Stochastic differential games with asymmetric information.

Pierre Cardaliaguet and Catherine Rainer

Département de Mathématiques, Université de Bretagne Occidentale,

6, avenue Victor-le-Gorgeu, B.P. 809, 29285 Brest cedex, France

e-mail : Pierre.Cardaliaguet@univ-brest.fr, Catherine.Rainer@univ-brest.fr

October 12, 2007

Abstract : We investigate a two-player zero-sum stochastic differential game in which the players have an asymmetric information on the random payoff. We prove that the game has a value and characterize this value in terms of *dual* solutions of some second order Hamilton-Jacobi equation.

Key-words : stochastic differential game, asymmetric information, viscosity solution.

A.M.S. classification : 49N70, 49L25, 91A23.

1 Introduction

This paper is devoted to a class of two-player zero-sum stochastic differential game in which the players have different information on the payoff. In this basic model, the terminal cost is chosen (at the initial time) randomly among a finite set of costs $\{g_{ij},\ i \in \{1,\dots,I\},\ j \in \{1,\dots,J\}\}$. More precisely, the indexes i and j are chosen independently according to a probability $p \otimes q$ on $\{1,\dots,I\} \times \{1,\dots,J\}$. Then the index i is announced to the first player and the index j to the second player. The players control the stochastic differential equation

$$\begin{aligned} &dX_s = b(s, X_s, u_s, v_s)ds + \sigma(s, X_s, u_s, v_s)dB_s,\ s \in [t, T], \\ &X_t = x, \end{aligned}$$

through their respective controls (u_s) and (v_s) in order, for the first player, to minimize $E[g_{ij}(X_T)]$ and, for the second player, to maximize this quantity. Note that the players do not really know which payoff they are actually optimizing because the first player, for instance, ignores which index j has been chosen. The key assumption in our model is that the players observe the evolving state (X_s). So they can deduce from this observation the behavior of their opponent and try to derive from it some knowledge on their missing data.

The formalization of such a game is quite involved: we refer to the second section of the paper where the notations are properly defined. In order to describe our results, let us introduce the upper and lower value functions V^+ and V^- of the game:

$$V^+(t,x,p,q) = \inf_{\hat{\alpha}\in(\mathcal{A}_r(t))^I} \sup_{\hat{\beta}\in\mathcal{B}_r(t))^J} J^{p,q}(t,x,\hat{\alpha},\hat{\beta}),$$

$$V^-(t,x,p,q) = \sup_{\hat{\beta}\in(\mathcal{B}_r(t))^J} \inf_{\hat{\alpha}\in(\mathcal{A}_r(t))^I} J^{p,q}(t,x,\hat{\alpha},\hat{\beta}).$$

where $J^{p,q}(t,x,\hat{\alpha},\hat{\beta})$ is the expectation under the probability $p\otimes q$ of the payoff associated with the strategies $\hat{\alpha}=(\alpha_i)_{i\in\{1,\dots,I\}}$ and $\hat{\beta}=(\beta_j)_{j\in\{1,\dots,J\}}$ of the players. The strategy $\hat{\alpha}$ takes into account the knowledge by the first player of the index i while $\hat{\beta}$ takes into account the knowledge of j by the second player. Our main result is that, under Isaacs'condition, the two value functions coincide: $V^+ = V^-$. Moreover, $\mathbf{V} := V^+ = V^-$ is the unique viscosity solution *in the dual sense* of some second order Hamilton-Jacobi equation. This means that

(i) $\mathbf{V}$ is convex with respect to p and concave with respect to q,

(ii) the convex conjugate of $\mathbf{V}$ with respect to p is a subsolution of some Hamilton-Jacobi-Isaacs (HJI) equation in the viscosity sense,

(iii) the concave conjugate of $\mathbf{V}$ with respect to q is a supersolution of a symmetric HJI equation,

(iv) $\mathbf{V}(T,x,p,q) = \sum_{i,j} p_i q_j g_{ij}(x)$ where $p=(p_i)_{i\in\{1,\dots,I\}}$ and $q=(q_j)_{j\in\{1,\dots,J\}}$.

We strongly underline that in general the value functions are *not* solution of the standard HJI equation: indeed $\mathbf{V}$ does not satisfy a dynamic programming principle in a classical sense.

An important current in Mathematical Finance is the modeling of insider trading (see for example Amendinger, Becherer, Schweizer [2] or Corcuera, Imkeller, Kohatsu-Higa, Nualart [7] and references therein). The basic question studied in these works is to evaluate how the addition of knowledge for a trader—i.e., mathematically, the addition to the original filtration of a variable depending on the future—shows up in his investing strategies, and an important tool is the theory of enlargement of filtrations. Our approach is completely different. Indeed, what is important in our game is not that the players have "more" information than what is contained in the filtration of the Brownian motion, but that their information differs from that of their opponent. In some sense we try to understand the strategic role of information in the game.

The model described above is strongly inspired by a similar one studied by Aumann and Maschler in the framework of repeated games. Since their seminal papers (reproduced in [3]), this model has attracted a lot of attention in game theory (see [11], [13], [15], [16]). However it is only recently that the first author has adapted the model to deterministic differential games (see [5], [6]).

The aim of this paper is to generalize the results of [5] to stochastic differential games and to game with integral payoffs. There are several difficulties towards this aim. First the notion of strategies for stochastic differential games is quite intricated (see [12], [14]). For our game it is all the more difficult that the players have to introduce additional noise in their strategies in order to confuse their opponent. One of the achievements of this paper is an important simplification of the notion of strategy which allows the introduction of the notion of random strategies. This also simplifies several proofs of [5]. Second the existence of a value for "classical" stochastic differential games relies on a comparison principle for some second order Hamilton-Jacobi equations. Here we have to be able to compare functions satisfying the condition (i,ii,iv) defined above with functions satisfying (i,iii,iv). While for deterministic differential games (i.e., first order HJI equations) we could do this without too much trouble (see [5]), for stochastic differential games (i.e., second order HJI equations) the proof is much more involved. In particular it requires a new maximum principle for lower semicontinuous functions (see the appendix) which is the most technical part of the paper.

The paper is organized in the following way: in section 2, we introduce the main notations and the notion of random strategies and we define the value functions of our game. In section 3 we prove that the value functions (and more precisely the convex and concave conjugates) are sub- and supersolutions of some HJ equation. Section 4 is devoted to the comparison principle and to the existence of the value. In Section 5 we investigate stochastic differential games with a running cost. The appendix is devoted to a new maximum principle.

Aknowledgement : We wish to thank the anonymous referee for pointing out a gap in a proof.

2 Definitions.

2.1 The dynamics.

Let $T > 0$ be a fixed finite time horizon. For $(t, x) \in [0, T] \times \mathbb{R}^n$, we consider the following doubly controlled stochastic system :

$$\begin{aligned} & dX_s = b(s, X_s, u_s, v_s)ds + \sigma(s, X_s, u_s, v_s)dB_s, \ s \in [t, T], \\ & X_t = x, \end{aligned} \tag{2.1}$$

where B is a d-dimensional standard Brownian motion on a given probability space $(\Omega, \mathcal{F}, P)$. For $s \in [t, T]$, we set

$$\mathcal{F}_{t,s} = \sigma\{B_r - B_t, r \in [t, s]\} \vee \mathcal{P},$$

where $\mathcal{P}$ is the set of all null-sets of P.

The processes u and v are assumed to take their values in some compact metric spaces U and V respectively. We suppose that the functions $b : [0, T] \times \mathbb{R}^n \times U \times V \to \mathbb{R}^n$ and $\sigma : [0, T] \times \mathbb{R}^n \times U \times V \to \mathbb{R}^{n \times d}$ are continuous and satisfy the assumption (H):

(H) b and σ are bounded and Lipschitz continuous with respect to (t, x), uniformly in $(u, v) \in U \times V$.

We also assume Isaacs' condition : for all $(t, x) \in [0, T] \times \mathbb{R}^n$, $p \in \mathbb{R}^n$, and all $A \in \mathcal{S}_n$ (where $\mathcal{S}_n$ is the set of symmetric $n \times n$ matrices) holds:

$$\begin{aligned} & \inf_u \sup_v \{< b(t, x, u, v), p > + \tfrac{1}{2} Tr(A\sigma(t, x, u, v)\sigma^*(t, x, u, v))\} = \\ & \qquad \sup_v \inf_u \{< b(t, x, u, v), p > + \tfrac{1}{2} Tr(A\sigma(t, x, u, v)\sigma^*(t, x, u, v))\} \end{aligned} \tag{2.2}$$

We set $H(t, x, p, A) = \inf_u \sup_v \{< b(t, x, u, v), p > + \frac{1}{2} Tr(A\sigma(t, x, u, v)\sigma^*(t, x, u, v))\}$.

For $t \in [0, T)$, we denote by $\mathcal{C}([t, T], \mathbb{R}^n)$ the set of continuous maps from $[t, T]$ to $\mathbb{R}^n$.

2.2 Admissible controls.

Definition 2.1 *An* admissible control *u for player I (resp. II) on $[t, T]$ is a process taking values in U (resp. V), progressively measurable with respect to the filtration $(\mathcal{F}_{t,s}, s \geq t)$. The set of admissible controls for player I (resp. II) on $[t, T]$ is denoted by $\mathcal{U}(t)$ (resp. $\mathcal{V}(t)$).*

We identify two processes u and $\overline{u}$ in $\mathcal{U}(t)$ if $P\{u = \overline{u}$ a.e. in $[t, T]\} = 1$.

Under assumption (H), for all $(t, x) \in [0, T] \times \mathbb{R}^n$ and $(u, v) \in \mathcal{U}(t) \times \mathcal{V}(t)$, there exists a unique solution to (2.1) that we denote by $X_\cdot^{t,x,u,v}$.

2.3 Strategies.

Definition 2.2 *A* strategy *for player I starting at time t is a Borel-measurable map $\alpha : [t,T] \times \mathcal{C}([t,T],\mathbb{R}^n) \to U$ for which there exists $\delta > 0$ such that, $\forall s \in [t,T], f, f' \in \mathcal{C}([t,T],\mathbb{R}^n)$, if $f = f'$ on $[t,s]$, then $\alpha(\cdot, f) = \alpha(\cdot, f')$ on $[t, s+\delta]$.*
We define strategies for player II in a symmetric way and denote by $\mathcal{A}(t)$ (resp. $\mathcal{B}(t)$) the set of strategies for player I (resp. player II).

We have the following existence result :

Lemma 2.1 *For all (t,x) in $[0,T] \times \mathbb{R}^n$, for all $(\alpha,\beta) \in \mathcal{A}(t) \times \mathcal{B}(t)$, there exists a unique couple of controls $(u,v) \in \mathcal{U}(t) \times \mathcal{V}(t)$ that satisfies $P-$a.s.*

$$(u,v) = (\alpha(\cdot, X_\cdot^{t,x,u,v}), \beta(\cdot, X_\cdot^{t,x,u,v})) \text{ on } [t,T]. \tag{2.3}$$

Proof: The controls u and v will be built step by step. Let $\delta > 0$ be a common delay for α and β. We can choose δ such that $T = t + N\delta$ for some $N \in \mathbb{N}^*$.
By definition, on $[t, t+\delta)$, for all $f \in \mathcal{C}([t,T],\mathbb{R}^n)$, $\alpha(s,f) = \alpha(s, f(t))$. Since, for all $(u,v) \in \mathcal{U}(t) \times \mathcal{V}(t)$, $X_t^{t,x,u,v} = x$, the control u is uniquely defined on $[t,t+\delta)$ by

$$\forall s \in [t, t+\delta), u(s) = \alpha(s,x).$$

The same holds for v, what permits us to define the process $X_\cdot^{t,x,u,v}$ on $[t,t+\delta)$ as a solution of the system (2.1) restricted on the interval $[t,t+\delta)$.
Now suppose that u, v and $X_\cdot^{t,x,u,v}$ are $P-$a.s. defined uniquely on some interval $[t, t+k\delta)$, $k \in \{1, \dots, N-1\}$. This allows us to set,

$$\forall s \in [t+k\delta, t+(k+1)\delta),\ u_s = \alpha(s, X_\cdot^{t,x,u^k,v^k}), v_s = \beta(s, X_\cdot^{t,x,u^k,v^k}),$$

where

$$(u^k, v^k) = \begin{cases} (u,v) \text{ on } [t, t+k\delta) \\ (u_0, v_0) \text{ else,} \end{cases}$$

for some arbitrary $(u_0,v_0) \in \mathcal{U}(t) \times \mathcal{V}(t)$.
Considering $X_\cdot^{t,x,u^k,v^k}$ as a random variable with values in the set of paths $\mathcal{C}([t,T),\mathbb{R}^n)$, it is clear that the map $(s,\omega) \to u_s(\omega)$ (defined on $[t+k\delta, t+(k+1)\delta) \times \Omega$) as the composition of the Borel measurable application α with the map $(s,\omega) \to (s, X_\cdot^{t,x,u^k,v^k}(\omega))$, is a process on $[t+k\delta, t+(k+1)\delta)$ with measurable paths. Further, the non anticipativity of α guaranties that, for all $s \in [t+k\delta, t+(k+1)\delta)$, u_s is $\mathcal{F}_{t,t+k\delta}$-measurable and the process $u|_{[t,t+(k+1)\delta)}$ is $(\mathcal{F}_{t,s})$-progressively measurable. The same holds of course for $v|_{[t,t+(k+1)\delta)}$.

With (u,v) defined on $[t, t+(k+1)\delta)$, we can now define the process $X^{t,x,u,v}_{\cdot}$ up to time $t+(k+1)\delta$. This completes the proof by induction. $\Box$

We denote by $X^{t,x,\alpha,\beta}_{\cdot}$ the process $X^{t,x,u,v}_{\cdot}$, with (u,v) associated to (α,β) by relation (2.3).

In the frame of incomplete information it is necessary to introduce random strategies. In contrast with [5] and [6], where the random probabilities are supposed to be absolutely continuous with respect to the Lebesgue measure, play a random strategy will consist here to choose some strategy in a finite set of possibilities, i.e. the involved probabilities are finite. It is not clear if this assumption is more realistic nor if the notation will be lighter, nevertheless this alternative allows us to avoid some technical steps of measure theory, in a paper that is already technical enough.

Notation: For $R \in \mathbb{N}^*$, let $\Delta(R)$ be the set of all $(r_1, \dots, r_R) \in [0,1]^R$ that satisfy $\sum_{n=1}^R r_n = 1$.
We define a random strategy $\overline{\alpha}$ for player I by $\overline{\alpha} = (\alpha^1, \dots \alpha^R; r^1, \dots, r^R)$, with $R \in \mathbb{N}^*$, $(\alpha^1, \dots \alpha^R) \in (\mathcal{A}(t))^R$, $(r^1, \dots, r^R) \in \Delta(R)$.
The heuristic interpretation of $\bar{\alpha}$ is that player I's strategy amounts to choose the pure strategy α^k with probability r^k.
We define in a similar way the random strategies for player II, and denote by $\mathcal{A}_r(t)$ (resp. $\mathcal{B}_r(t)$) the set of all random strategies for player I (resp. player II).
Finally, identifying $\alpha \in \mathcal{A}(t)$ with $(\alpha; 1) \in \mathcal{A}_r(t)$, we can write $\mathcal{A}(t) \subset \mathcal{A}_r(t)$, and the same holds for $\mathcal{B}(t)$ and $\mathcal{B}_r(t)$.

2.4 The payoff.

Fix $I, J \in \mathbb{N}^*$.
For $1 \le i \le I, 1 \le j \le J$, let $g_{ij} : \mathbb{R}^n \to \mathbb{R}$ be the terminal payoffs. We assume that

$$\text{For } 1 \le i \le I, 1 \le j \le J,\ g_{ij} \text{ are Lipschitz continuous and bounded.} \tag{2.4}$$

For $(p,q) \in \Delta(I) \times \Delta(J)$, with $p = (p_1, \dots, p_I)$, $q = (q_1, \dots q_J)$, we denote with a hat the elements of $(\mathcal{A}_r(t))^I$ (resp. $(\mathcal{B}_r(t))^J$): $\hat{\alpha} = (\overline{\alpha}_1, \dots, \overline{\alpha}_I)$, $\hat{\beta} = (\overline{\beta}_1, \dots, \overline{\beta}_J)$.
We adopt following notations :

For fixed $(i,j) \in \{1,\dots,I\} \times \{1,\dots,J\}$ and strategies $(\alpha,\beta) \in \mathcal{A}(t) \times \mathcal{B}(t)$, the payoff of the game with only one possible terminal payoff function g_{ij} will be denoted by

$$J_{ij}(t,x,\alpha,\beta) = E[g_{ij}(X_T^{t,x,\alpha,\beta})].$$

Now let $(\overline{\alpha},\overline{\beta}) \in \mathcal{A}_r(t) \times \mathcal{B}_r(t)$ be two random strategies, with $\overline{\alpha} = (\alpha^1,\dots,\alpha^R; r^1,\dots,r^R)$ and $\overline{\beta} = (\beta^1,\dots,\beta^S; s^1,\dots,s^S)$. The payoff associated with the pair $(\overline{\alpha},\overline{\beta}) \in \mathcal{A}_r(t)\times\mathcal{B}_r(t))$, is the average of the payoffs with respect to the probability distributions associated to the strategies:

$$J_{ij}(t,x,\overline{\alpha},\overline{\beta}) = \sum_{k=1}^{R}\sum_{l=1}^{S} r^k s^l E[g_{ij}(X_T^{t,x,\alpha^k,\beta^l})].$$

Further, for $p \in \Delta(I)$, $j \in \{1,\dots,J\}$, $\hat{\alpha} \in (\mathcal{A}_r(t))^I$ and $\overline{\beta} \in \mathcal{B}_r(t)$ we will use the notation

$$J_j^p(t,x,\hat{\alpha},\overline{\beta}) = \sum_{i=1}^{I} p_i J_{ij}(t,x,\overline{\alpha}_i,\overline{\beta}) = \sum_{i=1}^{I} p_i \sum_{k,l} r^k s^l E[g_{ij}(X_T^{t,x,\alpha_i^k,\beta^l})].$$

A symmetric notation holds for $\overline{\alpha} \in \mathcal{A}_r(t)$ and $\hat{\beta} \in (\mathcal{B}_r(t))^J$. Finally, the payoff of the game is, for $(\hat{\alpha},\hat{\beta}) \in (\mathcal{A}_r(t))^I \times (\mathcal{B}_r(t))^J$, $p \in \Delta(I)$, $q \in \Delta(J)$,

$$J^{p,q}(t,x,\hat{\alpha},\hat{\beta}) = \sum_{i=1}^{I}\sum_{j=1}^{J} p_i q_j J_{ij}(t,x,\overline{\alpha}_i,\overline{\beta}_j).$$

The reference to (t,x) in the notations is dropped when there is no possible confusion : we will write $J_{ij}(\alpha,\beta), J_{ij}(\overline{\alpha},\beta),\dots$.

We define the value functions for the game by

$$\begin{aligned} V^+(t,x,p,q) &= \inf\nolimits_{\hat{\alpha}\in(\mathcal{A}_r(t))^I} \sup\nolimits_{\hat{\beta}\in\mathcal{B}_r(t))^J} J^{p,q}(t,x,\hat{\alpha},\hat{\beta}), \\ V^-(t,x,p,q) &= \sup\nolimits_{\hat{\beta}\in(\mathcal{B}_r(t))^J} \inf\nolimits_{\hat{\alpha}\in(\mathcal{A}_r(t))^I} J^{p,q}(t,x,\hat{\alpha},\hat{\beta}). \end{aligned}$$

Again we will write $V^+(p,q)$ and $V^-(p,q)$ if there is no possible confusion on (t,x).

The following lemma follows easily from classical estimations for stochastic differential equations :

Lemma 2.2 *V^+ and V^- are bounded, Lipschitz continuous with respect to x,p,q and Hölder continuous with respect to t.*

Following [3] we now state one of the basic properties of the value functions. The technique of proof of this statement is known as the splitting method in repeated game theory (see [3], [16]).

Proposition 2.1 *For all $(t,x) \in [0,T] \times \mathbb{R}^n$, the maps $(p,q) \to V^+(t,x,p,q)$ and $(p,q) \to V^-(t,x,p,q)$ are convex in p and concave in q.*

Proof: We only prove the result for V^+, the proof for V^- is the same. First V^+ can be rewritten as

$$V^+(p,q) = \inf_{\hat{\alpha} \in (\mathcal{A}_r(t))^I} \sum_{j=1}^{J} q_j \sup_{\overline{\beta} \in \mathcal{B}_r(t)} J_j^p(\hat{\alpha}, \overline{\beta}).$$

It follows that V^+ is concave in q.

Now fix $q \in \Delta(J)$ and let $p, p' \in \Delta(I)$ and $a \in (0,1)$. Without loss of generality we can assume that, for all $i \in \{1, \dots, I\}$, p_i and p'_i are not simultaneously equal to zero.
We get a new element of $\Delta(I)$ if we set $p^a = ap + (1-a)p'$. For $\epsilon > 0$, let $\hat{\alpha} \in (\mathcal{A}_r(t))^I$ be ϵ-optimal for $V^+(p,q)$ (resp. $\hat{\alpha}' \in (\mathcal{A}_r(t))^I$ ϵ-optimal for $V^+(p',q)$).
We define a new strategy $\hat{\alpha}^a = (\overline{\alpha}_1^a, \dots, \overline{\alpha}_I^a)$ by

$$\overline{\alpha}_i^a = (\alpha_i^1, \dots, \alpha_i^R, \alpha_i'^1, \dots, \alpha_i'^{R'}; (r_i^a)^1, \dots, (r_i^a)^{(R+R')}), i \in \{1, \dots, I\},$$

with

$$(r_i^a)^k = \begin{cases} \frac{ap_i}{p_i^a} r_i^k & \text{for } k \in \{1, \dots, R\}, \\ \\ \frac{(1-a)p'_i}{p_i^a} r_i'^{k-R} & \text{for } k \in \{R+1, \dots, R+R'\} \end{cases}$$

(it is easy to check that $\hat{\alpha}^a \in (\mathcal{A}_r(t))^I$).
This means that, for all $\hat{\beta} \in (\mathcal{B}_r(t))^J$,

$$J^{p^a,q}(\hat{\alpha}^a, \hat{\beta}) = \sum_{i=1}^{I} \{ap_i \sum_{k=1}^{R} r_i^k J_i^q(\alpha_i^k, \hat{\beta}) + (1-a)p'_i \sum_{k=1}^{R'} r_i'^k J_i^q(\alpha_i'^k, \hat{\beta})\}$$

Thus

$$\sup_{\hat{\beta} \in (\mathcal{B}_r(t))^J} J^{p^a,q}(\hat{\alpha}^a, \hat{\beta}) \le a \sup_{\hat{\beta} \in (\mathcal{B}_r(t))^J} J^{p,q}(\hat{\alpha}, \hat{\beta}) + (1-a) \sup_{\hat{\beta} \in (\mathcal{B}_r(t))^J} J^{p',q}(\hat{\alpha}', \hat{\beta}).$$

It follows by the choice of $\hat{\alpha}$ and $\hat{\alpha}'$ that

$$V^+(p^a,q) \le aV^+(p,q) + (1-a)V^+(p',q).$$

□

3 Subdynamic programming and Hamilton-Jacobi-Bellman equations for the Fenchel conjugates.

Since V^+ and V^- are convex with respect to p and concave with respect to q, it is natural to introduce the Fenchel conjugates of these functions. For this we use the following notations. For any $w : [0,T] \times \mathbb{R}^n \times \Delta(I) \times \Delta(J) \to \mathbb{R}$, we define the Fenchel conjugate w^* of w with respect to p by

$$w^*(t,x,\hat{p},q) = \sup_{p\in\Delta(I)} \{\langle \hat{p}, p\rangle - w(t,x,p,q)\},\ (t,x,\hat{p},q) \in [0,T] \times \mathbb{R}^n \times \mathbb{R}^I \times \Delta(J).$$

For w defined on the dual space $[0,T] \times \mathbb{R}^n \times \mathbb{R}^I \times \Delta(J)$, we also set

$$w^*(t,x,p,q) = \sup_{\hat{p}\in\mathbb{R}^I} \{\langle \hat{p}, p\rangle - w(t,x,\hat{p},q)\},\ (t,x,p,q) \in [0,T] \times \mathbb{R}^n \times \Delta(I) \times \Delta(J).$$

It is well known that, if w is convex in p, we have $(w^*)^* = w$.
We also have to introduce the concave conjugate with respect to q of a map $w : [0,T] \times \mathbb{R}^n \times \Delta(I) \times \Delta(J) \to \mathbb{R}$:

$$w^\sharp(t,x,p,\hat{q}) = \inf_{q\in\Delta(J)} \{\langle \hat{q}, q\rangle - w(t,x,p,q)\},\ (t,x,p,\hat{q}) \in [0,T] \times \mathbb{R}^n \times \Delta(I) \times \mathbb{R}^J.$$

We use the following notations for the sub- and superdifferentials with respect to $\hat{p}$ and $\hat{q}$ respectively: if $w : [0,T] \times \mathbb{R}^n \times \mathbb{R}^I \times \Delta(J) \to \mathbb{R}$, we set

$$\partial^-_{\hat{p}} w(t,x,\hat{p},q) = \{p \in \mathbb{R}^I,\ w(t,x,\hat{p},q) + \langle p, \hat{p}' - \hat{p}\rangle \le w(t,x,\hat{p}',q)\ \forall \hat{p}' \in \mathbb{R}^I\}$$

and if $w : [0,T] \times \mathbb{R}^n \times \Delta(I) \times \mathbb{R}^J \to \mathbb{R}$

$$\partial^+_{\hat{q}} w(t,x,p,\hat{q}) = \{q \in \mathbb{R}^J,\ w(t,x,p,\hat{q}) + \langle q, \hat{q}' - \hat{q}\rangle \ge w(t,x,p,\hat{q}')\ \forall \hat{q}' \in \mathbb{R}^J\}.$$

In this chapter, we will show that $V^{+\sharp}$ and V^{-*} satisfy a subdynamic programming property. This part follows several ideas of [10], [11].

Lemma 3.1 *(Reformulation of V^{-*})*
For all $(t,x,\hat{p},q) \in [0,T] \times \mathbb{R}^n \times \mathbb{R}^I \times \Delta(J)$, we have

$$V^{-*}(t,x,\hat{p},q) = \inf_{\hat{\beta}\in(\mathcal{B}_r(t))^J} \sup_{\alpha\in\mathcal{A}(t)} \max_{i\in\{1,\ldots,I\}} \left\{\hat{p}_i - J_i^q(t,x,\alpha,\hat{\beta})\right\}. \tag{3.5}$$

Proof. We begin to establish a first expression for V^{-*}:

$$V^{-*}(\hat{p},q) = \inf_{\hat{\beta}\in(\mathcal{B}_r(t))^J} \sup_{\overline{\alpha}\in\mathcal{A}_r(t)} \max_{i\in\{1,\ldots,I\}} \left\{\hat{p}_i - J_i^q(\overline{\alpha},\hat{\beta})\right\} \tag{3.6}$$

(the difference with (3.5) is that player I here can use random strategies.)
Let's denote by $e = e(\hat{p}, q)$ the right hand term of (3.6). First we prove that e is convex with respect to $\hat{p}$:
Fix $q \in \Delta(J)$, $\hat{p}, \hat{p}' \in \mathbb{R}^I$ and $a \in (0,1)$.
For $\epsilon > 0$, let $\hat{\beta}$ (resp. $\hat{\beta}'$)$\in (\mathcal{B}_r(t))^J$ be some ϵ-optimal strategy for $e(\hat{p}, q)$ (resp. $e(\hat{p}', q)$).
Set $\hat{p}^a = a\hat{p} + (1-a)\hat{p}'$.
We define a new strategy $\hat{\beta}^a \in (\mathcal{B}_r(t))^J$ by

$$\overline{\beta}^a_j = (\beta^1_j, \ldots, \beta^S_j, \beta'^1_j, \ldots, \beta'^{S'}_j; (s^a_j)^1, \ldots, (s^a_j)^{S+S'}), \;\; j \in \{1, \ldots, J\},$$

with

$$(s^a)^k_j = \begin{cases} a s^k_j & \text{for } k \in \{1, \ldots, S\}, \\ \\ (1-a) s'^{k-S}_j & k \in \{S+1, \ldots, S+S'\}. \end{cases}$$

Let $\overline{\alpha} \in \mathcal{A}_r(t)$. Since the application $(x_1, \ldots, x_I) \to \max\{x_i, i = 1, \ldots, I\}$ is convex, we have

$$\begin{aligned} \max_i \left\{ \hat{p}^a_i - J^q_i(\overline{\alpha}, \hat{\beta}^a) \right\} = & \;\; \max_i \left\{ a(\hat{p}_i - J^q_i(\overline{\alpha}, \hat{\beta})) + (1-a)(\hat{p}'_i - J^{q'}_i(\overline{\alpha}, \hat{\beta}')) \right\} \\ \\ \leq & \;\; a \sup_{\overline{\alpha} \in \mathcal{A}_r(t)} \max_i (\hat{p}_i - J^q_i(\overline{\alpha}, \hat{\beta}^a)) \\ & + (1-a) \sup_{\overline{\alpha} \in \mathcal{A}_r(t)} \max_i (\hat{p}^1_i - J^{q'}_i(\overline{\alpha}, \hat{\beta}')) \\ \\ \leq & \;\; a e(\hat{p}, q) + (1-a) e(\hat{p}', q) + \epsilon. \end{aligned}$$

Since ϵ is arbitrary, we can deduce that e is convex with respect to $\hat{p}$.
The next step is to prove that $e^* = V^-$. By the convexity of e, this will imply that $V^{-*} = e$.
We can reorganize $e^*(p, q)$ as follows :

$$\begin{aligned} e^*(p,q) = & \;\; \sup_{\hat{p} \in \mathbb{R}^I} \left\{ \sum_{i=1}^I \hat{p}_i p_i + \sup_{\hat{\beta} \in (\mathcal{B}_r(t))^J} \inf_{\overline{\alpha} \in \mathcal{A}_r(t)} \min_{i' \in \{1, \ldots, I\}} \{ J^q_{i'}(\overline{\alpha}, \hat{\beta}) - \hat{p}_{i'} \} \right\} \\ \\ = & \;\; \sup_{\hat{\beta} \in (\mathcal{B}_r(t))^J} \sup_{\hat{p} \in \mathbb{R}^I} \sum_{i=1}^I p_i \min_{i' \in \{1, \ldots, I\}} \left\{ \inf_{\overline{\alpha} \in \mathcal{A}_r(t)} J^q_{i'}(\overline{\alpha}, \hat{\beta}) + (\hat{p}_i - \hat{p}_{i'}) \right\} \end{aligned}$$

The supremum over $\hat{p} \in \mathbb{R}^I$ is attained for $\hat{p}_{i'} = \inf_{\overline{\alpha} \in \mathcal{A}_r(t)} J^q_{i'}(\overline{\alpha}, \hat{\beta})$ and we get the claimed result.

Finally, to get (3.5), it remains to show that player I can use non random strategies. Indeed, writing V^{-*} as in (3.6) and since $\mathcal{A}(t) \subset \mathcal{A}_r(t)$, it is obvious that the left hand side

of (3.5) is not smaller than the right hand side.
Concerning the reverse inequality, we can write

$$\begin{aligned}\sup_{\overline{\alpha}\in\mathcal{A}_r(t)} \quad \max_i \left\{\hat{p}_i - J_i^q(\overline{\alpha},\overline{\beta})\right\} & \\ &\leq \sup_{R\in\mathbb{N}^*}\sup_{(\alpha^1,\ldots,\alpha^R)\in(\mathcal{A}(t))^R,(r^1,\ldots,r^R)\in\Delta(R)}\sum_{k=1}^R r^k \max_i\left\{\hat{p}_i - J_i^q(\alpha^k,\hat{\beta})\right\} \\ &\leq \sup_{R\in\mathbb{N}^*}\sup_{(r^1,\ldots,r^R)\in\Delta(R)}\sum_k r^k \sup_{\alpha\in\mathcal{A}(t)}\max_i\left\{\hat{p}_i - J_i^q(\alpha,\hat{\beta})\right\}.\end{aligned}$$

The result follows after one recalls that $\sum_{k=1}^R r^k = 1$. □

Proposition 3.1 *(Subdynamic programming for V^{-*})*
For all $0 \leq t_0 \leq t_1 \leq T, x_0 \in \mathbb{R}^n, \hat{p} \in \mathbb{R}^I, q \in \Delta(J)$*, it holds that*

$$V^{-*}(t_0, x_0, \hat{p}, q) \leq \inf_{\beta\in\mathcal{B}(t_0)} \sup_{\alpha\in\mathcal{A}(t_0)} E[V^{-*}(t_1, X_{t_1}^{t_0,x_0,\alpha,\beta}, \hat{p}, q)].$$

Proof : Set $V_1^{-*}(t_0, t_1, x_0, \hat{p}, q) = \inf_{\beta\in\mathcal{B}(t_0)} \sup_{\alpha\in\mathcal{A}(t_0)} E[V^{-*}(t_1, X_{t_1}^{t_0,x_0,\alpha,\beta}, \hat{p}, q)]$.
For $\epsilon > 0$, let $\beta^\epsilon \in \mathcal{B}(t_0)$ be ϵ-optimal for $V_1^{-*}(t_0, t_1, x_0, \hat{p}, q)$, and, for all $x \in \mathbb{R}^n$, let $\hat{\beta}^x \in (\mathcal{B}_r(t_1))^J$ be ϵ-optimal for $V^{-*}(t_1, x, \hat{p}, q)$. By the uniformly Lipschitz assumptions for the parameters of the dynamics, there exists $R > 0$ such that, for all $\alpha \in \mathcal{A}(t_0)$,

$$P[X_{t_1}^{t_0,x_0,\alpha,\beta^\epsilon} \in B(x_0, R)] \geq 1 - \epsilon,$$

where $B(x_0, R)$ denotes the ball in $\mathbb{R}^n$ of center x_0 and radius R.
Remark that J_i^q and V^{-*} are uniformly Lipschitz continuous in x. This implies that we can find $r > 0$ such that, for any $x \in \mathbb{R}^n$ and $y \in \mathcal{B}(x, r)$, $\hat{\beta}^x$ is 2ϵ-optimal for $V^{-*}(t_1, y, \hat{p}, q)$.
Now let $x_1, \ldots, x_M \in \mathbb{R}^n$ such that $\cup_{m=1}^M B(x_m, \frac{r}{2}) \supset B(x_0, R)$.
Set $\hat{\beta}^m = \hat{\beta}^{x_m}$ for $m = 1, \ldots, M$ and choose some arbitrary $\hat{\beta}^0 \in (\mathcal{B}_r(t_1))^J$.
Each $\hat{\beta}^m$ is detailed in the following way:

$$\hat{\beta}^m = (\overline{\beta}_1^m, \ldots, \overline{\beta}_J^m),$$

with

$$\overline{\beta}_j^m = (\beta_j^{m,1}, \ldots, \beta_j^{m,S_j^m}; s_j^{m,1}, \ldots, s_j^{m,S_j^m}).$$

Let δ be a common delay for $\hat{\beta}^0, \ldots, \hat{\beta}^M$ that we can choose as small as we need : $0 < \delta < \frac{r^2\epsilon}{4C} \wedge (t_1 - t_0)$, where $C > 0$ is defined through the parameters of the dynamics by

$$\forall \alpha \in \mathcal{A}(t), \beta \in \mathcal{B}(t), t, t' \in [t_0, T],\ E[|X_t^{t_0,x_0,\alpha,\beta} - X_{t'}^{t_0,x_0,\alpha,\beta}|^2] \leq C|t - t'|.$$

We then have in particular, for all $\alpha \in \mathcal{A}(t)$ and $\beta \in \mathcal{B}(t)$,

$$P[|X_{t_1}^{t_0,x_0,\alpha,\beta} - X_{t_1-\delta}^{t_0,x_0,\alpha,\beta}| > \frac{r}{2}] \leq \epsilon. \tag{3.7}$$

Let $(E_m)_{m=1,\dots,M}$ be a Borel measurable partition of $B(x_0,R)$, such that, for all $m \in \{1,\dots,M\}$, $E_m \subset B(x_m,\frac{r}{2})$. Set $E_0 = B(x_0,R)^c$.
We are now able to define a new strategy for player II, $\hat{\beta}^\epsilon \in (\mathcal{B}_r(t_0))^J$:
Fix $j \in \{1,\dots,J\}$. For $l=(l_0,\dots,l_M) \in L := \Pi_{m=0}^M\{1,\dots,S_j^m\}$, set $s_j^l = \Pi_{m=0}^M s_j^{m,l_m}$. Remark that $\{s_j^l, l \in L\} \in \Delta(\mathrm{Card}(L))$.
Then, for $l \in L, l=(l_0,\dots,l_M)$, we define $(\beta_j^\epsilon)^l \in \mathcal{B}(t_0)$ by

$$\forall f \in C([t_0,T],\mathbb{R}^n), \forall t \in [t_0,T],$$
$$(\beta_j^\epsilon)^l(t,f) = \begin{cases} \beta^\epsilon(t,f) & \text{if } t \in [t_0,t_1), \\ \beta_j^{m,l_m}(t,f|_{[t_1,T]}) & \text{if } t \in [t_1,T] \text{ and } f(t_1-\delta) \in E_m. \end{cases}$$

We set $\overline{\beta}_j^\epsilon := ((\beta_j^\epsilon)^l; s_j^l, l \in L) \in \mathcal{B}_r(t_0)$, and finally $\hat{\beta}^\epsilon = (\overline{\beta}_1^\epsilon,\dots,\overline{\beta}_J^\epsilon)$.

For some fixed $\alpha \in \mathcal{A}(t_0)$ and $f \in C([t_0,t_1],\mathbb{R}^n)$, we define a new strategy $\alpha_f \in \mathcal{A}(t_1)$ by: for all $t \in [0,T]$ and $f' \in C([t_1,T],\mathbb{R}^n)$,

$$\alpha_f(t,f') = \alpha(t,\tilde{f}), \text{ with } \tilde{f}(t) = \begin{cases} f(t) \text{ for } t \in [t_0,t_1], \\ f'(t) - f'(t_1) + f(t_1), \text{ for } t \in (t_1,T]. \end{cases}$$

Set $X_\cdot^\epsilon = X_\cdot^{t_0,x_0,\alpha,\beta^\epsilon}$ and, for $m \in \{0,\dots,M\}$, $A_m = \{X_{t_1-\delta}^\epsilon \in E_m\}$. Set further $A = \{|X_{t_1}^\epsilon - X_{t_1-\delta}^\epsilon| \leq \frac{r}{2}\}$. By (3.7), it holds that $P[A^c] \leq \epsilon$. Remark also that, on each $A \cap A_m$, $X_{t_1}^\epsilon$ belongs to $B(x_m,r)$ and consequently, still on $A \cap A_m$, $\hat{\beta}^m$ is 2ϵ-optimal for $V^{-*}(t_1,X_{t_1}^\epsilon,\hat{p},q)$.
For all $i \in \{1,\dots,I\}, j \in \{1,\dots,J\}$ and $l \in L$, we have

$$E[g_{ij}(X_T^{t_0,x_0,\alpha,(\beta_j^\epsilon)^l})|\mathcal{F}_{t_1}] = \sum_{m=0}^M \mathbf{1}_{A_m} E[g_{ij}(X_T^{t_1,y,\alpha_f,\beta_j^{m,l_m}})]|_{y=X_{t_1}^\epsilon, f=X^\epsilon_\cdot|_{[t_0,t_1]}}.$$

It follows that

$$\begin{aligned} J_i^q(t_0,x_0,\alpha,\hat{\beta}^\epsilon) = & \textstyle\sum_{j=1}^J q_j \sum_{l\in L} s_j^l E[g_{ij}(X_T^{t_0,x_0,\alpha,(\beta_j^\epsilon)^l})] \\ = & E[\textstyle\sum_{m=0}^M \mathbf{1}_{A_m} J_i^q(t_1,X_{t_1}^\epsilon,\alpha_{X^\epsilon|_{[t_0,t_1]}},\hat{\beta}^m)]. \end{aligned}$$

And

$$\begin{aligned}
\max_{i\in\{1,\dots,I\}} \left\{\hat{p}_i - \right. & \left. J_i^q(t_0,x_0,\alpha,\hat{\beta}^\epsilon)\right\} \\
\leq & \ E[\textstyle\sum_{m=0}^M \mathbf{1}_{A_m} \max_{i\in\{1,\dots,I\}}\{\hat{p}_i - J_i^q(t_1,X_{t_1}^\epsilon,\alpha_{X_\cdot^\epsilon|_{[t_0,t_1]}},\hat{\beta}^m)\}] \\
\leq & \ E[\textstyle\sum_{m=0}^M \mathbf{1}_{A_m} (\sup_{\alpha\in\mathcal{A}(t_1)} \max_{i\in\{1,\dots,I\}}\{\hat{p}_i - J_i^q(t_1,X_{t_1}^\epsilon,\alpha,\hat{\beta}^m)\})] \\
\leq & \ E[(V^{-*}(t_1,X_{t_1}^\epsilon,\hat{p},q)+2\epsilon)\mathbf{1}_{A\cap\{X_{t_1}^\epsilon\in B(x_0,R)\}}] \\
& \qquad + \max_{i\in\{1,\dots,I\}}\{|\hat{p}_i|+K\}(P[A^c]+P[X_{t_1}^\epsilon\notin B(x_0,R)]),
\end{aligned}$$

by the choice of $(\hat{\beta}^m, m\in\{1,\dots,M\})$ and where K is an upper bound of $|g|$.
By the choice of R and with the notation $K(\hat{p}) = 4\max_{i\in\{1,\dots,I\}}\{|\hat{p}_i|+K\}+\epsilon$, we get

$$\begin{aligned}
\max_{i\in\{1,\dots,I\}} \left\{\hat{p}_i - J_i^q(t_0,x_0,\alpha,\hat{\beta}^\epsilon)\right\} \leq & \ E[V^{-*}(t_1,X_{t_1}^\epsilon,\hat{p},q)+2\epsilon]+K(\hat{p})\epsilon \\
\leq & \ \sup_{\alpha\in\mathcal{A}(t_0)} E[V^{-*}(t_1,X_{t_1}^{t_0,x_0,\alpha,\beta^\epsilon},\hat{p},q)]+2\epsilon(1+K(\hat{p})) \\
\leq & \ V_1^{-*}(t_0,t_1,x_0,\hat{p},q)+\epsilon(3+2K(\hat{p}))
\end{aligned}$$

(for the last inequality, recall that β^ϵ was chosen ϵ-optimal for $V_1^{-*}(t_0,t_1,x_0,\hat{p},q)$).
We can deduce the result. □

A classical consequence of the subdynamic programming principle for V^{-*} is that this function is a subsolution of some associated Hamilton-Jacobi equation. We give a proof of that result for sake of completeness.

Corollary 3.1 *For any $(\hat{p},q)\in\mathbb{R}^I\times\Delta(J)$, $V^{-*}(\cdot,\cdot,\hat{p},q)$ is a subsolution in the viscosity sense of*

$$w_t + H^{-*}(t,x,Dw,D^2w) = 0, \qquad (t,x)\in(0,T)\times\mathbb{R}^n,$$

with

$$\begin{aligned}
H^{-*}(t,x,p,A) = -H^-(t,x,-p,-A) = \\
\inf_{v\in V}\sup_{u\in U}\{\langle b(t,x,u,v),p\rangle + \tfrac{1}{2}Tr(A\sigma(t,x,u,v)\sigma^*(t,x,u,v))\}.
\end{aligned} \tag{3.8}$$

<u>Proof</u> : For $(t_0,x_0)\in[0,T]\times\mathbb{R}^n, \hat{p}\in\mathbb{R}^I, q\in\Delta(J)$ fixed, let $\phi\in C^{1,2}$ such that $\phi(t_0,x_0) = V^{-*}(t_0,x_0,\hat{p},q)$ and, for all $(s,y)\in[0,T]\times\mathbb{R}^n$, $\phi(s,y)\geq V^{-*}(s,y,\hat{p},q)$.
We have to prove that

$$\phi_t(t_0,x_0) + H^{-*}(t_0,x_0,D\phi(t_0,x_0),D^2(t_0,x_0)) \geq 0.$$

Suppose that this is false and consider $\theta > 0$ such that

$$\phi_t(t_0, x_0) + H^{-*}(t_0, x_0, D\phi(t_0, x_0), D^2(t_0, x_0)) \leq -\theta < 0. \tag{3.9}$$

Set $\Lambda(t,x,u,v) = \phi_t(t,x) + \langle b(t,x,u,v), D\phi(t,x)\rangle + \mathrm{Tr}(D^2\phi(t,x)\sigma(t,x,u,v)\sigma^*(t,x,u,v))$. Since, for fixed $\hat{p}$, V^{-*} is bounded, we can choose ϕ such that ϕ_t and $D^2\phi$ are also bounded. It follows that, for some $K > 0$, we have $|\Lambda(t,x,u,v)| \leq K$.
Now the relation (3.9) is equivalent to

$$\inf_{v\in V}\sup_{u\in U} \Lambda(t_0, x_0, u, v) \leq -\theta\ .$$

This implies the existence of a control $v_0 \in V$ such that, for all $u \in U$,

$$\Lambda(t_0, x_0, u, v_0) \leq -\frac{2\theta}{3}.$$

Moreover, since Λ is continuous in (t,x), uniformly in u, v, we can find $R > 0$ such that,

$$\forall (t,x) \in [t_0, T] \times \mathrm{I\!R}^n, |t - t_0| \vee \|x - x_0\| < R, \forall u \in U, \Lambda(t, x, u, v_0) \leq -\frac{\theta}{2}. \tag{3.10}$$

Now define a strategy for player II by $\beta_0(t,f) = v_0$ for all $(t,f) \in [t_0,T] \times C([t_0,T], \mathrm{I\!R}^n)$. Fix $\epsilon > 0$ and $t \in (t_0, R)$. Because of the subdynamical programming (Proposition 3.1), there exists $\alpha_{\epsilon,t} \in \mathcal{A}(t_0)$ such that

$$E[V^{-*}(t_1, X_{t_1}^{t_0,x_0,\alpha_{\epsilon,t},\beta_0}, \hat{p}, q)] - V^{-*}(t_0, x_0, \hat{p}, q) \geq -\epsilon(t - t_0). \tag{3.11}$$

Let $(u_s, v_s) \in \mathcal{U}(t_0) \times \mathcal{V}(t_0)$ the controls associated to $(\alpha_{\epsilon,t}, \beta_0)$ by the relation (2.3) and set $X_\cdot = X_\cdot^{t_0,x_0,\alpha_{\epsilon,t},\beta_0} = X_\cdot^{t_0,x_0,u,v}$. (Remark that, by the choice of β_0, (v_s) is constant and equal to v_0.)
Now we write Itô's formula for $\phi(t, X_t)$:

$$\begin{aligned} \phi(t, X_t) - \phi(t_0, x_0) = \ & \textstyle\int_{t_0}^t \Lambda(s, X_s, u_s, v_s) ds \\ & + \textstyle\int_{t_0}^t \langle D\phi(s, X_s), b(s, X_s, u_s, v_s)\rangle dB_s. \end{aligned} \tag{3.12}$$

By (3.11), (3.12) and the definition of ϕ, we have

$$E[\int_{t_0}^t \Lambda(s, X_s, u_s, v_s) ds] \geq -\epsilon(t - t_0). \tag{3.13}$$

In the other hand, there exists a constant $C > 0$ depending only on the parameters of X, such that

$$P[\|X_\cdot - x_0\|_t > R] \leq \frac{C(t - t_0)^2}{R^4},$$

with the notation $\|f\|_t = \sup_{s\in[t_0,t]} \|f(s)\|$.
Following (3.10), this implies that, for all $t \in [t_0, T\wedge(t_0+R)]$,

$$E\left[I_{\{\|X_\cdot - x_0\|_t < R\}} \int_{t_0}^t \Lambda(s, X_s, u_s, v_s)ds\right] \le -\frac{\theta}{2}(t-t_0). \tag{3.14}$$

By (3.13) and (3.14), we now have

$$\begin{aligned}
-\epsilon(t-t_0) \le\ & E[\textstyle\int_{t_0}^t \Lambda(s, X_s, u_s, v_s)dsI_{\{\|X_\cdot - x_0\|_t > R\}}] + E[\int_{t_0}^t \Lambda(s, X_s, u_s, v_s)dsI_{\{\|X_\cdot - x_0\|_t \le R\}}] \\
\le\ & \tfrac{KC}{R^4}(t-t_0)^2 - \tfrac{\theta}{2}(t-t_0),
\end{aligned}$$

or, equivalently,

$$\frac{\theta}{2} \le \frac{KC}{R^4}(t-t_0) + \epsilon.$$

Since $t - t_0$ and ϵ can be chosen arbitrarily small, we get a contradiction. □

For V^+ we have:

Proposition 3.2 *(Superdynamic programming and HJI equation for $V^{+\sharp}$)*
For all $0 \le t_0 \le t_1 \le T, x_0 \in \mathbb{R}^n, p \in \Delta(I), \hat{q} \in \mathbb{R}^J$, it holds that

$$V^{+\sharp}(t_0, x_0, p, \hat{q}) \ge \inf_{\beta\in\mathcal{B}(t_0)} \sup_{\alpha\in\mathcal{A}(t_0)} E[V^{+\sharp}(t_1, X_{t_1}^{t_0,x_0,\alpha,\beta}, p, \hat{q})].$$

As a consequence, for any $(p,\hat{q}) \in \Delta(I)\times\mathbb{R}^J$, $V^{+\sharp}(\cdot,\cdot,p,\hat{q})$ is a supersolution in viscosity sense of

$$w_t + H^{+*}(t, x, Dw, D^2w)) = 0, \qquad (t,x) \in (0,T)\times\mathbb{R}^n,$$

where

$$\begin{aligned}
&H^{+*}(t,x,p,A) = -H^+(t,x,-p,-A) = \\
&\quad \sup_{u\in U}\inf_{v\in V}\{\langle b(t,x,u,v), p\rangle + \tfrac{1}{2}Tr(A\sigma(t,x,u,v)\sigma^*(t,x,u,v))\}.
\end{aligned} \tag{3.15}$$

Proof : We note that V^+ is equal to the opposite of the lower value of the game in which we replace g_{ij} by $-g_{ij}$, Player I is the maximizer and in which the respective roles of p and q are exchanged. Using Proposition 3.1 in this framework gives the superdynamic programming principle. Now Corollary 3.1 shows that, for any $(p,\hat{q}) \in \Delta(I)\times\mathbb{R}^J$, $(-V^+)^*(\cdot,\cdot,p,\hat{q}) = -V^{+\sharp}(\cdot,\cdot,p,-\hat{q})$ is a subsolution of

$$w_t + H^+(t,x,Dw,D^2w)) = 0, \qquad (t,x)\in(0,T)\times\mathbb{R}^n.$$

Hence $V^{+\sharp}(\cdot,\cdot,p,-\hat{q})$ is a supersolution of

$$w_t + H^{+*}(t,x,Dw,D^2w)) = 0, \qquad (t,x) \in (0,T) \times \mathbb{R}^n.$$

Since this holds true for any $(p,\hat{q})$, this proves our claim. $\Box$

4 Comparison principle and existence of a value

In this section we first state a new comparison principle and apply it to get the existence and the characterization of the value. Then we give a proof for the comparison principle.

4.1 Statement of the comparison principle and existence of a value

Let $H : [0,T] \times \mathbb{R}^n \times \mathbb{R}^n \times \mathcal{S}_n \times \Delta(I) \times \Delta(J) \to \mathbb{R}$ be continuous and satisfy

$$\begin{aligned} &H(s,y,\xi_2,X_2,p,q) - H(t,x,\xi_1,X_1,p,q) \ \geq \\ &\quad -\omega\left(|\xi_1-\xi_2| + a|(t,x)-(s,y)|^2 + b + |(t,x)-(s,y)|(1+|\xi_1|+|\xi_2|)\right) \end{aligned} \tag{4.16}$$

where ω is continuous and non decreasing with $\omega(0)=0$, for any $a,b \geq 0$, $(p,q) \in \Delta(I) \times \Delta(J)$, $s,t \in [0,T]$, $x,y,\xi_1,x_2 \in \mathbb{R}^n$ and $X_1, X_2 \in \mathcal{S}_n$ such that

$$\begin{pmatrix} -X_1 & 0 \\ 0 & X_2 \end{pmatrix} \leq a \begin{pmatrix} I & -I \\ -I & I \end{pmatrix} + bI$$

Definition 4.1 *We say that a map $w : (0,T) \times \mathbb{R}^n \times \Delta(I) \times \Delta(J) \to \mathbb{R}$ is a* supersolution in the dual sense *of equation*

$$w_t + H(t,x,Dw,D^2w,p,q) = 0 \tag{4.17}$$

if $w = w(t,x,p,q)$ is lower semicontinuous, concave with respect to q and if, for any $\mathcal{C}^2((0,T)\times\mathbb{R}^n)$ function ϕ such that $(t,x) \to w^*(t,x,\hat{p},\bar{q}) - \phi(t,x)$ has a maximum at some point $(\bar{t},\bar{x})$ for some $(\hat{p},\bar{q}) \in \mathbb{R}^I \times \Delta(J)$ at which $\frac{\partial w^*}{\partial \hat{p}}$ exists, we have

$$\phi_t(\bar{t},\bar{x}) - H(\bar{t},\bar{x},-D\phi(\bar{t},\bar{x}),-D^2\phi(\bar{t},\bar{x}),\bar{p},\bar{q}) \geq 0 \qquad \text{where } \bar{p} = \frac{\partial w^*}{\partial \hat{p}}(\bar{t},\bar{x},\hat{p},\bar{q})\ .$$

We say that w is a subsolution of (4.17) in the dual sense *if w is upper semicontinuous, convex with respect to p and if, for any $\mathcal{C}^2((0,T)\times\mathbb{R}^n)$ function ϕ such that $(t,x) \to w^\sharp(t,x,\bar{p},\hat{q}) - \phi(t,x)$ has a minimum at some point $(\bar{t},\bar{x})$ for some $(\bar{p},\hat{q}) \in \Delta(I) \times \mathbb{R}^J$ at which $\frac{\partial w^\sharp}{\partial \hat{q}}$ exists, we have*

$$\phi_t(\bar{t},\bar{x}) - H(\bar{t},\bar{x},-D\phi(\bar{t},\bar{x}),-D^2\phi(\bar{t},\bar{x}),\bar{p},\bar{q}) \leq 0 \qquad \text{where } \bar{q} = \frac{\partial w^\sharp}{\partial \hat{q}}(\bar{t},\bar{x},\bar{p},\hat{q})\ .$$

A solution of (4.17) in the dual sense is a map which is sub- and supersolution in the dual sense.

Remarks :

1. We have proved in Corollary 3.1 that V^- is a dual supersolution of the HJ equation

$$w_t + H^-(t, x, Dw, D^2w) = 0 ,$$

where H^- is defined by (3.8), while Proposition 3.2 shows that V^+ is a dual subsolution of the HJ equation

$$w_t + H^+(t, x, Dw, D^2w) = 0 ,$$

where H^- is defined by (3.15).

2. The necessity to deal with a Hamiltonian H with a (p, q) dependence will become clear in the next section where we study differential games with running costs.

3. When H does not depend on (p, q), one can omit the condition "at which $\frac{\partial w^*}{\partial \hat{p}}$ exists" (resp. " at which $\frac{\partial w^\sharp}{\partial \hat{q}}$ exists") in the definition of supersolution (resp. subsolution). See below (Lemma 4.3) for the proof as well as for an equivalent definition of solutions.

The main result of this section is the following:

Theorem 4.1 (Comparison principle) *Let us assume that H satisfies the structure condition (4.16). Let w_1 be a bounded, Hölder continuous subsolution of (4.17) in the dual sense which is uniformly Lipschitz continuous w.r. to q and w_2 be a bounded, Hölder continuous supersolution of (4.17) in the dual sense which is uniformly Lipschitz continuous w.r. to p. Assume that*

$$w_1(T, x, p, q) \leq w_2(T, x, p, q) \qquad \forall (x, p, q) \in \mathbb{R}^n \times \Delta(I) \times \Delta(J) . \tag{4.18}$$

Then

$$w_1(t, x, p, q) \leq w_2(t, x, p, q) \qquad \forall (t, x, p, q) \in [0, T] \times \mathbb{R}^n \times \Delta(I) \times \Delta(J) .$$

Remark : For simplicity we are assuming here that w_1 and w_2 are Hölder continuous and bounded. These assumptions could be relaxed by standard (but painfull) techniques. We do not know if the uniform Lipschitz continuity assumption on w_1 with respect to q and on w_2 with respect to p can be relaxed.

As a consequence we have

Theorem 4.2 (Existence of a value) *Under assumptions (H), (2.4) and (2.2), the game has a value:*

$$V^+(t,x,p,q) = V^-(t,x,p,q) \qquad \forall (t,x,p,q) \in (0,T) \times \mathbb{R}^n \times \Delta(I) \times \Delta(J) \ .$$

Furthermore $V^+ = V^-$ is the unique solution in the dual sense of HJI equation (4.17) with terminal condition

$$V^+(T,x,p,q) = V^-(T,x,p,q) = \sum_{i=1}^{I}\sum_{j=1}^{J} p_i q_j g_{ij}(x) \qquad \forall (x,p,q) \in \mathbb{R}^n \times \Delta(I) \times \Delta(J) \ .$$

Proof of Theorem 4.2 : The Hamiltonian H defined by (2.2) is known to satisfy (4.16) (see [9] for instance). From the definition of V^+ and V^- we have $V^- \leq V^+$. We have proved in Lemma 2.2 and Proposition 2.1 that V^+ and V^- are Hölder continuous, Lipschitz continuous with respect to p and q, convex w.r. to p and concave w.r. to q. From Corollary 3.1 we know that V^- is a supersolution of (4.17) in the dual sense while Proposition 3.2 states that V^+ is a supersolution of that same equation in the dual sense. The comparison principle then states that $V^+ \leq V^-$, whence the existence and the characterization of the value: $V^+ = V^-$ is the unique solution in the dual sense of HJI equation (4.17). $\square$

4.2 Proof of the comparison principle

The proof of Theorem 4.1 relies on several arguments: first on an equivalent definition and on a reformulation of the notions of sub- and supersolutions by using sub- and superjets; second on a new maximum principle described in the appendix.

Here is an equivalent definition of the notion of supersolution.

Lemma 4.3 *Let $w : (0,T) \times \mathbb{R}^n \times \Delta(I) \times \Delta(J) \to \mathbb{R}$ be a lower semicontinuous map which is concave with respect to q. Then w is a supersolution of (4.17) in the dual sense if and only if for any $\mathcal{C}^2((0,T) \times \mathbb{R}^n)$ function ϕ such that $(t,x) \to w^*(t,x,\hat{p},\bar{q}) - \phi(t,x)$ has a maximum at some point $(\bar{t},\bar{x})$ for some $(\hat{p},\bar{q}) \in \mathbb{R}^I \times \Delta(J)$, we have*

$$\phi_t(\bar{t},\bar{x}) - H(\bar{t},\bar{x},-D\phi(\bar{t},\bar{x}),-D^2\phi(\bar{t},\bar{x}),\bar{p},\bar{q}) \geq 0 \qquad \text{for some } \bar{p} \in \partial^-_{\hat{p}} w^*(\bar{t},\bar{x},\hat{p},\bar{q}) \ .$$

Remarks :

1. In particular, if the Hamiltonian H does not depend on the variables (p,q), then w is a supersolution if and only if for any ϕ such that $(t,x) \to w^*(t,x,\hat{p},\bar{q}) - \phi(t,x)$ has a maximum at some point $(\bar{t},\bar{x})$ for some $(\hat{p},\bar{q}) \in \mathbb{R}^I \times \Delta(J)$, we have

$$\phi_t(\bar{t},\bar{x}) - H(\bar{t},\bar{x},-D\phi(\bar{t},\bar{x}),-D^2\phi(\bar{t},\bar{x})) \geq 0 \; .$$

2. A symmetric result holds true for supersolutions.

Proof : If w satisfies the condition, then it is clear that w is a supersolution in the dual sense because, if w^* has a derivative with respect to $\hat{p}$, then $\partial^-_{\hat{p}} w^*(\bar{t},\bar{x},\hat{p},\bar{q}) = \{\frac{\partial w^*}{\partial \hat{p}}(\bar{t},\bar{x},\hat{p},\bar{q})\}$.

Conversely, let us assume that w is a supersolution in the dual sense. Let ϕ be such that $(t,x) \to w^*(t,x,\hat{p},\bar{q}) - \phi(t,x)$ has a maximum at some point $(\bar{t},\bar{x})$ for some $(\hat{p},\bar{q}) \in \mathbb{R}^I \times \Delta(J)$. Without loss of generality we can assume that this maximum is strict. From the definition of w^*, the map $(t,x,p) \to \langle \hat{p}, p\rangle - w(t,x,p,\bar{q}) - \phi(t,x)$ has a maximum at $(\bar{t},\bar{x},\bar{p})$ for any $\bar{p} \in \partial^-_{\hat{p}} w^*(\bar{t},\bar{x},\hat{p},\bar{q})$.

Let us now fix $\epsilon > 0$ and consider a maximum point $(t_\epsilon, x_\epsilon, p_\epsilon)$ of the map

$$(t,x,p) \to \langle \hat{p}, p\rangle - w(t,x,p,\bar{q}) - \phi(t,x) + \frac{\epsilon}{2}|p|^2 \; .$$

We note that, up to a subsequence, the $(t_\epsilon, x_\epsilon, p_\epsilon)$'s converge to $(\bar{t},\bar{x},\bar{p})$ for some $\bar{p} \in \partial^-_{\hat{p}} w^*(\bar{t},\bar{x},\hat{p},\bar{q})$. Let $\hat{p}_\epsilon = \hat{p} + \epsilon p_\epsilon$. We claim that $\frac{\partial w^*}{\partial \hat{p}}$ exists at $(t_\epsilon, x_\epsilon, \hat{p}_\epsilon)$ and is equal to p_ϵ. Indeed, the map $(t,x,p) \to \langle \hat{p}_\epsilon, p\rangle - w(t,x,p,\bar{q}) - \phi(t,x) + \frac{\epsilon}{2}|p-p_\epsilon|^2$ has a maximum at $(t_\epsilon, x_\epsilon, p_\epsilon)$, which implies that $p \to \langle \hat{p}_\epsilon, p\rangle - w(t_\epsilon, x_\epsilon, p, \bar{q})$ has a unique maximum at p_ϵ. Thus $\frac{\partial w^*}{\partial \hat{p}}(t_\epsilon, x_\epsilon, \hat{p}_\epsilon) = p_\epsilon$. Since the map $(t,x) \to w^*(t,x,\hat{p}_\epsilon,\bar{q}) - \phi(t,x)$ has a maximum at (t_ϵ, x_ϵ) and since w is a supersolution in the dual sense, we get

$$\phi_t - H(t_\epsilon, x_\epsilon, -D\phi, -D^2\phi, p_\epsilon, \bar{q}) \geq 0$$

at $(t_\epsilon, x_\epsilon, p_\epsilon, \bar{q})$, from which we deduce the desired inequality as $\epsilon \to 0$. □

For $I' \subset \{1,\ldots,I\}$ let us denote by $\Delta(I') = \{p = (p_i) \in \Delta(I)\ ,\ p_i = 0 \text{ if } i \notin I'\}$.

Corollary 4.4 *Let w be a supersolution (resp. subsolution) of (4.17) in the dual sense. Let I' and J' be a subsets of $\{1,\ldots,I\}$ and $\{1,\ldots,J\}$. Then the restriction of w to $\Delta(I')$ and $\Delta(J')$ is still a supersolution (resp. subsolution) of (4.17) (in $\mathbb{R}^n \times \Delta(I') \times \Delta(J')$).*

Proof : Let w' be the restriction of w and assume that ϕ is such that $(t,x) \to w'^*(t,x,\hat{p}',\bar{q}') - \phi(t,x)$ has a maximum at some point $(\bar{t},\bar{x})$ for some $(\hat{p}',\bar{q}') \in \mathbb{R}^{I'} \times \Delta(J')$ (where by abuse of notations I' also denotes the cardinal of I'). Let us set

$$\hat{p}_i = \hat{p}'_i \text{ and } p_i = p'_i \text{ if } i \in I' \qquad \text{and} \qquad \hat{p}_i = -M \text{ and } p_i = 0 \text{ if } i \notin I',$$

where $M = |\hat{p}'|_\infty + 2\|D_p w\|_\infty$ and $\bar{q}_j = \bar{q}'_j$ if $j \in J'$ and $\bar{q}_j = 0$ otherwise. We claim that $p \in \partial^-_{\hat{p}} w^*(\bar{t}, \bar{x}, \hat{p}, \bar{q})$. Indeed, let $p_1 \in \Delta(I)$ and set $\delta = \sum_{i \in I'} p_{1,i}$ and, if $\delta > 0$, let $p'_{1,i} = p_{1,i}/\delta$ if $i \in I'$ and $p'_{1,i} = 0$ otherwise (if $\delta = 0$, just set $p'_1 = p'$). Let $p' \in \partial^-_{\hat{p}'} w'((\bar{t}, \bar{x}, \hat{p}', \bar{q}')$. We note that, if $\delta \neq 0$, $p'_1 \in \Delta(I')$. Then we have

$$\begin{aligned}
\langle \hat{p}, p_1 \rangle - w(p_1) &\leq -M(1-\delta) + \delta\langle \hat{p}, p'_1\rangle - w(p'_1) + \|D_p w\|_\infty |p_1 - p'_1| \\
&\leq -M(1-\delta) + \langle \hat{p}', p'_1\rangle + |\hat{p}'|_\infty (1-\delta) - w(p'_1) + 2\|D_p w\|_\infty (1-\delta) \\
&\leq (1-\delta)(-M + |\hat{p}'|_\infty + 2\|D_p w\|_\infty) + w^*(\hat{p}') \qquad \leq w^*(\hat{p}')
\end{aligned}$$

(where we have omitted the dependance with respect to $(\bar{t}, \bar{x}, \bar{q})$). So $w^*(\hat{p}) \leq w^{'*}(\hat{p}')$. Since the reverse inequality $w^{'*} \leq w^*$ always holds, we have the equality. Hence $p \in \partial^-_{\hat{p}} w^*(\bar{t}, \bar{x}, \hat{p}, \bar{q})$ and, since w is a dual supersolution,

$$\phi_t(\bar{t},\bar{x}) - H(\bar{t},\bar{x},-D\phi(\bar{t},\bar{x}),-D^2\phi(\bar{t},\bar{x}),\bar{p}',\bar{q}') = \phi_t(\bar{t},\bar{x}) - H(\bar{t},\bar{x},-D\phi(\bar{t},\bar{x}),-D^2\phi(\bar{t},\bar{x}),\bar{p},\bar{q}) \geq 0$$

holds at $(\bar{t}, \bar{x}, \hat{p}', \bar{q}')$. □

Let us recall the notions of sub- and superjets of a function $w : (0,T) \times \mathbb{R}^n \to \mathbb{R}$: the subjet $D^{2,-}w(\bar{t},\bar{x})$ is the set of $(\xi_t, \xi_x, X) \in \mathbb{R}^{n+1} \times \mathcal{S}_n$ such that

$$w(t,x) \geq w(\bar{t},\bar{x}) + \xi_t(t-\bar{t}) + \xi_x.(x-\bar{x}) + \frac{1}{2}X(x-\bar{x}).(x-\bar{x}) + o(|t-\bar{t}| + |x-\bar{x}|^2)$$

and the superjet $D^{2,+}w$ is given by

$$D^{2,+}w(\bar{t},\bar{x}) = -D^{2,-}(-w)(\bar{t},\bar{x})$$

In order to deal with viscosity solutions in the dual sense, we introduce some new notations. For $w = w(t,x,p,\hat{q})$ defined on the dual space $\mathbb{R}^n \times \Delta(I) \times \mathbb{R}^J$, we set

$$\overline{D}^{2,-}_q w(\bar{t},\bar{x},\bar{p},\hat{q}) = \left\{ \begin{array}{c} (\xi_t,\xi_x,X,q) \in \mathbb{R}^{n+1} \times \mathcal{S}_n \times \Delta(J)\ ,\ \exists (t_n,x_n,p_n,\hat{q}_n) \to (\bar{t},\bar{x},\bar{p},\hat{q}), \\ \exists (\xi^n_t, \xi^n_x, X^n) \in D^{2,-}w(t_n,x_n,p_n,\hat{q}_n) \text{ and } \exists q^n := \frac{\partial w}{\partial \hat{q}}(t_n,x_n,p_n,\hat{q}_n) \\ \text{with } (\xi^n_t,\xi^n_x,X^n,q^n) \to (\xi_t,\xi_x,X,q) \end{array} \right\}.$$

We use a symmetric notation for $\overline{D}^{2,+}_p w(\bar{t},\bar{x},\hat{p},\bar{q})$, which is a subset of $\mathbb{R}^{n+1} \times \mathcal{S}_n \times \Delta(I)$.

The following equivalent formulation of the notion of sub- and supersolution is standard in viscosity solution theory, so we omit the proof:

Proposition 4.5 *A map $w : (0,T) \times \mathbb{R}^n \times \Delta(I) \times \Delta(J) \to \mathbb{R}$ is a supersolution of equation (4.17) in the dual sense if and only if $w = w(t,x,p,q)$ is lower semicontinuous, concave with respect to q and if, for any $(\bar{t},\bar{x},\hat{p},\bar{q})$ and any $(\xi_t,\xi_x,X,p) \in \overline{D}^{2,+}_p w^*(\bar{t},\bar{x},\hat{p},\bar{q})$ we have*

$$\xi_t - H(\bar{t},\bar{x},-\xi_x,-X,p,\bar{q}) \geq 0\ .$$

Symmetrically w is a subsolution of (4.17) in the dual sense if and only if w is upper semicontinuous, convex with respect to p and if, for any $(\bar t,\bar x,\bar p,\hat q)$ and any $(\xi_t,\xi_x,X,q)\in \overline{D}^{2,-}_q w^\sharp(\bar t,\bar x,\bar p,\hat q)$ we have

$$\xi_t - H(\bar t,\bar x,-\xi_x,-X,\bar p,q)\le 0\ .$$

Proof of Theorem 4.1 : Let us assume that

$$\sup_{t,x,p,q}(w_1-w_2)>0\ .$$

Since w_1 and w_2 are Hölder continuous and bounded, classical arguments show that, for $\epsilon,\eta,\alpha>0$,

$$M_{\epsilon,\eta,\alpha}:=\sup_{t,x,s,y,p,q}\left\{w_1(t,x,p,q)-w_2(s,y,p,q)-(\frac{|(t,x)-(s,y)|^2}{2\epsilon}+\frac{\alpha}{2}(|x|^2+|y|^2))+\eta t\right\} \tag{4.19}$$

is finite and achieved at a point $(\bar t,\bar x,\bar s,\bar y,\bar p_0,\bar q_0)$. One can also show that

$$\lim_{\epsilon,\eta,\alpha\to 0^+} M_{\epsilon,\eta,\alpha}=\sup_{t,x,p,q}(w_1-w_2) \tag{4.20}$$

and that

$$\frac{|(\bar t,\bar x)-(\bar s,\bar y)|^2}{\epsilon^2}\ ,\ \alpha|\bar x|^2\ ,\ \alpha|\bar y|^2\le 2M_\infty \tag{4.21}$$

where $M_\infty=|w_1|_\infty+|w_2|_\infty$. Using (4.18) and the Hölder continuity of w_1 and w_2 shows that $\bar t<T$ and $\bar s<T$ as soon as ϵ, η and α are small enough.

We now fix ϵ, η and α as above and claim that it is possible to find some restriction I' and J' of I and J such that: (i) for any $(\bar p',\bar q')$ such that $(\bar t,\bar x,\bar s,\bar y,\bar p',\bar q')$ is a maximum point of

$$M'_{\epsilon,\eta,\alpha}:= \tag{4.22}$$

$$\max_{\substack{(t,x)\in[0,T]\times \mathbb{R}^N\\ (p',q')\in\Delta(I')\times\Delta(J')}} w_1(t,x,p',q')-w_2(s,y,p',q')-(\frac{|(t,x)-(s,y)|^2}{2\epsilon}+\frac{\alpha}{2}(|x|^2+|y|^2))+\eta t$$

one has $(p',q')\in Int(\Delta(I')\times\Delta(J'))$; (ii) $M'_{\epsilon,\eta,\alpha}=M_{\epsilon,\eta,\alpha}$.

Indeed let $(\bar p,\bar q)\in\Delta(I)\times\Delta(J)$ be a pair for which the sum of the number of indices i such that $\bar p_i=0$ and of indices j such that $\bar q_j=0$ is maximal among all the pairs (p,q) such that $(\bar t,\bar x,\bar s,\bar y,p,q)$ is a maximum point in (4.19). Let I' and J' be the sets of i and j for which $\bar p_i>0$ and $\bar q_j>0$. We note that $(\bar p,\bar q)\in\Delta(I')\times\Delta(J')$ and so $(\bar t,\bar x,\bar s,\bar y,\bar p,\bar q)$ is also a maximum point of (4.22). Therefore $M'_{\epsilon,\eta,\alpha}=M_{\epsilon,\eta,\alpha}$. To prove (ii), we assume that contrary to our claim there is some $(\bar p',\bar q')$ such that $(\bar t,\bar x,\bar s,\bar y,\bar p',\bar q')$ is maximum point of (4.22) and there is some $i_0 \in I'$ with $\bar{p}'_{i_0} = 0$ (for instance). Then $(\bar{t}, \bar{x}, \bar{s}, \bar{y}, \bar{p}', \bar{q}')$ is a also maximum point in (4.19) and the total number of indices of i or j for which $\bar{p}'_i = 0$ or $\bar{q}'_j = 0$ is larger than the corresponding number for $(\bar{p}, \bar{q})$. This is in contradiction with the definition of $(\bar{p}, \bar{q})$.

From now on, for fixed ϵ, η and α, we replace the sets I and J by I' and J', the maps w_1 and w_2 by their restrictions to $I' \times J'$ and problem (4.19) by (4.22). We note that the new w_1 and w_2 are respectively sub- and supersolutions (from Corollary 4.4) and satisfy: for any $(\bar{p}, \bar{q})$ such that $(\bar{t}, \bar{x}, \bar{s}, \bar{y}, \bar{p}, \bar{q})$ is a maximum point of (4.22), one has $(\bar{p}, \bar{q}) \in Int(\Delta(I) \times \Delta(J))$.

From the maximum principle (Theorem 6.1 stated in the Appendix), there are $(\bar{p}, \bar{q})$, $(\hat{p}, \hat{q})$ and $X_1, X_2 \in \mathcal{S}_n$ such that

$$(-\frac{(\bar{t}-\bar{s})}{\epsilon} + \eta, -\frac{(\bar{x}-\bar{y})}{\epsilon} - \alpha\bar{x}, X_1, \bar{q}) \in \overline{D}_q^{2,-} w_1^\sharp(\bar{t}, \bar{x}, \bar{p}, \hat{q}) \ ,$$

$$(\frac{(\bar{s}-\bar{t})}{\epsilon}, \frac{(\bar{y}-\bar{x})}{\epsilon} + \alpha\bar{y}, X_2, \bar{p}) \in \overline{D}_p^{2,+} w_2^*(\bar{s}, \bar{y}, \hat{p}, \bar{q})$$

and

$$\begin{pmatrix} -X_1 & 0 \\ 0 & X_2 \end{pmatrix} \le (\frac{3}{\epsilon} + 2\alpha) \begin{pmatrix} I & -I \\ -I & I \end{pmatrix} + (\alpha + \alpha^2\epsilon)I \tag{4.23}$$

Since w_1 is a subsolution of (4.17) in the dual sense, Proposition 4.5 states that

$$\eta - \frac{\bar{t}-\bar{s}}{\epsilon} - H\left(\bar{t}, \bar{x}, \frac{\bar{x}-\bar{y}}{\epsilon} + \alpha\bar{x}, -X_1, \bar{p}, \bar{q}\right) \le 0 \ . \tag{4.24}$$

In the same way, since w_2 is a supersolution of (4.17) in the dual sense, we have

$$\frac{(\bar{s}-\bar{t})}{\epsilon} - H\left(\bar{s}, \bar{y}, -\frac{(\bar{y}-\bar{x})}{\epsilon} + \alpha\bar{y}, -X_2, \bar{p}, \bar{q}\right) \ge 0 \ , \tag{4.25}$$

Using the structure condition (4.16) on H, and plugging estimates (4.20), (4.21) and (4.23) into (4.24) and (4.25) yields to a contradiction for ϵ, α and η sufficiently small as in [9]. □

5 Games with running cost

We now investigate differential games with asymmetric information on the running cost and on the terminal cost. The framework is basically the same as before. At the initial time, the cost (now consisting in a running cost and a terminal one) is chosen at random among

$I \times J$ possible costs. The index i is announced to Player I while the index j is announced to Player II. Then the players play the game in order, for Player I to minimize the payoff and for Player II to maximize it.

In this section we keep the same terminology and the same notations as in the previous part. There is however a main difference: as we shall see later, in a game with a running cost, each player needs the knowledge of this running cost to build his strategy. Since we assume that the running cost depends on the control of both players, this means that the players have to observe the control of their opponent. This was not the case of the game before where the players only observed the state of the system. For this reason we have to change the notion of strategy: in this section the notion of strategy introduced in Definition 2.2 is replaced by the following one:

Definition 5.1 *A* strategy *for player I starting at time t is a Borel-measurable map $\alpha : [t,T] \times \mathcal{C}([t,T], \mathbb{R}^n) \times L^2([t,T], V) \to U$ for which there exists $\delta > 0$ such that, for all $s \in [t,T]$, $f, f' \in \mathcal{C}([t,T], \mathbb{R}^n)$ and $g, g' \in L^2([t,T], V)$, if $f = f'$ and $g = g'$ a.e. on $[t,s]$, then $\alpha(\cdot, f, g) = \alpha(\cdot, f', g')$ on $[t, s+\delta]$.*
We define strategies for player II in a symmetric way and denote by $\mathcal{A}(t)$ (resp. $\mathcal{B}(t)$) the set of strategies for player I (resp. player II).

We define random strategies as before (but with the modified notion of strategies) and still denote by $\mathcal{A}_r(t)$ (resp. $\mathcal{B}_r(t)$) the set of random strategies for player I (resp. player II).
We have an analogue of Lemma 2.1 :

Lemma 5.1 *For all (t,x) in $[0,T] \times \mathbb{R}^n$, for all $(\alpha, \beta) \in \mathcal{A}(t) \times \mathcal{B}(t)$, there exists a unique couple of controls $(u,v) \in \mathcal{U}(t) \times \mathcal{V}(t)$ that satisfies $P-$a.s.*

$$(u,v) = (\alpha(\cdot, X_\cdot^{t,x,u,v}, v.), \beta(\cdot, X_\cdot^{t,x,u,v}, u.)) \text{ a.e. on } [t,T]. \tag{5.26}$$

One can easily check that the results of the previous parts (i.e., 2.2, Proposition 2.1, Corollary 3.1 and Proposition 3.2) still hold true with the modified notion of strategy. In particular, the game with terminal payoff studied before has a value.
Let us fix $I, J \in \mathbb{N}$. For $1 \leq i \leq I$ and $1 \leq j \leq J$ we consider the terminal cost $g_{ij} : \mathbb{R}^n \to \mathbb{R}$ and the running cost $\ell_{ij} : [0,T] \times \mathbb{R}^n \times U \times V \to \mathbb{R}$ on which we do the following assumptions:

$$\begin{array}{c}\text{For any } 1 \leq i \leq I \text{ and } 1 \leq j \leq J,\ \ell_{ij} \text{ and } g_{ij} \text{ are continuous in all variables,}\\ \text{uniformly Lipschitz continuous with respect to } x \text{ and bounded.}\end{array} \tag{5.27}$$

For fixed $(i,j) \in \{1,\ldots,I\} \times \{1,\ldots,J\}$ and strategies $(\alpha,\beta) \in \mathcal{A}(t) \times \mathcal{B}(t)$, we set

$$J_{ij}(t,x,\alpha,\beta) = E[\int_t^T \ell_{ij}(s, X_s^{t,x,\alpha,\beta}, \alpha_s, \beta_s)ds + g_{ij}(X_T^{t,x,\alpha,\beta})] ,$$

where as before (α,β) denotes the unique pair of controls such that (5.26) holds.
The payoff of two random strategies $(\overline{\alpha},\overline{\beta}) \in \mathcal{A}_r(t) \times \mathcal{B}_r(t)$, with $\overline{\alpha} = (\alpha^1,\ldots,\alpha^R; r^1,\ldots,r^R)$ and $\overline{\beta} = (\beta^1,\ldots,\beta^S; s^1,\ldots,s^S)$, is the average of the payoffs with respect to the probability distributions associated to the strategies:

$$J_{ij}(t,x,\overline{\alpha},\overline{\beta}) = \sum_{k=1}^{R}\sum_{l=1}^{S} r^k s^l E[\int_t^T \ell_{ij}(s, X_s^{t,x,\alpha^k,\beta^l}, \alpha_s^k, \beta_s^l)ds + g_{ij}(X_T^{t,x,\alpha^k,\beta^l})].$$

Finally, the payoff of the game is, for $(\hat{\alpha},\hat{\beta}) = ((\bar{\alpha}_i)_{1\le i\le I}, (\bar{\beta}_j)_{1\le j\le J}) \in (\mathcal{A}_r(t))^I \times (\mathcal{B}_r(t))^J$,

$$J^{p,q}(t,x,\hat{\alpha},\hat{\beta}) = \sum_{i=1}^{I}\sum_{j=1}^{J} p_i q_j J_{ij}(t,x,\overline{\alpha}_i,\overline{\beta}_j).$$

We define the value functions for the game with running cost as before by

$$\begin{array}{c} V^+(t,x,p,q) = \inf_{\hat{\alpha}\in(\mathcal{A}_r(t))^I} \sup_{\hat{\beta}\in\mathcal{B}_r(t))^J} J^{p,q}(t,x,\hat{\alpha},\hat{\beta}), \\ V^-(t,x,p,q) = \sup_{\hat{\beta}\in(\mathcal{B}_r(t))^J} \inf_{\hat{\alpha}\in(\mathcal{A}_r(t))^I} J^{p,q}(t,x,\hat{\alpha},\hat{\beta}). \end{array}$$

In our game with running cost, Isaacs' assumption takes the following form: for all $(t,x) \in [0,T] \times \mathbb{R}^n$, $(p,q) \in \Delta(I) \times \Delta(J)$, $\xi \in \mathbb{R}^n$, and all $A \in \mathcal{S}_n$:

$$\begin{array}{r} \inf_u \sup_v \{< b(t,x,u,v),\xi > + \frac{1}{2}Tr(A\sigma(t,x,u,v)\sigma^*(t,x,u,v)) + sum_{i,j}\ell_{ij}(t,x,u,v)p_iq_j\} = \\ \sup_v \inf_u \{< b(t,x,u,v),\xi > + \frac{1}{2}Tr(A\sigma(t,x,u,v)\sigma^*(t,x,u,v)) + sum_{i,j}\ell_{ij}(t,x,u,v)p_iq_j\} \end{array} \tag{5.28}$$

We set

$$\begin{array}{rl} H(t,x,\xi,A,p,q) = & \inf_u \sup_v \{< b(t,x,u,v),\xi > \\ & + \frac{1}{2}Tr(A\sigma(t,x,u,v)\sigma^*(t,x,u,v)) - \sum_{i,j}\ell_{ij}(t,x,u,v)p_iq_j\} \ . \end{array}$$

Theorem 5.2 *Assume that (H), (5.27) and (5.28) hold. Then the game has a value: $V^+ = V^-$, which is the unique solution in the dual sense of*

$$w_t + H(t,x,Dw,D^2w,p,q) = 0 \tag{5.29}$$

with terminal condition

$$V^+(T,x,p,q) = V^-(T,x,p,q) = \sum_{i=1}^{I}\sum_{j=1}^{J} p_i q_j g_{ij}(x) \qquad \forall (x,p,q) \in \mathbb{R}^n \times \Delta(I) \times \Delta(J) \ .$$

Proof of Theorem 5.2 : Following standard arguments, one first checks that V^+ and V^- are globally Hölder continuous, and uniformly Lipschitz continuous with respect to p and q. We now show that V^- is a dual supersolution of the HJ equation

$$w_t + H^-(t,x,Dw,D^2w,p,q) = 0 \tag{5.30}$$

where

$$\begin{array}{rl} H^-(t,x,\xi,A,p,q) = & \sup_{v\in V}\inf_{u\in U}\{< b(t,x,u,v),\xi > \\ & +\frac{1}{2}Tr(A\sigma(t,x,u,v)\sigma^*(t,x,u,v)) + sum_{i,j}\ell_{ij}(t,x,u,v)p_iq_j\} \end{array}$$

The proof that V^+ is a dual subsolution of the HJ equation

$$w_t + H^+(t,x,Dw,D^2w,p,q) = 0$$

where

$$\begin{array}{rl} H^+(t,x,\xi,A,p,q) = & \inf_{u\in U}\sup_{v\in V}\{< b(t,x,u,v),\xi > \\ & +\frac{1}{2}Tr(A\sigma(t,x,u,v)\sigma^*(t,x,u,v)) + sum_{i,j}\ell_{ij}(t,x,u,v)p_iq_j\} \end{array}$$

can be achieved in the same way.

In order to prove that V^- is a dual supersolution, we introduce an extended differential game in $\mathbb{R}^{n+IJ}$. This game with asymmetric information and terminal payoff is defined by the dynamics

$$\begin{array}{l} dX_s = b(s,X_s,u_s,v_s)ds + \sigma(s,X_s,u_s,v_s)dB_s,\ s\in[t,T], \\ dZ_{ij,s} = \ell_{ij}(s,X_s,u_s,v_s)ds, \\ X_t = x,\ Z_{ij,t} = z_{ij}, \end{array} \tag{5.31}$$

where $(t,x,z)\in[0,T]\times\mathbb{R}^n\times\mathbb{R}^{IJ}$, with $z=(z_{ij})$. The terminal payoffs of this new game are $\tilde{g}_{ij}(x,z) = z_{ij} + g_{ij}(x)$. We denote by $\tilde{V}^-$ the lower value of this game. We note that

$$\tilde{V}^-(t,x,z,p,q) = V^-(t,x,p,q) + \sum_{ij} z_{ij}p_iq_j\ . \tag{5.32}$$

Following the proofs of Proposition 2.1, one can check that $\tilde{V}^-$ is convex in p and concave in q. Hence so is V^-. As in Corollary 3.1, one can also show that $\tilde{V}^-$ is a dual supersolution of the HJ equation

$$\tilde{w}_t + \tilde{H}^-(t,x,z,D_{x,z}w,D_x^2w) = 0$$

where, for $(t,x,z)\in\mathbb{R}^{n+IJ}$, $\xi_x\in\mathbb{R}^n$, $\xi_z\in\mathbb{R}^{IJ}$ and $A\in\mathcal{S}_n$,

$$\begin{array}{rl} \tilde{H}^-(t,x,z,\xi_x,\xi_z,A) = & \sup_{v\in V}\inf_{u\in U}\{< b(t,x,u,v),\xi_x > \\ & +\frac{1}{2}Tr(A\sigma(t,x,u,v)\sigma^*(t,x,u,v)) + \sum_{i,j}\ell_{ij}(t,x,u,v)\xi_{z,ij}\}\ . \end{array}$$

Note that it is precisely at this point that the players have to use the new definition of strategies. Indeed, in order to build their strategies in the sub- and superdynamic programming, they have to compute the state of the system and the running costs Z_{ij} (see the proof of Proposition 3.1). This is possible since, at time s, they know the controls $u.$ and $v.$ and the trajectory $X.$ up to time $s-\delta$, and therefore can compute $Z_{ij,r} = z_{ij} + \int_{t_0}^{r} \ell_{ij}(\tau, X_\tau, u_\tau, v_\tau)d\tau$ for $r \in (t_0, s-\delta)$.

Let now ϕ be a test function such that $(t,x) \to V^{-*}(t,x,\hat{p},\bar{q}) - \phi(t,x)$ has a maximum at some point $(\bar{t},\bar{x})$ for some $(\hat{p},\bar{q}) \in \mathbb{R}^I \times \Delta(J)$ at which $\frac{\partial V^{-*}}{\partial \hat{p}}$ exists. Without loss of generality we can assume that this maximum is global and strict and that $\phi \to +\infty$ as $|x| \to +\infty$. From (5.32) we have

$$V^{-*}(t,x,\hat{p}',\bar{q}) = \tilde{V}^{-*}\left(t,x,z,\hat{p}' + (\sum_j \bar{q}_j z_{ij})_i, \bar{q}\right) \qquad \forall (t,x,z,\hat{p}') \ . \tag{5.33}$$

In particular, $\frac{\partial \tilde{V}^{-*}}{\partial \hat{p}}$ exists at $(\bar{t},\bar{x},0,\hat{p},\bar{q})$ and is equal to $\bar{p} := \frac{\partial V^{-*}}{\partial \hat{p}}(\bar{t},\bar{x},\hat{p},\bar{q})$. Equality (5.33) also implies that the map

$$(t,x,z) \to \tilde{V}^{-*}\left(t,x,z,\hat{p} + (\sum_j \bar{q}_j z_{ij})_i, \bar{q}\right) - \phi(t,x)$$

has a maximum at $(\bar{t},\bar{x},0)$ in $(0,T)\times\mathbb{R}^n\times\mathbb{R}^{IJ}$. Fix $\varepsilon > 0$ and let $(t_\varepsilon, x_\varepsilon, z_\varepsilon)$ be a maximum point of the map $(t,x,z) \to \tilde{V}^{-*}(t,x,z,\hat{p},\bar{q}) - \phi(t,x) - \frac{1}{2\varepsilon}|z|^2$ (we note that this maximum exists because $\tilde{V}^{-*}$ has at most a linear growth and $\phi \to +\infty$ as $|x| \to +\infty$). From standard perturbation arguments, $\frac{1}{\varepsilon}|z_\varepsilon|^2$ converges to 0 as $\varepsilon \to 0$. Therefore, up to subsequences, $(t_\varepsilon, x_\varepsilon)$ converge to a maximum point of the map $(t,x) \to \tilde{V}^{-*}(t,x,0,\hat{p},\bar{q}) - \phi(t,x)$, i.e., to $(\bar{t},\bar{x})$. So we have $(t_\varepsilon, x_\varepsilon) \to (\bar{t},\bar{x})$.

Since $\tilde{V}^{-*}$ is a subsolution of the dual equation, we have at $(t_\varepsilon, x_\varepsilon, z_\varepsilon, \hat{p}, \bar{q})$

$$\phi_t - \inf_u \sup_v \left\{ -\langle b, D\phi\rangle - \frac{1}{2}Tr(D^2\phi\sigma\sigma^*) - \sum_{i,j} \ell_{ij}\frac{z_{\varepsilon,ij}}{\varepsilon} \right\} \geq 0 \ . \tag{5.34}$$

We now claim that

$$\lim_{\varepsilon\to 0} \frac{z_{\varepsilon,ij}}{\varepsilon} = -\bar{p}_i \bar{q}_j \tag{5.35}$$

Indeed, from (5.33) and the optimality of $(t_\varepsilon, x_\varepsilon, z_\varepsilon)$, we have for any z

$$\begin{aligned} \tilde{V}^{-*}\left(t_\varepsilon, x_\varepsilon, z_\varepsilon, \hat{p} - (\textstyle\sum_j \bar{q}_j (z_{ij} - z_{\varepsilon,ij}))_i, \bar{q}\right) &= \tilde{V}^{-*}(t_\varepsilon, x_\varepsilon, z, \hat{p}, \bar{q}) \\ &\leq \ \tilde{V}^{-*}(t_\varepsilon, x_\varepsilon, z_\varepsilon, \hat{p}, \bar{q}) + \tfrac{1}{2\varepsilon}\left(|z|^2 - |z_\varepsilon|^2\right) \ . \end{aligned}$$

Let j_0 with $\bar{q}_{j_0} > 0$ and $h \in \mathbb{R}^I$. If we choose $z = (z_{ij})$ such that

$$z_{ij} = z_{\varepsilon,ij} \text{ if } j \neq j_0 \qquad \text{and} \qquad z_{ij_0} = z_{\varepsilon,ij_0} - \frac{h_i}{\bar{q}_{j_0}} \ ,$$

then we get

$$\tilde{V}^{-*}(t_\varepsilon, x_\varepsilon, z_\varepsilon, \hat{p}+h, \bar{q}) \leq \tilde{V}^{-*}(t_\varepsilon, x_\varepsilon, z, \hat{p}, \bar{q}) + \frac{1}{2\varepsilon}\left(-2\sum_i z_{\varepsilon,ij_0}\frac{h_i}{\bar{q}_{j_0}} + \frac{|h|^2}{\bar{q}_{j_0}^2}\right) .$$

Since $\tilde{V}^{-*}$ is convex with respect to $\hat{p}$, this shows that $\tilde{V}^{-*}$ has a derivative with respect to $\hat{p}$ at $(t_\varepsilon, x_\varepsilon, z_\varepsilon, \hat{p}, \bar{q})$ with

$$\frac{\partial \tilde{V}^{-*}}{\partial \hat{p}_i}(t_\varepsilon, x_\varepsilon, z_\varepsilon, \bar{q}) = -\frac{z_{\varepsilon,ij_0}}{\varepsilon \bar{q}_{j_0}} .$$

As $\varepsilon \to 0$, $\frac{\partial \tilde{V}^{-*}}{\partial \hat{p}}(t_\varepsilon, x_\varepsilon, z_\varepsilon, \hat{p}, \bar{q})$ converges (up to some subsequence) to some element of $\partial^-_{\hat{q}} \tilde{V}^{-*}(\bar{t}, \bar{x}, 0, \hat{p}, \bar{q})$, which is reduced to $\{\bar{p}\}$ since $\frac{\partial \tilde{V}^{-*}}{\partial \hat{p}}$ exists at $(\bar{t}, \bar{x}, 0, \hat{p}, \bar{q})$ and equal to $\bar{p}$. Hence $\frac{z_{\varepsilon,ij_0}}{\varepsilon} \to -\bar{p}_i \bar{q}_{j_0}$. Next we assume that $\bar{q}_{j_0} = 0$. Choosing

$$z_{ij} = z_{\varepsilon,ij} \text{ if } j \neq j_0 \text{ and } z_{ij_0} = 0 \ ,$$

we get from (5.30)

$$\tilde{V}^{-*}(t_\varepsilon, x_\varepsilon, z_\varepsilon, \hat{p}, \bar{q}) \leq \tilde{V}^{-*}(t_\varepsilon, x_\varepsilon, z_\varepsilon, \hat{p}, \bar{q}) + \frac{1}{2\varepsilon}\left(-\sum_i z_{\varepsilon,ij_0}^2\right) .$$

Hence $z_{\varepsilon,ij_0} = 0$ and $\frac{z_{\varepsilon,ij_0}}{\varepsilon} \to -\bar{p}_i \bar{q}_{j_0}$. So in any case (5.35) holds.

Plugging (5.35) into (5.34) and then letting $\varepsilon \to 0$ finally yields to the desired inequality:

$$\phi_t - H^-(t, x, -Dw, -D^2 w, \bar{p}, \bar{q}) \geq 0 \text{ at } (\bar{t}, \bar{x}, \bar{p}, \bar{q}) .$$

So V^- is a dual supersolution of (5.30).

Combining Isaacs' assumption, which states that $H := H^+ = H^-$, the fact that H satisfies assumption (4.16) and the comparison principle (Theorem 4.1) shows that $V^+ = V^-$ is the unique dual solution of (5.29). □

6 Appendix : A maximum principle

The following result—used in a crucial way in the proof of the comparison principle—is an adaptation to our framework of the maximum principle for semicontinuous functions (see Theorem 3.2 of [9]):

Theorem 6.1 (Maximum principle) *For $k = 1, 2$, let $\mathcal{O}_k$ be open subsets of $\mathbb{R}^{n_k}$ and $w_k : \mathcal{O}_k \times \Delta(I) \times \Delta(J) \to \mathbb{R}$ be such that*

(i) $w_1 = w_1(x, p, q)$ is upper semicontinuous in all variables, convex with respect to p and uniformly Lipschitz continuous with respect to q,

(ii) $w_2 = w_2(y, p, q)$ is lower semicontinuous in all its variables, concave with respect to q and uniformly Lipschitz continuous with respect to p,

(iii) there is some $\mathcal{C}^2$ map $\phi : \mathcal{O}_1 \times \mathcal{O}_2 \to \mathbb{R}$ and some point $(\bar{x}, \bar{y}) \in \mathcal{O}_1 \times \mathcal{O}_2$ such that the map

$$(x, y) \to \max_{p,q} \{w_1(x, p, q) - w_2(y, p, q) - \phi(x, y)\} \tag{6.36}$$

has a maximum at $(\bar{x}, \bar{y})$,

(iv) any maximum point (p, q) in (6.36) with $(x, y) = (\bar{x}, \bar{y})$ belongs to the interior of $\Delta(I) \times \Delta(J)$.

Then, for any $\epsilon > 0$ small, there are $(\bar{p}, \bar{q}) \in \Delta(I) \times \Delta(J)$, $(\hat{p}, \hat{q}) \in \mathbb{R}^I \times \mathbb{R}^J$ and $(X_1, X_2) \in \mathcal{S}_{n_1} \times \mathcal{S}_{n_2}$ such that the map

$$(x, y, p, q) \to w_1(x, p, q) - w_2(y, p, q) - \phi(x, y)$$

has a maximum at $(\bar{x}, \bar{y}, \bar{p}, \bar{q})$,

$$(-D_x\phi(\bar{x}, \bar{y}), X_1, \bar{q}) \in \overline{D}_q^{2,-} w_1^\sharp(\bar{x}, \bar{p}, \hat{q}), \ (D_y\phi(\bar{x}, \bar{y}), X_2, \bar{p}) \in \overline{D}_p^{2,+} w_2^*(\bar{y}, \hat{p}, \bar{q}) \tag{6.37}$$

and

$$\left(\frac{1}{\epsilon} + \|A\|\right) I \le \begin{pmatrix} -X_1 & 0 \\ 0 & X_2 \end{pmatrix} \le A + \epsilon A^2 \tag{6.38}$$

with $A = D^2\phi(\bar{x}, \bar{y})$.

Remark : Compared with the classical maximum principle, the additional difficulty is the fact that we need elements of $\overline{D}_q^{2,-} w_1^\sharp$ and of $\overline{D}_p^{2,+} w_2^*$ while we have only information on the behavior of the difference $w_1 - w_2 - \phi$.

Proof of Theorem 6.1 : We follow closely the proof of Theorem 3.2 of [9]. Let us start by some reductions:

Reductions : As in [9], we can assume without loss of generality that $\mathcal{O}_k = \mathrm{I\!R}^{n_k}$, w_1 is bounded from above and w_2 is bounded from below. We can also assume that $\bar{x} = \bar{y} = 0$ and $\phi(x,y) = \langle A(x,y),(x,y)\rangle$ and

$$\max_{x,y,p,q} \{w_1(x,p,q) - w_2(y,p,q) - \langle A(x,y),(x,y)\rangle\} = 0\ . \tag{6.39}$$

Step 1 : introduction of the inf- and supconvolutions. As in [9], we have

$$(w_1(x',p,q) - \frac{\lambda}{2}|x'-x|^2) - (w_2(y',p,q) - \frac{\lambda}{2}|y'-y|^2) \leq \langle (A+\epsilon A^2)(x,y),(x,y)\rangle \tag{6.40}$$

for any (x,x',y,y',p,q), where $\lambda = \frac{1}{\epsilon} + \|A\|$. Let us set for $\lambda' > \lambda$,

$$\hat{w}_1(x,p,q) = \max_{x' \in \mathrm{I\!R}^{n_1},\, q' \in \Delta(J)} (w_1(x',p,q') - \frac{\lambda}{2}|x'-x|^2 - \frac{\lambda'}{2}|q'-q|^2)$$

where $(x,p,q) \in \mathrm{I\!R}^{n_1} \times \Delta(I) \times \mathrm{I\!R}^J$ and

$$\hat{w}_2(y,p,q) = \min_{y' \in \mathrm{I\!R}^{n_2},\, p' \in \Delta(I)} (w_2(y',p',q) + \frac{\lambda}{2}|y'-y|^2 + \frac{\lambda'}{2}|p'-p|^2)$$

where $(y,p,q) \in \mathrm{I\!R}^{n_2} \times \mathrm{I\!R}^I \times \Delta(J)$. Note that $\hat{w}_1$ and $\hat{w}_2$ are define in a larger space than w_1 and w_2 and that

$$\hat{w}_1(q) \leq C - \frac{\lambda'}{2}|q|^2 \qquad \text{and} \qquad \hat{w}_2(p) \geq -C + \frac{\lambda'}{2}|p|^2$$

for some constant C. In particular,

$$\hat{w}_1^\sharp(x,p,\hat{q}) = \inf_{q \in \mathrm{I\!R}^J} \{\hat{q}.q - \hat{w}_1(x,p,q)\}$$

is well defined and so is $\hat{w}_1^*$.

We also note that $\hat{w}_1$ is semiconvex in all its variables with a modulus λ', semiconvex in x with a modulus λ and convex in p (because w_1 is convex in p by assumption). In the same way, $\hat{w}_2$ is semiconcave in all its variables with a modulus λ', semiconvex in y with a modulus λ and concave in q.

Step 2 : localization of the maxima of a perturbed problem. Let us now introduce some small pertubation of the maximization problem (6.39): for $\alpha, \beta > 0$ and $\zeta = (\zeta_x, \zeta_y, \zeta_p, \zeta_q) \in \mathrm{I\!R}^{n_1+n_2+I+J}$, we set

$$\begin{aligned} z_\zeta(x,y,p,q) \;=\; & \hat{w}_1(x,p,q) - \hat{w}_2(y,p,q) - \langle (A+\epsilon A^2)(x,y),(x,y)\rangle \\ & -\alpha(|p|^2 - |q|^2) - \beta(|x|^2 + |y|^2) - \langle \zeta, (x,y,p,q)\rangle\ . \end{aligned}$$

We claim that,

$$\begin{array}{c}\text{for any } \eta > 0, \text{ we can choose } \lambda' \text{ large and } \gamma \text{ and } \alpha \text{ small such that,}\\ \text{for any } \zeta \text{ with } |\zeta| \leq \gamma, \text{ any maximum point } (x, y, p, q) \text{ of } z_\zeta \text{ in } \mathbb{R}^{n_1+n_2} \times \Delta(I) \times \Delta(J)\\ \text{satisfies } (x, y) \in B_\eta \text{ and } (p, q) \in Int(\Delta(I) \times \Delta(J)).\end{array} \tag{6.41}$$

To prove our claim, we introduce the maps

$$\hat{w}_1'(x, p, q) = \max_{x' \in \mathbb{R}^{n_1}} (w_1(x', p, q) - \frac{\lambda}{2}|x' - x|^2)$$

and

$$\hat{w}_2'(y, p, q) = \min_{y' \in \mathbb{R}^{n_2}} (w_2(y', p, q) + \frac{\lambda}{2}|y' - y|^2) \ .$$

From (6.40) we have

$$\hat{w}_1'(x, p, q) - \hat{w}_2'(y, p, q) - \langle (A + \epsilon A^2)(x, y), (x, y) \rangle \leq 0 \qquad \forall (x, y, p, q) \ . \tag{6.42}$$

Since w_1 is Lipschitz continuous with respect to q while w_2 is Lipschitz continous with respect to p, we have

$$\hat{w}_1(x, p, q) \leq \hat{w}_1'(x, p, q) + \frac{C}{\lambda'} \qquad \text{and} \qquad \hat{w}_2(x, p, q) \geq \hat{w}_2'(x, p, q) - \frac{C}{\lambda'} \ ,$$

for some constant C, so that, in view of (6.42),

$$z_\zeta(x, y, p, q) \leq \frac{2C}{\lambda'} - \alpha(|p|^2 - |q|^2) - \beta(|x|^2 + |y|^2) - \langle \zeta, (x, y, p, q) \rangle \ . \tag{6.43}$$

Let $\lambda_n' \to +\infty$, $\zeta_n, \alpha_n \to 0$ and (x_n, y_n, p_n, q_n) be a maximum point of z_{ζ_n}. From (6.43) the sequence (x_n, y_n, p_n, q_n) is bounded and we can assume that it converges to some $(\bar{x}, \bar{y}, \bar{p}, \bar{q})$ which is a maximum point of

$$z_0(x, y, p, q) := \hat{w}_1'(x, p, q) - \hat{w}_2'(y, p, q) - \langle (A + \epsilon A^2)(x, y), (x, y) \rangle - \beta(|x|^2 + |y|^2)$$

We are going to show that $(\bar{x}, \bar{y}) = (0, 0)$ and that $(\bar{p}, \bar{q}) \in Int(\Delta(I) \times \Delta(J))$, which proves claim (6.41). Let $(0, 0, p, q)$ be a maximum point in (6.39). Since $\hat{w}_1' \geq w_1$ and $\hat{w}_2' \leq w_2$, we have,

$$z_0(0, 0, p, q) \geq w_1(0, p, q) - w_2(0, p, q) - \phi(0, 0) = 0 \ .$$

Combining this inequality with (6.42) shows that

$$\max z_0 = 0 \qquad \text{and} \qquad (\bar{x}, \bar{y}) = (0, 0) \ .$$

We now show that $w_1(0,\bar p,\bar q)=\hat w_1'(0,\bar p,\bar q)$ and $w_2(0,\bar p,\bar q)=\hat w_2'(0,\bar p,\bar q)$. Indeed, let x',y' be such that

$$\hat w_1'(0,\bar p,\bar q)=w_1(x',\bar p,\bar q)-\frac{\lambda}{2}|x'|^2 \qquad \text{and} \qquad \hat w_2'(0,\bar p,\bar q)=w_2(y',\bar p,\bar q)+\frac{\lambda}{2}|y'|^2 \ .$$

Then

$$\hat w_1'(x',\bar p,\bar q)-\hat w_2'(y',\bar p,\bar q)\geq w_1(x',\bar p,\bar q)-w_2(y',\bar p,\bar q)=\hat w_1'(0,\bar p,\bar q)-\hat w_2'(0,\bar p,\bar q)+\frac{\lambda}{2}(|x'|^2+|y'|^2)$$

But from the maximality of $(0,0,\bar p,\bar q)$ we also have

$$\hat w_1'(0,\bar p,\bar q)-\hat w_2'(0,\bar p,\bar q)\geq \hat w_1'(x',\bar p,\bar q)-\hat w_2'(y',\bar p,\bar q)-\langle (A+\epsilon A^2)(x',y'),(x',y')\rangle-\beta(|x'|^2+|y'|^2)$$

Putting together the two inequalities above shows that $x'=y'=0$ as soon as $\lambda>2(\|A\|+\epsilon\|A\|^2+\beta)$, i.e., from the definition of λ, as soon as ϵ is small enough. Then $w_1(0,\bar p,\bar q)=\hat w_1'(0,\bar p,\bar q)$ and $w_2(0,\bar p,\bar q)=\hat w_2'(0,\bar p,\bar q)$ and the maximality of $(0,0,\bar p,\bar q)$ gives

$$w_1(0,\bar p,\bar q)-w_2(0,\bar p,\bar q)=\hat w_1'(0,\bar p,\bar q)-\hat w_2'(0,\bar p,\bar q)\geq \hat w_1'(0,p,q)-\hat w_2'(0,p,q)\geq w_1(0,p,q)-w_2(0,p,q)$$

for any (p,q). Hence $(0,0,\bar p,\bar q)$ is a maximum point in (6.39). This shows from assumption (iv) that $(\bar p,\bar q)\in Int(\Delta(I)\times\Delta(J))$ and completes the proof of claim (6.41).

Step 3 : use of Jensen maximum principle. Let λ', γ and α as above. Since z_0 is semiconvex, has a maximum at $(0,0,p,q)$, Jensen maximum principle (see Lemma A.3 of [9] for instance) states that the set

$$E_\gamma=\left\{\begin{array}{l}(x,y,p,q)\in B_\eta\times Int(\Delta(I)\times\Delta(J))\ ,\ \exists\zeta\ ,\ |\zeta|\leq\gamma\ ,\ \text{such that}\\ \qquad (i)\quad z_\zeta \text{ has a maximum at } (x,y,p,q) \text{ and}\\ \qquad (ii)\quad \hat w_1 \text{ and } \hat w_2 \text{ have a derivative at } (x,y,p,q)\end{array}\right\}$$

has a positive measure. We note that in the quoted Lemma A.3, the maximum is required to be strict ; this assumption is only used in [9] to localize the maximum points, which is not needed here.

We also note for later use that, if $(x,y,p,q)\in E_\gamma$, there is some $\zeta=(\zeta_x,\zeta_y,\zeta_p,\zeta_q)$ with $|\zeta|\leq\gamma$ such that z_ζ has a maximum at (x,y,p,q). In particular, this implies that

$$q'\to \hat w_1(x,p,q')-\hat w_2(y,p,q')+\alpha|q'|^2-\langle\zeta_q,q'\rangle$$

has a maximum at q. Since $\hat w_2$ is concave in q, $\hat w_1$ coincides with its concave hull with respect to q at (x,p,q). Hence, if we set $\hat q=\frac{\partial \hat w_1(x,p,q)}{\partial q}$, then

$$\hat w_1(x,p,q)+\hat w_1^\sharp(x,p,\hat q)=q.\hat q \qquad \text{and} \qquad q\in\partial^+_{\hat q}\hat w_1^\sharp(x,p,\hat q)\ . \tag{6.44}$$

In the same way, if we set $\hat{p} = \frac{\partial \hat{w}_2(y,p,q)}{\partial p}$, then we have

$$\hat{w}_2(x,p,q) + \hat{w}_2^*(y,\hat{p},q) = p.\hat{p} \qquad \text{and} \qquad p \in \partial^-_{\hat{p}} \hat{w}_2^*(x,\hat{p},q) \ . \tag{6.45}$$

Step 4 : measure estimate of a subset of E_γ. Let E'_γ be the set of points $(x,y,p,q) \in E_\gamma$ such that $\hat{w}_1^\sharp$ has a second order Taylor expansion at $(x,p,\frac{\partial \hat{w}_1}{\partial q}(x,p,q))$ and $\hat{w}_2^*$ has a second order Taylor expansion at $(y,\frac{\partial \hat{w}_2}{\partial p}(x,p,q),q)$. Our aim is to show that E'_γ has a full measure in E_γ.

For this we note that $E'_\gamma = E^1_\gamma \cap E^2_\gamma$ where

$$E^1_\gamma = \left\{ \begin{array}{c} (x,y,p,q) \in E_\gamma \ , \ \hat{w}_1^\sharp \text{ has a second order Taylor expansion} \\ \text{at } (x,p,\frac{\partial \hat{w}_1}{\partial q}(x,p,q)) \end{array} \right\}$$

and

$$E^2_\gamma = \left\{ \begin{array}{c} (x,y,p,q) \in E_\gamma \ , \ \hat{w}_2^* \text{ has a second order Taylor expansion} \\ \text{at } (y,\frac{\partial \hat{w}_2}{\partial p}(x,p,q),q) \end{array} \right\}$$

It is enough to show that E^1_γ and E^2_γ have a full measure in E_γ. We only do the proof for E^1_γ, the proof for E^2_γ being symmetric.

Let us set, for any (x,y,p),

$$E_\gamma(x,y,p) = \{q \in \Delta(J) \ , \ (x,y,p,q) \in E_\gamma\}$$

and

$$E^1_\gamma(x,y,p) = \{q \in \Delta(J) \ , \ (x,y,p,q) \in E^1_\gamma\}$$

Since E_γ has a positive measure, from Fubini Theorem we have to show that, for any (x,y,p) such that the set $E_\gamma(x,y,p)$ has a positive measure, the set $E^1_\gamma(x,y,p)$ has a full measure in $E_\gamma(x,y,p)$.

For this, let us introduce the map $\Phi : q \to \frac{\partial \hat{w}_1(x,p,q)}{\partial q}$ defined on $E_\gamma(x,y,p)$. We are going to show that

$$\forall q_1, q_2 \in E_\gamma(x,y,p), \ |q_1 - q_2| \leq \frac{1}{2\alpha}|\Phi(q_1) - \Phi(q_2)| \ , \tag{6.46}$$

which will imply that

$$\forall E \subset E_\gamma(x,y,p) \text{ measurable}, \ \mathcal{L}^J(E) \leq \frac{1}{(2\alpha)^I}\mathcal{L}^J(\Phi(E)) \ , \tag{6.47}$$

where $\mathcal{L}^J$ denotes the Lebesgue measure in $\mathbb{R}^J$. Then we will prove that (6.47) implies our claim.

Proof of (6.46) : Let $q_1, q_2 \in E_\gamma(x,y,p)$. There are ζ_1 and ζ_2 such that z_{ζ_k} has a maximum at (x,y,p,q_k) for $k=1,2$. The first order optimality conditions imply that

$$\Phi(q_k) = \frac{\partial \hat{w}_2(y,p,q_k)}{\partial q} - 2\alpha q_k + \zeta_{k,q} \qquad \text{for } k=1,2.$$

Using again the optimality of z_{ζ_1} at q_1 and the fact that $q \to \hat{w}_2(y,p,q)$ is concave, we have

$$\begin{aligned}\hat{w}_1(x,p,q_2) &\leq \hat{w}_1(x,p,q_1) + \langle \left(\tfrac{\partial \hat{w}_2(y,p,q_k)}{\partial q} - 2\alpha q_1 + \zeta_{1,q}\right), (q_2-q_1)\rangle - \alpha|q_2-q_1|^2 \\ &\leq \hat{w}_1(x,p,q_1) + \langle \Phi(q_1), (q_2-q_1)\rangle - \alpha|q_2-q_1|^2\end{aligned}$$

Reversing the role of q_1 and q_2 gives

$$\hat{w}_1(x,p,q_1) \leq \hat{w}_1(x,p,q_2) + \langle \Phi(q_2), (q_1-q_2)\rangle - \alpha|q_2-q_1|^2$$

Adding the two previous inequalities then leads to

$$0 \leq \langle \Phi(q_2) - \Phi(q_1), q_1 - q_2\rangle - 2\alpha|q_2-q_1|^2 \ .$$

Whence (6.46).

Proof of (6.47) : Let E be a measurable subset of $E_\gamma(x,y,p)$. We note that (6.46) states that Φ is a bijection between E and its image, with a $\frac{1}{2\alpha}-$Lipschitz continuous inverse. Hence

$$\mathcal{L}^I(E) = \mathcal{L}^I(\Phi^{-1}(\Phi(E))) \leq \frac{1}{(2\alpha)^I}\mathcal{L}^I(\Phi(E)) \ ,$$

i.e., (6.47) holds.

We finally show that $E^1_\gamma(x,y,p)$ has a full measure in $E_\gamma(x,y,p)$ for any (x,y,p) such that $E_\gamma(x,y,p)$ has a positive measure. Let F be the set of $(x,p,\hat{q})$ such that $\hat{w}_1^\sharp$ has a second order Taylor expansion at $(x,p,\hat{q})$. Since F has a full measure, for almost all $(x,p) \in \mathbb{R}^n \times \Delta(I)$, the set $F(x,p) = \{\hat{q} \in \mathbb{R}^J \ , \ (x,p,\hat{q}) \in F\}$ has a full measure in $\mathbb{R}^J$. Let (x,p) be such a pair and such that $E_\gamma(x,y,p)$ has a positive measure. Then $\Phi(E_\gamma(x,y,p))$ also has a positive measure from (6.47). Since $\Phi(E_\gamma(x,y,p))\backslash F(x,p)$ has a zero measure and since

$$\Phi^{-1}\left(\Phi(E_\gamma(x,y,p))\backslash F(x,p)\right) = E_\gamma(x,y,p)\backslash E^1_\gamma(x,y,p) \ ,$$

using again (6.47) shows that $E_\gamma(x,y,p)\backslash E^1_\gamma(x,y,p)$ has a zero measure. This completes our claim.

Step 5 : (further) magic properties of sup-convolution. We now explain that one can use second order Taylor expansions of $\hat{w}_1^\sharp$ and $\hat{w}_2^*$ to get elements of $D^{2,-}w_1^\sharp$, $D^{2,+}\hat{w}_1$, $D^{2,+}w_2^*$ and $D^{2,-}\hat{w}_1$.

From the definition of $\hat{w}_1$ and $\hat{w}_1^\sharp$ we have

$$\hat{w}_1^\sharp(x,p,\hat{q}) \;=\; \min_{x',q',q"}(q'.\hat{q}+\frac{\lambda}{2}|x'-x|^2+\frac{\lambda'}{2}|q"-q'|^2-w_1(x',p,q")) \tag{6.48}$$

where the minimum is taken over the $(x',q',q") \in \mathbb{R}^{n_1}\times\mathbb{R}^J\times\Delta(J)$. In particular $q' = q" - \hat{q}/\lambda'$ is always optimal in (6.48) and we have

$$\hat{w}_1^\sharp(x,p,\hat{q}) \;=\; -\frac{|\hat{q}|^2}{2\lambda'}+\min_{x',q"}(q".\hat{q}+\frac{\lambda}{2}|x'-x|^2-w_1(x',p,q"))$$

So

$$\hat{w}_1^\sharp(x,p,\hat{q}) \;=\; -\frac{|\hat{q}|^2}{2\lambda'}+\min_{x'}(\frac{\lambda}{2}|x'-x|^2+w_1^\sharp(x',p,\hat{q})) \tag{6.49}$$

and $\hat{w}_1^\sharp$ is nothing but the inf-convolution in space of the map $w_1^\sharp$.

Let us now assume that $(x,y,p,q) \in E'_\gamma$, set $\hat{q} = \frac{\partial\hat{w}_1(x,p,q)}{\partial q}$ and let $(\bar{x}',\bar{q}',\bar{q}")$ be a minimum point in (6.48). Since $\hat{w}_1^\sharp$ has a second order Taylor expansion at $(x,p,\hat{q})$, we have, following the proof of the "magic properties of sup-convolution" (Lemma A.4 of [9]):

$$\xi := D\hat{w}_1^\sharp(x,p,\hat{q}) = \lambda(x-\bar{x}')\ ,\ q = \frac{\partial\hat{w}_1^\sharp}{\partial\hat{q}}(x,p,\hat{q}) = \bar{q}' \in Int(\Delta(I))$$

and $\hat{q}+\lambda'(\bar{q}'-\bar{q}") = 0$. This implies that the minimum problem in (6.48) has a unique solution $(\bar{x}',\bar{q}',\bar{q}") = (x-\xi/\lambda, q, q+\hat{q}/\lambda')$.

Using (6.49) we see that $q"$ is optimal in (6.48) if and only if $q" \in \partial^+_{\hat{q}}w_1^\sharp(\bar{x}',p,\hat{q})$. Since there is a unique optimal solution in (6.48), the set $\partial^+_{\hat{q}}w_1^\sharp(\bar{x}',p,\hat{q})$ is reduced to the singleton $\{q+\hat{q}/\lambda'\}$. So $\frac{\partial w_1^\sharp}{\partial\hat{q}}$ exists at $(x-\xi/\lambda,p,\hat{q})$ and we have

$$\frac{\partial w_1^\sharp}{\partial\hat{q}}(x-\xi/\lambda,p,\hat{q}) = q+\hat{q}/\lambda'\ . \tag{6.50}$$

Furthermore, since $\hat{w}_1^\sharp$ has a second order Taylor expansion in x at the point $(x,p,\hat{q})$, the classical "magic properties" of inf-convolution (Lemma A.4 of [9]) applied in (6.49) state that

$$(D\hat{w}_1^\sharp(x,p,\hat{q}), D^2\hat{w}_1^\sharp(x,p,\hat{q})) \in D^{2,-}w_1^\sharp(x-\xi/\lambda,p,\hat{q})\ . \tag{6.51}$$

Following [1] we also note that for any x' close to x, we have

$$\hat{w}_1(x',p,q) \le q.\hat{q}-\hat{w}_1^\sharp(x',p,\hat{q}) = \hat{w}_1(x,p,\hat{q})+\hat{w}_1^\sharp(x,p,\hat{q})-\hat{w}_1^\sharp(x',p,\hat{q})$$

because $\hat{w}_1(x,p,q)+\hat{w}_1^\sharp(x,p,\hat{q}) = q.\hat{q}$. Since $\hat{w}_1^\sharp$ has a second order Taylor expansion at x, this gives

$$-(D\hat{w}_1^\sharp(x,p,\hat{q}), D^2\hat{w}_1^\sharp(x,p,\hat{q})) \in D^{2,+}\hat{w}_1(x,p,q)\ . \tag{6.52}$$

In a symmetric way, if $(x,y,p,q)\in E'_\gamma$ and $\hat{p} = \frac{\partial \hat{w}_2(y,p,q)}{\partial p}$, then

$$(D\hat{w}_2^*(y,\hat{p},q), D^2\hat{w}_2^*(y,\hat{p},q)) \in D^{2,+}w_2^*(y-\xi/\lambda,\hat{p},q)\ , \tag{6.53}$$

where $\xi = D\hat{w}_2^*(y,\hat{p},q)$,

$$\frac{\partial w_2^*}{\partial \hat{p}}(y-\xi/\lambda,\hat{p},q) = p + \frac{1}{\lambda'}\hat{p} \tag{6.54}$$

and

$$-(D\hat{w}_2^*(y,\hat{p},q), D^2\hat{w}_2^*(y,\hat{p},q)) \in D^{2,-}\hat{w}_2(y,p,q)\ . \tag{6.55}$$

Step 6 : conclusion. From the previous steps, we know that the set E'_γ defined in step 4 has a positive measure for any λ' sufficiently large, any $\alpha,\ \gamma>0$ sufficiently small and any $\beta>0$. Hence we can find sequences $\lambda'_n\to+\infty$, $\alpha_n,\ \beta_n,\ \gamma_n\to 0^+$, $\zeta_n=(\zeta_x^n,\zeta_y^n,\zeta_p^n,\zeta_q^n)\to 0$, (x_n,y_n,p_n,q_n) converging to some $(0,0,\bar{p},\bar{q})$ such that $(x_n,y_n,p_n,q_n)\in E'_\gamma$ and such that the map z_{ζ_n} has a maximum at (x_n,y_n,p_n,q_n).

Let us set

$$\hat{p}_n = \frac{\partial \hat{w}_2(y_n,p_n,q_n)}{\partial p}\ ,\qquad \hat{q}_n = \frac{\partial \hat{w}_1(x_n,p_n,q_n)}{\partial q}\ , \tag{6.56}$$

$$(\xi_1^n, X_1^n) = (D\hat{w}_1^\sharp(x_n,p_n,\hat{q}_n), D^2\hat{w}_1^\sharp(x_n,p_n,\hat{q}_n))$$

and

$$(\xi_2^n, X_2^n) = (D\hat{w}_2^*(y_n,\hat{p}_n,q_n), D^2\hat{w}_2^*(y_n,\hat{p}_n,q_n))\ .$$

From (6.50) and (6.54) we have

$$p_n + \frac{1}{\lambda'_n}\hat{p}_n = \frac{\partial w_2^*}{\partial \hat{p}}(y_n - \xi_2^n/\lambda, \hat{p}_n, q_n) \text{ and } q_n + \frac{1}{\lambda'_n}\hat{q}_n = \frac{\partial w_1^\sharp}{\partial \hat{q}}(x_n-\xi_n^1/\lambda, p_n,\hat{q}_n)\ . \tag{6.57}$$

From (6.52) and (6.55) we have

$$-(\xi_1^n, X_1^n)\in D^{2,+}\hat{w}_1(x_n,p_n,q_n) \text{ and } -(\xi_2^n,X_2^n)\in D^{2,-}\hat{w}_2(x_n,p_n,q_n)\ .$$

Since furthermore $(x,y)\to z_{\zeta_n}(x,y,p_n,q_n)$ has a maximum at (x_n,y_n,p_n,q_n), the first and second order optimality conditions imply that

$$(-\xi_1^n,\xi_2^n) = (A+\epsilon A^2)(x_n,y_n) + 2\beta_n(x_n,y_n) + (\zeta_x^n,\zeta_y^n) \tag{6.58}$$

and

$$\left(\frac{1}{\epsilon}+\|A\|\right)I \le \begin{pmatrix} -X_1^n & 0 \\ 0 & X_2^n \end{pmatrix} \le A+\epsilon A^2 + 2\beta_n I \tag{6.59}$$

The left-hand side inequality is due to the fact that $\hat{w}_1$ and $\hat{w}_2$ are semiconvex and semiconcave w.r. to x and y respectively with a modulus $\lambda = \frac{1}{\epsilon} + \|A\|$. Using (6.51) and (6.53) gives

$$(\xi_1^n, X_1^n) \in D^{2,-} w_2^\sharp(x_n - \xi_1^n/\lambda, p_n, \hat{q}_n) \text{ and } (\xi_2^n, X_2^n) \in D^{2,+} w_1^*(y_n - \xi_2^n/\lambda, \hat{p}_n, q_n) \qquad (6.60)$$

We now note that (X_1^n), (X_2^n), $(\hat{p}_n)$ and $(\hat{q}_n)$ are bounded. For (X_1^n), (X_2^n) this is an obvious consequence of (6.59). For $(\hat{p}_n)$ and $(\hat{q}_n)$ this comes from (6.56), from the Lipschitz continuity assumption of w_2 and w_1 with respect to p and q respectively and from the definition of $\hat{w}_1$ and $\hat{w}_2$.

We now let $n \to +\infty$. From (6.58), we have $\xi_1^n, \xi_2^n \to 0$. We can assume that $(\hat{p}_n, \hat{q}_n) \to (\hat{p}, \hat{q})$, $X_1^n \to X_1$ and $X_2^n \to X_2$. Then we have from (6.57), (6.60) and (6.59) that:

$$(0, X_1, \bar{q}) \in \overline{D}_q^{2,-} w_2^\sharp(0, \bar{p}, \hat{q}) \text{ and } (0, X_2, \bar{p}) \in \overline{D}_p^{2,+} w_1^*(0, \hat{p}, \bar{q})$$

and

$$\left(\frac{1}{\epsilon} + \|A\|\right) I \le \begin{pmatrix} -X_1 & 0 \\ 0 & X_2 \end{pmatrix} \le A + \epsilon A^2 \, .$$

□

References

[1] Alvarez, O.; Lasry J.-M.; Lions P.-L. (1997) *Convex viscosity solutions and state-constraints.* J. Math. Pures Appl. 9, 76(3), 265-288.

[2] Amendinger, D.; Becherer, A.; Schweizer, M. (2003) *A monetary value for initial information in portfolio optimization.* Finance and Stochastics 7(1) 29-46.

[3] Aumann, R. J.; Maschler, M. B. REPEATED GAMES WITH INCOMPLETE INFORMATION. With the collaboration of Richard E. Stearns. MIT Press, Cambridge, MA, 1995.

[4] Bensoussan, A. and Frehse, J. (2000) *Stochastic games for N players.* J. Optimization Theory Appl. 105, No.3, 543-565.

[5] Cardaliaguet P. (2006) *Differential games with asymmetric information.* To appear in SIAM J. Control Optim.

[6] Cardaliaguet P. (2006) *Representation formulas for differential games with asymmetric information.* To appear in JOTA.

[7] Corcuera, J.M.; Imkeller, P.; Kohatsu-Higa, A.; Nualart, D. (2004) *Additional utility of insiders with imperfect dynamical information.* Finance and Stochastics 8 437-450.

[8] Crandall M.G., Lions P.-L. *Viscosity solutions of Hamilton-Jacobi Equations.* Trans. Amer. Math. Soc. (1983), 277, 1-42.

[9] Crandall M.G., Ishii H., Lions P.-L. (1992) *User's guide to viscosity solutions of second order Partial Differential Equations.* Bull. Amer. Soc., 27, pp. 1-67.

[10] De Meyer, B. *Repeated games, duality and the central limit theorem.* Math. Oper. Res. 21 (1996), no. 1, 237–251.

[11] De Meyer, B.; Rosenberg, D. *"Cav u" and the dual game.* Math. Oper. Res. 24 (1999), no. 3, 619–626.

[12] Fleming, W.H.; Souganidis, P.E. *On the existence of value functions of two-player, zero-sum stochastic differential games.* Indiana Univ. Math. J. 38 (1989), No.2, 293-314.

[13] Mertens, J.F.; Zamir S. *The value of two person zero sum repeated games with lack of information on both sides*, Int. J. Game Theory (1994), 1, pp. 39-64.

[14] Nisio, M. (1988) *Stochastic differential games and viscosity solutions of Isaacs equations.* Nagoya Math. J. 110, 163-184 (1988)

[15] Rosenberg; Solan E.; Vieille N. *Stochastic games with a singler controller and incomplete information.* SIAM J. Control Optim. Vol. 43 (2004), N. 1, pp. 86-110

[16] Sorin, S. A FIRST COURSE ON ZERO-SUM REPEATED GAMES. Mathématiques & Applications (Berlin), 37. Springer-Verlag, Berlin, 2002.