**Notes on the course on MFG - Course 2**

Comments on Section 4.2 of the course on MFG.

The aim of this presentation is to simplify the lecture of Section 4.2 of the course devoted to the existence of a solution for the (relatively simple) MFG system

$$(1)\qquad \left\{\begin{array}{ll} (i) & -\partial_t u-\Delta u+\frac{1}{2}|Du|^2=F(x,m(t)) \qquad \text{in } \mathbb{R}^d\times(0,T)\\ (ii) & \partial_t m-\Delta m-\operatorname{div}(m\ Du)=0 \qquad \text{in } \mathbb{R}^d\times(0,T)\\ (iii) & m(0)=m_0\ ,\ u(x,T)=G(x,m(T)) \qquad \text{in } \mathbb{R}^d \end{array}\right.$$

The assumptions for this are explained at the beginning of Section 4.2 and are assumed to hold.

1. Explain why, formally, the MFG system (1) corresponds to a control problem with infinitely many agents where the dynamics for each agent is

$$X_s=x+\int_t^s \alpha_s ds+B_s-B_t$$

and where the cost (given the flow of probability measures $(m(t))$) is

$$J(t,x,\alpha)=\mathbb{E}\left[\int_t^T (\frac{1}{2}|\alpha_s|^2+F(X_s,m(s)))ds+G(X_T,m(T))\right].$$

The proof of the existence of a solution to (1) is based on a fixed point argument. Given a flow of probability measures $t\to\mu(t)$, one first solves the Hamilton-Jacobi equation (with unknown $u=u(t,x)$, depending on $\mu$)

$$(2)\qquad \left\{\begin{array}{l} -\partial_t u-\Delta u+\frac{1}{2}|Du|^2=F(x,\mu(t)) \qquad \text{in } \mathbb{R}^d\times(0,T)\\ u(x,T)=G(x,\mu(T)) \qquad \text{in } \mathbb{R}^d \end{array}\right.$$

and then the Fokker-Planck (or Kolmogorov) equation (with unknown $m=m(t,x)$, still depending on $\mu$, through $u$)

$$(3)\qquad \left\{\begin{array}{l} \partial_t m-\Delta m-\operatorname{div}(m\ Du)=0 \qquad \text{in } \mathbb{R}^d\times(0,T)\\ m(0)=m_0 \qquad \text{in } \mathbb{R}^d \end{array}\right.$$

This defines a map $\Psi$ which, to "any" $\mu$ associates some $m$. The problem is to understand in what extend the map $\Psi$ has a fixed point.

2. Note that a fixed point of $\Psi$ is (formally) a solution of the MFG system (1).

   There are a huge number of theorems ensuring the existence of a fixed point for a map $\Psi$. We will be mostly interested in two of them:

   (a) the more standard one is *Banach fixed point Theorem*, which asserts the existence of a unique fixed point of a map $\Psi$ provided that $\Psi$ is defined on a complete metric space into itself and is a contraction. When applied to MFGs, this fixed point theorem requires some "smallness conditions" on the data (for instance $T$), which are seldom met in practice (see, for instance, Exercise 1 in the previous notes on the course).

   (b) *Schauder fixed point Theorem* (a generalization to infinite dimensional spaces of Brouwer fixed point Theorem). It says that, if $K$ is a nonempty convex closed subset of a Hausdorff topological vector space $X$ and $\Psi$ is a continuous mapping of $K$ into itself such that $\Psi(K)$ is contained in a compact subset of $K$, then $\Psi$ has a fixed point.

   (c) Let us also mention *Schaefer fixed point Theorem*, which says that, if $\Psi$ is a continuous and compact mapping of a Banach space $X$ into itself, such that the set $\{x\in X,\ x=\lambda\Psi(x),\ \lambda\in[0,1]\}$ is bounded, then $\Psi$ has a fixed point. We won't use it here but it is a possible alternative to handle the problem.

Let us first build a compact convex space as required by Schauder fixed point Theorem. Recall that $\mathcal{P}_1$ is a set of Borel probability measures on $\mathbb{R}^d$ with finite first order moment endowed with the Monge-Kantorovitch distance $\mathrm{d}_1$ (see Chap. 3.2). We fix a constant $C_1 > 0$ to be chosen below and let $\mathcal{C}$ be the set of maps[1] $\mu \in \mathcal{C}^0([0,T], \mathcal{P}_1)$ such that

$$\sup_{s \neq t} \frac{\mathbf{d}_1(\mu(s), \mu(t))}{|t-s|^{\frac{1}{2}}} \leq C_1 \qquad \text{and} \qquad \sup_{t \in [0,T]} \int_{\mathbb{R}^d} |x|^2 m(t, dx) \leq C_1 \ .$$

3. Prove that $\mathcal{C}$ is a convex compact subset of $\mathcal{C}^0([0,T], \mathcal{P}_1)$ (which is itself embedded in the topological space of signed measures on $[0,T] \times \mathbb{R}^d$).

   The main step is to show that the map $\Psi$ is well-defined from $\mathcal{C}$ into itself and continuous. For this let $\mu \in \mathcal{C}$ and $u$ be the classical solution to (2) (by classical solution, we mean that $D^2 u$ and $\partial_t u$ exist and are continuous). We will assume for a while that this solution exists, is unique, with $D^2 u$ and $\partial_t u$ bounded, and that there exists a constant $C_2 > 0$, independent of $\mu$ and of $C_1$, such that

   $$\|Du\|_\infty \leq C_2. \tag{4}$$

   We will explain this later (in step (10)).

   Let $X_0$ be a random variable of law $m_0$ and $X$ be the solution to

   $$X_t = X_0 - \int_0^t Du(s, X_s) ds + \sqrt{2} B_t$$

   where $B$ is a $d-$dimensional BM. We denote by $m(t)$ the law of $X_t$.

4. Show that there exists a constant $C_3$, depending on $C_2$ and on $d$ only (in particular independent of $\mu$ and $C_1$) such that

   $$\sup_{t \in [0,T]} \int_{\mathbb{R}^d} |x|^2 m(t, dx) = \sup_{t \in [0,T]} \mathbb{E}\left[|X_t|^2\right] \leq C_3,$$

   and

   $$\sup_{s \neq t} \frac{\mathbf{d}_1(m(t), m(s))}{|s-t|^{1/2}} \leq \sup_{s \neq t} \frac{\mathbb{E}\left[|X_t - X_s|\right]}{|s-t|^{1/2}} \leq C_3.$$

   (Hint: have a look at Section 3.1.3 of the course).

5. Show that the law of $X$ is a solution in the sense of distribution of (3).

   We will admit that the solution to (3) in the sense of distribution is unique.

6. How can one choose $C_1$ in order to ensure that $\Psi$ is defined from $\mathcal{C}$ into itself?

   We now turn to the analysis of the Hamilton-Jacobi equations of the form (2) and show the existence of a solution satisfying (4). It could be a very delicate issue for a general Hamiltonian $H$, which would largely exceed the scope of these notes. However, the specific choice of the quadratic Hamiltonian makes things much simpler.

7. Set $w = e^{u/2}$ and check that $u$ is a solution of (2) if and only if $w$ is a solution of the linear (backward) equation

   $$\left\{ \begin{array}{ll} -\partial_t w(t,x) - \Delta w(t,x) = w(t,x) F(x, \mu(t)) & \text{in } \mathbb{R}^d \times (0,T) \\ w(x,T) = e^{G(x,\mu(T))/2} & \text{in } \mathbb{R}^d \end{array} \right. \tag{5}$$

   We will admit the following result (the so-called Schauder estimates). We consider the following second order parabolic equation

   $$\left\{ \begin{array}{ll} w_t - \Delta w + \langle a(x,t), Dw \rangle + b(x,t) w = f(x,t) & \text{in } \mathbb{R}^d \times [0,T] \\ w(x,0) = w_0(x) & \text{in } \mathbb{R}^d \end{array} \right. \tag{6}$$

[1] $\mathcal{C}^0([0,T], \mathcal{P}_1)$ is the set of continuous maps from $[0,T]$ to $\mathcal{P}_1$.

We denote by $\mathcal{C}^{s+\alpha}$ (for an integer $s \geq 0$ and $\alpha \in (0,1)$) the set of maps $z : [0,T] \times \mathbb{R}^d \to \mathbb{R}$ such that the derivatives $\partial_t^k D_x^l z$ exist for any pair $(k,l)$ with $2k + l \leq s$ and such that these derivatives are bounded and $\alpha-$Hölder continuous in space and $(\alpha/2)-$Hölder continuous in time. It is known that, if $a : \mathbb{R}^d \times [0,T] \to \mathbb{R}$, $b, f : [0,T] \times \mathbb{R}^d \to \mathbb{R}$ and $w_0 : \mathbb{R}^d \to \mathbb{R}$ belong to $\mathcal{C}^\alpha$ for some $\alpha \in (0,1)$, then the above heat equation is has a unique weak solution and this solution belongs to $\mathcal{C}^{2+\alpha}$ (references in the course).

Another estimates which will be needed is the following: if $a = b = 0$ and $f$ is continuous and bounded, any classical, bounded solution $w$ of (6) satisfies, for any compact set $K \subset (0,T) \times \mathbb{R}^d$,

$$(7) \qquad \sup_{(t,x),(s,y) \in K} \frac{|Dw(t,x) - Dw(s,y)|}{|x-y|^\beta + |t-s|^{\beta/2}} \leq C(K, \|w\|_\infty) \|f\|_\infty ,$$

where $\beta \in (0,1)$ depends only on the dimension $d$ while $C(K, \|w\|_\infty)$ depends on the compact set $K$, on $\|w\|_\infty$ and on $d$ (Reference in the course).

8. Show that there exists a unique classical solution $w$ to (5). Derive from this that (2) has a unique classical solution $u$ and that $D^2 u$ and $\partial_t u$ are bounded.

   An important property of equations of the form (2) is the comparison principle.

9. (difficult - can be omitted) Let $f : [0,T] \times \mathbb{R}^d$ and $g : \mathbb{R}^d \to \mathbb{R}$ belong to $\mathcal{C}^\alpha$. Let $u$ and $v$ be in $\mathcal{C}^{2+\alpha}$ verify

$$\begin{cases} -\partial_t u(t,x) - \Delta u(t,x) + \frac{1}{2}|Du(t,x)|^2 \leq f(t,x) & \text{in } \mathbb{R}^d \times (0,T) \\ u(x,T) \leq g(x) & \text{in } \mathbb{R}^d \end{cases}$$

$$\begin{cases} -\partial_t v(t,x) - \Delta v(t,x) + \frac{1}{2}|Dv(t,x)|^2 \geq f(t,x) & \text{in } \mathbb{R}^d \times (0,T) \\ v(x,T) \geq g(x) & \text{in } \mathbb{R}^d \end{cases}$$

   (one says that $u$ is a subsolution while $v$ is a super solution of the equation). Show that $u \leq v$. This result is called a comparison principle.

10. Infer from the comparison principle that the solution $u$ to (2) is uniformly bounded independently of $\mu$ and $C_1$ and uniformly Lipschitz continuous in space independently of $\mu$ and $C_1$. This shows the existence of $C_2$ in (4).
    (Hint: for the bound, find constant $C$ (independent of $\mu$ and $C_1$) such that $v^+(t) = C(T-t+1)$ is a supersolution and $v^-(t) = -C(T-t+1)$ is subsolution of (2). For the Lipschitz regularity, fix $h \in \mathbb{R}^d$ and find a constant $C > 0$ such that $v^+(t,x) = u(t,x+h) + C|h|(T-t+1)$ is a supersolution while $v^-(t,x) = u(t,x+h) - C|h|(T-t+1)$ is a subsolution of (2).)

So far we have proved that $\Psi : \mathcal{C} \to \mathcal{C}$ is well-defined. It remains to check that it is continuous. Assume that $(\mu_n)$ converges to to some $\mu$ in $\mathcal{C}$. Let $(u_n, m_n)$ and $(u, m)$ be the corresponding solutions of (2) and (3) associated with $\mu_n$ and $\mu$ respectively.

11. Use the comparison principle to prove that $(u_n)$ converges to $u$ uniformly on $[0,T] \times \mathbb{R}^d$.

12. Use the estimate in (7) as well as (4) to prove that $Du_n$ is locally equicontinuous and bounded.

13. Infer from this that any limit of a subsequence of the compact sequence $(m_n)$ in $\mathcal{C}$ is a solution of (3) in the sense of distribution.

14. Conclude that $\Psi$ has a fixed point.

15. Use Schauder estimates to show that, if $m$ is the solution to (3), then $m$ has a density which is in $\mathcal{C}^{2+\alpha}$.