Notes on the course on MFG - Course 1

Comments and Exercises on Section 4.1 of the course on MFG.

Warning: this part is very formal! rigorous analysis shall be led in the next parts. In Section 4.1, we consider a game with a very large number of players. This game is played on the time interval $[0,T]$ and the state of each player is in $\mathbb{R}^d$. We denote by $m(t)$ the empirical distribution of the players at time t. It is a probability measure on $\mathbb{R}^d$. As the state of each players is random, it should be a random measure. But we know from Section 3.3 that, if the noise of the players are independent, then $m(t)$ converges to a deterministic measure as the number of players tends to infinity. So we can assume in a first approximation that the number of players is so large that $m(t)$ is deterministic and continuous in time for the weak-* convergence.

We describe the game through the dynamics of a small player and its cost function. For this player, the time-dependent empirical measure is given.

The dynamics:

$$X_s = x + \int_t^s b(r, X_r, \alpha_r, m(r))dr + \int_t^s \sigma(r, X_r, \alpha_r, m(r))dB_r, \tag{1}$$

where X lives in $\mathbb{R}^d$, α is the control (taking its values in a fixed set A) and B is a given $M-$dimensional Brownian motion. Here the coefficients $b : [0,T]\times\mathbb{R}^d\times A\times\mathcal{P}(\mathbb{R}^d) \to \mathbb{R}^d$ and $\sigma : [0,T]\times\mathbb{R}^d\times A\times\mathcal{P}(\mathbb{R}^d) \to \mathbb{R}^{d\times M}$ are assumed to be smooth enough for the solution (X_t) to exist.

The cost function:

$$J(t, x, \alpha) = \mathbb{E}\left[\int_t^T L(s, X_s, \alpha_s, m(s))ds + g(X_T, m(T))\right]. \tag{2}$$

Here $T > 0$ is the finite horizon of the problem, $L : [0, T \times \mathbb{R}^d \times A \times \mathcal{P}(\mathbb{R}^d) \to \mathbb{R}$ and $g : \mathbb{R}^d \times \mathcal{P}(\mathbb{R}^d) \to \mathbb{R}$ are given continuous maps. The small player is minimizing.

1. What is the value function $u(t, x)$ of the problem? We assume in the rest of this part that u is smooth.
2. What is the dynamic programming principle and the Hamilton-Jacobi equation satisfied by u ?
3. What is the optimal feedback $\alpha^* = \alpha^*(t, x)$ for the small player in function of u and the data ?
4. We now assume (this is another modeling assumption) that the initial positions of the small players are independent and distributed according to a measure $m_0 \in \mathcal{P}(\mathbb{R}^d)$. Let X_0^* be the initial condition of a given small player (the law of X_0^* is m_0 and X_0^* is independent of B). By the argument above, we understand that each small player, if she plays in an optimal way, has for dynamics
$$X_s^* = X_0^* + \int_t^s b(r, X_r^*, \alpha^*(r, X_r^*), m(r))dr + \int_t^s \sigma(r, X_r^*, \alpha^*(r, X_r^*), m(r))dB_r,$$
5. For simplicity, we assume that $\sigma = I_d$ (B being a $d-$dimensional Brownian motion). Check that the law $\tilde{m}(t)$ of X^* satisfies in the sense of distribution the equation
$$\begin{cases} \partial_t \tilde{m} - \dfrac{1}{2}\Delta\tilde{m}(t, x) + \operatorname{div}(\tilde{m}(t, x)b(s, x, \alpha^*(s, x), m(s))) = 0 \quad \text{in } (0, T) \times \mathbb{R}^d, \\ \tilde{m}(0) = \bar{m}_0 \qquad \text{in } \mathbb{R}^d, \end{cases}$$
(recall that $\Delta\phi = \sum_{i=1}^d \frac{\partial^2\phi}{\partial x_i^2}$ and that, if $F = (F_i) : \mathbb{R}^d \to \mathbb{R}^d$ is a vector field, then $\operatorname{div}(F) = \sum_{i=1}^d \frac{\partial F_i}{\partial x_i}$). In a first step, it is convenient to do the proof for $d = 1$.
As explained in Section 3.3, if all the small players play the feedback α^*, have independent Brownian motions and independent initial conditions of law m_0, then their empirical distribution (for a large number of players) is the same as the law $\tilde{m}$ of the distribution of a given player. We say that we are in an equilibrium configuration if $\tilde{m}(t) = m(t)$ for all $t \in [0, T]$.

The interpretation of the equilibrium configuration can be the following: this is a game with infinitely many players. In this game, the players decide collectively to play the optimal feedback α^*. If they all do this, their empirical distribution is then $\tilde{m} = m$. On the other hand, if one (and only one) of the small player deviates, the empirical measure does not change become she is very small. Her optimal control problem is therefore the one described above, with dynamics (1) and cost (2). So the optimal feedback for her is precisely α^*. This means that α^* is a Nash equilibrium for this game with infinitely many players: no player has interest to deviate unilaterally.

6. Still in the case $\sigma = I_d$, show that the MFG problem reduces to the system

$$\begin{cases} -\partial_t u(t,x) - \frac{1}{2}\Delta u(t,x) + H(t,x,Du(t,x),m(t)) = 0 & \text{in } (0,T)\times\mathbb{R}^d, \\ \partial_t m - \frac{1}{2}\Delta m(t,x) + \mathrm{div}(m(t,x)b(s,x,\alpha^*(s,x),m(s))) = 0 & \text{in } (0,T)\times\mathbb{R}^d, \\ m(0) = \bar{m}_0,\ u(T,x) = g(x,m(T)) \quad \text{in } \mathbb{R}^d, \end{cases}$$

where α^* satisfies, for any s, x,

$$-b(s,x,\alpha^*(s,x),m(s)))\cdot Du(s,x) - L(s,x,\alpha^*(s,x),m(s)) = \sup_{\alpha\in\mathbb{R}^d} -b(s,x,\alpha,m(s)))\cdot Du(s,x) - L(s,x,\alpha,m(s))$$

is the optimal feedback. The above system is called the MFG system.

Exercice 1 (A simple explicit example)**.** We consider a simple problem in dimension $d = 1$, in which the dynamic is of the form

$$X_s = x + \int_t^s \alpha_r dr + B_t - B_s$$

and the cost is

$$J(t,x,\alpha) = \mathbb{E}\left[\int_t^T \frac{1}{2}\left[(\alpha_s)^2 + (X_s + e(s))^2\right] ds + \frac{1}{2}(X_T + e(T))^2\right].$$

where $e(s) = \int_{\mathbb{R}} x m(s,dx)$. We assume that the initial distribution m_0 for the players has a smooth density with a compact support.

1. Give a heuristic interpretation of the problem.

2. Write the MFG system in this setting.

3. One looks for a solution u in the form $u(t,x) = a(t)x^2 + b(t)x + c(t)$. Assume that u is such a solution. Show that a, b and c then satisfy, for any $t \in [0,T]$,

$$a(t) = \frac{1}{2},\ -b'(t) + 2b(t) = e(t),\ -c'(t) - 1 - \frac{1}{2}(b(t))^2 = \frac{1}{2}(e(t))^2,\ b(T) = e(T),\ c(T) = \frac{1}{2}(e(T))^2.$$

4. In view of the equation for m and assuming that $\lim_{|x|\to\pm\infty} m(t,x) = \lim_{|x|\to\pm\infty} x^2|\partial_x m(t,x)| = 0$, show that, formally, $e(t)$ satisfies for any $t \in [0,T]$,

$$e'(t) = -e(t) - b(t), \qquad e(0) = \int_{\mathbb{R}} x m_0(dx).$$

5. Show that, if $T > 0$ is small enough,then the MFG system has at least a solution.

Exercice 2 (Infinite horizon problem)**.** We consider here an infinite horizon problem. For simplicity of notation, we assume that the dynamics of a small player is given by

$$X_s = x + \int_t^s b(r, X_r, \alpha_r, m(r))dr + B_s - B_t, \qquad s \geq t,$$

while the cost function is

$$J(t,x,\alpha) = \mathbb{E}\left[\int_t^{+\infty} e^{-r(s-t)} L(s, X_s, \alpha_s, m(s))ds\right]$$

where $r > 0$ is the instantaneous discount rate (representing the preference of a player for the present). As usual, $m(t)$ is the distribution of the players at time t (probability on $\mathbb{R}^d$). The initial distribution is denoted by m_0. The map L is supposed to be bounded (so that the integral converges). Arguing formally as above, write the MFG system associated with this problem.

Hint for the exercises

Solution of Exercise 1.

1. Players want to be as close as possible from the opposite of the average of the crowd (the terms $(X_t + e(t))^2$) while not paying a loo large control (the term α_t^2).

2. The MFG system in this setting takes the form:

$$\begin{cases} -\partial_t u(t,x) - \frac{1}{2}\partial_{xx}u(t,x) + \frac{1}{2}|\partial_x u(t,x)|^2 - \frac{1}{2}(x+e(t))^2 = 0 \\ \partial_t m(t,x) - \frac{1}{2}\partial_{xx}m(t,x) - \partial_x(m(t,x)\partial_x u(t,x)) = 0 \\ e(t) = \int_{\mathbb{R}} x m(t,dx) \\ m(0) = m_0, \ u(T,x) = \frac{1}{2}(x+e(T))^2 \end{cases}$$

3. It is a straightforward verification.

4. The equation for m should be understood in the sense of distribution. Let us compute however formally

$$e'(t) = \int_{\mathbb{R}} x\partial_t m(t,dx) = \int_{\mathbb{R}} x(\frac{1}{2}\partial_{xx}m(t,x) + \partial_x(m(t,x)\partial_x u(t,x)))dx.$$

We integrate by parts to obtain, since $\lim_{|x|\to\pm\infty} m(t,x) = \lim_{|x|\to\pm\infty} x^2|\partial_x m(t,x)| = 0$:

$$\begin{aligned} e'(t) &= -\int_{\mathbb{R}} (\frac{1}{2}\partial_x m(t,x) + (m(t,x)\partial_x u(t,x)))dx \\ &= 0 - \int_{\mathbb{R}} (2a(t)x + b(t))m(t,x)dx = -e(t) - b(t), \end{aligned}$$

since $a(t) = 1/2$ and $\int_{\mathbb{R}} m(t,dx) = 1$.

5. The system satisfied by (b,e) is

$$\begin{pmatrix} b'(t) \\ e'(t) \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ -1 & -1 \end{pmatrix} = \begin{pmatrix} b(t) \\ e(t) \end{pmatrix}, \qquad e(0) = e_0 := \int_{\mathbb{R}} x m_0(dx), \ b(T) = e(T).$$

The solutions of the system are of the form

$$\begin{pmatrix} b(t) \\ e(t) \end{pmatrix} = Av_1 \exp\{\lambda_1 t\} + Bv_2 \exp\{\lambda_2 t\}$$

with $A, B \in \mathbb{R}$ are to be found, $\lambda_1 = (1+\sqrt{13})/2$, $\lambda_2 = (1-\sqrt{13})/2$ and v_1 and v_2 are fixed eigenvectors associated with these eigenvalues λ_1 and λ_2 of the matrix. Note that $\{v_1, v_2\}$ is a basis of $\mathbb{R}^2$. The initial and terminal conditions read

$$Av_{1,2} + Bv_{2,2} = e_0, \ Av_{1,1}\exp\{\lambda_1 T\} + Bv_{2,1}\exp\{\lambda_2 T\} = Av_{1,2}\exp\{\lambda_1 T\} + Bv_{2,2}\exp\{\lambda_2 T\},$$

which determines completely A and B provided

$$v_{1,2}(v_{2,1} - v_{2,2})\exp\{\lambda_2 T\} - v_{2,2}(v_{1,1} - v_{1,2})\exp\{\lambda_1 T\} \neq 0,$$

which is the case for $T = 0$ (since $\{v_1, v_2\}$ is a basis) and therefore for $T > 0$ small enough. We can then find $c(t)$ easily. This gives a solution to the system because a solution the equation for m can be obtained as the law of the solution to $dX_t = -(X_t - b(t))dt + dB_t$ with an initial condition X_0 with law m_0.

Hint of the solution of Exercise 2. One finds that the pair (u,m) solves the system:

$$\begin{cases} (i) \quad -\partial_t u(t,x) + ru(t,x) - \frac{1}{2}\Delta u(t,x) + H(x, Du(t,x), m(t)) = 0 & \text{in } (0,+\infty)\times\mathbb{R}^d \\ (ii) \quad \partial_t m(t,x) - \frac{1}{2}\Delta m(t,x) - \operatorname{div}(m \ D_pH(x, Du(t,x), m(t))) = 0 & \text{in } (0,+\infty)\times\mathbb{R}^d \\ (iii) \quad m(0) = m_0 \text{ in } \mathbb{R}^d, \qquad u \text{ bounded} \end{cases}$$