Notes on the course on MFG - Course 3

Comments on Sections 4.2.3 and 4.2.5 of the notes on MFG.

The aim of this presentation is to simplify the lecture of Section 4.2.3 of the notes devoted to the uniqueness of a solution for the (relatively simple) MFG system

$$(1)\qquad \begin{cases} (i) & -\partial_t u - \Delta u + \frac{1}{2}|Du|^2 = F(x, m(t)) \qquad \text{in } \mathbb{R}^d \times (0,T) \\ (ii) & \partial_t m - \Delta m - \operatorname{div}(m\ Du) = 0 \qquad \text{in } \mathbb{R}^d \times (0,T) \\ (iii) & m(0) = m_0\ ,\ u(x,T) = G(x, m(T)) \qquad \text{in } \mathbb{R}^d \end{cases}$$

and to the interpretation for games with finitely many players (Section 4.2.5 of the notes). The assumptions on F and G are discussed at the beginning of Section 4.2 and are assumed to hold here.

1. Here we explain why there exists at most a unique classical solution to (1) under the monotonicity condition

$$(2)\qquad \int_{\mathbb{R}^d} (F(x,m_1) - F(x,m_2))d(m_1 - m_2)(x) \geq 0 \qquad \forall m_1, m_2 \in \mathcal{P}_1,$$

and

$$(3)\qquad \int_{\mathbb{R}^d} (G(x,m_1) - G(x,m_2))d(m_1 - m_2)(x) \geq 0 \qquad \forall m_1, m_2 \in \mathcal{P}_1\ .$$

Let (u_1, m_1) and (u_2, m_2) be two classical solutions of (1).

(a) Show that

$$\begin{aligned} &\frac{d}{dt}\int_{\mathbb{R}^d} (u_1(t,x) - u_2(t,x))(m_1(t,x) - m_2(t,x))dx \\ &\quad = -\int_{\mathbb{R}^d} (F(x, m_1(t)) - F(x, m_2(t))(m_1(t,x) - m_2(t,x))dx \\ &\qquad - \int_{\mathbb{R}^d} \frac{1}{2}((m_1(t,x) + m_2(t,x))|Du_1(t,x) - Du_2(t,x)|^2 dx \end{aligned}$$

(b) Use the monotonicity of F and G to infer that $Du_1 = Du_2$ in $\{m_1 > 0\} \cup \{m_2 > 0\}$.

(c) Prove then that m_1 and m_2 solves the same Kolmogorov equation and therefore have to be equal.
(Hint : use the representation of m_1 and m_2 in terms of SDE in the Course 2)

(d) Using the comparison principle (see Course 2), conclude that $u_1 = u_2$.

Exercice 1. Using the same technique, show that the slightly more general MFG system

$$\begin{cases} -\partial_t u(t,x) - \Delta u(t,x) + H(t,x,Du(t,x)) = F(x, m(t)) \qquad \text{in } (0,T) \times \mathbb{R}^d, \\ \partial_t m - \Delta m(t,x) - \operatorname{div}(m(t,x) D_p H(t,x,Du(t,x)) = 0 \qquad \text{in } (0,T) \times \mathbb{R}^d, \\ m(0) = \bar{m}_0,\ u(T,x) = g(x, m(T)) \qquad \text{in } \mathbb{R}^d, \end{cases}$$

has at most a classical solution if $H = H(x,p)$ is convex in the p variable and (2) and (3) hold.

2. Before addressing the mean field limit, let us understand better the MFG system (1). Fix (u, m) a solution of the MFG system (1). We consider a single player, with a random initial position X_0 with law m_0. He faces the following minimization problem

$$(4)\qquad \inf_{\alpha} \mathcal{J}(\alpha) \qquad \text{where} \qquad \mathcal{J}(\alpha) = \mathbb{E}\left[\int_0^T \frac{1}{2}|\alpha_s|^2 + F\left(X_s, m(s)\right)\ ds + G\left(X_T, m(T)\right)\right].$$

In the above formula, $X_t = X_0 + \int_0^t \alpha_s ds + \sqrt{2}B_s$, X_0 is a fixed random intial condition with law m_0 and the control α is adapted to some filtration $(\mathcal{F}_t)$. We assume that (B_t) is an $d-$dimensional Brownian motion adapted to $(\mathcal{F}_t)$ and that X_0 and (B_t) are independent.

Exercice 2. (a) Show that $\bar{\alpha}(x,t) := -Du(x,t)$ is optimal for this optimal stochastic control problem: namely, that, if $(\bar{X}_t)$ be the solution of the stochastic differential equation

$$\begin{cases} d\bar{X}_t = \bar{\alpha}(\bar{X}_t, t)dt + \sqrt{2}dB_t \\ \bar{X}_0 = X_0 \end{cases}$$

and $\tilde{\alpha}(t) = \bar{\alpha}(\bar{X}_t, t)$. Then

$$\inf_{\alpha} \mathcal{J}(\alpha) = \mathcal{J}(\tilde{\alpha}) = \int_{\mathbb{R}^N} u(x,0)\ dm_0(x)\ .$$

(Hint: as in a usual a verification theorem, compute $u(t, X_t)$ along a standard solution X_t)

(b) Check that $m(t)$ is the law of $\bar{X}_t$ for any $t \in [0,T]$.
(Hint: compute the equation satisfied by the law of X_t and conclude by uniqueness of the weak solution of the Fokker-Planck equation).

3. We now study the mean field limit. We consider a differential game with N players which consists in a kind of discrete version of the mean field game. In this game player i ($i = 1, \dots, N$) is controlling through his control α^i a dynamics of the form

$$dX_t^i = \alpha_t^i dt + \sqrt{2}dB_t^i \tag{5}$$

where (B_t^i) is a $d-$dimensional brownian motion. The initial condition X_0^i for this system is also random and has for law m_0. We assume that the all X_0^i and all the brownian motions (B_t^i) ($i = 1, \dots, N$) are independent. However player i can choose his control α^i adapted to the filtration ($\mathcal{F}_t = \sigma(X_0^j,\ B_s^j,\ s \le t,\ j = 1, \dots, N\}$). His payoff is then given by

$$\begin{aligned} &\mathcal{J}_i^N(\alpha^1, \dots, \alpha^N) \\ &\quad = \mathbb{E}\left[\int_0^T \frac{1}{2}|\alpha_s^i|^2 + F\left(X_s^i, \frac{1}{N-1}\sum_{j \ne i} \delta_{X_s^j}\right)\ ds + G\left(X_T^i, \frac{1}{N-1}\sum_{j \ne i} \delta_{X_T^j}\right)\right] \end{aligned}$$

Our aim is to explain that the strategy given by the mean field game is suitable for this problem. More precisely, let (u, m) be one classical solution to (1) and let us set $\bar{\alpha}(x,t) = -Du(x,t)$. With the closed loop strategy $\bar{\alpha}$ one can associate the open-loop control $\tilde{\alpha}^i$ obtained by solving the SDE

$$d\bar{X}_t^i = \bar{\alpha}(\bar{X}_t^i, t)dt + \sqrt{2}dB_t^i \tag{6}$$

with random initial condition X_0^i and setting $\tilde{\alpha}_t^i = \bar{\alpha}(\bar{X}_t^i, t)$. The main result of Section 4.2.5 (Theorem 4.2.9) says that, for any $\epsilon > 0$, there exists N_0 such that, if $N \ge N_0$, then the family $(\tilde{\alpha}^i)$ is an $\epsilon-$Nash equilibrium in the $N-$player game:

$$\mathcal{J}_i^N(\tilde{\alpha}^1, \dots, \tilde{\alpha}^N) \le \mathcal{J}_i^N((\tilde{\alpha}^j)_{j \ne i}, \alpha) + \epsilon$$

for any control α adapted to the filtration $(\mathcal{F}_t)$ and any $i \in \{1, \dots, N\}$. The proof is a little bit technical and we discuss here only a few points through questions (and we do it for $i = 1$)

Exercice 3. (a) Let us denote by $\bar{X}_t^j$ the solution of the stochastic differential equation (6) with initial condition X_0^j. Show that the $(\bar{X}_t^j)$ are independent and identically distributed with law $m(t)$ (see Exercise 2).

(b) Using the law of large number, show that the (random) empirical measure $m_t^N = \dfrac{1}{N-1}\displaystyle\sum_{j=2}^N \delta_{X_t^j}$ converges to $m(t)$ $\mathbb{P}-$a.s.

(c) Infer that, for any α adapted to the filtration $(\mathcal{F}_t)$,

$$\lim_{N \to +\infty} \mathcal{J}_1^N(\alpha^1, \tilde{\alpha}^2, \dots, \tilde{\alpha}^N) = \mathcal{J}(\alpha),$$

where $\mathcal{J}$ is defined in (4).

The computation above explains that, roughly speaking, one can replace de cost $\mathcal{J}_1^N$ by $\mathcal{J}$, which simplifies of course the problem.