8.2.13. *The value function $\mathcal{U}$ defined in (8.4) is the unique uniformly continuous viscosity solution to (8.6).*

The proof of the proposition relies on the following remark:

Lemma 8.2.14. *Define $U(t, X) = \mathcal{U}(t, \mathcal{L}(X))$ for $(t, X) \in [0, T] \times L^2$. Then*

$$U(t_0, X_0) = \inf_X \int_{t_0}^T \frac{1}{2} \mathbb{E} [|\dot{X}(t)|^2] dt + \int_{t_0}^T \mathcal{F}(\mathcal{L}(X(t))) dt$$

where the infimum is taken over the a.c. curves $X : [t_0, T] \rightarrow L^2$ such that $\int_{t_0}^T \mathbb{E} [|\dot{X}(t)|^2] dt < \infty$.

Proof. Let $V = V(t_0, m_0)$ be the right-hand side of the above equality. Fix (m, α) be a solution of the continuity equation starting from m_0 . As often we assume that α is a smooth and bounded vector field, since otherwise we argue by regularization. Denote by $\xi(t, x)$ its integral curves:

$$\frac{d}{dt} \xi(t, x) = \alpha(t, \xi(t, x)), \quad \xi(\bar{t}, x) = x.$$

Then $m(t) = \xi(t, \cdot) \# m$. Set also $X(t) = \xi(t, X_0)$ and note that X is a C^1 curve in L^2 with $\dot{X}(t) = \alpha(t, X(t))$. Then

$$\begin{aligned} V(t_0, X_0) &\leq \int_{t_0}^T \frac{1}{2} \mathbb{E} [|\dot{X}(t)|^2] dt + \int_{t_0}^T \mathcal{F}(\mathcal{L}(X(t))) dt \\ &= \int_{t_0}^T \frac{1}{2} \mathbb{E} [|\alpha(t, X(t))|^2] dt + \int_{t_0}^T \mathcal{F}(\mathcal{L}(X(t))) dt \\ &= \int_{t_0}^T \int_{\mathbb{R}^d} \frac{1}{2} |\alpha(t, x)|^2 m(t, dx) dt + \int_{t_0}^T \mathcal{F}(m(t)) dt. \end{aligned}$$

Taking the infimum over the pairs (m, α) leads to $V \leq U$.

Conversely, let $X : [t_0, T] \rightarrow L^2$ be an a.c. curve such that $\int_{t_0}^T \mathbb{E} [|\dot{X}(t)|^2] dt < \infty$. Proceeding by approximation, we can assume without loss of generality that X is actually C^1 . Set $m(t) = \mathcal{L}(X(t))$. We prove below that $(m(t))$ is an absolutely continuous curve in $\mathcal{P}_2$, namely

$$\mathbf{d}_2(m(s), m(t)) \leq \int_s^t \|\dot{X}(\tau)\|_{L^2} d\tau.$$

Let us admit the claim for a while and complete the proof. Following [3, Theorem 8.3.1], this implies the existence of a map $\alpha \in L^2_m$ such that (m, α) solves the continuity equation with

$$\|\alpha(t, \cdot)\|_{L^2_{m(t)}} \leq \|\dot{X}(t)\|_{L^2} \text{ a.e. } t.$$

Then

$$\begin{aligned} \mathcal{U}(t_0, m_0) &\leq J(t_0, m_0, m, \alpha) = \int_{t_0}^T \int_{\mathbb{R}^d} \frac{1}{2} |\alpha|^2 m + \int_{t_0}^T \mathcal{F}(m(t)) dt \\ &\leq \int_{t_0}^T \int_{\mathbb{R}^d} \frac{1}{2} \|\dot{X}(t)\|_{L^2}^2 dt + \int_{t_0}^T \mathcal{F}(\mathcal{L}(X(t))) dt \end{aligned}$$

Taking the infimum over X then shows that $U(t_0, X_0) = \mathcal{U}(t_0, m_0) \leq V(t_0, X_0)$.

It remains to check the claim. We have

$$\mathbf{d}_2^2(m(s), m(t)) \leq \mathbb{E} [|X(t) - X(s)|^2] = \mathbb{E} \left[\left| \int_s^t \dot{X}(\tau) d\tau \right|^2 \right] \leq (t-s) \int_s^t \|\dot{X}(\tau)\|_{L^2}^2 d\tau.$$

Hence, for any partition $t_0 = s < t_1 < \dots < t_n = t$, we have

$$\mathbf{d}_2(m(s), m(t)) \leq \sum_{i=0}^{n-1} \mathbf{d}_2(m(t_i), m(t_{i+1})) \leq \sum_{i=0}^{n-1} (t_{i+1} - t_i) \sup_{t \in [t_i, t_{i+1}]} \|\dot{X}(\tau)\|_{L^2}.$$

Letting the step of the partition tend to 0 proves the claim since X is $C^1([0, T], L^2)$. $\square$

Proof of Proposition 8.2.13. As there is at most one viscosity solution, we just need to check that the map $U(t, X) = \mathcal{U}(t, \mathcal{L}(X))$ is a viscosity solution (we just simplified the notation by setting $U = \hat{\mathcal{U}}$).

The rest of the proof relies on the representation formula given in Lemma 8.2.14, dynamic programming for this formula and then the same argument as in finite dimension. $\square$

8.2.4 Mean field limit: proof by viscosity solutions

Proof of Theorem 8.2.1 by viscosity solution argument. We borrow the proof from Gangbo et al. [32]. We work on the probability space $\Omega = [0, 1]$, endowed with its Borel σ -algebra and $\mathbb{P}$ is simply the Lebesgue measure.

The first remark is that, as $\mathcal{V}^N$ is uniformly Lipschitz and bounded (Lemma 8.2.2), there is a subsequence, denoted in the same way, and a continuous bounded function $\mathcal{V} : [0, T] \times \mathcal{P}_1 \rightarrow \mathbb{R}$ such that, for any $R > 0$,

$$\sup_{t \in [0, T], \mathbf{x} \in (\mathbb{R}^d)^N, M_2(m_{\mathbf{x}}^N) \leq R} |\mathcal{V}^N(t, \mathbf{x}) - \mathcal{V}(t, m_{\mathbf{x}}^N)| \rightarrow 0.$$

(this is a small adaptation of Theorem 3.1.6 to a noncompact setting). The goal is to prove that $\mathcal{V} = \mathcal{U}$, which shows the convergence of the whole sequence. For this it is enough to check that $\mathcal{V}$ is a viscosity solution to (8.6), or, by definition, that $V(t, X) := \mathcal{V}(t, \mathcal{L}(X))$ is a (classical) viscosity of (8.7).

Let ϕ be a smooth test function such that $V - \phi$ has a strict maximum at $(t_0, X_0) \in [0, T] \times L^2$. Let $(A^{N,i} = [i - 1/N, i/N])_{i=1, \dots, N}$ be a partition of Ω such that $\mathbb{P}(A^{N,i}) = 1/N$. We consider

$$\phi^N(t, \mathbf{x}) = \phi(t, \sum_{i=1}^N x^i \mathbf{1}_{A^{N,i}}), \quad \forall \mathbf{x} = (x^i) \in (\mathbb{R}^d)^N.$$

Then $\mathcal{V}^N - \phi^N$ has a maximum at some $(t_N, \mathbf{x}_N)$ which is such that

$$X_N := \sum_{i=1}^N x^i \mathbf{1}_{A^{N,i}} \rightarrow X_0,$$

as $N \rightarrow \infty$ as the maximum of $V - \phi$ at (t_0, X_0) is strict. Note that ϕ^N is C^1 , with

$$D_{x^i} \phi^N(t, \mathbf{x}) = \mathbb{E} \left[\nabla_X \phi(t, \sum_{i=1}^N x^i \mathbf{1}_{A^{N,i}}) \mathbf{1}_{A^{N,i}} \right], \quad \partial_t \phi^N(t, \mathbf{x}) = \partial_t \phi(t, \sum_{i=1}^N x^i \mathbf{1}_{A^{N,i}}).$$

As $\mathcal{V}^N$ is a viscosity solution to the HJ equation

$$\begin{cases} -\partial_t \mathcal{V}^N + N \sum_{i=1}^N \frac{1}{2} |D_{x^i} \mathcal{V}^N|^2 = \mathcal{F}(m_{\mathbf{x}}^N) \\ \mathcal{V}^N(T, \mathbf{x}) = 0 \end{cases}$$

and $\mathcal{V}^N - \phi^N$ has a maximum at (t_N, X_N) , we get

$$-\partial_t \phi^N(t_N, \mathbf{x}_N) + N \sum_{i=1}^N \frac{1}{2} |D_{x^i} \phi^N(t_N, \mathbf{x}_N)|^2 \leq \mathcal{F}(m_{\mathbf{x}_N}^N)$$

Thus

$$(8.10) \quad -\partial_t \phi(t_N, X_N) + N \sum_{i=1}^N \frac{1}{2} |\mathbb{E}[\nabla_X \phi(t_N, X_N) \mathbf{1}_{A^{N,i}}]|^2 \leq \mathcal{F}(\mathcal{L}(X_N)).$$

Let us note that, for any $\zeta, \xi \in L^2$,

$$\begin{aligned} \left| N \sum_{i=1}^N |\mathbb{E}[\xi \mathbf{1}_{A^{N,i}}]|^2 - N \sum_{i=1}^N |\mathbb{E}[\zeta \mathbf{1}_{A^{N,i}}]|^2 \right| &\leq N \sum_{i=1}^N \left| |\mathbb{E}[\xi \mathbf{1}_{A^{N,i}}]|^2 - |\mathbb{E}[\zeta \mathbf{1}_{A^{N,i}}]|^2 \right| \\ &\leq N \sum_{i=1}^N (|\mathbb{E}[\xi \mathbf{1}_{A^{N,i}}]| + |\mathbb{E}[\zeta \mathbf{1}_{A^{N,i}}]|) |\mathbb{E}[(\zeta - \xi) \mathbf{1}_{A^{N,i}}]| \\ &\leq CN \left(\sum_{i=1}^N (|\mathbb{E}[\xi \mathbf{1}_{A^{N,i}}]|^2 + |\mathbb{E}[\zeta \mathbf{1}_{A^{N,i}}]|^2) \right)^{1/2} \left(\sum_{i=1}^N |\mathbb{E}[(\zeta - \xi) \mathbf{1}_{A^{N,i}}]|^2 \right)^{1/2} \\ &\leq CN \left(\sum_{i=1}^N \mathbb{P}(A^{N,i}) (\mathbb{E}[|\xi|^2 \mathbf{1}_{A^{N,i}}] + \mathbb{E}[|\zeta|^2 \mathbf{1}_{A^{N,i}}]) \right)^{1/2} \left(\sum_{i=1}^N \mathbb{P}(A^{N,i}) \mathbb{E}[|\zeta - \xi|^2 \mathbf{1}_{A^{N,i}}] \right)^{1/2} \end{aligned}$$

As $\mathbb{P}(A^{N,i}) = 1/N$, we get

$$\left| N \sum_{i=1}^N |\mathbb{E}[\xi \mathbf{1}_{A^{N,i}}]|^2 - N \sum_{i=1}^N |\mathbb{E}[\zeta \mathbf{1}_{A^{N,i}}]|^2 \right| \leq C(\|\xi\|_{L^2} + \|\zeta\|_{L^2}) \|\xi - \zeta\|_{L^2}.$$

We also note that for any *continuous* map $\zeta : \Omega \rightarrow \mathbb{R}$, we have

$$\lim_N N \sum_{i=1}^N |\mathbb{E}[\zeta \mathbf{1}_{A^{N,i}}]|^2 = \mathbb{E}[|\zeta|^2].$$

So, coming back to (8.10), using the C^1 continuity of ϕ and the convergence of X_N to X_0 , we have, for any continuous map $\zeta : \Omega \rightarrow \mathbb{R}$,

$$-\partial_t \phi(t_0, X_0) - C(\|\zeta\|_{L^2} + \|\nabla_X \phi(t_0, X_0)\|_{L^2}) \|\zeta - \nabla_X \phi(t_0, X_0)\|_{L^2} + \frac{1}{2} \mathbb{E}[|\zeta|^2] \leq \mathcal{F}(\mathcal{L}(X_0)).$$

We now use a sequence of continuous maps (ζ_n) converging to $\nabla_X \phi(t_0, X_0)$ in L^2 to conclude that

$$-\partial_t \phi(t_0, X_0) + \frac{1}{2} \mathbb{E} [|\nabla_X \phi(t_0, X_0)|^2] \leq \mathcal{F}(\mathcal{L}(X_0)),$$

which shows that $\mathcal{V}$ is a subsolution. The proof of the supersolution can be achieved in a symmetric way. $\square$

8.3 Further reading

The theory of HJ equations in the Wasserstein space—and more generally in metric spaces—has known a large development in the recent years. It is impossible in these short notes to describe the various methods used in this topic. We just now quote a few selected references. The first analysis of the value function of the limit problem and of the associated dynamic programming can be found in [47] (see also [51]). For the characterization of the value function in terms of viscosity solutions, see, among many other approaches, [32], [22], [57], [25, 24] [6], [21, 20], [17]. Section 8.2 is largely borrowed from [32]. Let us recall that [42] discussed in a very general set-up the mean field limit for controlled McKean-Vlasov dynamics.

Chapter 9

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