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Introduction to Time series
TD1 - Stationary process

Exercice 1 (Stationarity and strict Stationarity) Suppose X is a random variable with a standard normal distribution $\mathcal{N}(0,1)$, and $Y = X\mathbf{1}_{U=1} - X\mathbf{1}_{U=0}$ where U is a Bernoulli random variable with parameter $1/2$, and independent of X.

1. Show that X and Y have the same distribution;
2. Show that $\mathrm{Cov}(X,Y) = 0$, but X and Y are not independent;
3. Construct a process which is a weak-sense white noise but not a strong-sense white noise.

Exercice 2 (Random walk) Consider a random walk $X = (X_t)_{t\in\mathbb{N}}$ with drift μ : $X_t = \mu + X_{t-1} + Z_t$ for $t \geq 1$ where $X_0 = 0$ and $(Z_t)_{t\in\mathbb{N}}$ is a strong-sense white noise.

1. Calculate the autocovariance function γ_X of X. Is X stationary ?
2. Is the process $(\Delta X_t)_{t\in\mathbb{N}}$ stationary ?

Exercice 3 (Sum of stationary processes) Suppose that $X = (X_t)_{t\in\mathbb{Z}}$ and $Y = (Y_t)_{t\in\mathbb{Z}}$ are two stationary processes and uncorrelated (i.e. $\mathrm{cov}(X_t, Y_s) = 0$ for all s, t). Show that $Z = (Z_t)_{t\in\mathbb{Z}}$ defined by $Z_t = X_t + Y_t$ for all $t \in \mathbb{Z}$ is also stationary. Then find the expression of its autocovariance as a function of those of X and Y.

Exercice 4 (Stationarity of process) Find the stationary processes among the following processes:

1. $X_t = Z_t$ if t is even, $X_t = Z_t + 1$ if t is odd, where $(Z_t)_{t\in\mathbb{Z}}$ is stationary;
2. $X_t = Z_1 + \cdots + Z_t$ where $(Z_t)_{t\in\mathbb{Z}}$ is a strong-sense white noise;
3. $X_t = Z_t + \theta Z_{t-1}$ where $(Z_t)_{t\in\mathbb{Z}}$ is a white noise and $\theta \in \mathbb{R}$ is a constant;
4. $X_t = Z_t Z_{t-1}$ where $(Z_t)_{t\in\mathbb{Z}}$ is a strong-sense white noise;
5. $Y_t = (-1)^t Z_t$ and $X_t = Y_t + Z_t$ where $(Z_t)_{t\in\mathbb{Z}}$ is a strong-sense white noise.

Exercice 5 (Harmonic processes) Consider a process $X = (X_t)_{t\in\mathbb{Z}}$ defined by $X_t = A\cos(\theta t) + B\sin(\theta t)$ for all $t \in \mathbb{Z}$, where A and B are two independent random variables, with zero mean and variance σ^2, and $\theta \in \mathbb{R}$ is a constant.
Is the process X stationary ? Calculate its autocovariance function.

Exercice 6 (Law of large numbers) Consider a stationary process (X_t) with mean μ and the autocovariance γ. Suppose that $\lim_{h\to\infty} \gamma(h) = 0$ (the process is said to be uncorrelated at infinity).
Show that the limit in L^2 of $\frac{1}{T}\sum_{t=1}^{T} X_t$ is equal to μ.

Exercice 7 (Finite difference, trend and moving average) Denote Δ the finite difference operator, which maps a process (X_t) to (ΔX_t) defined by $\Delta X_t = X_t - X_{t-1}$ for all $t \in \mathbb{Z}$. And denote M the moving average operator, which maps a process (X_t) to another process (MX_t) defined by $MX_t = \frac{1}{3}(X_{t-1} + X_t + X_{t+1})$ for all $t \in \mathbb{Z}$.

1. If (X_t) is stationary, show that (ΔX_t) and (MX_t) are also stationary and give the expression of their autocovariance as a function of that of (X_t).
2. (X_t) is said to be a process with polynomial trend of degree d ($d \in \mathbb{N}$), if there exists a polynomial P of degree d and a stationary process (U_t) such that $X_t = P(t) + U_t$ for all $t \in \mathbb{Z}$. Show that if (X_t) is a process with polynomial trend of degree d, then (ΔX_t) is a process with polynomial trend of degree $d-1$.
3. Suppose that (X_t) is written as $X_t = P(t) + S(t) + U_t$, where P is a polynomial of degree $d \in \mathbb{N}$, $S : \mathbb{Z} \to \mathbb{R}$ a $R-$periodic function (with $R \in \mathbb{N}^*$) and (U_t) is a stationary process. Give a simple operation transforming (X_t) into a stationary process.
4. What is the effect of M on a process with affine trend (=polynomial trend of degree 1)? And what about the effect of M on a process $X_t = S(t) + U_t$ where $S : \mathbb{Z} \to \mathbb{R}$ is a periodic function with period 3 and (U_t) is stationary ?
5. Give an analogous construction of M which eliminates periodic sequences of degree 3 while keeping polynomials of degree 2 invariant.

Exercice 8 (Property of autocovariance function)

1. Show that the function γ, defined by $\gamma(0) = 1$ and $\gamma(h) = \rho$ for $|h| = 1$ and $\gamma(h) = 0$ otherwise, is an autocovariance function if and only if $|\rho| \leq 1/2$. Give an example of a stationary Gaussian process possessing such an autocovariance function.
2. Same question for γ defined by $\gamma(0) = 1$ and $\gamma(h) = \rho$ if $h \neq 0$.
3. Are the following functions autocovariance functions of a stationary process?
   (a) $\gamma(h) = 1$ if $h = 0$ and $\gamma(h) = 1/h$ if $h \neq 0$;
   (b) $\gamma(h) = 1 + \cos(h\pi/2)$;
   (c) $\gamma(h) = (-1)^{|h|}$.

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TD2 - Filtering

Exercice 9 Show that the autocorrelation function of a linear process is summable, in particular a linear process is decorrelated at infinity.

Exercice 10 (Weak filtering) Suppose (Z_t) is a standardized white noise (i.e. $\mathbb{E}[Z_t] = 0, \mathbb{V}ar[Z_t] = 1$) and $(a_n) \in \ell^2(\mathbb{Z})$. We want to define the process X by

$$X_t = \sum_{k \in \mathbb{Z}} a_k Z_{t-k} \qquad \forall t \in \mathbb{Z}.$$

1. Explain why the filtering theorem does not work here.
2. Nevertheless show that, for all $t \in \mathbb{Z}$, the series $\sum_{k \in \mathbb{Z}} a_k Z_{t-k}$ converges in L^2. We call X_t its limit.
3. Show further that the process (X_t) is stationary.

Exercice 11 (Auto-regressive equation) Let $\phi \in \mathbb{R}^*$ and $Z = (Z_t)$ be a standardized white noise. We are interested in the stochastic processes $X = (X_t)$, solutions of the following auto-regressive equation :

$$X_t = \phi X_{t-1} + Z_t, \qquad t \in \mathbb{Z}.$$

1. Show that, when $|\phi| < 1$, the equation admits a unique stationary solution. Is it causal ? (i.e., Does it verify $X_t \in \overline{\text{Vect}(Z_t, Z_{t-1}, Z_{t-2}, \dots)}$ for all $t \in \mathbb{Z}$? The closure taking place in L^2).
2. Same question when $|\phi| > 1$.
3. Conversely, if $\phi = \pm 1$, show that the equation does not admit a stationary solution.
4. More generally, if $a_1, \dots, a_n$ is a sequence of real numbers satisfying

   $$\sum_{i=1}^{n} a_i = 1 \qquad \text{or} \qquad \sum_{i=1}^{n} (-1)^i a_i = 1,$$

   show that the auto-regressive equation

   $$X_t = \sum_{i=1}^{n} a_i X_{t-i} + Z_t, \qquad t \in \mathbb{Z}$$

   does not admit a stationary solution.

Exercice 12 (Invertibility) In each following case, compute the inverse of the filter $\alpha \in \ell^1(\mathbb{Z})$ if it exists.

1. $\alpha_0 = 2$, $\alpha_1 = -1$, and $\alpha_k = 0$ if $k \notin \{0, 1\}$.

2. $\alpha_0 = 1$, $\alpha_1 = 2$, and $\alpha_k = 0$ if $k \notin \{0, 1\}$.

3. $\alpha_0 = 1$, $\alpha_1 = -1$, and $\alpha_k = 0$ if $k \notin \{0, 1\}$.

Exercice 13 (Abstract version of the filtering theorem) We denote by $\ell^1(\mathbb{Z})$ the space of real and summable sequences, with the norm $\|\alpha\|_1 := \sum_{k\in\mathbb{Z}} |a_n|$. Let E be the space of processes (X_t) bounded in L^2 equipped with the norm $\|X\|_E = sup_{t\in\mathbb{Z}}\|X_t\|_2$. We denote by $\mathcal{L}(E)$ the space of continuous linear maps from E to E. Let us admit that E and $\mathcal{L}(E)$ equipped with their respective norms are both Banach spaces.

Define the lag operator (or backshift operator) $B \in \mathcal{L}(E)$ by $BX = (X_{t-1})_{t\in\mathbb{Z}}$.

1. Show that B is an isometric operator on E, i.e. B is invertible linear operator and satisfies $\|BX\|_E = \|X\|_E$ for all X of E.

2. Deduce that if $\alpha \in \ell^1(\mathbb{Z})$, then the series $\sum_{n\in\mathbb{Z}} \alpha_n B^n$ converges in $\mathcal{L}(E)$. We note $\phi(\alpha)$ this sum.

3. Show that $\phi(\alpha \star \beta) = \phi(\alpha) \circ \phi(\beta)$ for all $\alpha, \beta \in \ell^1(\mathbb{Z})$ (ϕ is said to be an algebraic morphism). If α is invertible, deduce that $\phi(\alpha)$ is also.

4. Show that ϕ is injective.
   Hint : We can start by showing that given a white noise $(Z_t)_{t\in\mathbb{Z}}$, the map
   $$\alpha \longrightarrow \sum_{n\in\mathbb{Z}} \alpha_n B^n Z \in E$$
   is injective.

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TD3 - ARMA Process

Exercice 14 1. Solve the following ARMA equation:

$$X_t = 3X_{t-1} + Z_t - \frac{10}{3} Z_{t-1} + Z_{t-2} \qquad t \in \mathbb{Z},$$

where Z is white noise with zero mean and variance σ^2. Specify the mean and the auto-covariance function of the solution.

2. Same question for

$$X_t = X_{t-1} - \frac{1}{4} X_{t-2} + Z_t + Z_{t-1} \qquad t \in \mathbb{Z}.$$

Exercice 15 (ARMA$(1,1)$) Consider the equation

$$X_t - \phi X_{t-1} = Z_t + \theta Z_{t-1}$$

where (Z_t) is white noise with zero mean and variance σ^2 and $\phi, \theta \in \mathbb{R}$

1. If $\phi \neq \pm 1$, show that there exists a unique stationary solution; calculate it and find its mean and its auto-covariance function.
2. if $\phi = 1$ and X is the solution, show that, for all $t \geq 1$,

$$X_t = X_0 + \theta Z_0 + (1+\theta) \sum_{s=1}^{t-1} Z_s + Z_t.$$

   Deduce that, if $\theta \neq -1$, there is no stationary solution.
3. Show similarly that there is no stationary solution if $\phi = -1$ and $\theta \neq 1$.
4. We now assume that $\phi = 1$ and $\theta = -1$. Show that the solutions of the equation are processes of the form $X_t = Z_t + \xi$, where ξ is a random variable. Show that such a process is stationary, if and only if, ξ is square integrable and uncorrelated from Z.
5. Find in the same way the stationary solutions when $\phi = \theta = 1$.

Exercice 16 (MA(1)) Suppose (Z_t) is white noise with mean μ and variance σ^2, θ is a real number and X is a process given by $X_t = Z_t + \theta Z_{t-1}$ for $t \in \mathbb{Z}$. Show that X is a stationary process and its autocovariance function is

$$\gamma_X(h) = \begin{cases} a & \text{si } h = 0 \\ b & \text{si } h = \pm 1 \\ 0 & \text{sinon} \end{cases}$$

where a and b are real numbers that will be determined, with $|b| \leq a/2$.

Now we want to show that, if X is a stationary process possessing autocovariance function of this form, then there is a white noise Z and a real number θ such that $X_t = Z_t + \theta Z_{t-1}$ for all $t \in \mathbb{Z}$.

1. For now we suppose that $|b| < a/2$. Show that we can solve the problem by choosing $|\theta| < 1$ and Z a causal filter of X. But if $|b| = a/2$, what happens ?

2. Now we suppose $b = -a/2$. We set $Y_n = \sum_{k=1}^{n} X_k$ for all $n \geq 1$.

   (a) Show that $\text{Var}(Y_n) = a$ and $\text{Cov}(Y_n, Y_m) = a/2$ for all different integers n and m.

   (b) Deduce that $\frac{1}{n}\sum_{k=1}^{n} Y_k$ converges to a random variable U in L^2.

   (c) Show that the random variables $Y_n - U$ are uncorrelated and then conclude.

3. Similarly solve the problem when $b = a/2$.

Exercice 17 Let Z and W be uncorrelated standard white noises and let $\phi, \psi \in [0, 1)$. Consider the stationary solutions of X and Y of

$$\begin{cases} Y_t = \phi Y_{t-1} + X_t + W_t \\ X_t = \psi X_{t-1} + Z_t \end{cases}$$

1. Show that the second equation defines a unique stationary solution X. Express the solution and calculate the mean and the autocovariance function.

2. Show that $X + W$ is stationary, calculate its mean and autocovariance function.

3. Deduce that the first equation defines a unique stationary process Y.

4. For all $t \in \mathbb{Z}$, set $V_t = Y_t - (\phi + \psi)Y_{t-1} + \phi\psi Y_{t-2}$.

5. Pour tout $t \in \mathbb{Z}$, on pose $V_t = Y_t - (\phi + \psi)Y_{t-1} + \phi\psi Y_{t-2}$. Using Exercise 16, show that there exists a real number θ and a white noise H such that $V_t = H_t + \theta H_{t-1}$ for all $t \in \mathbb{Z}$.

6. Deduce that Y satisfies some ARMA equation driven by the white noise H.

7. Solve this ARMA equation (we can distinguish the two cases $\phi = \psi$ and $\phi \neq \psi$).

Exercice 18 (Lack of uniqueness of the ARMA solution) Let P be a polynomial with real coefficients and P has a root of modulus 1.

1. We denote by B the lag operator. Show that there exists a harmonic process Y solution of $P(B)Y = 0$ (i.e., there exist $\theta \in \mathbb{R}$, U and V of centered random variables with variance 1 and uncorrelated, such that $Y = U\cos(\theta t) + V\sin(\theta t)$ satisfies $P(B)Y = 0$).

2. Let Q be a polynomial whose roots of modulus 1 compensate those of P (i.e. all the roots of P of modulus 1 are also roots of Q). Show that the ARMA equation $P(B)X = Q(B)Z$ has a stationary solution which is not a filter of Z.

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TD4 - Spectral Analysis

Exercice 19 Suppose X and Y are two uncorrelated stationary process. Give the spectral measure of $X+Y$ as a function of those of X and Y.

Exercice 20 Determine the spectral measure of a MA(1) process, i.e. a stationary process X defined by $X_t = Z_t + \theta Z_{t-1}$ where $\theta \in \mathbb{R}$ and Z is a white noise.

Exercice 21

1. Determine the spectral measure of a AR(1) process, i.e. a stationary process X verifying $X_t = \phi X_{t-1} + Z_t$ where Z is a white noise with mean 0 and variance σ^2 and $\phi \notin \{-1, 1\}$.
2. Show that, if $\phi \neq \{-1, 1, 0\}$, there is a white noise $\tilde{Z}$ such that the stationary solution of $\tilde{X}_t = \phi^{-1}\tilde{X}_{t-1} + \tilde{Z}_t$ has the same autocovariance function as X. Give the variance of $\tilde{Z}$.

Exercice 22 Let $(f(h))_{h\in\mathbb{Z}}$ and $(g(h))_{h\in\mathbb{Z}}$ be two positive definite real sequence. Prove that fg is also positive definite. We can proceed in two ways:

1. Build a stationary process for which fg is the autocovariance function.
2. Build a finite measure on $[-\pi, \pi[$ such that fg is the Fourier transform of this measure (we can assume that $(f(h))$ is in ℓ^1).

   Let $\rho \in]0, 1[$, $\theta \in \mathbb{R}$ and $\gamma(h) = \rho^{|h|}\cos(\theta h)$ for all $h \in \mathbb{Z}$.
3. Show that $(\gamma(h))_{h\in\mathbb{Z}}$ is positive definite.
4. Specify its spectral measure and construct an explicit process having $(\gamma(h))_{h\in\mathbb{Z}}$ as function of autocovariance.

Exercice 23 Consider an ARMA equation

$$Q(B)X = P(B)Z$$

with P and Q two polynomials without common roots, and Q without roots of modulus 1, so that the equation admits a unique stationary solution (with $Z \in BB(0, 1)$).

1. Under what condition on Q is the solution causal?
2. The solution X is said to be invertible when there exists a filter $(a_n) \in \ell^1$ such that

$$Z_t = \sum_{k=0}^{\infty} a_n X_{t-k} \qquad \forall t \in \mathbb{Z}.$$

   Under what condition on P is the solution invertible ? And under what additional condition is the filter (a_n) causal ?

3. We assume in this question that neither P nor Q have roots of modulus 1. Give an ARMA equation
$$\tilde{Q}(B)X = \tilde{P}(B)Z$$
whose solution has the same autocorrelation as that of the original equation, i.e. causal, invertible and inversely causal.

4. Apply this principle to the equation
$$X_t - 2X_{t-1} = Z_t + 3Z_{t-1}.$$

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TD5 - Linear projection

Exercice 24 Let X be a stationary process with mean $\mu \in \mathbb{R}$. We set $Y_t = X_t - \mu$. Show that, for all $t \in \mathbb{Z}$ and for all $p \in \mathbb{N}^*$,

$$\text{proj}(X_t, \text{Vect}\{1, X_{t-1}, \dots, X_{t-p}\}) = \mu + \text{proj}(Y_t, \text{Vect}\{Y_{t-1}, \dots, Y_{t-p}\}).$$

Exercice 25 Consider the following AR(2) equation:

$$X_t = Z_t + \frac{3}{4}X_{t-1} - \frac{1}{8}X_{t-2} \qquad \forall t \in \mathbb{Z}.$$

1. Show that there is a unique stationary solution X to this equation, which is causal and invertible.
2. Determine the progressive predictor of order p of X, for all $p \geq 1$.

Exercice 26 We consider the following ARMA(2,1) equation:

$$X_t = Z_t - \frac{1}{2}Z_{t-1} + X_{t-1} - \frac{1}{4}X_{t-2} \qquad \forall t \in \mathbb{Z}.$$

1. Show that this equation has a unique stationary solution X.
2. Show that X is the solution of an AR(1) equation and give its explicit formula.
3. Determine the progressive predictor of order p of X, for all $p \geq 1$.

Exercice 27 Let (X_t) be a centered stationary process and $p \in \mathbb{N}^*$. For all $t \in \mathbb{Z}$ and $h \in \mathbb{N}^*$ we set

$$\widehat{X}_{t,h} = \text{proj}(X_t, \text{Vect}\{X_{t-h}, \dots, X_{t-h-p+1}\}).$$

1. Determine the equation that the coefficients $\theta_1, \dots, \theta_p$ satisfy such that $\widehat{X}_t = \sum_{k=1}^{p} \theta_k X_{t-h-k+1}$.
2. Show that $\|X_t - \widehat{X}_{t,h}\|_2$ does not depend on t.
3. We assume that (X_t) is decorrelated at infinity (i.e., $\gamma_X(h) \to 0$ when $h \to +\infty$). Show that $(\widehat{X}_{t,h})$ tends to 0 in L^2 when $h \to +\infty$.

Exercice 28 (Deterministic process) A centered stationary process X is said to be deterministic if

$$X_t \in \overline{Vect\{X_s,\ s < t\}} \qquad \forall t \in \mathbb{Z}.$$

We recall that, for $n \in \mathbb{N}^*$, $\sigma_n^2 = \text{Var}(X_t - \text{proj}(X_t, H_{t-1,n})$ where $H_{t-1,n} = \text{Vect}(X_{t-k},\ k \in \{1, \dots, n\}$. Let $\sigma_\infty := \lim_{n\ to \infty} \sigma_n$ (this limit exists because (σ_n) is decreasing).

1. Show that X is deterministic, if and only if, $\sigma_\infty = 0$.

2. Give an example of a deterministic process and an example of a non-deterministic process.

3. Are AR(p) processes deterministic?

Exercice 29 We consider the general equation ARMA(p,q)

$$X_t = Z_t + \sum_{k=1}^{p} a_k X_{t-k} + \sum_{k=1}^{q} b_k Z_{t-k}$$

Assume that the polynomials $A(z) = 1 - \sum_{k=1}^{p} a_k z^k$ and $B(z) = 1 + \sum_{k=1}^{q} b_k z^k$ do not have a root inside the unit disk (all the roots are of modulus > 1).

1. Show that there exists $(\alpha_n)_{n\geq 0}$ and $(\beta_n)_{n\geq 0}$ in ℓ^1 such that, for all $t \in \mathbb{Z}$,

$$X_t = \sum_{k=0}^{\infty} \beta_k Z_{t-k} \text{ et } Z_t = \sum_{k=0}^{\infty} \alpha_k X_{t-k}$$

2. For a given $n \in \mathbb{Z}$, we denote $H_n^X = \overline{\text{Vect}\{X_k,\ k \in \mathbb{Z} \text{ and } k \leq n\}}$. Show that

$$H_n^X = \overline{\text{Vect}\{Z_k,\ k \in \mathbb{Z} \text{ and } k \leq n\}}.$$

3. For $n \in \mathbb{N}^*$, set $\hat{X}_t = \text{proj}(X_t, H_n^X)$. Show that, for all $t \geq 1$, we have

$$\hat{X}_{t+n} = -\sum_{j=1}^{\infty} \alpha_j \hat{X}_{t+n-j} = \sum_{j=t}^{\infty} \beta_j Z_{t+n-j}$$

and

$$\text{var}(X_{t+n} - \hat{X}_{t+n}) = \sum_{j=0}^{t-1} \beta_j^2.$$

4. Compute $\hat{X}_{n+1}$ in the particular case AR(p) and MA(1).

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TD6 - Estimation

Exercice 30 (Estimation of an AR(1)) Let (X_t) be an AR(1) process, i.e. $X_t = \phi X_{t-1} + Z_t$ where (Z_t) is a centered strong white noise having a moment of order 4. We suppose that $|\phi| < 1$.

1. Show that $\phi = \gamma_X(1)/\gamma_X(0)$.
2. Deduce a natural estimator for ϕ.
3. Show that it converges in L^2 and the speed of the convergence is on the order of $1/\sqrt{n}$.

Exercice 31 (Estimation of an MA(1)) Let (X_t) be an MA(1) process, i.e. $X_t = \theta Z_{t-1} + Z_t$ where (Z_t) is a strong white noise centered and having a moment of order 4. We assume that $|\theta| < 1$ and we will try to estimate it.

1. Show that θ can be expressed as a function of the correlation coefficient $\rho_X(1) = \gamma_X(1)/\gamma_X(0)$.
2. Give an estimator for $\rho_X(1)$ and show that it converges in L^2.
   Hint: The argument is the same as for the previous exercise.
3. Deduce an estimator of θ and show that it is also consistent in L^2.
   Hint: Use the fact that the function that expresses θ as a function of $\rho_X(1)$ is Lipschitz in a neighborhood $\rho_X(1)$ and bounded.

Exercice 32 (Estimation of the mean and confidence interval) Let $Y_t = \theta + X_t$, where (X_t) is an AR(1) defined by $X_t - \phi X_{t-1} = Z_t$, where $|\phi| < 1$ and the Z_t are independent and identically distributed with law $\mathcal{N}(0, \sigma^2)$. We seek to estimate θ from $Y_0, Y_1, \ldots, Y_{n-1}$. We note $\widehat{\theta}_n$ the empirical mean of $Y_0, Y_1, \ldots, Y_{n-1}$ defined by

$$\widehat{\theta}_n = \frac{1}{n} \sum_{i=0}^{n-1} Y_i.$$

1. Calculate $\lim_{n\to\infty} n\mathrm{var}(\widehat{\theta}_n)$ and give the expression of γ defined by
$$\sqrt{n}(\widehat{\theta}_n - \theta) \xrightarrow[n\to\infty]{\text{law}} \mathcal{N}(0, \gamma).$$
2. We choose $\phi = 0.6$ and $\sigma^2 = 2$. When we observe $n = 100$ values, we get $\widehat{\theta}_n = 0.271$. Build an interval asymptotic confidence at 95% for θ. Can we say that θ =0?
3. We propose another estimator of θ defined by $\widetilde{\theta}_n = (\mathbf{1}_n^\top \gamma_n^{-1} \mathbf{1}_n)^{-1} \mathbf{1}_n^\top \gamma_n^{-1} Y^{(n)}$ where $Y^{(n)} = (Y_0, Y_1, \ldots, Y_{n-1})^\top$, $\mathbf{1}_n = (1, \ldots, 1)^\top$ and γ_n is the matrix of covariance of $Y^{(n)}$. Justify the choice of this estimator;
4. Calculate $\lim_{n\to\infty} n\mathrm{Var}(\widetilde{\theta}_n)$. What do you conclude ?