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Notes on the course "Stochastic control"

with former exams

Pierre Cardaliaguet

Contents

1. Feynman-Kac formula 2
1.1. Stochastic differential equations 2
1.2. The value function and a Feynman-Kac formula 5
2. A finite horizon problem 7
2.1. Statement of the problem 7
2.2. Dynamic programming 8
2.3. HJ equation and verification theorem 11
3. The infinite horizon problem 17
3.1. Value function and dynamic programming 17
3.2. The Hamilton-Jacobi equation and the verification theorem 19
4. Viscosity solutions of the Hamilton-Jacobi equation 23
4.1. Structure of the equation and classical solutions 23
4.2. The notion of viscosity solutions 25
4.3. The value function is a viscosity solution 27
5. Comparison principles and applications 29
5.1. Classical solution to the Hamilton-Jacobi equations 29
5.2. Comparison principle for first order HJ equations 30
5.3. Comparison principle for second order equation 34
References 38

In this course we are interested in optimization problems involving a stochastic and controlled dynamical system. For instance, in math finance, it can be the problem faced by a trader to buy or sell a certain amount of a risky asset over a given period (optimal trading): the dynamical system in this case the made of the price of the asset (in general random) and the amount of this asset owned by the trader. The control is the amount of risky asset the trader buys or sells at each time. The dynamical system can evolve either in discrete or continuous time: for instance, in economy, models are most often written in discrete time, while in finance or engineering the continuous time setting dominates. We will focus here on the continuous time formalism.

In terms of math, optimal control is at the intersection of stochastic calculus, optimization and partial differential equations (PDEs): it studies stochastic differential equations (SDEs) parametrized by a "control" which is chosen at each time in order to optimize a payoff function. To study this optimization problem, we will introduce a "value function", which satisfies a dynamic programming principle and thus a PDE, the Hamilton-Jacobi equation (HJ equation). One of the interests of this formulation is that it is possible to derive the optimal control from the HJ equation—at least in sufficiently smooth settings (verification principle). When the value function is not smooth, it is convenient to characterize it as a weak solution to the HJ equation (viscosity solutions): this gives tools to manipulate the PDE in an efficient way (for instance, to analyze numerical schemes to approximate the value function).

Standard monographs on the subject:

- Fleming W.H. & Rischel R.W. (1975) Deterministic and stochastic optimal control. Springer-Verlag, New York.
  A classical reference (but without viscosity solutions)
- Fleming W.H. & Soner H.M. (1993) Controlled Markov processes and viscosity solution. Springer-Verlag, New-York.
  A classical monograph, one of the first ones on viscosity solutions
- Lamberton, D., & Lapeyre, B. (2011). Introduction to stochastic calculus applied to finance. Chapman and Hall/CRC.
  One of the most classical (French) references on optimal control problems applied to finance
- Pham, H. (2007). Optimisation et contrôle stochastique appliqués à la finance (Vol. 61). Berlin: Springer.
  Another classical reference on optimal control problems applied to finance
- Yong, J., & Zhou, X. Y. (1999). Stochastic controls: Hamiltonian systems and HJB equations (Vol. 43). Springer Science & Business Media.
  A classical reference to control theory, with a good development on the theory of viscosity solutions and on optimality conditions (through backward SDEs).

The reader can also find a lot of material in the online courses:

- Bruno Bouchard "Introduction to stochastic control of mixed diffusion processes, viscosity solutions and applications in finance and insurance"
  https://www.ceremade.dauphine.fr/%7Ebouchard/bouchard.htm
- Nizar Touzi "Deterministic and Stochastic Control, Application to Finance"
  http://www.cmap.polytechnique.fr/ touzi/♯Lecture

1. Feynman-Kac formula

We start with the analysis of an uncontrolled system. It might seem strange to start a course on stochastic control by a problem in which there is no control. The reason is, that in this simple case, ideas and proofs are relatively easy.

1.1. **Stochastic differential equations.** We fix throughout these notes a stochastic basis $(\Omega, \mathcal{F}, \mathbb{P})$, supporting a $m-$dimensional standard Brownian motion W. We denote by $(\mathcal{F}_t)_{t\in[0,T]}$ the complete filtration generated by W. Our aim in this section is to study the SDE

$$dX_t = b_t(X_t)dt + \sigma_t(X_t)dW_t,\ t \in [t_0, T], \qquad X_{t_0} = x_0, \tag{1.1}$$

In this equation, $T > 0$ is a fixed horizon. The initial condition $(t_0, x_0) \in [0, T] \times \mathbb{R}^d$ is given ($d \in \mathbb{N}$, $d \geq 1$). The random drift $b : \Omega \times [0, T] \times \mathbb{R}^d \to \mathbb{R}^d$ and the random volatility $\sigma : \Omega \times [0, T] \times \mathbb{R}^d \to \mathbb{R}^{d\times m}$ are Borel measurable and are supposed to satisfy the following conditions:

$$\begin{aligned}
&(i) && \text{(Measurability) } b \text{ and } \sigma \text{ are progressively measurable w.r. to } (\mathcal{F}_t)_{t\in[0,T]}\\
&(ii) && \text{(Lipschitz continuity in space) } \exists L > 0 \text{ such that}\\
&&& \quad |b_t(y) - b_t(x)| + |\sigma_t(y) - \sigma_t(x)| \leq L|y - x| \qquad \forall t \in [0, T],\ x, y \in \mathbb{R}^d \text{ and a.s.}\\
&(iii) && \text{(Growth condition) } \exists M > 0 \text{ such that}\\
&&& \quad \mathbb{E}\left[\int_0^T |b_t(0)| + |\sigma_t(0)|^2\ dt\right] \leq M.
\end{aligned} \tag{1.2}$$

By a solution X to (1.1) we mean that $X : \Omega \times [t_0, T] \to \mathbb{R}^d$ is progressively measurable with respect to the filtration $(\mathcal{F}_t)_{t\in[0,T]}$, is integrable:

$$\mathbb{E}\left[\int_{t_0}^T |X_t|dt\right] < \infty$$

and satisfies

$$X_t = x_0 + \int_{t_0}^t b_s(X_s)ds + \int_{t_0}^t \sigma_s(X_s)dW_s \qquad \forall t \in [t_0, T],\ \text{a.s.}$$

Note that, by classical properties of the stochastic integral with respect to a Brownian motion, the path of X are a.s. continuous.

Theorem 1.1. *Under Assumption* (1.2)*, there is a unique solution* X^{t_0,x_0} *to* (1.1)*. It satisfies the following estimates:*

(1) *(Growth)*

$$\mathbb{E}\left[\sup_{t\in[t_0,T]}|X_t^{t_0,x_0}|^2\right] \leq C(|x_0|^2+1) \qquad \forall (t_0,x_0)\in[0,T]\times\mathbb{R}^d,$$

where C *depends on* T*,* L *and* M *only.*

(2) *(dependence with respect to the initial condition)*

$$\mathbb{E}\left[\sup_{t\in[t_0,T]}|X_t^{t_0,x_0}-X_t^{t_0,y_0}|^2\right] \leq C|x_0-y_0|^2 \qquad (t_0,x_0,y_0)\in[0,T]\times\mathbb{R}^d\times\mathbb{R}^d,$$

where C *depends on* T *and* L *only.*

(3) *(regularity in time)*

$$\mathbb{E}\left[|X_t^{t_0,x_0}-X_s^{t_0,x_0}|^2\right] \leq C|t-s| \qquad (t_0,t,s,x_0,y_0)\in[0,T]^3\times\mathbb{R}^d,\ st,\geq t_0,$$

where C *now depends on* x_0*,* T*,* L *and* M*.*

Remark 1.2. In fact similar argument as the one developed below show that, when σ is also bounded, then

$$\mathbb{E}\left[\sup_{t\in[t_0,T]}|X_t^{t_0,x_0}-X_t^{t_0,y_0}|\right] \leq |x_0-y_0|e^{CT} \qquad (t_0,x_0,y_0)\in[0,T]\times\mathbb{R}^d\times\mathbb{R}^d,$$

Proof. The existence of a solution has been established in the course of Stochastic Calculus. Let us now show the various estimates. We only do the proof in the case where the coefficients b and σ are bounded by M as this simplifies (substantially) the computation. We set $X_t = X_t^{t_0,x_0}$.

$$|X_t|^2 \leq 3\left(|x_0|^2+\left|\int_{t_0}^t b_s(X_s)ds\right|^2+\left|\int_{t_0}^t \sigma_s(X_s)dW_s\right|^2\right)$$

Thus

$$\begin{aligned}\mathbb{E}\left[\sup_{t\in[t_0,T]}|X_t|^2\right] &\leq 3|x_0|^2+3\mathbb{E}\left[\sup_{t\in[t_0,T]}\left|\int_{t_0}^t b_s(X_s)ds\right|^2\right]+3\mathbb{E}\left[\sup_{t\in[t_0,T]}\left|\int_{t_0}^t \sigma_s(X_s)dW_s\right|^2\right]\\ &\leq 3|x_0|^2+3\mathbb{E}\left[\sup_{t\in[t_0,T]}(t-t_0)\int_{t_0}^t|b_s(X_s)|^2ds\right]+12\mathbb{E}\left[\int_{t_0}^T|\sigma_s(X_s)|^2ds\right]\\ &\leq 3|x_0|^2+3M^2T^2+12M^2T,\end{aligned}$$

where we used Cauchy-Schwarz and Doob inequalities in the second line.

We now check the dependence with respect to the initial condition. We set $X_t = X_t^{t_0,x_0}$ and $Y_t = X_t^{t_0,y_0}$. As

$$X_t - Y_t = x_0 - y_0 + \int_{t_0}^t b_s(X_s)-b_s(Y_s)\ ds + \int_{t_0}^t(\sigma_s(X_s)-\sigma_s(Y_s))\ dW_s,$$

we obtain

$$|X_s-Y_s|^2 \leq 3\left(|x_0-y_0|^2+\left|\int_{t_0}^s b_r(X_r)-b_r(Y_r)\ dr\right|^2+\left|\int_{t_0}^s(\sigma_r(X_r)-\sigma_r(Y_r))\ dW_r\right|^2\right)$$

Thus

$$\mathbb{E}\left[\sup_{s\in[t_0,t]}|X_s-Y_s|^2\right]\leq 3|x_0-y_0|^2+3\mathbb{E}\left[\sup_{s\in[t_0,t]}\left|\int_{t_0}^s b_r(X_r)-b_r(Y_r)\ dr\right|^2\right]$$
$$+3\mathbb{E}\left[\sup_{s\in[t_0,t]}\left|\int_{t_0}^s(\sigma_r(X_r)-\sigma_r(Y_r))\ dW_r\right|^2\right]$$

We note that, by Cauchy-Schwarz inequality and the Lipschitz continuity of b,

$$\sup_{s\in[t_0,t]}\left|\int_{t_0}^s b_r(X_r)-b_r(Y_r)\ dr\right|^2\leq T\sup_{s\in[t_0,t]}\int_{t_0}^s|b_r(X_r)-b_r(Y_r)|^2\ dr\leq TL^2\int_{t_0}^t|X_r-Y_r|^2\ dr$$
$$\leq TL^2\int_{t_0}^t\sup_{r\in[t_0,s]}|X_r-Y_r|^2\ ds.$$

On the other hand, by Doob's inequality

$$\mathbb{E}\left[\sup_{s\in[t_0,t]}\left|\int_{t_0}^s(\sigma_r(X_r)-\sigma_r(Y_r))dW_r\right|^2\right]\leq 4\mathbb{E}\left[\int_{t_0}^t|\sigma_r(X_r)-\sigma_r(Y_r)|^2\,dr\right]$$
$$\leq 4\mathbb{E}\left[\int_{t_0}^t L^2\,|X_r-Y_r|^2\,dr\right]\leq 4L^2\mathbb{E}\left[\int_{t_0}^t\sup_{r\in[t_0,s]}|X_r-Y_r|^2\ ds\right].$$

We infer that

$$\mathbb{E}\left[\sup_{s\in[t_0,t]}|X_s-Y_s|^2\right]\leq 3|x_0-y_0|^2+3L^2(T+4)\int_{t_0}^t\mathbb{E}\left[\sup_{r\in[t_0,s]}|X_r-Y_r|^2\right]\ ds.$$

We conclude by Gronwall's Lemma (see below).

Next we prove the regularity in time. Here again we assume that the coefficients b and σ are bounded by M to simplify the computation. Assume that $t_0\leq s\leq t$. Then

$$X_t-X_s=\int_s^t b_r(X_r)dr+\int_s^t\sigma_r(X_r)dW_r,$$

so that

$$\mathbb{E}\left[|X_t-X_s|^2\right]\leq 3\mathbb{E}\left[\left|\int_s^t b_r(X_r)dr\right|^2\right]+3\mathbb{E}\left[\left|\int_s^t\sigma_r(X_r)dW_r\right|^2\right]$$
$$\leq 3\mathbb{E}\left[\left|\int_s^t|b_r(X_r)|dr\right|^2\right]+6\mathbb{E}\left[\int_s^t|\sigma_r(X_r)|^2dr\right]\qquad\text{(Doob)}$$
$$\leq 3M^2\left((t-s)^2+2(t-s)\right)\leq 3M^2(T+2)(t-s).$$

□

We used above the

Lemma 1.3 (Gronwall's Lemma)**.** *Let $T>0$ and let g be a nonnegative measurable map which is bounded on the time interval $[0,T]$. Assume that there exists two constants $a>0$ and $b>0$ such that*

$$g(t)\leq a+b\int_0^t g(s)ds\qquad\forall t\in[0,T].$$

Then we have

$$g(t)\leq a\exp(bt)\qquad\forall t\in[0,T].$$

We note that the above result extends to the case where the initial condition is random.

Proposition 1.4. *Fix $t_0 \in [0,T]$ and assume that the random variable Z belongs to L^2 and is $\mathcal{F}_{t_0}-$measurable. Then the solution X to*

$$dX_t = b_t(X_t)dt + \sigma_t(X_t)dW_t,\ t \in [t_1,T], \qquad X_{t_1} = Z,$$

exists, is unique, and given by $X^{t_0,Z}$.

Proof. The existence and the uniqueness of the solution can be proved as for the case where the initial condition is deterministic. The fact that the solution is given by $X^{t_0,Z}$ is due to the continuous dependence of $X^{t_0,x}$ with respect to x (see Theorem 1.1) and that $(W_t - W_{t_0})_{t\geq t_0}$ is independent of $\mathcal{F}_{t_0}$. □

Let us now underline an important property of the solution.

Proposition 1.5 (Semi-group property)**.** *Fix $0 \leq t_0 \leq t_1 \leq T$ and set $Z = X_{t_1}^{t_0,x_0}$. Then*

$$X_t^{t_0,x_0} = X_t^{t_1,Z} \qquad \forall t \geq t_1,\ a.s..$$

Proof. This is obvious because $Z \in \mathcal{F}_{t_1}$ and thus, from the previous proposition, $X^{t_1,Z}$ is the unique solution to

$$dX_t = b_t(X_t)dt + \sigma_t(X_t)dW_t,\ t \in [t_1,T], \qquad X_{t_1} = Z,$$

while $(X_t^{t_0,x_0})_{t\in[t_1,T]}$ is another solution to this equation. □

1.2. **The value function and a Feynman-Kac formula.** From now on, we assume that, in addition to (1.2),

$$\text{the maps } b \text{ and } \sigma \text{ are deterministic and continuous.} \tag{1.3}$$

The main consequence of this assumption is the fact that, for any $t_0 \in [0,T]$ and any $x \in \mathbb{R}^d$, the process $X^{t_0,x}$ is independent of $\mathcal{F}_{t_0}$. This can be proved by using the fact that $X^{t_0,x}$ is fixed point of the operator

$$X \to (t \to x_0 + \int_{t_0}^t b_s(X_s)ds + \int_{t_0}^t \sigma_s(X_s)dW_s)$$

which maps processes progressively measurable for the completed filtration generated by $(W_t - W_{t_0})_{t\geq t_0}$ into itself. This filtration is independent of $\mathcal{F}_{t_0}$.

We fix a terminal payoff $g : \mathbb{R}^d \to \mathbb{R}$ which we assume to be Lipschitz continuous and bounded, and define the value function

$$u(t_0,x_0) = \mathbb{E}\left[g(X_T^{t_0,x_0})\right]$$

where X^{t_0,x_0} is the solution to (1.1).

Lemma 1.6. *The map $u = u(t,x)$ is bounded, Lipschitz continuous in the space variable x, uniformly with respect to t and Holder continuous in t, uniformly in x.*

Proof. The fact that u is bounded and uniformly Lispschitz continuous in space derives easily from the assumption and of the regularity of the map $x \to X^{t_0,x}$. For the proof of the time regularity, we use the dynamic programming, proved below (fortunately without using the regularity of u in time !): for $t_0 \leq t_1 \leq T$,

$$u(t_0,x_0) = \mathbb{E}\left[u(t_1,X_{t_1}^{t_0,x_0})\right]$$

Thus

$$|u(t_0,x_0) - u(t_1,x_0)| \leq \mathbb{E}\left[|u(t_1,X_{t_1}^{t_0,x_0}) - u(t_1,x_0)|\right] \leq C\mathbb{E}\left[|X_{t_1}^{t_0,x_0} - x_0|\right] \leq C|t_1 - t_0|^{1/2},$$

where we first used the Lipschitz regularity of u in space and then the time regularity of the process X (Theorem 1.1). □

Proposition 1.7 (Dynamic programming)**.** *Let τ be an $(\mathcal{F}_t)-$stopping time with $\tau \in [t_0,T]$. Then*

$$u(t_0,x_0) = \mathbb{E}\left[u(\tau,X_\tau^{t_0,x_0})\right].$$

Proof. Let us start the proof when the stopping time is a constant $t_1 \in (t_0, T]$. By semi-group property,

$$u(t_0, x_0) = \mathbb{E}\left[\mathbb{E}\left[g(X_T^{t_1,Z}) \mid \mathcal{F}_{t_1}\right]\right],$$

where $Z = X_{t_1}^{t_0,x_0}$. As, for any $x \in \mathbb{R}^d$, $X_T^{t_1,x}$ is independent of $\mathcal{F}_{t_1}$ while Z is $\mathcal{F}_{t_1}-$measurable, we have

$$\mathbb{E}\left[g(X_T^{t_1,Z}) \mid \mathcal{F}_{t_1}\right] = \mathbb{E}\left[g(X_T^{t_1,x})\right]_{x=Z} = u(t_1, Z).$$

(the proof of the first equality can be justified first when Z is discrete, and then by passing to the limit). Taking expectation proves the result.

The case of a general stopping time can be obtained first for stopping times taking only a finite number of values and then by approximation. □

A first consequence of the dynamic programming above is that u solves a PDE—at least if it is smooth enough:

Theorem 1.8 (Feynman-Kac formula)**.** *Assume that u is $C^{1,2}$. Then u satisfies the equation (in w)*

$$\begin{cases} -\partial_t w - \mathcal{L} w = 0 & \text{in } (0,T) \times \mathbb{R}^d \\ v(T,x) = g(x) & \text{in } \mathbb{R}^d \end{cases} \tag{1.4}$$

where $\mathcal{L}\phi(t,x) = b_t(x) \cdot D\phi(t,x) + \frac{1}{2}\mathrm{tr}(\sigma_t(x)\sigma_t^T(x)D^2\phi(t,x))$ for any ϕ C^2.

Under the above assumptions, u is not $C^{1,2}$ in general. However, we will see below that u satisfies the equation in a weak sense (viscosity sense). On the other hand, in this simple setting where there is no control, u is $C^{1,2}$ as soon as the data (b, σ and g) are smooth enough (say C^3).

Proof. To do the proof we first assume that σ and u and its first and second order derivatives are bounded. The general case is left as an exercise, using a localization argument explained for Theorem 1.9. Fix t_0 and $h > 0$ small. Then by dynamic programming, we have

$$u(t_0, x_0) = \mathbb{E}\left[u(t_0 + h, X_{t_0+h})\right], \tag{1.5}$$

where we have set $X = X^{t_0,x_0}$. By Ito's formula, we have

$$u(t_0 + h, X_{t_0+h}) = u(t_0, x_0) + \int_{t_0}^{t_0+h} (\partial_t u(s, X_s) + \mathcal{L}u(s, X_s))ds + \int_{t_0}^{t_0+h} Du(s, X_s) \cdot \sigma_s(X_s) dW_s.$$

As Du and σ are bounded, the last term is a martingale. Thus pugging this equality into (1.5), we get (after simplification by $u(t_0, x_0)$),

$$0 = \mathbb{E}\left[\int_{t_0}^{t_0+h} (\partial_t u(s, X_s) + \mathcal{L}u(s, X_s))ds\right].$$

We divide by $h > 0$ and let $h \to 0$. As the integrand is bounded and a.s. continuous in s, with $X_{t_0} = x_0$, we find

$$0 = \mathbb{E}\left[\partial_t u(t_0, x_0) + \mathcal{L}u(t_0, x_0)\right].$$

The term in the expectation being deterministic, we conclude that u solves the equation. The fact that u satisfies the terminal condition is straightforward. □

Theorem 1.9 (Verification)**.** *Assume that a bounded $C^{1,2}$ map v satisfies* (1.4)*. Then $v = u$.*

In other words, if u is $C^{1,2}$, then it is the unique solution to (1.4). Note that the boundedness assumption on g and v can be relaxed. However some growth condition is required for the result to holds (typically, a polynomial growth is OK).

Proof. Assume first that σ and Dv are bounded. Set $X = X^{t_0,x_0}$. Then, by Itô's formula,

$$v(T, X_T) = v(t_0, x_0) + \int_{t_0}^{T} (\partial_s v(s, X_s) + \mathcal{L}v(s, X_s))ds + \int_{t_0}^{T} Dv(s, X_s) \cdot \sigma_s(X_s)dW_s.$$

By the equation for v, the first integral vanishes. Moreover, the last term in the RHS is a martingale because Dv and σ are bounded. Thus taking expectation,

$$\mathbb{E}\left[v(T, X_T)\right] = v(t_0, x_0).$$

As $v(T, \cdot) = g$ we conclude that $v(t_0, x_0) = u(t_0, x_0)$.

We now explain how to remove the extra condition. Fix $R > 0$ large and let τ_R be the stopping time defined by

$$\tau_R = \inf\{t \geq t_0,\ |X_t - x_0| > R\},$$

with $\tau_R = T$ if the RHS is empty. We write Itô's formula between t_0 and τ_R:

$$v(\tau_R, X_{\tau_R}) = v(t_0, x_0) + \int_{t_0}^{\tau_R} (\partial_s v(s, X_s) + \mathcal{L}v(s, X_s))ds + \int_{t_0}^{\tau_R} Dv(s, X_s) \cdot \sigma_s(X_s)dW_s.$$

As before, the first integral vanishes. In the set $[t_0, T] \times B(x_0, R)$ the maps Dv and σ are bounded by a constant M. Thus, on the interval $[[t_0, \tau_R]]$, $Dv(s, X_s)$ and $\sigma_s(X_s)$ are bounded by M. Thus $t \to \int_{t_0}^{\tau_R \wedge t} Dv(s, X_s) \cdot \sigma_s(X_s)dW_s$ is a martingale. So taking expectation, we obtain

$$\mathbb{E}\left[v(\tau_R, X_{\tau_R})\right] = v(t_0, x_0)$$

We now let $R \to \infty$. Then, by a.s. continuity of the path $t \to X_t$, $\tau_R \to T$ a.s. As v is bounded, we can use the dominate convergence theorem to pass to the limit as $R \to \infty$ and obtain the equality

$$\mathbb{E}\left[v(T, X_T)\right] = v(t_0, x_0).$$

We then conclude as before. □

2. A finite horizon problem

2.1. **Statement of the problem.** We consider an optimal control problem of the form: for any $(t_0, x_0) \in \mathbb{R}^d \times [0, T]$,

$$u(t_0, x_0) = \inf_{\alpha \in \mathcal{A}(t_0)} J(t_0, x_0, \alpha) \qquad \text{where} \qquad J(t_0, x_0, \alpha) = \mathbb{E}\left[\int_{t_0}^{T} L(X_t^\alpha, \alpha_t)dt + g(X_T^\alpha)\right],$$

where $X^\alpha = X^{t_0,x_0,\alpha}$ is the unique solution to

$$dX_t^\alpha = b(X_t^\alpha, \alpha_t)dt + \sigma(X_t^\alpha, \alpha_t)dW_t,\ t \in [t_0, T], \qquad X_{t_0}^\alpha = x_0,$$

and

$$\mathcal{A}(t_0) = \{\alpha : [t_0, T] \times \Omega \to A,\ \text{progressively measurable with respect to } \sigma(W_s,\ s \in [t_0, t])\}.$$

We work under our standing assumptions:

$$\begin{aligned} &A \text{ is a compact subset of a finite dimensional space,} \\ &(b, \sigma, L) : \mathbb{R}^d \times A \to \mathbb{R}^d \times \mathbb{R}^{d \times m} \times \mathbb{R} \text{ is continuous, bounded,} \\ &\qquad \text{and Lipschitz continuous in } x \text{ unif. in } a, \\ &g : \mathbb{R}^d \to \mathbb{R} \text{ is bounded and Lipschitz continuous.} \end{aligned}$$

These conditions are much too strong for any practical application. However, on the one hand, there is no "general framework" fitting a large number of applications, and, on the other hand, working in an elementary set-up allows to point out the key argument without bothering with technical details.

It will be convenient below to have some regularity property of the value function.

Proposition 2.1. *Under our standing assumptions, the value function u is Lipschitz continuous in space uniformly in time. In addition it is Holder continuous in time, uniformly in space.*

Proof. Throughout the proof, C denotes a positive constant which might vary from line to line but depends only on b, σ, L and g. Let us start with the regularity in space. Fix $t_0 \in [0,T]$ and $x_0, x_1 \in \mathbb{R}^d$. Let $\alpha \in \mathcal{A}(t_0)$. We set $X_t^0 = X_t^{t_0,x_0,\alpha}$ and $X_t^1 = X_t^{t_0,x_1,\alpha}$ for $t \in [t_0,T]$. Then by Theorem 1.1-(2)

$$\mathbb{E}\left[\sup_{t\in[t_0,T]} |X_t^0 - X_t^1|^2\right] \leq C|x_0 - x_1|^2.$$

Thus

$$|J(t_0,x_0,\alpha) - J(t_1,x_1,\alpha)| \leq C|x_0 - x_1|,$$

which means that the map $x \to J(t_0,x_0,\alpha)$ is Lipschitz continuous uniformly in α. Taking the infimum in α shows that

$$|u(t_0,x_0) - u(t_0,x_1)| \ \leq C|x_0 - x_1|.$$

We now prove the regularity in time. Let $x_0 \in \mathbb{R}^d$ and $0 \leq t_0 \leq t_1 \leq T$ be fixed and let $\alpha \in \mathcal{A}(t_0)$. Note that $\alpha_{|_{[t_1,T]}} \in \mathcal{A}(t_0)$. We set $X_t^0 = X_t^{t_0,x_0,\alpha}$ for $t \in [t_0,T]$ and $X_t^1 = X_t^{t_1,x_0,\alpha_{|_{[t_1,T]}}}$ for $t \in [t_1,T]$. We claim that

$$\mathbb{E}\left[\sup_{t\in[t_1,T]} |X_t^0 - X_t^1|^2\right] \leq C|t_0 - t_1|.$$

Indeed, by semi-group property, we have $X_t^0 = X_t^{t_1,X_{t_1}^0,\alpha_{|_{[t_1,T]}}}$ for $t \in [t_1,T]$. Hence, by Theorem 1.1-(2) (applied with a random initial condition) and then Theorem 1.1-(3),

$$\mathbb{E}\left[\sup_{t\in[t_1,T]} |X_t^0 - X_t^1|^2\right] \leq C\mathbb{E}\left[|X_{t_1}^0 - x_0|^2\right] \leq C|t_1 - t_0|.$$

This easily implies that

$$|J(t_0,x_0,\alpha) - J(t_1,x_0,\alpha_{|_{[t_1,T]}})| \ \leq C|t_1 - t_0|^{1/2}$$

and then we can conclude as above that

$$|u(t_0,x_0) - u(t_1,x_0)| \ \leq C|t_1 - t_0|^{1/2}.$$

□

2.2. **Dynamic programming.**

Theorem 2.2. *Let $\mathcal{T}(t_0)$ be the set of stopping times between t_0 and T. Then*

$$\begin{aligned} u(t_0,x_0) &= \inf_{\alpha\in\mathcal{A}(t_0)} \sup_{\tau\in\mathcal{T}(t_0)} \mathbb{E}\left[\int_{t_0}^{\tau} L(X_s^{t_0,x_0,\alpha},\alpha_s)ds + u(\tau, X_\tau^{t_0,x_0,\alpha})\right] \\ &= \inf_{\alpha\in\mathcal{A}(t_0)} \inf_{\tau\in\mathcal{T}(t_0)} \mathbb{E}\left[\int_{t_0}^{\tau} L(X_s^{t_0,x_0,\alpha},\alpha_s)ds + u(\tau, X_\tau^{t_0,x_0,\alpha})\right]. \end{aligned}$$

The result says that the problem is strongly Markovian: at time τ, we can forget all about the past and start again the problem with initial condition $X_\tau^{t_0,x_0,\alpha}$. This is surprising because, by definition, controls are allowed to depend on all the past. The proof of the dynamic programming in the framework of stochastic control is very delicate and shall not be completely justified here. Different argument can be found in the references quote above on stochastic control. The argument developed below encounters several measurability issue that will be overlooked here.

Sketch of proof. We only sketch the argument when the stopping time τ is reduced to a deterministic time $t_1 \in (t_0,T)$. Throughout the proof, the constant C changes from line to line. We will admit the following claims: let

$$\tilde{\mathcal{A}}(t_0) = \{\alpha : [t_0,T]\times\Omega \to A, \text{ progressively measurable with respect to } \sigma(W_s - W_{t_0},\ s \in [t_0,t])\}.$$

Then, for any (t_0, x_0) and any random variable $Z \in L^2$ which is $\mathcal{F}_{t_0}-$measurable,

$$u(t_0, x_0) = \inf_{\alpha \in \tilde{\mathcal{A}}(t_0)} J(t_0, x_0, \alpha). \tag{2.1}$$

while

$$u(t_0, Z) = \text{ess} - \inf_{\alpha \in \mathcal{A}(t_0)} \mathbb{E}\left[\int_{t_0}^{T} f(X_t^{t_0, Z, \alpha}, \alpha_t) dt + g(X_T^{t_0, Z, \alpha}) \mid \mathcal{F}_{t_0} \right]. \tag{2.2}$$

These two claims are two different ways to express the fact that, at time t_0, we can forget all about the Brownian motion before time t_0. Note for later use that, if $\alpha \in \tilde{\mathcal{A}}(t_0)$, then $X^{t_0, x_0, \alpha}$ is independent of $\mathcal{F}_{t_0}$. The proof of these statement require some analysis and strongly rely on the fact that the underlying filtration is the one associated to the Brownian motion.

Let us set

$$v(t_0, x_0) = \inf_{\alpha \in \mathcal{A}(t_0)} \mathbb{E}\left[\int_{t_0}^{t_1} L(X_s^{t_0, x_0, \alpha}, \alpha_s) ds + u(t_1, X_{t_1}^{t_0, x_0, \alpha}) \right].$$

We first prove that $u \geq v$. Fix $\varepsilon > 0$ and let $\alpha \in \mathcal{A}(t_0)$ be $\varepsilon-$optimal for $u(t_0, x_0)$. Recall that $\hat{\alpha} := \alpha_{|_{[t_1, T]}} \in \mathcal{A}(t_1)$. Then by the semi-group property and (2.2),

$$\begin{aligned} u(t_0, x_0) + \varepsilon &\geq \mathbb{E}\left[\int_{t_0}^{t_1} f(X_t^{t_0, x_0, \alpha}, \alpha_t) dt \right] + \mathbb{E}\left[\mathbb{E}\left[\int_{t_1}^{T} f(X_t^{t_1, X_{t_1}^{t_0, x_0, \alpha}, \hat{\alpha}}, \hat{\alpha}_t) dt + g(X_T^{t_1, X_{t_1}^{t_0, x_0, \alpha}}) \mid \mathcal{F}_{t_1} \right] \right] \\ &\geq \mathbb{E}\left[\int_{t_0}^{t_1} f(X_t^{t_0, x_0, \alpha}, \alpha_t) dt \right] + \mathbb{E}\left[u(t_1, X_{t_1}^{t_0, x_0, \alpha}) \right] \;\geq\; v(t_0, x_0). \end{aligned}$$

This implies that $u \geq v$.

We now turn to the reverse inequality $u \leq v$. Let $(Q_k)_{k \in \mathbb{N}}$ be a partition of $\mathbb{R}^d$ with sets of diameter less than ε and let $x_k \in Q_k$ for any k. We also fix $\alpha_k \in \tilde{\mathcal{A}}(t_1)$ which is $\varepsilon-$optimal for $u(t_1, x_k)$ (recall equality (2.1)):

$$u(t_1, x_k) + \varepsilon \geq \mathbb{E}\left[\int_{t_1}^{T} f(X_t^{t_1, x_k, \alpha_k}, \alpha_{k,t}) dt + g(X_T^{t_1, x_k, \alpha_k}) \right].$$

Finally, let $\tilde{\alpha} \in \mathcal{A}(t_0)$ be $\varepsilon-$optimal for $v(t_0, x_0)$:

$$v(t_0, x_0) + \varepsilon \geq \mathbb{E}\left[\int_{t_0}^{t_1} L(X_s^{t_0, x_0, \tilde{\alpha}}, \tilde{\alpha}_s) ds + u(t_1, X_{t_1}^{t_0, x_0, \tilde{\alpha}}) \right].$$

We set

$$\alpha_t = \begin{cases} \tilde{\alpha}_t & \text{if } t \in [t_0, t_1), \\ \alpha_k & \text{if } t \in [t_1, T] \text{ and } X_{t_1}^{t_0, x_0, \tilde{\alpha}} \in Q_k, \end{cases}$$

and $X_t = X_t^{t_0, x_0, \alpha}$. We note that, on $[t_0, t_1]$, $X_t = X_t^{t_0, x_0, \tilde{\alpha}}$. Thus

$$u(t_0, x_0) \leq \mathbb{E}\left[\int_{t_0}^{t_1} f(X_t^{t_0, x_0, \tilde{\alpha}}, \tilde{\alpha}_t) dt + \sum_{k \in \mathbb{N}} \mathbf{1}_{\{X_{t_1} \in Q_k\}} \left(\int_{t_1}^{T} f(X_t, \alpha_{k,t}) dt + g(X_T) \right) \right].$$

By (a small extension of) Theorem 1.1-(2) and since, in $\{X_{t_1} \in Q_k\}$, $X_t = X_t^{t_1, X_{t_1}, \alpha_{k,t}}$ for $t \in [t_1, T]$, we have

$$\mathbb{E}\left[\sup_{t \in [t_1, T]} |X_t - X_t^{t_1, x_k, \alpha_k}| \mathbf{1}_{\{X_{t_1} \in Q_k\}} \mid \mathcal{F}_{t_1} \right] \leq |X_{t_1} - x_k| \mathbf{1}_{\{X_{t_1} \in Q_k\}} \leq \varepsilon \mathbf{1}_{\{X_{t_1} \in Q_k\}},$$

since the diameter of Q_k is less than ε. Thus

$$\mathbb{E}\left[\mathbf{1}_{\{X_{t_1} \in Q_k\}} \left| \int_{t_1}^{T} f(X_t, \alpha_{k,t}) dt + g(X_T) - \left(\int_{t_1}^{T} f(X_t^{t_1, x_k, \alpha_k}, \alpha_{k,t}) dt + g(X^{t_1, x_k, \alpha_k}) \right) \right| \mid \mathcal{F}_{t_1} \right] \leq C \varepsilon \mathbf{1}_{\{X_{t_1} \in Q_k\}}$$

and therefore, recalling the definition of α_k which belongs to $\tilde{\mathcal{A}}(t_1)$,

$$\begin{aligned}
&\mathbb{E}\left[\mathbf{1}_{\{X_{t_1}\in Q_k\}}(\int_{t_1}^T f(X_t,\alpha_{k,t})dt + g(X_T)) \mid \mathcal{F}_{t_1}\right] \\
&\quad\le \mathbb{E}\left[\mathbf{1}_{\{X_{t_1}\in Q_k\}}(\int_{t_1}^T f(X_t^{t_1,x_k,\alpha_k},\alpha_{k,t})dt + g(X_T^{t_1,x_k,\alpha_k})) \mid \mathcal{F}_{t_1}\right] + C\varepsilon\mathbf{1}_{\{X_{t_1}\in Q_k\}} \\
&\quad= \mathbf{1}_{\{X_{t_1}\in Q_k\}}\mathbb{E}\left[\int_{t_1}^T f(X_t^{t_1,x_k,\alpha_k},\alpha_{k,t})dt + g(X_T^{t_1,x_k,\alpha_k}))\right] + C\varepsilon\mathbf{1}_{\{X_{t_1}\in Q_k\}} \\
&\quad\le (u(t_1,x_k)+C\varepsilon)\mathbf{1}_{\{X_{t_1}\in Q_k\}} \;\le\; (u(t_1,X_{t_1})+C\varepsilon)\mathbf{1}_{\{X_{t_1}\in Q_k\}}
\end{aligned}$$

This implies that

$$\begin{aligned}
u(t_0,x_0) &\le \mathbb{E}\left[\int_{t_0}^{t_1} f(X_t^{t_0,x_0,\tilde\alpha},\tilde\alpha_t)dt + \sum_{k\in\mathbb{N}}(u(t_1,X_{t_1})+C\varepsilon)\mathbf{1}_{\{X_{t_1}\in Q_k\}}\right] \\
&= \mathbb{E}\left[\int_{t_0}^{t_1} f(X_t^{t_0,x_0,\tilde\alpha},\tilde\alpha_t)dt + u(t_1,X_{t_1}^{t_0,x_0,\tilde\alpha})\right] + C\varepsilon \;\le\; v(t_0,x_0)+C\varepsilon,
\end{aligned}$$

where we used the definition of $\tilde\alpha$ in the last inequality. This shows that $u\le v$.

We now briefly justify (2.1) and (2.2). We work on the canonical space Ω for the Brownian motion: $\omega\in\Omega$ if $\omega:[0,T]\to\mathbb{R}^m$ is continuous with $\omega(0)=0$. We let $\Omega_0=\{x:[0,t_0]\to\mathbb{R}^m$, continuous with $x(0)=0\}$ and $\Omega_1=\{x:[t_0,T]\to\mathbb{R}^m$, continuous with $x(t_0)=0\}$. All these spaces are endowed with the topology of uniform convergence. Note that there is a bi-continuous bijection between $\Omega_0\times\Omega_1$ and Ω given by $j(\omega_0,\omega_1)=\omega_0\mathbf{1}_{[0,t_0)}+(\omega_1+\omega_0(t_0))\mathbf{1}_{[t_0,T]}$. In addition, if $\mathbb{P}_0$ and $\mathbb{P}_1$ are the Wiener measures on Ω_0 and Ω_1 respectively, then $\mathbb{P}$ is the image by j of $\mathbb{P}_0\otimes\mathbb{P}_1$: for any bounded Borel measurable map $f:\Omega\to\mathbb{R}$,

$$\int_\Omega f(\omega)\mathbb{P}(d\omega)=\int_{\Omega_0\times\Omega_1} f(j(\omega_0,\omega_1))\mathbb{P}_0(d\omega_0)\mathbb{P}_1(d\omega_1).$$

With these notations, the conditional expectation with respect to $\mathcal{F}_{t_0}$ is nothing but the integration with respect to $\mathbb{P}_1$:

$$\mathbb{E}^{\mathbb{P}}\left[f\mid\mathcal{F}_{t_0}\right]=\mathbb{E}^{\mathbb{P}_1}\left[f(j(\omega_0,\cdot))\right]\qquad\text{for }\mathbb{P}_0\text{−a.e }\omega_0\in\Omega_0.$$

For simplicity, we do not write j anymore in the equalities. Given $\alpha\in\mathcal{A}(t_0)$ and $Z\in L^2$ an $\mathcal{F}_{t_0}$−measurable random variable, one can check that $t\to\alpha(t,\omega_0,\cdot)$ belongs to $\tilde{\mathcal{A}}(t_0)$ and that $X^{t_0,Z,\alpha}(t,\omega_0,\cdot)=X^{t_0,Z(\omega_0),\alpha(\omega_0,\cdot)}(t,\cdot)$ $\mathbb{P}_1$−a.s., for $\mathbb{P}_0$−a.e. ω_0. Moreover,

$$\mathbb{E}\left[\int_{t_0}^T f(X_t^{t_0,Z,\alpha},\alpha_t)dt+g(X_T^{t_0,Z,\alpha})\mid\mathcal{F}_{t_0}\right](\omega_0)=J(t_0,Z(\omega_0),\alpha(\omega_0,\cdot))\qquad\text{for }\mathbb{P}_0\text{−a.e. }\omega_0. \quad (2.3)$$

Let

$$\tilde u(t_0,x_0)=\inf_{\tilde\alpha\in\tilde{\mathcal{A}}(t_0)} J(t_0,x_0,\tilde\alpha).$$

Note that $\tilde u\ge u$ because $\tilde{\mathcal{A}}(t_0)\subset\mathcal{A}(t_0)$. As, given $\alpha\in\mathcal{A}(t_0)$, $t\to\alpha(t,\omega_0,\cdot)$ belongs to $\tilde{\mathcal{A}}(t_0)$ for $\mathbb{P}_0$−a.e. ω_0, we have

$$J(t_0,x_0,\alpha(\omega_0,\cdot))\ge\tilde u(t_0,x_0)\qquad\text{for }\mathbb{P}_0\text{−a.e. }\omega_0.$$

Thus

$$u(t_0,x_0)=\inf_{\alpha\in\mathcal{A}(t_0)}J(t_0,x_0,\alpha)=\inf_{\alpha\in\mathcal{A}(t_0)}\mathbb{E}\left[J(t_0,x_0,\alpha(\omega_0,\cdot))\right]\ge\tilde u(t_0,x_0).$$

This shows (2.1). Fix $\varepsilon>0$. We note that, by the continuity of $x\to u(t_0,x)$ and of $x\to J(t_0,x,\alpha)$ (uniform in α) it is possible to find a measurable family $x\to\tilde\alpha_x$ of elements of $\tilde{\mathcal{A}}(t_0)$ such that

$$u(t_0,x)+\varepsilon\ge J(t_0,x,\tilde\alpha_x)\qquad\forall x\in\mathbb{R}^d.$$

Thus, for Z as above,

$$u(t_0, Z(\omega_0)) + \varepsilon \geq J(t_0, Z(\omega_0), \tilde{\alpha}_{Z(\omega_0)}) \qquad \text{for } \mathbb{P}_0\text{−a.e. } \omega_0.$$

Note that $\tilde{\alpha}_{Z(\omega_0)} \in \mathcal{A}(t_0)$ and, by (2.3),

$$J(t_0, Z(\omega_0), \tilde{\alpha}_{Z(\omega_0)}) = \mathbb{E}\left[\int_{t_0}^{T} f(X_t^{t_0,Z,\alpha}, \alpha_t)dt + g(X_T^{t_0,Z,\alpha}) \mid \mathcal{F}_{t_0}\right](\omega_0),$$

so that (2.2) holds. □

2.3. **HJ equation and verification theorem.**

Theorem 2.3. *Under the above conditions, assume that u is of class $C^{1,2}$ on $[0,T] \times \mathbb{R}^d$. Then u solves the Hamilton-Jacobi (HJ) equation*

$$\begin{cases} -\partial_t u(t,x) + H(x, Du(t,x), D^2u(t,x)) = 0 & \text{in } (0,T) \times \mathbb{R}^d, \\ u(T,x) = g(x) & \text{in } \mathbb{R}^d, \end{cases}$$

where, for any $(x,p,S) \in \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^{d\times d}$,

$$H(x,p,S) = \max_{a\in A}\left\{-L(x,a) - b(x,a)\cdot p - \frac{1}{2}\mathrm{Tr}(\sigma\sigma^T(x,a)S)\right\}.$$

The HJ equation is backward in time: one has a terminal condition at time T, and one infers from the equation the values of the function at earlier times.

Theorem 2.3 is not very useful in practice because it is almost impossible to prove the $C^{1,2}$ regularity of u just from its definition (and in general, this regularity does not hold). The interest of the result is that it allows to guess the HJ equation.

Proof. The proof is done in two steps: first we prove the (relatively easier) inequality

$$-\partial_t u(t,x) + H(x, Du(t,x), D^2u(t,x)) \leq 0 \tag{2.4}$$

and then the reverse one. Note that the terminal condition holds by the definition of $u(T,x)$.

Proof of (2.4): Fix $(t_0,x_0) \in (0,T)\times\mathbb{R}^d$, $a \in A$ and let us consider the constant control $\alpha_t^a = a$ for any $t \in [t_0,T]$. Note that $\alpha^a \in \mathcal{A}(t_0)$. Given a radius $R > 0$ and $h \in (0, T-t_0)$, we consider the stopping times

$$\tau_R = \inf\{t \geq t_0,\ |X_t^{\alpha^a} - x_0| \geq R\},\ \tau_{R,h} = \tau_R \wedge (t_0 + h),$$

(with $\tau_R = T$ if there is no such t). By Dynamic Programming we have (where $\mathcal{S}(t_0)$ is the set of stopping times $\tau \geq t_0$):

$$\begin{aligned} u(t_0,x_0) &= \inf_{\alpha\in\mathcal{A}(t_0)} \inf_{\tau\in\mathcal{S}(t_0)} \mathbb{E}\left[\int_{t_0}^{\tau} L(X_t^\alpha, \alpha_t)dt + u(\tau, (X_\tau^\alpha))\right] \\ &\leq \mathbb{E}\left[\int_{t_0}^{\tau_{R,h}} L(X_t^{\alpha^a}, \alpha_t^a)dt + u(\tau_{R,h}, (X_{\tau_{R,h}}^{\alpha^a}))\right]. \end{aligned}$$

Using Itô's formula and the definition of α^a, we get

$$\begin{aligned} u(t_0,x_0) \leq \mathbb{E}\Big[&\int_{t_0}^{\tau_{R,h}} (L + \partial_t u + b\cdot Du + \frac{1}{2}\mathrm{Tr}(\sigma\sigma^T D^2u))(t, X_t^{\alpha^a}, a)dt \\ &+ \int_{t_0}^{\tau_{R,h}} (\sigma^T Du)(t, X_t^{\alpha^a}, a)\cdot dW_t + u(t_0,x_0)\Big]. \end{aligned}$$

On $[[t_0, \tau_{R,h}]]$, X_t^α remains in the ball $B_R(x_0)$ where Du and σ are bounded. Therefore the local martingale term is a martingale. Hence, simplifying by $u(t_0,x_0)$ and dividing by $h > 0$ gives

$$0 \leq \mathbb{E}\left[\frac{1}{h}\int_{t_0}^{\tau_{R,h}} (L + \partial_t u + b\cdot Du + \frac{1}{2}\mathrm{Tr}(\sigma\sigma^T D^2u))(t, X_t^{\alpha^a}, a)dt\right].$$

We now let $h \to 0^+$. Note that the integrand is bounded and converges a.s. to $(L + \partial_t u + b \cdot Du + \frac{1}{2}\mathrm{Tr}(\sigma\sigma^T D^2 u))(t_0, x_0)$ as X^{α^a} has a continuous path and equals x_0 at t_0. Note also that, as $\tau_R > t_0$ a.s.,

$$\mathbf{1}_{\{\tau_{R,h} = t_0 + h\}} \to 1 \qquad \text{as } h \to 0^+,$$

and thus, a.s.,

$$\frac{1}{h}\int_{t_0}^{\tau_{R,h}} (L + \partial_t u + b \cdot Du + \frac{1}{2}\mathrm{Tr}(\sigma\sigma^T D^2 u))(t, X_t^{\alpha^a}, a)dt \to (L + \partial_t u + b \cdot Du + \frac{1}{2}\mathrm{Tr}(\sigma\sigma^T D^2 u))(t_0, x_0).$$

By Lebesgue Theorem, this implies that

$$0 \leq (L + \partial_t u + b \cdot Du + \frac{1}{2}\mathrm{Tr}(\sigma\sigma^T D^2 u))(t_0, x_0, a).$$

As $a \in A$ is arbitrary, and after changing sign, this inequality can be rewritten as

$$-\partial_t u(t_0, x_0) + \sup_{a \in A}\left\{-L(x_0, a) - b(x_0, a) \cdot Du(t_0, x_0) - \frac{1}{2}\mathrm{Tr}(\sigma\sigma^T(x_0, a) D^2 u(t_0, x_0))\right\} \leq 0.$$

This is (2.4) by the definition of H.

Proof of the reverse inequality:

$$-\partial_t u(t, x) + H(x, Du(t, x), D^2 u(t, x)) \geq 0 \qquad \text{in } (0, T) \times \mathbb{R}^d. \tag{2.5}$$

The proof is slightly more technical than the previous one. We argue by contradiction, assuming that there exists $(t_0, x_0) \in (0, T) \times \mathbb{R}^d$ and $\eta > 0$ such that

$$-\partial_t u(t_0, x_0) + H(x_0, Du(t_0, x_0), D^2 u(t_0, x_0)) \leq -2\eta.$$

We show below (Lemma 2.4) that H is a continuous map. Thus there exists $r > 0$ such that, for any $(t, x) \in [t_0, t_0 + r] \times B_r(x_0)$,

$$-\partial_t u(t, x) + H(x, Du(t, x), D^2 u(t, x)) \leq -\eta.$$

By the definition of H this inequality says that, for any $(t, x, a) \in [t_0, t_0 + r] \times B_r(x_0) \times A$,

$$-\partial_t u(t, x) - L(x, a) - b(x, a) \cdot Du(t, x) - \frac{1}{2}\mathrm{Tr}(\sigma\sigma^T(x, a) D^2 u(t, x)) \leq -\eta.$$

Changing sign, we get:

$$\partial_t u(t, x) + L(x, a) + b(x, a) \cdot Du(t, x) + \frac{1}{2}\mathrm{Tr}(\sigma\sigma^T(x, a) D^2 u(t, x)) \geq \eta. \tag{2.6}$$

Given a control $\alpha \in \mathcal{A}(t_0)$ and $h > 0$, we define the stopping times

$$\tau_\alpha = \inf\{t \geq t_0,\ |X_t^\alpha - x_0| \geq r\},\ \tau_{\alpha,h} = \tau_\alpha \wedge (t_0 + h),$$

(with $\tau_\alpha = T$ if there is no such t). By dynamic programming we get

$$\begin{aligned} u(t_0, x_0) &= \inf_{\alpha \in \mathcal{A}(t_0)} \sup_{\tau \in \mathcal{S}(t_0)} \mathbb{E}\left[\int_{t_0}^{\tau} L(X_t^\alpha, \alpha_t)dt + u(\tau, (X_\tau^\alpha))\right] \\ &\geq \inf_{\alpha \in \mathcal{A}(t_0)} \mathbb{E}\left[\int_{t_0}^{\tau_{\alpha,h}} L(X_t^\alpha, \alpha_t)dt + u(\tau_{\alpha,h}, (X_{\tau_{\alpha,h}}^\alpha))\right]. \end{aligned}$$

Fixing $\varepsilon \in (0, \eta)$, let α_h be $(\varepsilon h)-$optimal in the right-hand side of the inequality above. Using Itô's formula (and the usual argument to get rid of the local martingale part) we obtain:

$$\begin{aligned} u(t_0, x_0) + \varepsilon h &\geq \mathbb{E}\left[\int_{t_0}^{\tau_{\alpha_h,h}} L(X_t^{\alpha_h}, \alpha_{h,t})dt + u(\tau_{\alpha_h,h}, (X_{\tau_{\alpha_h,h}}^{\alpha_h}))\right] \\ &\geq u(t_0, x_0) + \mathbb{E}\left[\int_{t_0}^{\tau_{\alpha_h,h}} (L + \partial_t u + b \cdot Du + \frac{1}{2}\mathrm{Tr}(\sigma\sigma^T D^2 u))(t, X_t^{\alpha_h}, \alpha_{h,t})dt\right]. \end{aligned}$$

For $h \in (0,r)$ and $t \in [[t_0, \tau_{\alpha_h,h}]]$, we have that $X_t^{\alpha_h} \in B_r(x_0)$ by the definition of $\tau_{\alpha_h,h}$. Hence we can use inequality (2.6) to obtain, after simplification by $u(t_0,x_0)$:

$$\varepsilon h \geq \mathbb{E}\left[\int_{t_0}^{\tau_{\alpha_h,h}} \eta dt\right] = \eta \mathbb{E}\left[\tau_{\alpha_h,h} - t_0\right].$$

We divide by $h > 0$ and rewrite this inequality as

$$\begin{aligned}\varepsilon &\geq \eta \mathbb{E}\left[\mathbf{1}_{\{\tau_{\alpha_h,h}=t_0+h\}} + \frac{\tau_{\alpha_h,h}-t_0}{h}\mathbf{1}_{\{\tau_{\alpha_h,h}<t_0+h\}}\right] \\ &\geq \eta \mathbb{P}\left[\tau_{\alpha_h,h} = t_0+h\right]. \end{aligned} \tag{2.7}$$

Note that, by Markov inequality,

$$\mathbb{P}\left[\tau_{\alpha_h,h} < t_0+h\right] = \mathbb{P}\left[\sup_{t\in[t_0,t_0+h]} |X_t^{\alpha_h} - x_0| > r\right] \leq \frac{\mathbb{E}\left[\sup_{t\in[t_0,t_0+h]} |X_t^{\alpha_h}-x_0|^2\right]}{r^2} \leq \frac{Ch}{r^2},$$

where the constant C depends only on $\|b\|_\infty + \|\sigma\|_\infty$. Thus

$$\mathbb{P}\left[\tau_{\alpha_h,h} = t_0+h\right] \to 1 \text{ as } h \to 0^+.$$

Letting $h \to 0^+$ in (2.7) we find a contradiction with $\varepsilon < \eta$. This proves (2.5). □

It remains to check the continuity of H.

Lemma 2.4. *Under our standing assumption, $H : [0,T] \times \mathbb{R}^d \times \mathbb{R}^{d\times d} \to \mathbb{R}$ is continuous.*

Proof. Let $(x_n, p_n, S_n) \to (x,p,S)$. Fix $a \in A$. Then

$$H(x_n,p_n,S_n) \geq -L(x_n,a) - b(x_n,a)\cdot p_n - \frac{1}{2}\mathrm{Tr}(\sigma\sigma^T(x_n,a)S_n).$$

Taking the lim inf and using the continuity of the data we get

$$\liminf_n H(x_n,p_n,S_n) \geq -L(x,a) - b(x,a)\cdot p - \frac{1}{2}\mathrm{Tr}(\sigma\sigma^T(x,a)S).$$

As this holds for any $a \in A$, we obtain

$$\liminf_n H(x_n,p_n,S_n) \geq \sup_{a\in A}\left\{-L(x,a) - b(x,a)\cdot p - \frac{1}{2}\mathrm{Tr}(\sigma\sigma^T(x,a)S)\right\} = H(x,p,S).$$

For any $n \in \mathbb{N}$, there exists $a_n \in A$ optimal for $H(x_n,p_n,S_n)$:

$$H(x_n,p_n,S_n) = -L(x_n,a_n) - b(x_n,a_n)\cdot p_n - \frac{1}{2}\mathrm{Tr}(\sigma\sigma^T(x_n,a_n)S_n).$$

Let (n_k) be a subsequence such that

$$\lim_k H(x_{n_k},p_{n_k},S_{n_k}) = \limsup_n H(x_n,p_n,S_n)$$

and the compact sequence (a_{n_k}) has a limit a (recall that A is compact). Then, by continuity of the data,

$$\begin{aligned}\limsup_n H(x_n,p_n,S_n) &= \lim_k -L(x_{n_k},a_{n_k}) - b(x_{n_k},a_{n_k})\cdot p_{n_k} - \frac{1}{2}\mathrm{Tr}(\sigma\sigma^T(x_{n_k},a_{n_k})S_{n_k}) \\ &= -L(x,a) - b(x,a)\cdot p - \frac{1}{2}\mathrm{Tr}(\sigma\sigma^T(x,a)S) \leq H(x,p,S).\end{aligned}$$

So we have proved that

$$H(x,p,S) \leq \liminf_n H(x_n,p_n,S_n) \leq \limsup_n H(x_n,p_n,S_n) \leq H(x,p,S),$$

thus showing the continuity of H at (x,p,S). □

We now turn to a much more interesting statement: the Verification Theorem. This result says that smooth solutions of the HJ equation are equal to the value function and allow to characterize the optimal control.

Theorem 2.5. *Assume that w is a $C^{1,2}$ bounded solution to the HJ equation*

$$\begin{cases} -\partial_t w(t,x) + H(x, Dw(t,x), D^2 w(t,x)) = 0 & \text{in } (0,T)\times\mathbb{R}^d, \\ w(T,x) = g(x) & \text{in } \mathbb{R}^d, \end{cases}$$

(1) *Then $w \le u$ in $[0,T]\times\mathbb{R}^d$.*
(2) *If in addition, if, given $(t_0,x_0)\in(0,T)\times\mathbb{R}^d$, there exists $\alpha^*\in\mathcal{A}(t_0)$ such that, a.s. and for a.e. $t\in(t_0,T)$,*

$$\begin{aligned} &H(X_t^{\alpha^*}, Dw(t,X_t^{\alpha^*}), D^2w(t,X_t^{\alpha^*})) \\ &\quad = -L(X_t^{\alpha^*},\alpha_t^*) - b(X_t^{\alpha^*},\alpha_t^*)\cdot Dw(t,X_t^{\alpha^*}) - \frac{1}{2}\mathrm{Tr}(\sigma\sigma^T(X_t^{\alpha^*},\alpha_t^*)D^2w(t,X_t^{\alpha^*}), \end{aligned} \tag{2.8}$$

then α^* is optimal for $u(t_0,x_0)$ and $u(t_0,x_0)=w(t_0,x_0)$.

Equality (2.8) says that α_t^* is a maximum point over A of

$$a \to -L(X_t^{\alpha^*},a) - b(X_t^{\alpha^*},a)\cdot Dw(t,X_t^{\alpha^*}) - \frac{1}{2}\mathrm{Tr}(\sigma\sigma^T(X_t^{\alpha^*},a)D^2w(t,X_t^{\alpha^*})$$

The second part of the result can also be stated as follows: Assume that there exists a Borel measurable map $a^*:[0,T]\times\mathbb{R}^d\to A$ satisfying

$$H(x,Dw(t,x),D^2w(t,x)) = -L(x,a^*(t,x)) - b(x,a^*(t,x))\cdot Dw(t,x) - \frac{1}{2}\mathrm{Tr}(\sigma\sigma^T(x,a^*(t,x))D^2w(t,x))$$

for any $(t,x)\in(0,T)\times\mathbb{R}^d$ and that the SDE

$$dX_t = b(X_t,a^*(t,X_t))dt + \sigma(t,X_t)dW_t,\ t\in(t_0,T) \qquad X_{t_0}=x_0,$$

has a solution X^*. Then the control $(\alpha_t^* = a^*(t,X_t^*))$ is optimal for $u(t_0,x_0)$ and $u(t_0,x_0) = w(t_0,x_0)$. The map a^* is called an optimal feedback: it allows to take an optimal action in view of the current time t and the current position x.

An example. Before giving the proof, let us discuss a simple example. This example (a linear-quadratic problem) does not fit our assumptions. It is not difficult, however, to adapt the proof of the verification theorem to this setting. We consider an optimal control problem of the form: for any $(t_0,x_0)\in\mathbb{R}^d\times[0,T]$,

$$u(t_0,x_0) = \inf_{\alpha\in\mathcal{A}(t_0)} \mathbb{E}\left[\int_{t_0}^T (\frac{1}{2}|\alpha_t|^2 + X_t^\alpha\cdot c)dt + \frac{1}{2}|X_T^\alpha|^2\right],$$

where $X^\alpha = X^{t_0,x_0,\alpha}$ is the unique solution to

$$dX_t^\alpha = \alpha_t dt + \sigma dW_t,\ t\in[t_0,T], \qquad X_{t_0}^\alpha = x_0,$$

and

$$\begin{aligned} \mathcal{A}(t_0) = \Big\{&\alpha:[t_0,T]\times\Omega\to A,\ \text{progressively measurable with respect to } \sigma(W_s - W_{t_0},\ s\in[t_0,t]), \\ &\text{and } \mathbb{E}\left[\textstyle\int_{t_0}^T |\alpha_t|^2 dt\right] < \infty\Big\}. \end{aligned}$$

Above $c\in\mathbb{R}^d$ is fixed, the set of controls A is $\mathbb{R}^d$ $(A=\mathbb{R}^d)$, $\sigma\in\mathbb{R}$ is a constant and (W_t) is a $d-$dimensional Brownian motion. This corresponds to our previous optimal control problem with

$$L(x,a) = \frac{1}{2}|a|^2 + c\cdot x,\ b(x,a)=a,\ \sigma(x,a)=\sigma,\ g(x)=\frac{1}{2}|x|^2.$$

Let us again underline that our standing assumption are not met. However, as we changed slightly the definition of $\mathcal{A}(t_0)$, for any $\alpha \in \mathcal{A}(t_0)$ there is a unique solution X^α to the SDE and the cost is well defined. We have

$$\begin{aligned} H(x,p,S) &= \sup_{a\in\mathbb{R}^d} -L(x,a) - b(x,a)\cdot p - \frac{1}{2}\mathrm{Tr}(\sigma\sigma^T(t,x)S) \\ &= \sup_{a\in\mathbb{R}^d} -\frac{1}{2}|a|^2 - c\cdot p - a\cdot p - \frac{1}{2}\sigma^2\mathrm{Tr}(S) \\ &= \frac{1}{2}|p|^2 - c\cdot p - \frac{1}{2}\sigma^2\mathrm{Tr}(S) \end{aligned}$$

where the sup is reached by $\bar{a}(x,p,S) = \bar{a}(p) = -p$. In order to apply the Verification Theorem we need to compute a smooth solution to the HJ equation

$$\begin{cases} -\partial_t w(t,x) - \frac{\sigma^2}{2}\Delta w(t,x) + \frac{1}{2}|Dw(t,x)|^2 - c\cdot Dw(t,x) = 0 & \text{in } (0,T)\times\mathbb{R}^d, \\ w(T,x) = \frac{1}{2}|x|^2 & \text{in } \mathbb{R}^d, \end{cases}$$

In view of the specific form of the equation, we look for a solution in the form of a second order polynomial in x:

$$w(t,x) = \alpha(t) + \beta(t)\cdot x + \frac{1}{2}\gamma(t)|x|^2,$$

where $\alpha,\gamma : [0,T]\to\mathbb{R}$ and $\beta : [0,T]\to\mathbb{R}^d$ are C^1 maps to determine. Note that

$$Dw(t,x) = \beta(t) + \gamma(t)x, \qquad D^2w(t,x) = \gamma(t)I_d.$$

Plugging this choice of w into the HJ equation, we see that (α,β,γ) has to satisfy:

$$\begin{cases} -(\alpha'(t) + \beta'(t)\cdot x + \frac{1}{2}\gamma'(t)|x|^2) - \frac{\sigma^2}{2}\gamma(t)d + \frac{1}{2}|\beta(t)+\gamma(t)x|^2 - c\cdot(\beta(t)+\gamma(t)x) = 0 & \text{in } (0,T)\times\mathbb{R}^d, \\ \alpha(T) + \beta(T)\cdot x + \frac{1}{2}\gamma(T)|x|^2 = \frac{1}{2}|x|^2 & \text{in } \mathbb{R}^d, \end{cases}$$

Identifying the constant (in x) terms, the linear (in x) terms and the quadratic (in x) ones, we find that (α,β,γ) has to satisfy:

$$\begin{aligned} &-\frac{1}{2}\gamma'(t) + \frac{1}{2}(\gamma(t))^2 = 0 \qquad \text{(quadratic terms)} \\ &-\beta'(t) + \beta(t)\gamma(t) - \gamma(t)c = 0 \qquad \text{(linear terms)} \\ &-\alpha'(t) - \frac{\sigma^2}{2}\gamma(t)d + \frac{1}{2}|\beta(t)|^2 - c\cdot\beta(t) = 0 \qquad \text{(constant terms)} \\ &\alpha(T) = 0,\ \beta(T) = 0,\ \gamma(T) = \frac{1}{2} \qquad \text{(terminal condition)} \end{aligned}$$

One easily checks the existence of a C^1 solution to these ODEs: the first one can be solved explicitly: $\gamma(t) = (T-t+1)^{-1}$, while the second one gives the existence of β and the third one of α. This implies that there is a classical solution to the HJ equation, and the optimal feedback is given by

$$a^*(t,x) = \bar{a}(x, Du(t,x), D^2u(t,x)) = -Du(t,x) = -(\beta(t)+\gamma(t)x).$$

Thus, for any initial condition $(t_0,x_0)\in(0,T)\times\mathbb{R}^d$, the SDE

$$dX_t = -(\beta(t)+\gamma(t)X_t)dt + \sigma dW_t,\ t\in(t_0,T), \qquad X_{t_0} = x_0,$$

has a unique solution, which proves that w is the value function of the problem.

Proof of the Verification Theorem. It is (again) a simple application of Itô's formula. Fix $(t_0,x_0)\in(0,T)\times\mathbb{R}^d$ and let $\alpha\in\mathcal{A}(t_0)$ be an arbitrary control. Let $R>0$ be large and let us define the stopping time

$$\tau_R = \inf\{t\in[t_0,T],\ |X_t^\alpha - x_0|\geq R\},$$

(with $\tau_R = T$ if the right-hand side is empty). We compute by Itô's formula:

$$w(\tau_R, X^\alpha_{\tau_R}) = w(t_0,x_0) + \int_{t_0}^{\tau_R} (\partial_t u + b\cdot Du + \frac{1}{2}\mathrm{Tr}(\sigma\sigma^T D^2 u))(t, X^\alpha_t, \alpha_t)dt + \int_{t_0}^{\tau_R} (\sigma^T Du)(t, X^\alpha_t, \alpha_t)\cdot dW_t.$$

On $[[t_0, \tau_R]]$, X^α_t remains in $B_R(x_0)$, and thus $(\sigma^T Du)(t, X^\alpha_t, \alpha_t)$ is bounded. We take expectation to infer that

$$\mathbb{E}\left[w(\tau_R, X^\alpha_{\tau_R})\right] = w(t_0,x_0) + \mathbb{E}\left[\int_{t_0}^{\tau_R} (\partial_t u + b\cdot Du + \frac{1}{2}\mathrm{Tr}(\sigma\sigma^T D^2 u))(t, X^\alpha_t, \alpha_t)dt\right]. \tag{2.9}$$

By the HJ equation satisfied by w and definition of H, we have that, for any $(t,x,a) \in (t_0,T)\times \mathbb{R}^d \times A$,

$$\partial_t w(t,x) = H(x, Dw(t,x), D^2 w(t,x)) \geq -L(x,a) - b(x,a)\cdot Dw(t,x) - \frac{1}{2}\mathrm{Tr}(\sigma\sigma^T(x,a)D^2 w(t,x)),$$

which can be rewritten as

$$\partial_t w(t,x) + b(x,a)\cdot Dw(t,x) + \frac{1}{2}\mathrm{Tr}(\sigma\sigma^T(x,a)D^2 w(t,x)) \geq -L(x,a).$$

Plugging this inequality applied at $(t, X^\alpha_t, \alpha_t)$ into (2.9) we obtain

$$\mathbb{E}\left[w(\tau_R, X^\alpha_{\tau_R})\right] \geq w(t_0,x_0) - \mathbb{E}\left[\int_{t_0}^{\tau_R} L(X^\alpha_t, \alpha_t)dt\right].$$

Our aim is to let $R \to \infty$. Then $\tau_R \to T$ a.s. As w is bounded (this is here that this assumption is used) and continuous, and that the same holds for L, we obtain by Lebesgue Theorem:

$$w(t_0,x_0) \leq \mathbb{E}\left[\int_{t_0}^{T} L(X^\alpha_t, \alpha_t)dt + w(T, X^\alpha_T)\right].$$

Recalling the terminal condition for w: $w(T\cdot) = g$ and the fact that $\alpha \in \mathcal{A}(t_0)$ is arbitrary, we get

$$w(t_0,x_0) \leq \inf_{\alpha\in\mathcal{A}(t_0)} \mathbb{E}\left[\int_{t_0}^{T} L(X^\alpha_t, \alpha_t)dt + g(X^\alpha_T)\right] = u(t_0,x_0).$$

This proves the first statement of the theorem.

Let now α^* be as in the second statement of the Theorem. Using the same argument as above we obtain:

$$\mathbb{E}\left[w(\tau_R, X^{\alpha^*}_{\tau_R})\right] = w(t_0,x_0) + \mathbb{E}\left[\int_{t_0}^{\tau_R} (\partial_t u + b\cdot Du + \frac{1}{2}\mathrm{Tr}(\sigma\sigma^T D^2 u))(t, X^{\alpha^*}, \alpha^*)dt\right].$$

The key point is that now there is an equality in the rest of the formulas: indeed, by the HJ equation satisfied by w and the property (2.8) of α^*, we have

$$\begin{aligned}\partial_t w(t, X^{\alpha^*}_t) &= H(X^{\alpha^*}_t, Dw(t, X^{\alpha^*}_t), D^2 w(t, X^{\alpha^*}_t)) \\ &= -L(X^{\alpha^*}_t, {\alpha^*}_t) - b(X^{\alpha^*}_t, {\alpha^*}_t)\cdot Dw(t, X^{\alpha^*}_t) - \frac{1}{2}\mathrm{Tr}(\sigma\sigma^T(X^{\alpha^*}_t, {\alpha^*}_t)D^2 w(t, X^{\alpha^*}_t)),\end{aligned}$$

which can be rewritten as

$$(\partial_t w + b\cdot Dw + \frac{1}{2}\mathrm{Tr}(\sigma\sigma^T D^2 w))(t, X^{\alpha^*}_t, {\alpha^*}_t) = -L(X^{\alpha^*}_t, {\alpha^*}_t).$$

Arguing as above, this leads to the equality

$$w(t_0,x_0) = \mathbb{E}\left[\int_{t_0}^{T} L(X^{\alpha^*}_t, \alpha^*_t)dt + g(X^{\alpha^*}_T)\right].$$

As $\alpha^* \in \mathcal{A}(t_0)$, we have by the definition of $u(t_0,x_0)$ and the first part of the proof:

$$w(t_0,x_0) = \mathbb{E}\left[\int_{t_0}^{T} L(X^{\alpha^*}_t, {\alpha^*}_t)dt + g(X^{\alpha^*}_T)\right] \leq u(t_0,x_0) \leq w(t_0,x_0),$$

which implies that $w(t_0,x_0)=u(t_0,x_0)$ and that α^* is optimal for $u(t_0,x_0)$. □

3. The infinite horizon problem

3.1. **Value function and dynamic programming.** We consider an optimal control problem of the form: for any $x_0\in\mathbb{R}^d$,

$$u(x_0)=\inf_{\alpha\in\mathcal{A}}\mathbb{E}\left[\int_0^\infty e^{-rt}L(X_t^\alpha,\alpha_t)dt\right], \tag{3.1}$$

where $X^\alpha=X^{x_0,\alpha}$ is the unique solution to

$$dX_t^\alpha=b(X_t^\alpha,\alpha_t)dt+\sigma(X_t^\alpha,\alpha_t)dW_t,\ t\in[0,\infty),\qquad X_0^\alpha=x_0,$$

and

$$\mathcal{A}=\{\alpha:[0,\infty)\times\Omega\to A,\ \text{progressively measurable with respect to }\sigma(W_s,\ s\in[0,t])\}.$$

We work under our standing assumptions:

(i) the discount rate r is positive,

(ii) A is a compact subset of a finite dimensional space,

(iii) $(b,\sigma,L):\mathbb{R}^d\times A\to\mathbb{R}^d\times\mathbb{R}^{d\times m}\times\mathbb{R}$ is continuous, bounded,
and Lipschitz continuous in x unif. in a,

(iv) W is the $m-$dimensional Brownian motion defined on the canonical space $\Omega=C^0([0,\infty),\mathbb{R}^m)$ (endowed with the topology of local uniform convergence), $\mathcal{F}$ is the Borel $\sigma-$algebra and $\mathbb{P}$ is the Wiener measure on Ω. We denote by $(\mathcal{F}_t)_{t\geq0}$ the (completed) canonical filtration generated by W.

Let us start with a technical remark on the regularity of the value function u.

Lemma 3.1. *The value function $u:\mathbb{R}^d\to\mathbb{R}$ is bounded and (Hölder) continuous.*

Proof. As L is bounded,

$$\left|\int_0^\infty e^{-rt}L(X_t^\alpha,\alpha_t)dt\right|\ \leq\|L\|_\infty\int_0^\infty e^{-rt}dt=\|L\|_\infty/r.$$

Thus $\|u\|_\infty\leq\|L\|_\infty/r$.

In order to prove the Hölder regularity of u, let us recall (see Remark 1.2) that there exists a constant $C>0$, depending on the Lipschitz constant of b and σ and of $\|\sigma\|_\infty$, such that, for any $x_0,y_0\in\mathbb{R}^d$ and any control $\alpha\in\mathcal{A}$,

$$\mathbb{E}\left[\sup_{s\in[0,t]}|X_s^{x_0,\alpha}-X_s^{y_0,\alpha}|^2\right]\leq|x_0-y_0|^2e^{2Ct},$$

(to simplify the computation, w.l.o.g. we will assume $C>r$ below).

Next we fix $x_0,y_0\in\mathbb{R}^d$, $\varepsilon>0$ and α an $\varepsilon-$optimal control for $u(x_0)$:

$$u(x_0)+\varepsilon\geq\mathbb{E}\left[\int_0^\infty e^{-rt}L(X_t^{x_0,\alpha},\alpha_t)dt\right].$$

$$\begin{aligned}u(y_0)&\leq\mathbb{E}\left[\int_0^\infty e^{-rt}L(X_t^{y_0,\alpha},\alpha_t)dt\right]\leq\mathbb{E}\left[\int_0^\infty e^{-rt}L(X_t^{x_0,\alpha},\alpha_t)dt\right]+R\\&\leq u(x_0)+\varepsilon+R,\end{aligned}$$

where

$$R:=\mathbb{E}\left[\int_0^\infty e^{-rt}|L(X_t^{y_0,\alpha},\alpha_t)-L(X_t^{x_0,\alpha},\alpha_t)|dt\right]$$

For any $T > 0$ we have

$$\begin{aligned} R &\leq \mathbb{E}\left[\int_0^T e^{-rt}|L(X_t^{y_0,\alpha},\alpha_t) - L(X_t^{x_0,\alpha},\alpha_t)|dt\right] + \mathbb{E}\left[\int_T^\infty e^{-rt}|L(X_t^{y_0,\alpha},\alpha_t) - L(X_t^{x_0,\alpha},\alpha_t)|dt\right] \\ &\leq Lip(L)\mathbb{E}\left[\int_0^T e^{-rt}|X_t^{y_0,\alpha} - X_t^{x_0,\alpha}|dt\right] + 2\|L\|_\infty \int_T^\infty e^{-rt}dt \\ &\leq Lip(L)|x_0 - y_0| \int_0^T e^{-rt}e^{Ct}dt + \frac{2\|L\|_\infty}{r}e^{-rT}dt \leq \frac{Lip(L)}{C-r}|x_0 - y_0|e^{(C-r)T} + \frac{2\|L\|_\infty}{r}e^{-rT}. \end{aligned}$$

We assume without loss of generality that $|x_0 - y_0| < 1$. We choose $T > 0$ such that $|x_0 - y_0| = e^{-CT}$, i.e., $T = -\ln(|x_0 - y_0|)/C > 0$, to find that

$$R \leq (\frac{Lip(L)}{C-r} + \frac{2\|L\|_\infty}{r})|x_0 - y_0|^{r/C}.$$

This implies that

$$u(y_0) \leq u(x_0) + \varepsilon + C'|x_0 - y_0|^{r/C}.$$

Exchanging the roles of x_0 and y_0 and using the fact that $\varepsilon > 0$ is arbitrary then yields to

$$|u(y_0) - u(x_0)| \leq C'|x_0 - y_0|^{r/C}.$$

□

Theorem 3.2 (Dynamic programming)**.** *Let $\mathcal{T}$ be the collection of all stopping times (for the canonical filtration). Then*

$$\begin{aligned} u(x_0) &= \inf_{\alpha\in\mathcal{A}} \inf_{\tau\in\mathcal{T}} \mathbb{E}\left[\int_0^\tau e^{-rt}L(X_t^\alpha,\alpha_t)dt + e^{-r\tau}u(X_\tau^\alpha)\right] \\ &= \inf_{\alpha\in\mathcal{A}} \sup_{\tau\in\mathcal{T}} \mathbb{E}\left[\int_0^\tau e^{-rt}L(X_t^\alpha,\alpha_t)dt + e^{-r\tau}u(X_\tau^\alpha)\right], \end{aligned}$$

Remark 3.3. *In the finite horizon problem, all the analysis could have been carried out with time dependent coefficient, for the infinite horizon problem it is mandatory that the data (b, σ, L) do not depend on time. A crucial role is also played by the specific form of the discount factor.*

We only explain the result when there is no control at all (namely b, σ and L do not depend on a). The value function is then given by

$$u(x_0) = \mathbb{E}\left[\int_0^\infty e^{-rt}L(X^{x_0})dt\right],$$

where X^{x_0} is the unique solution to

$$dX = b(X_t,)dt + \sigma(X_t)dW_t,\ t \in [0,\infty), \qquad X_0 = x_0,$$

Fix $\tau \in \mathcal{T}$ and note that

$$\begin{aligned} \int_0^\infty e^{-rt}L(X_t^{x_0})dt &= \int_0^\tau e^{-rt}L(X_t^{x_0})dt + \int_\tau^\infty e^{-rt}L(X_t^{x_0})dt \\ &= \int_0^\tau e^{-rt}L(X_t^{x_0})dt + e^{-r\tau}\int_0^\infty e^{-rt}L(X_{\tau+t}^{x_0})dt. \end{aligned}$$

By the strong Markov property, given $\mathcal{F}_\tau$, the process $(X_{\tau+t}^{x_0})_{t\geq 0}$ has the same law as the process $(X_t^y)_{t\geq 0}$ started at $y = X_\tau^{x_0}$. Thus

$$\mathbb{E}\left[\int_0^\infty e^{-rt}L(X_{\tau+t}^{x_0})dt \mid \mathcal{F}_\tau\right] = \mathbb{E}\left[\int_0^\infty e^{-rt}L(X_t^y)dt\right]_{|_{y=X_\tau^{x_0}}} = u(X_\tau^{x_0}).$$

This implies, by the tower property, that

$$\begin{aligned} u(x_0) &= \mathbb{E}\left[\mathbb{E}\left[\int_0^\tau e^{-rt}L(X_t^{x_0})dt + e^{-r\tau}e^{-r\tau}\int_0^\infty e^{-rt}L(X_{\tau+t}^{x_0})dt \mid \mathcal{F}_\tau\right]\right] \\ &= \mathbb{E}\left[\int_0^\tau e^{-rt}L(X_t^{x_0})dt + e^{-r\tau}\mathbb{E}\left[\int_0^\infty e^{-rt}L(X_{\tau+t}^{x_0})dt \mid \mathcal{F}_\tau\right]\right] \\ &= \mathbb{E}\left[\int_0^\tau e^{-rt}L(X_t^{x_0})dt + e^{-r\tau}u(X_\tau^{x_0})\right]. \end{aligned}$$

3.2. **The Hamilton-Jacobi equation and the verification theorem.**

Theorem 3.4. *Under the above conditions, assume that u is of class C^2 on $\mathbb{R}^d$. Then u is a bounded solution to the Hamilton-Jacobi (HJ) equation*

$$ru(x) + H(x, Du(x), D^2u(x)) = 0 \qquad \text{in } \mathbb{R}^d,$$

where, for any $(x, p, S) \in \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^{d\times d}$,

$$H(x, p, S) = \max_{a\in A}\left\{-L(x, a) - b(x, a)\cdot p - \frac{1}{2}\mathrm{Tr}(\sigma\sigma^T(x, a)S)\right\}. \tag{3.2}$$

Note that the equation is stationary. It is stated on the all space $\mathbb{R}^d$. The "boundary condition" is the fact that u is bounded (actually polynomial growth would be enough).

As its counterpart in the finite horizon problem, the result is seldom useful in practice. Again the interest of the result is that it allows to guess the HJ equation.

Proof. The proof is done in two steps: first we prove the (relatively easier) inequality

$$ru(x) + H(x, Du(x), D^2u(x)) \le 0 \tag{3.3}$$

and then the reverse one.

Proof of (3.3)*:* Fix $(x_0) \in (0, T) \times \mathbb{R}^d$, $a \in A$ and let us consider the constant control $\alpha_t^a = a$ for any $t \in [0, \infty)$. Note that $\alpha^a \in \mathcal{A}$. Given a radius $R > 0$ and $h > 0$, we consider the stopping times

$$\tau_R = \inf\{t \ge 0,\ |X_t^{\alpha^a} - x_0| \ge R\},\ \tau_{R,h} = \tau_R \wedge h,$$

(with $\tau_R = \infty$ if there is no such t). By Dynamic Programming we have:

$$\begin{aligned} u(x_0) &= \inf_{\alpha\in\mathcal{A}}\inf_{\tau\in\mathcal{S}}\mathbb{E}\left[\int_0^\tau e^{-rt}L(X_t^\alpha, \alpha_t)dt + e^{-r\tau}u(X_\tau^\alpha)\right] \\ &\le \mathbb{E}\left[\int_0^{\tau_{R,h}} e^{-rt}L(X_t^{\alpha^a}, \alpha_t^a)dt + e^{-r\tau}u(X_{\tau_{R,h}}^{\alpha^a})\right]. \end{aligned}$$

Using Itô's formula and the definition of α^a, we get

$$\begin{aligned} u(x_0) \le u(x_0) + \mathbb{E}\Big[&\int_0^{\tau_{R,h}} e^{-rt}(L - ru + b\cdot Du + \frac{1}{2}\mathrm{Tr}(\sigma\sigma^T D^2u))(X_t^{\alpha^a}, a)dt \\ &+ \int_0^{\tau_{R,h}} e^{-rt}(\sigma^T Du)(X_t^{\alpha^a}, a)\cdot dW_t\Big]. \end{aligned}$$

On $[[0, \tau_{R,h}]]$, X_t^α remains in the ball $B_R(x_0)$ where Du and σ are bounded. Therefore the local martingale term is a martingale. Hence, simplifying by $u(x_0)$ and dividing by $h > 0$ gives

$$0 \le \mathbb{E}\left[\frac{1}{h}\int_0^{\tau_{R,h}} e^{-rt}(L - ru + b\cdot Du + \frac{1}{2}\mathrm{Tr}(\sigma\sigma^T D^2u))(X_t^{\alpha^a}, a)dt\right].$$

We now let $h \to 0^+$. Note that the integrand is bounded and converges a.s. to $(L - ru + b\cdot Du + \frac{1}{2}\mathrm{Tr}(\sigma\sigma^T D^2u))(x_0, a)$ as X^{α^a} has a continuous path and equals x_0 at $t = 0$. Note also that, as $\tau_R > 0$ a.s.,

$$\mathbf{1}_{\{\tau_{R,h}=h\}} \to 1 \qquad \text{as } h \to 0^+,$$

and thus, a.s.,

$$\frac{1}{h}\int_0^{\tau_{R,h}} e^{-rt}(L-ru+b\cdot Du+\frac{1}{2}\mathrm{Tr}(\sigma\sigma^T D^2u))(X_t^{\alpha^a},a)dt \to (L-ru+b\cdot Du+\frac{1}{2}\mathrm{Tr}(\sigma\sigma^T D^2u))(x_0,a).$$

By Lebesgue Theorem, this implies that

$$0 \le (L-ru+b\cdot Du+\frac{1}{2}\mathrm{Tr}(\sigma\sigma^T D^2u))(x_0,a).$$

As $a \in A$ is arbitrary, and after changing sign, this inequality can be rewritten as

$$ru(x_0)+\sup_{a\in A}\left\{-L(x_0,a)-b(x_0,a)\cdot Du(x_0)-\frac{1}{2}\mathrm{Tr}(\sigma\sigma^T(x_0,a)D^2u(x_0))\right\} \le 0.$$

Recalling the definition of H, this is (3.3).

Proof of the reverse inequality:

$$ru(x)+H(x,Du(x),D^2u(x)) \ge 0 \qquad \text{in } (0,T)\times\mathbb{R}^d. \tag{3.4}$$

The proof is slightly more technical than the previous one. We argue by contradiction, assuming that there exists $x_0 \in \mathbb{R}^d$ and $\eta > 0$ such that

$$ru(x_0)+H(x_0,Du(x_0),D^2u(x_0)) \le -2\eta.$$

We have seen in the previous lecture that H is a continuous map. Thus there exists $R > 0$ such that, for any $x \in B_R(x_0)$,

$$ru(x)+H(x,Du(x),D^2u(x)) \le -\eta.$$

By the definition of H this inequality says that, for any $(x,a) \in B_R(x_0)\times A$,

$$ru(x)-L(x,a)-b(x,a)\cdot Du(x)-\frac{1}{2}\mathrm{Tr}(\sigma\sigma^T(x,a)D^2u(x)) \le -\eta.$$

Changing sign, we get:

$$-ru(x)+L(x,a)+b(x,a)\cdot Du(x)+\frac{1}{2}\mathrm{Tr}(\sigma\sigma^T(x,a)D^2u(x)) \ge \eta. \tag{3.5}$$

Given a control $\alpha \in \mathcal{A}$ and $h > 0$, we define the stopping times

$$\tau_\alpha = \inf\{t \ge 0,\ |X_t^\alpha - x_0| \ge R\},\ \tau_{\alpha,h} = \tau_\alpha \wedge h,$$

(with $\tau_\alpha = \infty$ if there is no such t). By dynamic programming we get

$$\begin{aligned} u(x_0) &= \inf_{\alpha\in\mathcal{A}}\sup_{\tau\in\mathcal{S}}\mathbb{E}\left[\int_0^\tau e^{-rt}L(X_t^\alpha,\alpha_t)dt+e^{-r\tau}u(X_\tau^\alpha)\right] \\ &\ge \inf_{\alpha\in\mathcal{A}}\mathbb{E}\left[\int_0^{\tau_{\alpha,h}} e^{-rt}L(X_t^\alpha,\alpha_t)dt+e^{-r\tau}u(X_{\tau_{\alpha,h}}^\alpha)\right]. \end{aligned}$$

Fixing $\varepsilon \in (0,\eta)$, let α_h be $(\varepsilon h)-$optimal in the right-hand side of the inequality above. Using Itô's formula (and the usual argument to get rid of the local martingale part) we obtain:

$$\begin{aligned} u(x_0)+\varepsilon h &\ge \mathbb{E}\left[\int_0^{\tau_{\alpha_h,h}} e^{-rt}L(X_t^{\alpha_h},\alpha_{h,t})dt+e^{-r\tau_{\alpha_h,h}}u(X_{\tau_{\alpha_h,h}}^{\alpha_h})\right] \\ &\ge u(x_0)+\mathbb{E}\left[\int_0^{\tau_{\alpha_h,h}} e^{-rt}(L-ru+b\cdot Du+\frac{1}{2}\mathrm{Tr}(\sigma\sigma^T D^2u))(X_t^{\alpha_h},\alpha_{h,t})dt\right]. \end{aligned}$$

For $t \in [[0,\tau_{\alpha_h,h}]]$, we have that $X_t^{\alpha_h} \in B_R(x_0)$ by the definition of $\tau_{\alpha_h,h}$. Hence we can use inequality (3.5) to obtain, after simplification by $u(x_0)$:

$$\varepsilon h \ge \mathbb{E}\left[\int_0^{\tau_{\alpha_h,h}} e^{-rt}\eta dt\right] = \eta\mathbb{E}\left[\frac{1-e^{-r\tau_{\alpha_h,h}}}{r}\right].$$

We divide by $h > 0$ and obtain (omitting the event $\{\tau_{\alpha_h,h} < h\}$ since the integrand is positive)

$$\varepsilon \geq \eta \mathbb{P}\left[\tau_{\alpha_h,h} = h\right] \frac{1 - e^{-rh}}{rh}. \tag{3.6}$$

Note that, by Markov inequality,

$$\mathbb{P}\left[\tau_{\alpha_h,h} < h\right] = \mathbb{P}\left[\sup_{t\in[0,h]} |X_t^{\alpha_h} - x_0| \geq R\right] \leq \frac{\mathbb{E}\left[\sup_{t\in[0,h]} |X_t^{\alpha_h} - x_0|^2\right]}{R^2} \leq \frac{Ch}{R^2},$$

where the constant C depends only on $\|b\|_\infty$ and $\|\sigma\|_\infty$. Thus

$$\mathbb{P}\left[\tau_{\alpha_h,h} = h\right] \to 1 \text{ as } h \to 0^+.$$

Letting $h \to 0^+$ in (3.6) we find a contradiction with $\varepsilon < \eta$. This proves (3.4). □

We used above the regularity of the Hamiltonian:

Lemma 3.5. *Under our standing assumption, $H : [0,T] \times \mathbb{R}^d \times \mathbb{R}^{d\times d} \to \mathbb{R}$ is Lipschitz continuous in x uniformly in $(p,S) \in B_R \times B_R$ for any $R > 0$ and Lipschitz continuous in (p,S) uniformly in $x \in \mathbb{R}^d$.*

Proof. Fix $R > 0$ and note that, for any $p \in B_R$, $S \in B_R$ and $a \in A$, the map

$$x \to -L(x,a) - b(x,a)\cdot p - \frac{1}{2}\mathrm{Tr}(\sigma\sigma^T(x,a)S)$$

is Lipschitz continuous with a Lipschitz constant of the form $Lip(L) + RLip(b) + R\|\sigma\|_\infty Lip(\sigma)$. On the other hand, for any $x \in \mathbb{R}^d$ and $a \in A$, the map

$$(p,S) \to -L(x,a) - b(x,a)\cdot p - \frac{1}{2}\mathrm{Tr}(\sigma\sigma^T(x,a)S)$$

is Lipschitz continuous with a Lipschitz constant given by $\|b\|_\infty + \|\sigma\|_\infty^2/2$. Recalling the definition of H in (3.2) completes the proof. □

We now turn to a much more interesting statement: the Verification Theorem. This result says that smooth solutions of the HJ equation are equal to the value function and allow to characterize the optimal control.

Theorem 3.6. *Assume that w is a C^2 bounded solution to the HJ equation*

$$rw(t,x) + H(x, Dw(x), D^2w(x)) = 0 \qquad \text{in } \mathbb{R}^d.$$

(1) *Then $w \leq u$ in $[0,T] \times \mathbb{R}^d$.*
(2) *If in addition, if, given $x_0 \in \mathbb{R}^d$, there exists $\alpha^* \in \mathcal{A}$ such that, a.s. and for a.e. $t \in (0,\infty)$,*

$$\begin{aligned} &H(X_t^{\alpha^*}, Dw(X_t^{\alpha^*}), D^2w(X_t^{\alpha^*})) \\ &\quad = -L(X_t^{\alpha^*}, \alpha_t^*) - b(X_t^{\alpha^*}, \alpha_t^*)\cdot Dw(X_t^{\alpha^*}) - \frac{1}{2}\mathrm{Tr}(\sigma\sigma^T(X_t^{\alpha^*}, \alpha_t^*)D^2w(X_t^{\alpha^*}), \end{aligned} \tag{3.7}$$

then α^* is optimal for $u(x_0)$ and $u(x_0) = w(x_0)$.

Equality (3.7) says that α_t^* is a maximum point over A of

$$a \to -L(X_t^{\alpha^*}, a) - b(X_t^{\alpha^*}, a)\cdot Dw(X_t^{\alpha^*}) - \frac{1}{2}\mathrm{Tr}(\sigma\sigma^T(X_t^{\alpha^*}, a)D^2w(X_t^{\alpha^*})$$

The second part of the result can also be stated as follows: Assume that the measurable map $a^* : \mathbb{R}^d \to A$ satisfies

$$H(x, Dw(x), D^2w(x)) = -L(x, a^*(x)) - b(x, a^*(x))\cdot Dw(x) - \frac{1}{2}\mathrm{Tr}(\sigma\sigma^T(x, a^*(x))D^2w(x))$$

for any $x \in \mathbb{R}^d$ and that the SDE

$$dX_t = b(X_t, a^*(X_t))dt + \sigma(X_t, a^*(X_t))dW_t,\ t \in (0,\infty) \qquad X_0 = x_0,$$

has a solution X^*. Then the control $(\alpha_t^* = a^*(t, X_t^*))$ is optimal for $u(x_0)$ and $u(x_0) = w(x_0)$. The map a^* is called an optimal feedback: it allows to take an optimal action in view of the current position x.

Proof of the Verification Theorem. It is (again) a simple application of Itô's formula. Fix $x_0 \in \mathbb{R}^d$ and let $\alpha \in \mathcal{A}$ be an arbitrary control. Let $R > 0$ be large and let us define the stopping time

$$\tau_R = \inf\{t \in [0,T],\ |X_t^\alpha - x_0| \geq R\},$$

(with $\tau_R = T$ if the right-hand side is empty). We compute by Itô's formula:

$$e^{-r\tau_R} w(X_{\tau_R}^\alpha) = w(x_0) + \int_0^{\tau_R} e^{-rt}(-rw + b\cdot Dw + \frac{1}{2}\mathrm{Tr}(\sigma\sigma^T D^2 w))(X_t^\alpha, \alpha_t)dt + \int_0^{\tau_R} e^{-rt}(\sigma^T Dw)(X_t^\alpha, \alpha_t)\cdot dW_t.$$

On $[[0, \tau_R]]$, X_t^α remains in $B_R(x_0)$, and thus $(\sigma^T Du)(t, X_t^\alpha, \alpha_t)$ is bounded. We take expectation to infer that

$$\mathbb{E}\left[e^{-r\tau_R} w(X_{\tau_R}^\alpha)\right] = w(x_0) + \mathbb{E}\left[\int_0^{\tau_R} e^{-rt}(-ru + b\cdot Du + \frac{1}{2}\mathrm{Tr}(\sigma\sigma^T D^2 u))(t, X_t^\alpha, \alpha_t)dt\right]. \quad (3.8)$$

By the HJ equation satisfied by w and definition of H, we have that, for any $(t,x,a) \in (0,T)\times\mathbb{R}^d \times A$,

$$-rw(t,x) = H(x, Dw(t,x), D^2 w(t,x)) \geq -L(x,a) - b(x,a)\cdot Dw(x) - \frac{1}{2}\mathrm{Tr}(\sigma\sigma^T(x,a)D^2 w(x)),$$

which can be rewritten as

$$-rw(t,x) + b(x,a)\cdot Dw(x) + \frac{1}{2}\mathrm{Tr}(\sigma\sigma^T(x,a)D^2 w(x)) \geq -L(x,a).$$

Plugging this inequality applied at (X_t^α, α_t) into (3.8) we obtain

$$\mathbb{E}\left[e^{-r\tau_R} w(\tau_R, X_{\tau_R}^\alpha)\right] \geq w(x_0) - \mathbb{E}\left[\int_0^{\tau_R} e^{-rt} L(X_t^\alpha, \alpha_t)dt\right].$$

Our aim is to let $R \to \infty$. Then $\tau_R \to \infty$ a.s. As w and L are bounded (this is here that this assumption is used),we obtain by Lebesgue Theorem:

$$w(x_0) \leq \mathbb{E}\left[\int_0^{\infty} e^{-rt} L(X_t^\alpha, \alpha_t)dt\right].$$

Recalling that $\alpha \in \mathcal{A}$ is arbitrary, we get

$$w(x_0) \leq \inf_{\alpha\in\mathcal{A}} \mathbb{E}\left[\int_0^{\infty} e^{-rt} L(X_t^\alpha, \alpha_t)dt\right] = u(x_0).$$

This proves the first statement of the theorem.

Let now α^* be as in the second statement of the Theorem. Using the same argument as above we obtain:

$$\mathbb{E}\left[e^{-\tau_R} w(X_{\tau_R}^{\alpha^*})\right] = w(x_0) + \mathbb{E}\left[\int_0^{\tau_R} e^{-rt}(-ru + b\cdot Du + \frac{1}{2}\mathrm{Tr}(\sigma\sigma^T D^2 u))(t, X^{\alpha^*}, \alpha^*)dt\right].$$

The key point is that now there is an equality in the rest of the formulas: indeed, by the HJ equation satisfied by w and the property (3.7) of α^*, we have

$$\begin{aligned} -rw(X_t^{\alpha^*}) &= H(X_t^{\alpha^*}, Dw(X_t^{\alpha^*}), D^2 w(X_t^{\alpha^*})) \\ &= -L(X_t^{\alpha^*}, {\alpha^*}_t) - b(X_t^{\alpha^*}, {\alpha^*}_t)\cdot Dw(X_t^{\alpha^*}) - \frac{1}{2}\mathrm{Tr}(\sigma\sigma^T(X_t^{\alpha^*}, {\alpha^*}_t)D^2 w(X_t^{\alpha^*})), \end{aligned}$$

which can be rewritten as

$$(-rw + b\cdot Dw + \frac{1}{2}\mathrm{Tr}(\sigma\sigma^T D^2 w))(X_t^{\alpha^*}, {\alpha^*}_t) = -L(X_t^{\alpha^*}, {\alpha^*}_t).$$

Arguing as above, this leads to the equality

$$w(x_0) = \mathbb{E}\left[\int_0^\infty e^{-rt} L(X_t^{\alpha^*}, \alpha^*{}_t) dt\right].$$

As $\alpha^* \in \mathcal{A}$, we have by the definition of $u(x_0)$ and the first part of the proof:

$$w(x_0) = \mathbb{E}\left[\int_0^\infty e^{-rt} L(X_t^{\alpha^*}, \alpha^*{}_t) dt\right] \leq u(x_0) \leq w(x_0),$$

which implies that $w(x_0) = u(x_0)$ and that α^* is optimal for $u(x_0)$. □

4. Viscosity solutions of the Hamilton-Jacobi equation

We now investigate further the Hamilton-Jacobi equation associated with the infinite horizon problem:

$$rw(x) + H(x, Dw(x), D^2 w(x)) = 0 \qquad \text{in } \mathbb{R}^d. \tag{4.1}$$

4.1. **Structure of the equation and classical solutions.** The key point in this equation is that the Hamiltonian $H = H(x, p, S)$ is nonincreasing in the last variable with respect to the order of symmetric matrices. Let $\mathcal{S}_d$ be the set of symmetric matrices. Recall that $S_1 \leq S_2$ means that $S_2 - S_1$ is semidefinite positive (all its eigenvalues are nonnegative).

Lemma 4.1. *Let H be defined by (3.2). Then, for any $(x, p) \in \mathbb{R}^d \times \mathbb{R}^d$ and any $S_1, S_2 \in \mathcal{S}_d$,*

$$S_1 \leq S_2 \qquad \Longrightarrow \qquad H(x, p, S_1) \geq H(x, p, S_2).$$

Such a map H is called weakly elliptic.

Proof. Note that, for any $S \in \mathcal{S}_d$ with $S \geq 0$,

$$Tr(\sigma\sigma^T S) = Tr(\sigma^T S \sigma) = \sum_{j=1}^m \sigma_j^T S \sigma_j \geq 0,$$

where $\sigma_j \in \mathbb{R}^m$ are the columns of σ. Thus, if $S_1 \leq S_2$,

$$Tr(\sigma\sigma^T S_1) \leq Tr(\sigma\sigma^T S_2).$$

This implies that

$$\begin{aligned} H(x, p, S_1) &= \max_{a \in A} \left\{ -L(x, a) - b(x, a) \cdot p - \frac{1}{2} \mathrm{Tr}(\sigma\sigma^T(x, a) S_1) \right\} \\ &\geq \max_{a \in A} \left\{ -L(x, a) - b(x, a) \cdot p - \frac{1}{2} \mathrm{Tr}(\sigma\sigma^T(x, a) S_2) \right\} = H(x, p, S_2) \end{aligned}$$

□

We now check that the Hamilton-Jacobi equation has at most one classical bounded solution.

Proposition 4.2. *There is at most one C^2 and bounded map w which satisfies the Hamilton-Jacobi equation (4.1).*

Remark 4.3. *The boundedness assumption (or at least a polynomial growth assumption) is mandatory. For instance, in dimension $d = 1$, and for $H(x, p, S) = p$, the Hamilton-Jacobi equation*

$$ru + u' = 0$$

has obviously for solution $u = 0$. But $u(x) = e^{-rx}$ is also a solution (but not with a polynomial growth!).

The proof relies on comparison: we say that $w : \mathbb{R}^d \to \mathbb{R}$ is a classical subsolution of the HJ equation if w is of class C^2 and satisfies

$$rw(x) + H(x, Dw(x), D^2w(x)) \leq 0 \qquad \text{in } \mathbb{R}^d.$$

In the symmetric way, w is a classical supersolution to the HJ equation if w is of class C^2 and satisfies

$$rw(x) + H(x, Dw(x), D^2w(x)) \geq 0 \qquad \text{in } \mathbb{R}^d.$$

Thus a solution to HJ is just a map which is at the same time a sub and a supersolution.

Lemma 4.4 (Comparison)**.** *If $w_1 : \mathbb{R}^d \to \mathbb{R}$ is a bounded classical subsolution of the HJ equation while $w_2 : \mathbb{R}^d \to \mathbb{R}$ is a bounded classical supersolution of the HJ equation, then $w_1 \leq w_2$ in $\mathbb{R}^d$.*

Proof of Proposition 4.2. If w_1 and w_2 are two bounded and classical solutions to the HJ equation, then by comparison $w_1 \leq w_2$ and $w_2 \leq w_1$. Whence the equality. □

We now turn to the proof of the comparison. Note that the result is a kind of variant of the verification theorem. The proof we give here, however, is purely analytic. The heuristic argument is very simple. Suppose that w_1 is a C^2 subsolution while w_2 is a C^2 supersolutions of the equation. *Assume in addition* that the map $x \to w_1 - w_2$ has a maximum on $\mathbb{R}^d$ at a point x_0. Then by optimality conditions we have

$$Dw_1(x_0) = Dw_2(x_0), \qquad D^2w_1(x_0) \leq D^2w_1(x_0).$$

By the (in)equation satisfied by w_1 and w_2, we get

$$\sup w_1 - w_2 = (w_1 - w_2)(x_0) \leq -r^{-1}(H(x_0, Dw_1(x_0), D^2w_1(x_0)) - H(x_0, Dw_2(x_0), D^2w_2(x_0))) \leq 0,$$

where we used Lemma 4.1 for the last inequality. Thus $w_1 \leq w_2$.

Unfortunately, this elementary proof is not completely correct, as it assumes the existence of a maximum to $w_1 - w_2$. This is not a surprise, since we do not use at all the fact that w_1 and w_2 are bounded. To overcome this issue, we need to use a penalization argument.

Proof. Step 1: penalization. For $\alpha > 0$ small let us consider a maximum point x_α of the map

$$x \to w_1(x) - w_2(x) - \frac{\alpha}{2}|x|^2.$$

Such a maximum point exists since w_1 and w_2 are bounded. We claim that

$$\lim_{\alpha \to 0^+} \max_x w_1(x) - w_2(x) - \frac{\alpha}{2}|x|^2 = \sup_x w_1(x) - w_2(x).$$

Indeed,

$$\max_x w_1(x) - w_2(x) - \frac{\alpha}{2}|x|^2 \leq \sup_x w_1(x) - w_2(x).$$

On the other hand, for $\varepsilon > 0$ and x^ε $\varepsilon-$optimal for $\sup_x w_1(x) - w_2(x)$, we have

$$\max_x w_1(x) - w_2(x) - \frac{\alpha}{2}|x|^2 \geq w_1(x^\varepsilon) - w_2(x^\varepsilon) - \frac{\alpha}{2}|x^\varepsilon|^2 \geq \sup_x (w_1(x) - w_2(x)) - \varepsilon - \frac{\alpha}{2}|x^\varepsilon|^2,$$

so that

$$\liminf_{\alpha \to 0^+} \max_x w_1(x) - w_2(x) - \frac{\alpha}{2}|x|^2 \geq \sup_x (w_1(x) - w_2(x)) - \varepsilon.$$

We conclude by using the fact that ε is arbitrary.

Step 2: estimates. We now estimate $|x_\alpha|$. Note, by optimality of x_α,

$$w_1(x_\alpha) - w_2(x_\alpha) - \frac{\alpha}{2}|x_\alpha|^2 \geq w_1(0) - w_2(0),$$

so that, using the fact that w_1 and w_2 are bounded,

$$\alpha|x_\alpha|^2 \leq 4(\|w_1\|_\infty + \|w_2\|_\infty).$$

Thus

$$|x_\alpha| \leq C\alpha^{-1/2}.$$

Step 3: use of the equations. By optimality of x_α, we have

$$Dw_1(x_\alpha) - Dw_2(x_\alpha) - \alpha x_\alpha = 0, \qquad D^2w_1(x_\alpha) - D^2w_2(x_\alpha) - \alpha I_d \leq 0.$$

Using the fact that w_1 is a subsolution while w_2 is a supersolution we have:

$$w_1(x_\alpha) - w_2(x_\alpha) \leq r^{-1}(H(x_\alpha, Dw_1(x_\alpha), D^2w_1(x_\alpha)) - H(x_\alpha, Dw_2(x_\alpha), D^2w_2(x_\alpha)).$$

As H is Lipschitz continuous in (p, S) uniformly in x, we find

$$\begin{aligned}
&w_1(x_\alpha) - w_2(x_\alpha)\\
&\leq r^{-1}(H(x_\alpha, Dw_1(x_\alpha), D^2w_1(x_\alpha)) - H(x_\alpha, Dw_1(x_\alpha), D^2w_2(x_\alpha) + \alpha I_d) + C\alpha(|x_\alpha| + 1)\\
&\leq C\alpha^{1/2}(1 + \alpha^{1/2}),
\end{aligned}$$

where we used the optimality conditions in the first inequality and the weak ellipticity of H and our estimate on x_α in the last inequality.

This proves that

$$\sup(w_1 - w_2) = \lim_{\alpha \to 0^+} w_1(x_\alpha) - w_2(x_\alpha) - \frac{\alpha}{2}|x_\alpha|^2 \leq \lim_{\alpha \to 0^+} C\alpha^{1/2}(1 + \alpha^{1/2}) = 0.$$

Hence $w_1 \leq w_2$. □

4.2. **The notion of viscosity solutions.** The results above are unsatisfactory because in general we do not know that the value function is of class C^2 (and in general it is not, unless some specific structure conditions). For this reason we now introduce the notion of viscosity solution for the HJ equation.

Definition 4.5. *Fix $\mathcal{O}$ an open subset of $\mathbb{R}^d$. We say that a continuous map $w : \mathcal{O} \to \mathbb{R}$ is a viscosity subsolution (resp. supersolution) to the HJ equation*

$$rw + H(x, Dw, D^2w) = 0$$

in $\mathcal{O}$ if, for any C^2 map $\phi : \mathbb{R}^d \to \mathbb{R}$ such that $u - \phi$ has a local maximum (resp. local minimum) at some point $x_0 \in \mathcal{O}$, one has

$$rw(x_0) + H(x, D\phi(x_0), D^2\phi(x_0)) \leq 0. \qquad \textit{(resp. } \geq 0\textit{).}$$

We say that w is a a viscosity solution of the HJ equation in $\mathcal{O}$ if it is at the same time a viscosity sub- and super-solution of the HJ equation in $\mathcal{O}$.

The key point is the definition is to replace derivatives of w (which are not known to exist) by derivatives of the test function. Similar definitions can be stated for time dependent equations of the form

$$-\partial_t w + H(t, x, Dw, D^2w) = 0 \qquad \text{in } \mathcal{O}$$

where $\mathcal{O}$ is an open subset of $\mathbb{R} \times \mathbb{R}^d$. In this case, test function must also depend on time. (exercise).

We now discuss classical properties of viscosity solutions. The first one is the consistency with the notion of classical solution.

Proposition 4.6 (Consistency)**.** *Assume that the map $H : \mathbb{R}^d \times \mathbb{R}^d \times \mathcal{S}_d \to \mathbb{R}$ is degenerate elliptic. Let $w : \mathcal{O} \to \mathbb{R}$ be of class C^2 in $\mathcal{O}$. Then w is a classical subsolution of the HJ equation in $\mathcal{O}$ if an only if it is a viscosity subsolution in $\mathcal{O}$. The same holds for supersolutions and for solutions.*

Proof. If w is a viscosity subsolution then taking $\phi = w$, $w - w = 0$ has a local maximum at any point $x_0 \in \mathcal{O}$ and thus, by the definition of a viscosity subsolution,

$$rw(x_0) + H(x, Dw(x_0), D^2w(x_0)) \leq 0.$$

This shows that w is a classical subsolution. Assume now that w is a classical subsolution. Let ϕ be a C^2 test function such that $w-\phi$ has a local maximum at $x_0 \in \mathcal{O}$. The optimality conditions give

$$D(w-\phi)(x_0) = 0, \qquad D^2(w-\phi)(x_0) \leq 0.$$

Thus, by degenerate ellipticity of H,

$$rw(x_0) + H(x_0, D\phi(x_0), D^2\phi(x_0)) \leq rw(x_0) + H(x_0, Dw(x_0), D^2w(x_0)) \leq 0,$$

because w is classical subsolution. □

Exercise 4.1. *if w is a subsolution to the HJ equation in $\mathcal{O}$, then $u := -w$ is a supersolution of the HJ equation*

$$ru + \tilde{H}(x, Du, D^2u) = 0 \qquad \text{in } \mathcal{O},$$

where $\tilde{H}(x,p,S) = -H(x,-p,-S)$.

This result allows to transfer properties for subsolution to properties for supersolutions.

Proposition 4.7. *Assume that w_1 and w_2 are two viscosity subsolutions of the HJ equation in $\mathcal{O}$. Then $\sup\{w_1, w_2\}$ is also a subsolution of the HJ in $\mathcal{O}$.*

Proof. Set $w = \sup\{w_1, w_2\}$. Then w is continuous in $\mathcal{O}$ as so are w_1 and w_2. Let ϕ be a C^2 test function such that $w-\phi$ has a local maximum at some point x_0: there exists $R>0$ such that, for any $x \in B(x_0, R)$,

$$(w-\phi)(x) \leq (w-\phi)(x_0).$$

Replacing ϕ by $\phi - (w-\phi)(x_0)$ if necessary (which does change the inequality to prove since it only involves derivatives of ϕ), we can assume that $(w-\phi)(x_0) = 0$. Thus

$$w(x) \leq \phi(x) \qquad \forall x \in B(x_0, R), \text{ with an equality at } x_0.$$

Assume to fix the ideas that $w(x_0) = w_1(x_0)$ (the other case can be treated in a symmetric way). Then

$$w_1(x) \leq w(x) \leq \phi(x) \qquad \forall x \in B(x_0, R), \text{ with an equality at } x_0.$$

This means that $w_1 - \phi$ has a local maximum at x_0. Thus

$$rw_1(x_0) + H(x_0, D\phi(x_0), D^2\phi(x_0)) \leq 0,$$

which proves the result as $w(x_0) = w_1(x_0)$. □

Exercise 4.2. *State and prove the corresponding result for viscosity supersolutions.*

Proposition 4.8 (Stability)**.** *Assume that (w_n) is a sequence of viscosity subsolution of the HJ equation*

$$rw_n + H_n(x, Dw_n, D^2w_n) \leq 0$$

in $\mathcal{O}$. If H_n converges locally uniformly to H in $\mathcal{O} \times \mathbb{R}^d \times \mathcal{S}_d$ and if (w_n) converges locally uniformly to a continuous map w in $\mathcal{O}$, then w is a subsolution of the HJ equation

$$rw + H(x, Dw, D^2w) \leq 0$$

in $\mathcal{O}$.

The proof requires as a preliminary step an equivalent notion of viscosity solution.

Lemma 4.9. *w is a viscosity solution to the HJ equation in $\mathcal{O}$ iff for any C^2 map $\phi : \mathbb{R}^d \to \mathbb{R}$ such that $w-\phi$ has a* strict *local maximum at some point $x_0 \in \mathcal{O}$, one has*

$$rw(x_0) + H(x, D\phi(x_0), D^2\phi(x_0)) \leq 0.$$

Proof. If $w-\phi$ has a strict local maximum at some point $x_0 \in \mathcal{O}$, then it has a local maximum at x_0. Therefore if w is a viscosity solution, it satisfies the condition. Conversely, assume that w satisfies the property and let ϕ C^2 such that $w-\phi$ has a local maximum at some point $x_0 \in \mathcal{O}$: there exists $R>0$ such that

$$(w-\phi)(x) \leq (w-\phi)(x_0) \qquad \forall x \in B(x_0, R).$$

For $\varepsilon > 0$, set

$$\phi_\varepsilon(x) = \phi(x) + \varepsilon |x-x_0|^4.$$

Then

$$(w-\phi_\varepsilon)(x) = (w-\phi)(x) - \varepsilon|x-x_0|^4 \leq (w-\phi)(x_0) - \varepsilon|x-x_0|^4 < (w-\phi)(x_0) \qquad \forall x \in B(x_0,R),\ x \neq x_0.$$

Thus $w-\phi_\varepsilon$ has a strict local maximum at x_0 and

$$rw(x_0) + H(x, D\phi_\varepsilon(x_0), D^2\phi_\varepsilon(x_0)) \leq 0.$$

Note however that

$$D\phi_\varepsilon(x) = D\phi(x) + 4\varepsilon|x-x_0|^2(x-x_0),\ D^2\phi_\varepsilon(x) = D^2\phi(x) + 4\varepsilon|x-x_0|^2 I_d + 8(x-x_0)(x-x_0)^T,$$

so that $D\phi_\varepsilon(x_0) = D\phi(x_0)$ while $D^2\phi_\varepsilon(x_0) = D^2\phi(x_0)$. □

We are now ready to prove the stability property.

Proof. The map w is continuous, as local uniform limit of continuous maps. Let ϕ be C^2 and such that $w-\phi$ has a strict maximum at x_0: there exists $R>0$ such that

$$(*) \qquad (w-\phi)(x) < (w-\phi)(x_0) \qquad \forall x \in B(x_0,R),\ x \neq x_0.$$

(here $B(x_0,R)$ is the closed ball). Let x_n be a maximum point of $w_n - \phi$ in the ball $B(x_0,R)$.

We claim that (x_n) converges to x_0. Let $(x_{n'})$ be a converging subsequence of the compact sequence (x_n) with limit $\bar{x} \in B(x_0,R)$. We have, by definition,

$$(w_{n'}-\phi)(x) \leq (w_{n'}-\phi)(x_{n'}) \qquad \forall x \in B(x_0,R).$$

Letting $n \to \infty$ we have, by local uniform convergence of w_n,

$$(w-\phi)(x) \leq (w-\phi)(\bar{x}) \qquad \forall x \in B(x_0,R).$$

Thus $(w-\phi)(x_0) \leq (w-\phi)(\bar{x})$. Plugging $\bar{x}$ in $(*)$ gives on the other hand that $(w-\phi)(\bar{x}) < (w-\phi)(x_0)$. So $\bar{x} = x_0$. This shows that any converging subsequence of the compact sequence (x_n) converges to x_0: thus (x_n) converges to x_0.

As (x_n) converges to x_0, x_n belongs to the interior of $B(x_0,R)$ for any n large enough. Thus $w_n - \phi$ has a local maximum at x_n (without the constraint now). As w_n is a subsolution we get

$$rw_n(x_n) + H_n(x_n, D\phi(x_n), D^2\phi(x_n)) \leq 0.$$

Letting $n \to \infty$ and using the regularity of ϕ and the local uniform convergence of (H_n) and (w_n) we obtain

$$rw(x_0) + H(x_0, D\phi(x_0), D^2\phi(x_0)) \leq 0,$$

which shows that w is a subsolution. □

4.3. **The value function is a viscosity solution.** We now come back to our infinite horizon problem, with value function u defined by (3.1).

Theorem 4.10. *The value function u is a viscosity solution to the HJ equation*

$$rw + H(x, Dw, D^2w) = 0 \text{ in } \mathbb{R}^d,$$

where H is defined by (3.2).

Proof. We leave the proof of the fact that u is a subsolution to the reader: it relies on the same argument than the proof of the supersolution property and is easier.

We now prove that u is a supersolution. We already know that u is continuous. Fix now ϕ a C^2 map and assume that $u-\phi$ has a local minimum at $x_0 \in \mathbb{R}^d$: there exists $R>0$ such that

$$(u-\phi)(x) \geq (u-\phi)(x_0) \qquad \forall x \in B(x_0,R).$$

Replacing ϕ by $\phi - (u-\phi)(x_0)$ if necessary, we can assume that $(u-\phi)(x_0)=0$. Thus

$$u(x) \geq \phi(x) \qquad \forall x \in B(x_0,R), \qquad \text{with an equality at } x_0.$$

We now argue by contradiction, assuming that there exists $\eta>0$ such that

$$ru(x_0) + H(x_0, D\phi(x_0), D^2\phi(x_0)) \leq -2\eta.$$

As H is continuous, ϕ is C^2 and $u(x_0)=\phi(x_0)$, there is a neighborhood again denoted $B(x_0,R)$ (reducing R if necessary) such that

$$r\phi(x) + H(x, D\phi(x), D^2\phi(x)) \leq -\eta \qquad \forall x \in B(x_0,R).$$

Recalling the definition of H this means that

$$r\phi(x) - L(x,a) - b(x,a)\cdot D\phi(x) - \frac{1}{2}Tr(\sigma\sigma^T(x,a)D^2\phi(x)) \leq -\eta \qquad \forall x \in B(x_0,R). \tag{4.2}$$

Fix now $\alpha \in \mathcal{A}$ and, for $h>0$ let us introduce the stopping times

$$\tau_\alpha := \inf\{t \geq 0,\ |X_t^{x_0,\alpha} - x_0| \geq R\},\ \tau_{\alpha,h} := \tau_\alpha \wedge h.$$

We now use the dynamic programming: we get

$$\begin{aligned} u(x_0) &= \inf_{\alpha\in\mathcal{A}} \sup_{\tau\in\mathcal{S}} \mathbb{E}\left[\int_0^\tau e^{-rt}L(X_t^\alpha,\alpha_t)dt + e^{-r\tau}u(X_\tau^\alpha)\right] \\ &\geq \inf_{\alpha\in\mathcal{A}} \mathbb{E}\left[\int_0^{\tau_{\alpha,h}} e^{-rt}L(X_t^\alpha,\alpha_t)dt + e^{-r\tau}u(X_{\tau_{\alpha,h}}^\alpha)\right] \\ &\geq \inf_{\alpha\in\mathcal{A}} \mathbb{E}\left[\int_0^{\tau_{\alpha,h}} e^{-rt}L(X_t^\alpha,\alpha_t)dt + e^{-r\tau}\phi(X_{\tau_{\alpha,h}}^\alpha)\right], \end{aligned}$$

where we used the fact that $X_{\tau_{\alpha,h}}^\alpha \in B(x_0,R)$ in $[[0,\tau_{\alpha,h}]]$ and that $u\geq\phi$ in this ball. Fixing $\varepsilon \in (0,\eta)$, let α_h be $(\varepsilon h)-$optimal in the right-hand side of the inequality above. We set for simplicity $\tau_h := \tau_{\alpha_h,h}$. Using Itô's formula (and the usual argument to get rid of the local martingale part) we obtain (recalling that $\phi(x_0)=u(x_0)$):

$$\begin{aligned} u(x_0)+\varepsilon h &\geq \phi(x_0) + \mathbb{E}\left[\int_0^{\tau_h} e^{-rt}L(X_t^{\alpha_h},\alpha_{h,t})dt + e^{-r\tau_h}\phi(X_{\tau_h}^{\alpha_h})\right] \\ &\geq u(x_0) + \mathbb{E}\left[\int_0^{\tau_h} e^{-rt}(L - ru + b\cdot D\phi + \frac{1}{2}\mathrm{Tr}(\sigma\sigma^T D^2\phi))(X_t^{\alpha_h},\alpha_{h,t})dt\right]. \end{aligned}$$

For $t \in [[0,\tau_h]]$, we have that $X_t^{\alpha_h} \in B_R(x_0)$ by the definition of τ_h. Hence we can use inequality (4.2) to obtain, after simplification by $u(x_0)$:

$$\varepsilon h \geq \mathbb{E}\left[\int_0^{\tau_h} e^{-rt}\eta dt\right] = \eta\mathbb{E}\left[\frac{1-e^{-r\tau_h}}{r}\right].$$

We divide by $h>0$ and obtain (omitting the event $\{\tau_h<h\}$ since the integrand is positive)

$$\varepsilon \geq \eta\mathbb{P}\left[\tau_h = h\right]\frac{1-e^{-rh}}{rh}. \tag{4.3}$$

Note that, by Markov inequality,

$$\mathbb{P}\left[\tau_h < h\right] = \mathbb{P}\left[\sup_{t\in[0,h]} |X_t^{\alpha_h} - x_0| \geq R\right] \leq \frac{\mathbb{E}\left[\sup_{t\in[0,h]} |X_t^{\alpha_h} - x_0|^2\right]}{R^2} \leq \frac{Ch}{R^2},$$

where the constant C depends only on $\|b\|_\infty$ and $\|\sigma\|_\infty$. Thus

$$\mathbb{P}[\tau_h = h] \to 1 \text{ as } h \to 0^+.$$

Letting $h \to 0^+$ in (4.3) we find a contradiction with $\varepsilon < \eta$. This proves that u is a supersolution. □

5. Comparison principles and applications

This part is devoted to comparison principles for Hamilton-Jacobi equations. Let us first try to understand when the notion of viscosity solution is necessary.

5.1. **Classical solution to the Hamilton-Jacobi equations.**

5.1.1. *First order Hamilton-Jacobi equations.* We first start with first order Hamilton-Jacobi (HJ) equations to explain that, in general, we cannot expect to have a classical solution. The simplest example is in dimension $d = 1$ and the so-called Eikonal equation in the open set $(-1, 1)$:

$$(E) \qquad \begin{cases} |u_x|^2 = 1 & \text{in } (-1,1), \\ u(-1) = u(1) = 0. \end{cases}$$

A viscosity solution to (E) is a map $u : [-1, 1] \to \mathbb{R}$, which is continuous in $[-1, 1]$, satisfies the boundary conditions $u(-1) = u(1) = 0$ and is a viscosity solution to the equation in $(-1, 1)$. Next we check that there cannot be a classical solution to this equation:

Lemma 5.1. *There is no map $u : [-1, 1] \to \mathbb{R}$ which is continuous in $[-1, 1]$, satisfies the boundary conditions $u(-1) = u(1) = 0$ and is a C^1 classical solution to the equation in $(-1, 1)$.*

Proof. Indeed, let us assume that such a classical solution u exists and let x_0 be a maximum point of $|u|$ in $[-1, 1]$. If $|u(x_0)| = 0$, then u is identically 0, which is impossible since $u = 0$ does not solve the equation. Hence $|u(x_0)| > 0$. Assume to fix the ideas that $u(x_0) > 0$ (the case $u(x_0) < 0$ being treated in a symmetric way). Then $x_0 \notin \{-1, 1\}$ since u satisfies the boundary conditions. Thus u has a local maximum at x_0, which implies that $u_x(x_0) = 0$. This impossible in view of the equation satisfied by u. □

We now note that viscosity solutions of our Eikonal do exist.

Lemma 5.2. *The map $u(x) = 1 - |x|$ on $[-1, 1]$ is a viscosity solution to (E).*

We will present below a comparison principle for this kind of equation, which implies that the map u given in the lemma is the unique viscosity solution to (E).

Proof. The map $u(x) = 1 - |x|$ is continuous in $[-1, 1]$ and satisfies the boundary conditions $u(-1) = u(1) = 0$. Note that u is C^1 in $(-1, 1)\backslash\{0\}$, so that it is a viscosity solution at these points (Exercise). It remain to check that it is a viscosity solution at 0.

Let us check that it is a viscosity supersolution. Indeed the maps $v(x) = 1 - x$ and $w(x) = 1 + x$ are both classical solution to the equation $|u_x| = 1$ in $(-1, 1)$. Thus $u(x) = \min\{v(x), w(x)\}$ is a viscosity supersolution of this equation (as minimum of two supersolutions).

Let us now check that it is a viscosity subsolution at 0. Let ϕ be a C^2 test function such that $u - \phi$ has a maximum at 0. We have to check that $|\phi_x(0)|^2 \leq 1$. Without loss of generality we assume that $u(0) = \phi(0) = 1$ and thus $u(x) \leq \phi(x)$ with an equality at 0. For $h > 0$ small, we have

$$u(h) = 1 - h \leq \phi(h) = 1 + \phi_x(0)h + o(h),$$

so that, simplifying by 1, dividing by $h > 0$ and letting $h \to 0^+$ gives $-1 \leq \phi_x(0)$. If we use the same argument for $h < 0$ small: we get

$$u(h) = 1 + h \leq \phi(h) = 1 + \phi_x(0)h + o(h),$$

and thus $1 \geq \phi_x(0)$. So $|\phi_x(0)| \leq 1$, which is the required inequality. □

A classical extension of the Eikonal equation in higher dimension is discussed next as an exercise.

Exercise 5.1. *Let $\mathcal{O}$ be an open and bounded subset of $\mathbb{R}^d$. We consider the Eikonal equation in $\mathcal{O}$:*

$$(E) \qquad \begin{cases} |Du(x)|^2 = 1 & \text{in } \mathcal{O}, \\ u(x) = 0 & \text{in } \partial\mathcal{O}. \end{cases}$$

A viscosity solution of (E) is a map $u : \overline{\mathcal{O}} \to \mathbb{R}$ which is continuous in $\overline{\mathcal{O}}$, satisfies the boundary condition $u(x) = 0$ in $\partial\mathcal{O}$ and is a viscosity solution to the equation in $\mathcal{O}$.

(1) *Show that this equation has no C^1 solution.*
(2) *Let u be defined by*

$$u(x) = \min_{y\in\partial\mathcal{O}} |x - y|$$

Show that u is Lipschitz continuous in $\mathcal{O}$ with Lipschitz constant not larger than 1.
(3) *Show that, for any $y \in \partial O$, the map $x \to |x - y|$ solves the equation in a classical sense in $\mathcal{O}$ and infer that u is a viscosity supersolution of the HJ equation.*
(4) *Given $x \in \mathcal{O}$ and $y \in \partial\mathcal{O}$ such that $u(x) = |x-y|$, check that $u((1-t)x+ty) = (1-t)|x-y|$ for $t \in [0, 1]$. Infer from this that u is a viscosity subsolution to the equation.*

The conclusion of these examples is that, even when the equation has smooth coefficient, one cannot expect to have classical solution to these first order Hamilton-Jacobi equations.

5.1.2. *The uniformly elliptic case.* Let us define the matrix $A(x, a) = \sigma\sigma^T(x, t, a)$. The problem is said to be uniformly elliptic if there if a constant $c > 0$ such that

$$\xi^T A(x, a)\xi \geq c|\xi|^2 \qquad \forall (x, t, a, \xi) \in \mathbb{R}^d \times A \times \mathbb{R}^d.$$

In this case, and if the Hamiltonian is sufficiently smooth, one can expect the solution to the Hamilton-Jacobi equation to be C^2. The statement and proof of the results exceed the scope of this course. Classical references are:

- Gilbarg, D., Trudinger, N. S., Gilbarg, D., & Trudinger, N. S. (1977). Elliptic partial differential equations of second order (Vol. 224, No. 2). Berlin: springer.
- Krylov, N. V. (1987). Nonlinear elliptic and parabolic equations of the second order. Dordrecht, Holland: D. Reidel Publishing Company.
- Ladyzhenskaia, O. A., Solonnikov, V. A., & Ural'tseva, N. N. (1968). Linear and quasi-linear equations of parabolic type (Vol. 23). American Mathematical Soc..
- Lieberman, G. M. (1996). Second order parabolic differential equations. World scientific.

5.2. **Comparison principle for first order HJ equations.** Throughout this part, we assume that $H : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ is continuous and satisfies the following Lipschitz condition: there exists a constant $C > 0$ such that, for any $x, y, p, q \in \mathbb{R}^d$,

$$|H(x, p) - H(y, p)| \leq C|x - y|(1 + |p|), \qquad |H(x, p) - H(x, q)| \leq C|p - q|. \tag{5.1}$$

Example: If H corresponds to a deterministic infinite horizon control problem (i.e., with $\sigma \equiv 0$) and our standard assumptions hold, then

$$H(x, p) = \max_{a\in A} -L(x, a) - b(x, a) \cdot p \tag{5.2}$$

satisfies (5.2). Indeed, for any $x, y, p \in \mathbb{R}^d$ and if a is optimal for $H(x, p)$, we have

$$H(y, p) \geq -L(y, a)-b(y, a)\cdot p \geq -L(x, a)-b(x, a)\cdot p-Lip(L)|x-y|-Lip(b)|p||x-y| \geq -C|x-y|(1+|p|),$$

where $C = \max\{Lip(L), Lip(b)\}$. Exchanging the roles of x and y we find that H satisfies our condition (5.1) on H.

5.2.1. *Bounded domains and stationary equations.* Let $\mathcal{O}$ be an open and bounded subset of $\mathbb{R}^d$. We consider the equation:

$$\begin{cases} u + H(x, Du(x)) = 0 \text{ in } \mathcal{O}, \\ u(x) = 0 \text{ in } \partial\mathcal{O}. \end{cases} \tag{5.3}$$

This HJ equation corresponds (at least formally) to the following optimal control problem:

$$u(x_0) = \inf_{\alpha \in \mathcal{A}} \int_0^{\tau^\alpha} e^{-t} L(X_t^\alpha, \alpha_t) dt$$

where

$$\tau^\alpha = \inf\{t \geq 0,\ X_t^\alpha \in \partial\mathcal{O}\}$$

and, for $x_0 \in \mathcal{O}$, $X^\alpha = X^{x_0,\alpha}$ is the solution to

$$\frac{d}{dt} X_t^\alpha = b(X_t^\alpha, \alpha_t), \qquad X_0^\alpha = x_0.$$

One of the issue for this control problem is that u is discontinuous in general. However one can show that u is continuous if, for instance, the boundary $\partial\mathcal{O}$ is sufficiently smooth (i.e., locally the graph of a C^1 function) and if the following "controllability condition" holds: there exists $r > 0$ such that

$$B(0, r) \subset b(x, A) \qquad \forall x \in \mathcal{O}.$$

Theorem 5.3 (Comparison)**.** *Assume that $u : \overline{\mathcal{O}} \to \mathbb{R}$ is continuous and a viscosity subsolution of* (5.3) *in $\mathcal{O}$ while $v : \overline{\mathcal{O}} \to \mathbb{R}$ is continuous and a viscosity supersolution of* (5.3) *in $\mathcal{O}$ with $u \leq v$ in $\partial\mathcal{O}$. Then $u \leq v$ in $\mathcal{O}$*

This shows that the HJ (5.3) has at most one viscosity solution. If the value function u described above is continuous, then one can show that it is the unique viscosity solution to the equation.

Proof. Step 1: penalization. Our aim is to prove that $M := \sup(u - v) \leq 0$. We argue by contradiction and assume that $M > 0$. In order to use the notion of viscosity solutions, we penalize the problem by the so-called "doubling variable method". Given $\varepsilon > 0$, let

$$M_\varepsilon = \max_{x,y \in \overline{\mathcal{O}}} u(x) - v(y) - \frac{|x - y|^2}{2\varepsilon}.$$

Let $(x_\varepsilon, y_\varepsilon) \in \overline{\mathcal{O}} \times \overline{\mathcal{O}}$ be such that $M_\varepsilon = u(x_\varepsilon) - v(y_\varepsilon) - \frac{|x_\varepsilon - y_\varepsilon|^2}{2\varepsilon}$.

Step 2: estimates. We claim that

(i) $M_\varepsilon \to M$ as $\varepsilon \to 0^+$ and $\lim_{\varepsilon \to 0^+} \frac{|x_\varepsilon - y_\varepsilon|^2}{2\varepsilon} = 0$,
(ii) If $\varepsilon > 0$ is small enough, then $x_\varepsilon \in \mathcal{O}$ and $y_\varepsilon \in \mathcal{O}$.

Note that (i) implies that $|x_\varepsilon - y_\varepsilon| \to 0$.

Proof of the claim. Note that $M_\varepsilon \geq M$ since, for any $x \in \mathcal{O}$, $M_\varepsilon \geq u(x) - v(x)$. Note also that, by optimality of $(x_\varepsilon, y_\varepsilon)$, we have, for any $x \in \mathcal{O}$,

$$u(x_\varepsilon) - v(y_\varepsilon) - \frac{|x_\varepsilon - y_\varepsilon|^2}{2\varepsilon} \geq u(x) - v(x),$$

so that, as u and v are continuous in the compact set $\overline{\mathcal{O}}$ and thus bounded,

$$\frac{|x_\varepsilon - y_\varepsilon|^2}{2\varepsilon} \leq 2(\|u\|_\infty + \|v\|_\infty).$$

Thus $|x_\varepsilon - y_\varepsilon| \to 0$ as $\varepsilon \to 0$. So, denoting by ω a modulus of continuity of v in the compact set $\overline{\mathcal{O}}$,

$$M_\varepsilon = u(x_\varepsilon) - v(x_\varepsilon) - \frac{|x_\varepsilon - y_\varepsilon|^2}{2\varepsilon} + v(x_\varepsilon) - v(y_\varepsilon) \leq M - \frac{|x_\varepsilon - y_\varepsilon|^2}{2\varepsilon} + \omega(|x_\varepsilon - y_\varepsilon|). \tag{5.4}$$

Thus

$$\limsup_{\varepsilon \to 0} M_\varepsilon \leq \limsup_{\varepsilon \to 0} M + \omega(|x_\varepsilon - y_\varepsilon|) = M.$$

This proves that $M_\varepsilon \to 0$ and, coming back to (5.4), that

$$\limsup_{\varepsilon\to 0} \frac{|x_\varepsilon - y_\varepsilon|^2}{2\varepsilon} \le \limsup_{\varepsilon\to 0} M - M_\varepsilon + \omega(|x_\varepsilon - y_\varepsilon|) = 0.$$

So (i) holds.

We now show (ii) by contradiction. Assume (to fix the ideas) that there exists a subsequence (ε_n) which tend to 0 and such that $x_{\varepsilon_n} \in \partial\mathcal{O}$ for any n. Then, extracting a subsequence if necessary, there exists $x \in \partial\mathcal{O}$ such that (x_{ε_n}) tends to x. But we know from the first step that $|x_\varepsilon - y_\varepsilon| \to 0$, so that (y_{ε_n}) tends also to x. Hence, as $u \le v$ in $\partial\mathcal{O}$,

$$M = \lim_{\varepsilon\to 0} u(x_\varepsilon) - v(y_\varepsilon) - \frac{|x_\varepsilon - y_\varepsilon|^2}{2\varepsilon} \le \limsup_n u(x_{\varepsilon_n}) - v(y_{\varepsilon_n}) = u(x) - v(x) \le 0.$$

This contradicts our assumption $M > 0$ and shows that $x_\varepsilon \in \mathcal{O}$ for $\varepsilon > 0$ small. The symmetric argument works for y_ε, which completes the proof of (ii).

Step 3: using the HJ equation. We now choose $\varepsilon > 0$ small enough so that (ii) holds. As x_ε is a interior maximum point of

$$x \to u(x) - v(y_\varepsilon) - \frac{|x - y_\varepsilon|^2}{2\varepsilon},$$

using as C^2 test function the map $\phi(x) = v(y_\varepsilon) + \frac{|x-y_\varepsilon|^2}{2\varepsilon}$ in the definition of viscosity subsolution for u, we have

$$u(x_\varepsilon) + H(x_\varepsilon, p_\varepsilon) \le 0, \qquad \text{where } p_\varepsilon = D\phi(x_\varepsilon) = \frac{x_\varepsilon - y_\varepsilon}{\varepsilon}. \tag{5.5}$$

In the same way, as y_ε is a minimum point of

$$x \to v(y) - u(x_\varepsilon) + \frac{|x_\varepsilon - y|^2}{2\varepsilon}$$

and v is a supersolution, we obtain

$$v(y_\varepsilon) + H(y_\varepsilon, p_\varepsilon) \ge 0. \tag{5.6}$$

Thus, using the regularity assumption on H, (5.5) and (5.6),

$$M_\varepsilon \le u(x_\varepsilon) - v(y_\varepsilon) \le -H(x_\varepsilon, p_\varepsilon) + H(y_\varepsilon, p_\varepsilon) \le C|x_\varepsilon - y_\varepsilon|(1 + |p_\varepsilon|),$$

where, by (i),

$$\limsup |x_\varepsilon - y_\varepsilon|(1 + |p_\varepsilon|) = \limsup |x_\varepsilon - y_\varepsilon| + \frac{|x_\varepsilon - y_\varepsilon|^2}{\varepsilon} = 0.$$

So

$$M = \lim M_\varepsilon \le 0,$$

which is in contraction with our assumption $M > 0$. Thus $M = \sup(u - v) \le 0$. □

Exercise 5.2. *In this exercise, we explain how to obtain a comparison for an equation of the form*

$$\begin{cases} H(x, Du(x)) = 0 \text{ in } \mathcal{O} \\ u(x) = 0 \text{ in } \partial\mathcal{O}. \end{cases}$$

when $H(x,p)$ satisfies, in addition to (5.1), the condition: for any $\alpha \in (0,1)$, there exists $\eta_\alpha > 0$ such that, for any $x, p \in \mathbb{R}^d$,

$$H(x, \alpha p) \le \alpha H(x,p) - \eta_\alpha.$$

Note that this is the case when H is given by (5.2) with $L > 0$, or for the Eikonal equation discussed above.

(1) *Show that, for any $\alpha \in (0,1)$, $u_\alpha := \alpha u$ is a viscosity subsolution of the equation*

$$H(x, Du_\alpha) \le -\eta_\alpha \text{ in } \mathcal{O}.$$

(2) *Arguing as in the proof of Theorem 5.3, show that $u_\alpha \le v$ and conclude.*

5.2.2. *Unbounded domains and stationary equations.* We now turn to the HJ in the whole space:

$$u + H(x, Du(x)) = 0 \qquad \text{in } \mathbb{R}^d.$$

Theorem 5.4 (Comparison)**.** *Let $u : \mathbb{R}^d \to \mathbb{R}$ be a bounded viscosity subsolution and $v : \mathbb{R}^d \to \mathbb{R}$ be a bounded viscosity supersolution. Then $u \le v$.*

If H is given by an infinite horizon deterministic control problem, then we know that the value function is a continuous and bounded viscosity solution. Therefore it is the unique viscosity solution.

Proof. Step 1: The penalization. We argue by contradiction, assuming that $M = \sup(u - v) > 0$. For $\varepsilon > 0$ and $\alpha > 0$, let

$$M_{\varepsilon,\alpha} = \max_{x,y\in\mathbb{R}^d} u(x) - v(y) - \frac{|x-y|^2}{\varepsilon} - \alpha(|x|^2 + |y|^2).$$

We denote by $(x_{\varepsilon,\alpha}, y_{\varepsilon,\alpha})$ a maximum point in the expression above. Note that this penalization is a mixture between the doubling variable technique (used for viscosity solution) and the usual penalization to treat unbounded domains.

Step 2: estimates. We claim that

(i) $\lim_{\alpha\to 0} \lim_{\varepsilon\to 0} M_{\varepsilon,\alpha} = M$,

(ii) and $\lim_{\alpha\to 0} \limsup_{\varepsilon\to 0} \frac{|x_{\varepsilon,\alpha}-y_{\varepsilon,\alpha}|^2}{\varepsilon} = \lim_{\alpha\to 0} \limsup_{\varepsilon\to 0} \alpha(|x_{\varepsilon,\alpha}|^2 + |y_{\varepsilon,\alpha}|^2) = 0$.

Proof of the claim. We have, for any $x \in \mathbb{R}^d$, $M_{\varepsilon,\alpha} \ge u(x) - v(x) - 2\alpha|x|^2$. Thus

$$\liminf_{\alpha\to 0} \liminf_{\varepsilon\to 0} M_{\varepsilon,\alpha} \ge u(x) - v(x).$$

Taking the sup in the right-hand side gives

$$\liminf_{\alpha\to 0} \liminf_{\varepsilon\to 0} M_{\varepsilon,\alpha} \ge M.$$

Conversely, we note that, as u and v are bounded and comparing for instance with $x = y = 0$, we get

$$\frac{|x_{\varepsilon,\alpha} - y_{\varepsilon,\alpha}|^2}{\varepsilon} + \alpha(|x_{\varepsilon,\alpha}|^2 + |y_{\varepsilon,\alpha}|^2) \le 2(\|u\|_\infty + \|v\|_\infty) =: C.$$

Let ω_α be the modulus of continuity of v in $B(0, C/\sqrt{\alpha})$. Then

$$\begin{aligned} M_{\varepsilon,\alpha} &= u(x_{\varepsilon,\alpha}) - v(x_{\varepsilon,\alpha}) - \frac{|x_{\varepsilon,\alpha} - y_{\varepsilon,\alpha}|^2}{\varepsilon} - \alpha(|x_{\varepsilon,\alpha}|^2 + |y_{\varepsilon,\alpha}|^2) + v(x_{\varepsilon,\alpha}) - v(y_{\varepsilon,\alpha}) \\ &\le M - \frac{|x_{\varepsilon,\alpha} - y_{\varepsilon,\alpha}|^2}{\varepsilon} - \alpha(|x_{\varepsilon,\alpha}|^2 + |y_{\varepsilon,\alpha}|^2) - \omega(|x_{\varepsilon,\alpha} - y_{\varepsilon,\alpha}|). \end{aligned}$$

Thus

$$\limsup_{\alpha\to 0} \limsup_{\varepsilon\to 0} M_{\varepsilon,\alpha} \le M.$$

We infer that (i) holds. Therefore

$$\limsup_{\alpha\to 0} \limsup_{\varepsilon\to 0} \frac{|x_{\varepsilon,\alpha} - y_{\varepsilon,\alpha}|^2}{\varepsilon} + \alpha(|x_{\varepsilon,\alpha}|^2 + |y_{\varepsilon,\alpha}|^2) \le \limsup_{\alpha\to 0} \limsup_{\varepsilon\to 0} M - M_{\varepsilon,\alpha} - \omega(|x_{\varepsilon,\alpha} - y_{\varepsilon,\alpha}|) = 0.$$

This shows (ii).

Step 3: Use of the equation. As the map

$$x \to u(x) - v(y_{\varepsilon,\alpha}) - \frac{|x - y_{\varepsilon,\alpha}|^2}{2\varepsilon} - \alpha(|x|^2 + |y_{\varepsilon,\alpha}|^2)$$

has a maximum at $x_{\varepsilon,\alpha}$, and as u is a subsolution, we have

$$u(x_{\varepsilon,\alpha}) + H(x_{\varepsilon,\alpha}, p_{\varepsilon,\alpha} + 2\alpha x_{\varepsilon,\alpha}) \le 0, \qquad \text{where } p_{\varepsilon,\alpha} = \frac{x_{\varepsilon,\alpha} - y_{\varepsilon,\alpha}}{\varepsilon}.$$

In a symmetric way, as

$$y \to v(y) - u(x_{\varepsilon,\alpha}) + \frac{|x_{\varepsilon,\alpha} - y|^2}{2\varepsilon} + \alpha(|x_{\varepsilon,\alpha}|^2 + |y|^2)$$

has a mininum at $y_{\varepsilon,\alpha}$ and as v is a supersolution, we have

$$v(y_{\varepsilon,\alpha}) + H(y_{\varepsilon,\alpha}, p_{\varepsilon,\alpha} - 2\alpha y_{\varepsilon,\alpha}) \geq 0.$$

Thus

$$\begin{aligned} M_{\varepsilon,\alpha} &\leq u(x_{\varepsilon,\alpha}) - v(y_{\varepsilon,\alpha}) \leq -H(x_{\varepsilon,\alpha}, p_{\varepsilon,\alpha} + 2\alpha x_{\varepsilon,\alpha}) + H(y_{\varepsilon,\alpha}, p_{\varepsilon,\alpha} - 2\alpha y_{\varepsilon,\alpha}) \\ &\leq -H(x_{\varepsilon,\alpha}, p_{\varepsilon,\alpha}) + H(y_{\varepsilon,\alpha}, p_{\varepsilon,\alpha}) + 2C\alpha(|x_{\varepsilon,\alpha}| + |y_{\varepsilon,\alpha}|) \\ &\leq C|x_{\varepsilon,\alpha} - y_{\varepsilon,\alpha}|(1 + |p_{\varepsilon,\alpha}|) + 2C\alpha(|x_{\varepsilon,\alpha}| + |y_{\varepsilon,\alpha}|). \end{aligned}$$

We can then conclude as previously that

$$M = \lim_{\alpha \to 0} \lim_{\varepsilon \to 0} M_{\varepsilon,\alpha} \leq 0,$$

which contradicts our assumption $M > 0$. Thus $M \leq 0$. □

5.2.3. *First order evolution equations.* The ideas developed above can be adapted to the first order evolution equation

$$\begin{cases} -\partial_t u + H(x, Du) = 0 & \text{in } (0,T) \times \mathbb{R}^d, \\ u(T,x) = g(x) & \text{in } \mathbb{R}^d, \end{cases}$$

where H satisfies (5.1). A standard idea is to change variable, setting $v(t,x) = e^t u$, to obtain an equation with a zero order term of the form

$$\begin{cases} -\partial_t v + v + \tilde{H}(t, x, Dv) = 0 & \text{in } (0,T) \times \mathbb{R}^d, \\ v(T,x) = e^T g(x) & \text{in } \mathbb{R}^d. \end{cases}$$

where $\tilde{H}(t,x,p) = e^t H(x, e^{-t}p)$.

5.3. **Comparison principle for second order equation.** The second order equations are much more challenging to handle than the first order equations. The following exercise shows that, even for an elementary equation, the previous approach does not work.

Exercise 5.3. *We consider the equation (in dimension $d = 1$)*

$$u(x) - u''(x) = 0 \qquad \text{in } (0,1)$$

and try to adapt the techniques of the previous section to compare a viscosity subsolution u with a viscosity supersolution v such that $u \leq v$ at $x = 0$ and $x = 1$. For this we assume that $(x_\varepsilon, y_\varepsilon)$ is a maximum point of $(x,y) \to u(x) - v(y) + \frac{|x-y|^2}{2\varepsilon}$ with $x_\varepsilon, y_\varepsilon \in (0,1)$. Show that the notion of viscosity solution only gives

$$u(x_\varepsilon) - \frac{1}{\varepsilon} \leq 0, \qquad v(y_\varepsilon) + \frac{1}{\varepsilon} \geq 0.$$

To overcome this issue, we need an equivalent notion of viscosity solution, which is more "pointwise".

Definition 5.5 (subjet and superjet, stationary case)**.** *Let $\mathcal{O}$ be an open subset of $\mathbb{R}^d$, $u : \mathcal{O} \to \mathbb{R}$ be a continuous function and $x \in \mathcal{O}$. We say that a pair $(p,S) \in \mathbb{R}^d \times \mathcal{S}_d$ belongs to the subjet $\overline{J}^- u(x)$ to u at x if there exists $x_n \to x$, $p_n \to p$ and $S_n \to S$ such that*

$$u(x_n + h) - u(x_n) \geq p_n \cdot h + \frac{1}{2} h^T S_n h + |h|^2 \varepsilon_n(h),$$

where $\varepsilon_n(h) \to 0$ as $h \to 0$. A pair (p,S) belongs to the superjet $\overline{J}^+ u(x)$ to u at x if $(-p,-S) \in \overline{J}^-(-u)(x)$.

The notion of viscosity sub/super solution can be expressed in terms of sub/superjet.

Proposition 5.6. *A continuous map $u : \mathcal{O} \to \mathbb{R}$ if a viscosity subsolution to the equation*

$$u + H(x, Du, D^2u) \leq 0 \text{ in } \mathcal{O}$$

where H is continuous, if and only if,

$$\forall x \in \mathcal{O},\ \forall (p,S) \in \overline{J}^+ u(x), \qquad u(x) + H(x,p,S) \leq 0. \tag{5.7}$$

Proof. Assume first that (5.7) holds. Let ϕ be a C^2 test function such that $u - \phi$ has a local maximum at $x \in \mathcal{O}$. We assume without loss of generality that $\phi(x) = u(x)$. Then, by the second order expansion of ϕ,

$$u(x+h) \leq \phi(x+h) = u(x) + D\phi(x) \cdot h + \frac{1}{2} h^T D^2\phi(x) h + |h|^2 \varepsilon(h).$$

Thus $(D\phi(x), D^2\phi(x)) \in \overline{J}^+ u(x)$, which implies that

$$u(x) + H(x, D\phi(x), D^2\phi(x)) \leq 0.$$

So u is a viscosity subsolution.

Suppose now that u is a viscosity subsolution. Assume first that $(p,S) \in \mathbb{R}^d \times \mathcal{S}_d$ is such that

$$u(x+h) - u(x) \leq p \cdot h + \frac{1}{2} h^T S h + |h|^2 \varepsilon(h).$$

For $\varepsilon > 0$ small, let $r > 0$ be such that $B(x,r) \subset \mathcal{O}$ and $|\varepsilon(h)| \leq \varepsilon$ in the ball $B(0,r)$. Then

$$u(x+h) - u(x) \leq p \cdot h + \frac{1}{2} h^T (S + \varepsilon I_d) h,$$

in $B(0,r)$ and an equality at $h = 0$. Thus $u(y) - p \cdot (y-x) + \frac{1}{2}(y-x)^T (S + \varepsilon I_d)(y-x)$ has a local maximum at $y = x$, so that $u(x) + H(x, p, S + \varepsilon I_d) \leq 0$. Letting $\varepsilon \to 0$ gives:

$$u(x) + H(x,p,S) \leq 0.$$

If now (p,S) belongs to $\overline{J}^+ u(x)$, then there exists $x_n \to x$, $p_n \to p$ and $S_n \to S$ such that

$$u(x_n + h) - u(x_n) \leq p_n \cdot h + \frac{1}{2} h^T S_n h + |h|^2 \varepsilon_n(h).$$

By the previous step,

$$u(x_n) + H(x_n, p_n, S_n) \leq 0.$$

We obtain the desired inequality by letting $n \to \infty$. □

Exercise 5.4. *Characterize in the same way the notion of viscosity supersolution.*

The central result to obtain a comparison principle in for second order equations is the following:

Theorem 5.7 (Ishii Lemma)**.** *Assume that $u, v : \mathcal{O} \to \mathbb{R}$ are two continuous maps and that $\phi = \phi(x,y)$ is C^2 and such that $(x,y) \to u(x) - v(y) - \phi(x,y)$ has a local maximum at a point $(x_0, y_0) \in \mathcal{O} \times \mathcal{O}$. Then, for any $\eta > 0$ there exists $X, Y \in \mathcal{S}_d$ such that $(D_x\phi(x_0,y_0), X) \in \overline{J}^+ u(x_0)$, $(-D_y\phi(x_0,y_0), Y) \in \overline{J}^- u(y_0)$, and*

$$\begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \leq D^2\phi(x_0,y_0) + \eta (D^2\phi(x_0,y_0))^2.$$

Remark 5.8. *One often takes $\phi(x,y) = \frac{1}{2\varepsilon}|x-y|^2$. Then $D^2\phi(x,y) = \frac{1}{\varepsilon}\begin{pmatrix} I_d & -I_d \\ -I_d & I_d \end{pmatrix}$, and thus $(D^2\phi(x,y))^2 = \frac{2}{\varepsilon} D^2\phi(x,y)$. A good choice for η is then $\eta = \varepsilon$, which yields to*

$$\begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \leq \frac{3}{\varepsilon} \begin{pmatrix} I_d & -I_d \\ -I_d & I_d \end{pmatrix}. \tag{5.8}$$

The proof of the Ishii Lemma exceeds the scope of the course. It can be found in the standard references on viscosity solution [2, 4, 11]. We now explain how to prove the comparison in a very simple setting:

Proposition 5.9. *Assume that $\mathcal{O}$ is an open and bounded subset of $\mathbb{R}^d$. Let $H : \mathbb{R}^d \times \mathcal{S}_d \to \mathbb{R}$ be continuous and (degenerate) elliptic. If $u : \overline{\mathcal{O}} \to \mathbb{R}$ is a subsolution to*

$$u + H(Du, D^2u) = 0 \qquad \text{in } \mathcal{O}$$

and $v : \overline{\mathcal{O}} \to \mathbb{R}$ is a supersolution to that equation, with $u \leq v$ in $\partial\mathcal{O}$, then $u \leq v$ in $\mathcal{O}$.

Proof. The beginning of the proof is exactly as in the proof of Theorem 5.3: Given $\varepsilon > 0$, let

$$M_\varepsilon = \max_{x,y\in\overline{\mathcal{O}}} u(x) - v(y) - \frac{|x-y|^2}{2\varepsilon}.$$

Let $(x_\varepsilon, y_\varepsilon) \in \overline{\mathcal{O}} \times \overline{\mathcal{O}}$ be such that $M_\varepsilon = u(x_\varepsilon) - v(y_\varepsilon) - \frac{|x_\varepsilon - y_\varepsilon|^2}{2\varepsilon}$. Then we can show as in the proof of Theorem 5.3 that

(i) $M_\varepsilon \to M$ as $\varepsilon \to 0^+$ and $\lim_{\varepsilon\to 0^+} \frac{|x_\varepsilon - y_\varepsilon|^2}{2\varepsilon} = 0$,
(ii) If $\varepsilon > 0$ is small enough, then $x_\varepsilon \in \mathcal{O}$ and $y_\varepsilon \in \mathcal{O}$.

We now use the equations: we apply Ishii Lemma to the map $\phi_\varepsilon(x,y) = \frac{|x-y|^2}{2\varepsilon}$. As $(x_\varepsilon, y_\varepsilon)$ is a maximum point of $(x,y) \to u(x) - v(y) - \phi_\varepsilon(x,y)$, there exists $X_\varepsilon, Y_\varepsilon \in \mathcal{S}_d$ such that $(p_\varepsilon, X_\varepsilon) \in \overline{J}^- u(x_\varepsilon)$, $(p_\varepsilon, Y_\varepsilon) \in \overline{J}^+ v(y_\varepsilon)$ and (recall Remark 5.8)

$$\begin{pmatrix} X_\varepsilon & 0 \\ 0 & -Y_\varepsilon \end{pmatrix} \leq \frac{3}{\varepsilon} \begin{pmatrix} I_d & -I_d \\ -I_d & I_d \end{pmatrix}.$$

Applying this inequality to vectors of the form $(x,x) \in \mathbb{R}^{2d}$ shows that $X_\varepsilon \leq Y_\varepsilon$. As u is a subsolution while v is a supersolution, we have

$$u(x_\varepsilon) + H(p_\varepsilon, X_\varepsilon) \leq 0, \qquad v(y_\varepsilon) + H(p_\varepsilon, Y_\varepsilon) \geq 0.$$

As H is degenerate elliptic and $X_\varepsilon \leq Y_\varepsilon$, we obtain:

$$M_\varepsilon \leq u(x_\varepsilon) - v(y_\varepsilon) \leq H(p_\varepsilon, Y_\varepsilon) - H(p_\varepsilon, X_\varepsilon) \leq 0.$$

Thus $M = \sup(u - v) = \lim_{\varepsilon\to 0} M_\varepsilon \leq 0$. □

A classical use of inequality (5.8) is the following.

Proposition 5.10. *Assume that*

$$H(x,S) = \sup_{a\in A}\{-Tr(\sigma\sigma^T(x,a)S\}$$

where σ satisfies our standing conditions and that (X,Y) satisfies (5.8)*. Then*

$$H(x,X) - H(y,Y) \geq -\frac{3}{\varepsilon}|x-y|^2.$$

Proof. Use the vector $(\sigma(x,a), \sigma(y,a))$ in (5.8) and take the trace: we get

$$\mathrm{Tr}(\sigma\sigma^T(x,a)X) - \mathrm{Tr}(\sigma\sigma^T(y,a)Y) \leq \frac{3}{\varepsilon}|\sigma(x,a) - \sigma(y,a)|^2 \leq \frac{3Lip(\sigma)^2}{\varepsilon}|x-y|^2.$$

Thus

$$H(x,X) \geq H(y,Y) - \frac{3Lip(\sigma)^2}{\varepsilon}|x-y|^2.$$

□

Following the argument of Proposition 5.9 easily yields to a comparison for an equation of the form

$$u + H(x, D^2u) = 0 \qquad \text{in } \mathcal{O},$$

where $\mathcal{O}$ is an open bounded subset of $\mathbb{R}^d$.

Exercise 5.5. *Extend the proposition above to the Hamiltonian*

$$H(x,p,S) = \sup_{a\in A} -L(x,a) - b(x,a)\cdot p - \mathrm{Tr}(\sigma\sigma^T(x,a)S\}, \tag{5.9}$$

where L, b and σ satisfy our standing assumptions, and with $p = (x-y)/\varepsilon$. Then use the technique of Proposition 5.9 to prove a comparison principle for an equation of the form

$$u + H(x, Du, D^2u) = 0 \qquad \text{in } \mathcal{O},$$

where $\mathcal{O}$ is an open bounded subset of $\mathbb{R}^d$. Generalization to bounded solutions in $\mathbb{R}^d$.

The approach extends to parabolic equations. The sub- and superjets are replaced by the parabolic sub- and superjets: given $\mathcal{O}$ an open subset of $\mathbb{R}^d \times \mathbb{R}$ and $u : \mathcal{O} \to \mathbb{R}$, we say that $(p_t, p_x, X) \in \mathbb{R}\times\mathbb{R}^d\times\mathcal{S}_d$ belongs to the parabolic subjet $\overline{P}^-(t_0,x_0)$ to u at $(t_0,x_0)\in\mathcal{O}$ if there exists $(t_n, x_n, p_{t,n}, p_{x,n}, S_n) \to (t_0, x_0, p_t, p_x, S)$ and

$$u(t_n + r, x_0 + h) \le u(t_0,x_0) + p_{t,n}r + p_{x,n}\cdot h + \frac{1}{2}h^T S_n h + (|r| + |h|^2)\varepsilon_n(r,h).$$

We define $\overline{P}^- u(t_0,x_0) = -\overline{P}^+(-u)(t_0,x_0)$. As above, if H is continuous, one can show that u is a viscosity subsolution to

$$-\partial_t u + H(t,x,Du,D^2u) = 0 \qquad \text{in } \mathcal{O},$$

if and only if

$$\forall (t,x)\in\mathcal{O},\ \forall (p_t,p_x,X)\in\overline{P}^+u(t,x), \qquad -p_t + H(t,x,p_x,X)\le 0.$$

A version of Ishii Lemma in this case is the following:

Theorem 5.11 (Ishii Lemma)**.** *Let $\mathcal{O}$ be an open subset of $\mathbb{R}^d\times\mathbb{R}$. Assume that $u, v : \mathcal{O}\to\mathbb{R}$ are two continuous maps and that $\phi = \phi(s,t,x,y)$ is $C^{1,2}$ and such that $(s,t,x,y)\to u(s,x) - v(t,y) - \phi(s,t,x,y)$ has a local maximum at a point $(s_0,t_0,x_0,y_0)\in\mathcal{O}\times\mathcal{O}$. Then, for any $\eta > 0$ there exists $X, Y\in\mathcal{S}_d$ such that*

$$(\partial_s\phi(s_{0,0}, x_0, y_0), D_x\phi(s_{0,0}, x_0, y_0), X) \in \overline{P}^+u(x_0),$$

$$(-\partial_t\phi(s_{0,0}, x_0, y_0), D_y\phi(s_{0,0}, x_0, y_0), Y) \in \overline{P}^-u(y_0)$$

and

$$\begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \le D^2_{x,y}\phi(x_0,y_0) + \eta(D^2_{x,y}\phi(x_0,y_0))^2.$$

As a consequence, the comparison principle holds for parabolic equations of the form

$$\begin{cases} -\partial_t u + H(x,Du,D^2u) = 0 & \text{in } (0,T)\times\mathbb{R}^d, \\ u(T,x) = g(x) & \text{in } \mathbb{R}^d \end{cases} \tag{5.10}$$

Theorem 5.12. *If H is given by (5.9) and under our standing assumption, if $u : [0,T]\times\mathbb{R}^d\to\mathbb{R}$ is a continuous and bounded viscosity subsolution to*

$$-\partial_t u + H(x,Du,D^2u) = 0 \qquad \text{in } (0,T)\times\mathbb{R}^d$$

and $v : [0,T]\times\mathbb{R}^d\to\mathbb{R}$ is a continuous and bounded viscosity supersolution to that equation, then

$$u\le v \text{ in } \{T\}\times\mathbb{R}^d \qquad\Longrightarrow\qquad u\le v \text{ in } [0,T]\times\mathbb{R}^d.$$

As a consequence, the value function of the finite horizon problem is the unique viscosity solution to (5.10).

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