Non fattening condition for the generalized evolution by mean curvature and applications

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Abstract

We prove a non fattening condition for geometrical evolution described by the level set approach. This condition is close to those of Soner [19] and Barles, Soner and Souganidis [4] but applies to some unbounded hypersurfaces. It allows us to apply our result to prove the uniqueness of the evolution by mean curvature of graphs which are *convex at infinity*, without any restriction on their growth at infinity.

1 Introduction

We consider the evolution Γ_t of a given initial hypersurface Γ_0 of $I\!R^{N+1}$ moving according to the law

$$\mathcal{V}_{(x,t)} = h(n_x, Dn_x), \tag{1}$$

where $\mathcal{V}_{(x,t)}$, n_x and Dn_x stand respectively for the normal velocity, an oriented unit normal, the second fundamental form of Γ_t at $x \in \Gamma_t$ and h is the given evolution law. The hypothesis on h will be introduced later but the key assumption in this paper is the map h to be elliptic with respect to the second variable. Namely, if X, Y are symmetric matrices, then

$$X \leq Y \quad \Longrightarrow \quad h(n_x, X) \geq h(n_x, Y). \tag{2}$$

The most typical example we are interested in is the celebrated mean curvature evolution where

$$\mathcal{V}_{(x,t)} = h(Dn_x) = -\mathrm{Tr}(Dn_x). \tag{3}$$

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To describe the evolution of Γ_t according to (1), different ways have been proposed; see the references in the introduction of [10]. Here, we follow the level set approach introduced by Barles [1] and Osher and Sethian [18] and developed first independently by Evans and Spruck [10] and Chen, Giga and Goto [7].

The level set approach presents the advantage to be defined for all time $t \geq 0$, even past some singularities. We refer the reader to section 2 for the definition and recall here that the evolution Γ_t by the level set approach is given, at each time t, as the 0-level set of an auxiliary function, namely $\Gamma_t := \{z \in I\!R^{N+1} : v(z,t) = 0\}$, where $v : I\!R^{N+1} \times [0, +\infty) \to I\!R$ is the solution of a suitable parabolic pde under the form

$$\frac{\partial v}{\partial t} + F(Dv, D^2 v) = 0 \quad \text{in} \quad I\!R^{N+1} \times (0, +\infty). \tag{4}$$

In this approach, one of the main issues is the so-called *fattening phenomenon* which happens when the front $\bigcup_{t\geq 0} \Gamma_t \times \{t\}$ has nonempty interior in $I\!R^{N+1} \times [0, +\infty)$. Some examples are known for which such a phenomenon arises (see Ilmanen [15] or Soner [19]). This fattening phenomenon is closely related to the non-uniqueness of the geometrical evolution (1). We refer to Ilmanen [15], Soner [19], Barles, Soner and Souganidis [4] or Barles, Biton and Ley [3] for further details.

Nevertheless, in [19] (see also Barles, Soner and Souganidis [4] and Soner and Souganidis [20]), Soner shows that, in the case of compact hypersurfaces which are roughly speaking starshaped and if the normal velocity is linear in the second fundamental form, i.e,

$$\mathcal{V}_{(x,t)} = -\mathrm{Tr}(G(n_x) D n_x), \tag{5}$$

the front never fattens. Note that this kind of motion includes the mean curvature motion (3) as a particular case. Our aim in this article is to extend this method to unbounded hypersurfaces having in mind the case when the hypersurface is the graph of a function from $I\!R^N$ into $I\!R$.

To be more specific on our result, we prove that under suitable assumptions on the nonlineary F appearing in (4), the front never fattens if there exists a family of affine dilatations $(\mathcal{A}_\varepsilon)_{\varepsilon>0}$ going to identity as ε goes to 0 and such that

$$\mathrm{d}(\Gamma_0, \mathcal{A}_\varepsilon(\Gamma_0)) := \inf\{|a-b| : (a,b) \in \Gamma_0 \times \mathcal{A}_\varepsilon(\Gamma_0)\} > 0, \tag{6}$$

for any $\varepsilon > 0$.

The main result of this paper is an application of the previous condition to the evolution of entire graphs. We prove in Section 4 that this geometrical condition is fulfilled when Γ_0 is the graph of a function which is *convex at infinity* without any restriction on its growth at infinity. As a consequence, we prove the uniqueness of the solution of the mean curvature equation for graphs, namely

$$\frac{\partial u}{\partial t} - \Delta u + \frac{\langle D^2 u Du, Du\rangle}{1 + |Du|^2} = 0 \quad \text{in} \quad I\!R^N \times (0, +\infty), \tag{7}$$

with $u(\cdot,0)=u_0$ where $u_0 \in C(I\!\!R^N)$ is convex at infinity. Let us mention that the existence of a smooth solution was proved by Ecker and Huisken [9] (see also Chou and Kwong [8] and Barles, Biton and Ley [3]). The question of uniqueness in its whole generality is still open, see [3] for partial results and comments.

We point out that this uniqueness result holds in fact for more general quasilinear equations the class of which is described in [3] and generalizes the result for convex initial data of [3]. Moreover, we give an example (see Example 4.1) of application of (6) to initial data which are not convex at infinity. It follows first that "convex at infinity" functions is not the right class of uniqueness for equations like (7). Secondly, it emphasizes that our condition is of geometrical nature in the sense that we do not have idea how to prove such a result by pdes' methods.

The paper is organised as follows. In Section 2, we recall briefly the level-set approach. In Section 3, we state and prove the sufficient condition (6). The last section is devoted to applications.

2 Preliminary about the level set approach

In this section, we recall what we need about the level set approach and give the definition of the generalized evolution Γ_t.

We start by introducing some definitions and notations. An *admissible partition* $(\Gamma_0, \Omega_0^+, \Omega_0^-)$ is a partition of $I\!\!R^{N+1}$ such that $I\!\!R^{N+1} = \Gamma_0 \cup \Omega_0^+ \cup \Omega_0^-$, where Ω_0^+ and Ω_0^- are open subsets of $I\!\!R^{N+1}$.

If $(\Gamma_0, \Omega_0^+, \Omega_0^-)$ is an admissible partition, then the *signed distance* $d_s(\cdot,\Gamma_0)$ to Γ_0 is defined by

$$d_s(z,\Gamma_0) := \begin{cases} \mathrm{d}(z,\Gamma_0) & \text{if } z \in \Omega_0^+, \\ 0 & \text{if } z \in \Gamma_0, \\ -\mathrm{d}(z,\Gamma_0) & \text{if } z \in \Omega_0^-, \end{cases} \tag{8}$$

where d is the usual nonnegative distance in $I\!\!R^{N+1}$. Clearly $d_s(\cdot,\Gamma_0) \in UC(I\!\!R^{N+1})$, where "$UC$" denotes the uniformly continuous functions.

Looking for an auxiliary function $v : I\!\!R^{N+1} \times [0,+\infty) \to I\!\!R$ satisfying, for every $t \geq 0$, the conditions

$$\{v(\cdot,t)=0\} = \Gamma_t \quad \{v(\cdot,t)>0\} = \Omega_t^+ \quad \text{and} \quad \{v(\cdot,t)<0\} = \Omega_t^-,$$

we obtain that v has to be a solution, at least formally, of the so-called *level set equation* for (1),

$$\begin{cases} \dfrac{\partial v}{\partial t} + F(Dv, D^2v) = 0 & \text{in} \quad I\!\!R^{N+1} \times (0,+\infty) \\ v(\cdot,0) = v_0 & \text{in} \quad I\!\!R^{N+1}, \end{cases} \tag{9}$$

where, for instance $v_0 = d_s(\cdot, \Gamma_0)$, and

$$F(p, X) = -|p|\, h\left(-\frac{p}{|p|}, \left(-\frac{1}{|p|}(Id - \frac{p\otimes p}{|p|^2})X\right)_{|p^\perp}\right), \tag{10}$$

for $p \in D(F) \subset I\!R^{N+1}$, $X \in \mathcal{S}_{N+1}$ and where $M_{|p^\perp}$ is the restriction to the subspace $p^\perp$ of the linear map induced by $M \in \mathcal{S}_{N+1}$. Note that in general, F has singularities in the gradient variable and $D(F) \neq I\!R^{N+1}$. Here and in the sequel, $\mathcal{S}_{N+1}$ denotes the space of symmetric matrices of size $N+1$.

From the very definition of F, it follows that

$$F(\lambda p, \mu p\otimes p + \lambda M) = \lambda F(p, M) \quad \text{for all } p \in D(F), M \in \mathcal{S}_{N+1}, \lambda \geq 0, \mu \in I\!R, \tag{11}$$

and from (2), we have

$$M \geq N \implies F(p, M) \leq F(p, N) \quad \text{for all } p \in D(F), M, N \in \mathcal{S}_{N+1}. \tag{12}$$

It is worth noticing that (11) implies that the equation in (9) is invariant under changes of funtion $v \to \varphi\circ v$ with $\varphi' > 0$ and that (12) implies that (9) is degenerate elliptic and, at least formally, a maximum principle is expected. To state this maximum principle, we need additional assumptions we denote by **(H)** here to avoid technicalities. These assumptions are discussed at the end of the section.

Now, we can state the following theorem and define the generalized evolution of Γ_0 with normal velocity given by (1):

Theorem 2.1 *Suppose that F in (9) is given by (10) and that (2) and* **(H)** *hold. Let $v_0 \in UC(I\!R^{N+1})$. Then, there exists a unique viscosity solution of (9) with initial data v_0. In addition, if $w_0 \in UC(I\!R^{N+1})$ satisfies $w_0 \geq v_0$ in $I\!R^{N+1}$, then the associated solution w of (9) satisfies $w \geq v$ in $I\!R^{N+1}\times[0, +\infty)$.*

Moreover, let

$$(\Gamma_t, \Omega_t^+, \Omega_t^-) := \big(\{v(\cdot, t) = 0\}, \{v(\cdot, t) > 0\}, \{v(\cdot, t) < 0\}\big)$$

be the generalized evolution of the initial admissible partition

$$(\Gamma_0, \Omega_0^+, \Omega_0^-) := \big(\{v_0 = 0\}, \{v_0 > 0\}.\{v_0 < 0\}\big). \tag{13}$$

Then the family $(\Gamma_t, \Omega_t^+, \Omega_t^-)_{t\geq 0}$ is independent of the choice of $v_0 \in UC(I\!R^{N+1})$ satisfying (13). Hence, it allows to define $(\Gamma_t)_{t\geq 0}$ as the generalized evolution of Γ_0 with normal velocity (1).

We do not prove the theorem here. Various assumptions **(H)** on F such that the theorem holds true can be find in: Evans and Spruck [10], Chen, Giga and Goto [7], Giga, Goto, Ishii and Sato [13], Barles, Souganidis and Soner [4], Soner [19], Ishii & Souganidis [17], Ishii [16], Souganidis [21], Giga and Sato [14], Barles, Biton and Ley [3], Giga [12], etc. See Section 4 for explicit examples.

3 A sufficient condition for the non fattening

In this section we give a condition on an initial hypersurface Γ_0 under which its generalized evolution never fattens. To this end we need the following assumption: there exists a positive continuous real-valued function m such that $m(1) = 1$ and

$$F(p, \lambda M) = m(\lambda)F(p, M) \quad \text{for all } p \in D(F), M \in \mathcal{S}_{N+1}, \lambda > 0. \tag{14}$$

Let $\mathcal{T}$ be the group of affine dilatations of $I\!\!R^{N+1}$,

$$\mathcal{T} = \left\{\mathcal{A} : I\!\!R^{N+1} \to I\!\!R^{N+1} : \mathcal{A}(z) = \lambda z + z_0, \lambda \neq 0, z_0 \in I\!\!R^{N+1}\right\}.$$

Lemma 3.1 *Assume that the assumptions of Theorem 2.1 and (14) hold. Suppose that $D(F)$ is invariant under dilatations, that $(\Gamma_0, \Omega_0^+, \Omega_0^-)$ is an admissible initial partition and $\mathcal{A} \in \mathcal{T}$ with coefficient $\lambda \neq 0$. Let v (respectively $v_{\mathcal{A}}$) be the solution of (9) associated to $d_s(\cdot, \Gamma_0)$ (respectively $d_s(\cdot, \mathcal{A}(\Gamma_0))$). Then we have*

$$v_{\mathcal{A}}(z, t) = \lambda\, v\left(\mathcal{A}^{-1}z, \frac{t}{\lambda m(\lambda)}\right). \tag{15}$$

Proof of Lemma 3.1. Suppose that $\mathcal{A}z = \lambda z + z_0$. Let $v_0 = d_s(\cdot, \Gamma_0)$, (respectively $w_0 = d_s(\cdot, \mathcal{A}(\Gamma_0))$) and v (respectively w) be the unique uniformly continuous solution of (9) with initial data v_0 (respectively w_0). Observing that $w_0 = \lambda v_0 \circ \mathcal{A}^{-1}$ we set

$$w(z, t) = \lambda v\left(\frac{z - z_0}{\lambda}, \frac{t}{\lambda m(\lambda)}\right).$$

Using (11) and (14), one checks that w is a solution of (9) with initial data w_0. Therefore we get the Lemma by the uniqueness result for (9). $\square$

Let then two subsets A, B of $I\!\!R^{N+1}$. We recall that the minimum distance $\mathrm{d}(A, B)$ between A and B is defined by

$$\mathrm{d}(A, B) := \inf_{a \in A, b \in B} |a - b|.$$

Lemma 3.2 *For any admissible partitions $(\Gamma_0, \Omega_0^+, \Omega_0^-)$ and $(\tilde{\Gamma}_0, \tilde{\Omega}_0^+, \tilde{\Omega}_0^-)$, if $\tilde{\Omega}_0^+ \cup \tilde{\Gamma}_0 \subset \Omega_0^+$ then*

$$\mathrm{d}(\Gamma_0, \tilde{\Gamma}_0) \geq \eta \geq 0 \quad \Longrightarrow \quad d_s(\cdot, \Gamma_0) \geq d_s(\cdot, \tilde{\Gamma}_0) + \eta$$

Remark 3.1 A consequence of the lemma is an inclusion (or avoidance) principle. Namely, if Ω_0^+ and $\tilde{\Omega}_0^+$ are such that $\Omega_0^+ \subset \tilde{\Omega}_0^+$, then this inclusion remains true for all time: $\Omega_t^+ \subset \tilde{\Omega}_t^+$. The proof consists in noticing that the inequality given by the lemma still holds when taking the solutions to (9) associated respectively to $d_s(\cdot, \Gamma_0)$ and $d_s(\cdot, \tilde{\Gamma}_0)$. We point out that this inclusion principle is an important underlying property of geometrical evolutions satisfying (2). For a general study of geometrical evolutions which satisfies this principle, see Barles and Souganidis [5].

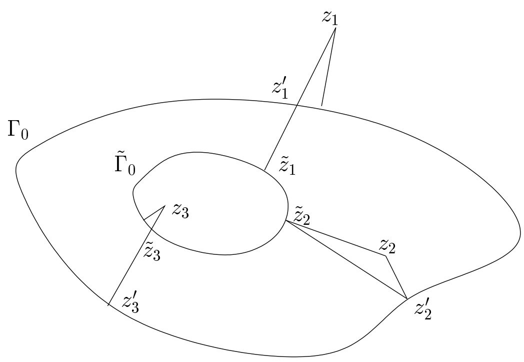

Figure 1: *The different cases under consideration in the proof of Lemma 3.2.*

Proof of Lemma 3.2. We distinguish several cases according to the position of z (see Figure 1).
Case 1.— $z_1 \in \Omega_0^- \cup \Gamma_0$.
Let $a = \mathrm{d}(z_1, \Gamma_0)$ and $b = \mathrm{d}(z_1, \tilde{\Gamma}_0) = |z_1 - \tilde{z}_1|$, with $\tilde{z}_1 \in \tilde{\Gamma}_0$. Consider a point $z_1' \in [z_1, \tilde{z}_1] \cap \Gamma_0$. We have

$$d_s(z_1, \Gamma_0) - d_s(z_1, \tilde{\Gamma}_0) = -a + b = -a + |z_1 - z_1'| + |z_1' - \tilde{z}_1|.$$

But $|z_1 - z_1'| \geq \mathrm{d}(z_1, \Gamma_0) = a$ and $|z_1' - \tilde{z}_1| \geq d_e(\Gamma_0, \tilde{\Gamma}_0) \geq \eta$; therefore $d_s(z_1, \Gamma_0) - d_s(z_1, \tilde{\Gamma}_0) \geq \eta$.
Case 2.— $z_2 \in \Omega_0^+ \cap \tilde{\Omega}_0^-$.
Let $a = \mathrm{d}(z_2, \Gamma_0) = |z_2 - z_2'|$ with $z_2' \in \Gamma_0$, and $b = \mathrm{d}(z_2, \tilde{\Gamma}_0) = |z_2 - \tilde{z}_2|$, with $\tilde{z}_2 \in \tilde{\Gamma}_0$. We have
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$$d_s(z_2, \Gamma_0) - d_s(z_2, \tilde{\Gamma}_0) = a + b \geq |z_2' - \tilde{z}_2| \geq \eta.$$

Case 3.— $z_3 \in \tilde{\Omega}_0^- \cup \tilde{\Gamma}_0$.
Let $a = \mathrm{d}(z_3, \Gamma_0) = |z_3 - z_3'|$ with $z_3' \in \Gamma_0$, and $b = \mathrm{d}(z_3, \tilde{\Gamma}_0)$. Consider a point $\tilde{z}_3 \in [z_3, z_3'] \cap \tilde{\Gamma}_0$. We have

$$d_s(z_3, \Gamma_0) - d_s(z_3, \tilde{\Gamma}_0) = a - b = |z_3 - \tilde{z}_3| + |\tilde{z}_3 - z_3'| - b.$$

But $|\tilde{z}_3 - z_3'| \geq d_e(\Gamma_0, \tilde{\Gamma}_0) \geq \eta$ and $|z_3 - \tilde{z}_3| \geq \mathrm{d}(z_3, \tilde{\Gamma}_0) = b$; thus $d_s(z_3, \Gamma_0) - d_s(z_3, \tilde{\Gamma}_0) \geq \eta$, which completes the proof of the Lemma 3.2. □

Now, we can state the main result of this section.

Theorem 3.1 *Let $(\Gamma_0, \Omega_0^+, \Omega_0^-)$ be an admissible initial partition. Suppose there exists a family $(\mathcal{A}_\varepsilon)_{\varepsilon>0} \subset \mathcal{T}$ and a sequence of positive numbers $(\eta_\varepsilon)_{\varepsilon>0}$ such that*

$$\mathcal{A}_\varepsilon \underset{\varepsilon \to 0}{\longrightarrow} Id \quad \textit{and} \quad \mathrm{d}(\Gamma_0, \mathcal{A}_\varepsilon(\Gamma_0)) \geq \eta_\varepsilon > 0 \textit{ for } \varepsilon > 0. \tag{16}$$

Then the front $\bigcup_{t\geq 0}\Gamma_t\times\{t\}$ *has empty interior in* $I\!R^{N+1}\times[0,+\infty)$.

This theorem is an adaptation of the one of Barles, Soner and Souganidis [4, Theorem 4.3] and Soner [19] which are more adaptated to the case of compact hypersurfaces. Our result applies when Γ_0 is an entire graph (see Section 4).

Proof of Theorem 3.1. Let v and $v_{\mathcal{A}_\varepsilon}$ be the uniformly continuous viscosity solutions of (9) associated to the initial data $d_s(\cdot,\Gamma_0)$ and $d_s(\cdot,\mathcal{A}_\varepsilon(\Gamma_0))$. From Lemma 3.1, for every $(z,t)\in I\!R^{N+1}\times[0,+\infty)$, we have

$$v_{\mathcal{A}_\varepsilon}(z,t)=\lambda_\varepsilon v\left(\mathcal{A}_\varepsilon^{-1}z,\frac{t}{\lambda_\varepsilon m(\lambda_\varepsilon)}\right), \tag{17}$$

where λ_ε is the coefficient of $\mathcal{A}_\varepsilon$. Next, from (16), Lemma 3.2 and the comparison result we get that, for any $\varepsilon>0$,

$$v(z,t)\geq v_{\mathcal{A}_\varepsilon}(z,t)+\eta_\varepsilon.$$

Therefore

$$v(z,t)\geq \lambda_\varepsilon v\left(\mathcal{A}_\varepsilon^{-1}z,\frac{t}{\lambda_\varepsilon m(\lambda_\varepsilon)}\right)+\eta_\varepsilon. \tag{18}$$

Assume now that the front $\bigcup_{t\geq 0}\Gamma_t\times\{t\}$ has nonempty interior in $I\!R^{N+1}\times[0,+\infty)$. It follows that there is some $(z_0,t_0)\in I\!R^{N+1}\times(0,+\infty)$ and some $r>0$ such that

$$v\equiv 0 \quad \text{in} \quad B(z_0,r)\times[t_0-r,t_0+r].$$

Since $\mathcal{A}_\varepsilon\to Id$, one has $\lambda_\varepsilon\to 1$ and then, for ε sufficiently small,

$$\left(\mathcal{A}_\varepsilon^{-1}z_0,\frac{t_0}{\lambda_\varepsilon m(\lambda_\varepsilon)}\right)\in B((z_0,t_0),r)\times[t_0-r,t_0+r].$$

Writing (18) at the point (z_0,t_0), we obtain a contradiction which ends the proof. $\square$

4 Application to uniqueness results.

In this section we give some applications to Theorem 3.1. The first application is known and concerns the evolution of compact sets. The second, which is the main result and the motivation of this work, gives new uniqueness results for quasilinear parabolic pdes.

4.1 Uniqueness of generalized evolutions of compact sets

In this section, we show that Theorem 3.1 applies to recover some results of [19] and [17].

Suppose h is like in (5) and assume that $G:S^N\to\mathcal{S}^+_{N+1}$ is continuous, where $S^N=\{\xi\in I\!R^{N+1}:|\xi|=1\}$ is the unit sphere and $\mathcal{S}^+_{N+1}$ is the space of the nonnegative symmetric matrices of size $N+1$. Noticing that (14) holds with $m(r)=r$, we have

Theorem 4.1 *Let $(\Gamma_0, \Omega_0^+, \Omega_0^-)$ be an admissible partition such that Γ_0 has empty interior. Suppose that $\overline{\Omega_0^+}$ is a compact subset which is starshaped, namely: there exists $z_0 \in \Omega_0^+$ such that, for all $z \in \overline{\Omega_0^+}$, $[z_0, z[\subset \Omega_0^+$.*

Then $\bigcup_{t\geq 0} \Gamma_t \times \{t\}$ has empty interior.

This theorem was proved by Soner [19, Theorem 9.3]. Our proof is basically the same so we only sketch it. Up to a translation, we can suppose that $z_0 = 0$ and we check that the family $\mathcal{A}_\varepsilon(z) = (1-\varepsilon)z$, for $\varepsilon \in (0,1)$ satisfies (16). We conclude by Theorem 3.1.

It is worth pointing out again that (5) includes the mean curvature motion, where $h(Dn_x) = -\mathrm{Tr}(Dn_x) = -\mathrm{div}(n_x)$. In this case, the level set equation (9) is the well-known mean curvature equation,

$$\frac{\partial v}{\partial t} - \Delta v + \frac{\langle D^2 v Dv, Dv\rangle}{|Dv|^2} = 0 \quad \text{in } I\!R^{N+1} \times (0, +\infty).$$

We present another type of motion studied in [17]. We set

$$h(Dn_x) = \kappa_1^+ \kappa_2^+ \cdots \kappa_N^+, \tag{19}$$

where $\kappa_1, \kappa_2, \cdots, \kappa_N$ are the principal curvatures of Γ_t (the eigenvalues of Dn_x) and $r^+ := \max\{r, 0\}$. Theorem 2.1 applies in this case and (14) holds with $m(r) = r^N$. Thus Theorem 4.1 holds true for h given by (19) (with the same proof). By this way, we recover [17, Proposition 3.4]; but contrary to the latter proposition, we do not require Γ_0 to be C^2.

It is worth noticing that non fattening means that, roughly speaking, Γ_t is the unique generalized solution of (1).

4.2 Uniqueness of solutions of quasilinear parabolic pdes

We turn to our main application. Consider the following pde,

$$\begin{cases} \dfrac{\partial u}{\partial t} - \mathrm{Tr}\left[b\left(Du\right) D^2 u\right] = 0 \quad \text{in} \quad I\!R^N \times (0, +\infty) \\ u(\cdot, 0) = u_0 \in C(I\!R^N), \end{cases} \tag{20}$$

where $b : I\!R^N \to \mathcal{S}_N^+$ and $\mathcal{S}_N^+$ is the set of the nonnegative symmetric matrices of size N. Note that Equation (20) is quasilinear parabolic (possibly degenerate). The typical example we have in mind is (7) where

$$b(p) = Id - \frac{p \otimes p}{1 + |p|^2}.$$

In [3] we show that under suitable assumptions on the diffusion matrix b, the graphs of the solutions of (20) are hypersurfaces of $I\!R^{N+1}$ moving accordind to a geometrical law of type (1). In this case, the level-set equation is (9) with

$$F(Dv, D^2v) = -\mathrm{Tr}\left[b\left(-\frac{D_xv}{D_yv}\right)\left(D^2_{xx}v - 2D^2_{xy}\otimes\frac{D_xv}{D_yv} + D_{yy}v\frac{D_xv}{D_yv}\otimes\frac{D_xv}{D_yv}\right)\right], \quad (21)$$

and $D(F) = \{p = (p_1, p_2, \cdots, p_{N+1}) \in I\!R^{N+1} : p_{N+1} = 0\}$. We do not recall these assumptions here to avoid technicality and refer to [3, (H1,2,3,4)]. Roughly speaking, these assumptions allow us to control the singularities of F in order to apply Theorem 2.1. It makes possible to define the generalized evolution Γ_t of $\Gamma_0 = \mathrm{Graph}(u_0)$ and we prove that the graphs of all the continuous viscosity solutions of (20) are contained in the front $\bigcup_{t\geq 0}\Gamma_t\times\{t\}$. It follows that the uniqueness of continuous viscosity solutions is equivalent to the non fattening of the front (see [3, Theorem 6.2]).

Existence of solutions to (20) is not the point here, we refer to [9], [8], [3] for quite general results. In particular, one has existence results for any continuous initial data u_0 without any growth restriction at infinity, an unusual result.

The question we address here is the uniqueness of these solutions without growth restriction at infinity. Several partial results are known in this direction: in dimension $N = 1$, the problem was completely solved independently by Chou and Kwong [8] and [2]. In any dimension, the uniqueness was proved in the following situations: (i) for (7) when u_0 is radially symmetric in $I\!R^N$ (see [6]); (ii) for (20) when u_0 is convex in $I\!R^N$ (see [3]).

Here, we extend the result (ii) to initial data which are convex at infinity (see the precise definition below). One could be think that generalization is of little interest but we point out that a very small perturbation of u_0 even on a compact set can modify the solution everywhere.

A continuous function $f : I\!R^N \to I\!R$ is said to be *convex at infinity* if there exists $R > 0$ such that, for any convex set $\mathcal{C} \subset I\!R^N - B(0, R)$, the restriction $f : \mathcal{C} \to I\!R$ is convex. Note that, if f is twice differentiable in $I\!R^N - B(0, R)$, f is convex at infinity if and only if $D^2f(x) \geq 0$ for every $x \in I\!R^N - B(0, R)$.

Theorem 4.2 *Assume that Theorem 2.1 holds for (9) with F defined by (21). If $u_0 \in C(I\!R^N)$ is convex at infinity, then (20) has at most one continuous viscosity solution. In particular, for any u_0 which is convex at infinity there exists a unique classical solution of the mean curvature equation for graphs (7).*

Before giving the proof of the theorem, we state an example and do some comments about it.

Example 4.1 *A function u_0 which satisfies the assumptions of Theorem 3.1 but is not convex at infinity.*

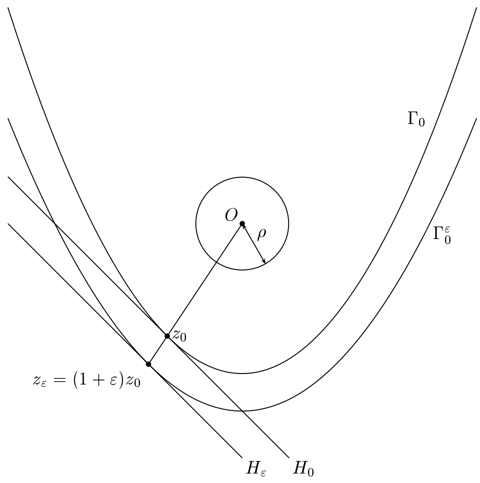

Figure 2: $\Gamma_0 = \mathrm{Graph}(u_0)$ *with* u_0 *convex.*

Let $f, g \in C(I\!R^N)$ such that f is convex in $I\!R^N$ and $g(x) \to 0$ as $|x| \to +\infty$. Set $u_0 = \max\{f(x), g(x)\}$. Then $u_0 \in C(I\!R^N)$ is not necessary convex at infinity (in the case $N = 1$, take for instance f, g defined by $f(x) = \mathrm{e}^{x^3}$ and $g(x) = \sin x/x$). But u_0 satisfies (16). We do not give the proof since it is close to the proof of Theorem 4.2.

Firstly, this example shows that Theorem 3.1 applies to a larger class of graphs than convex at infinity ones. Secondly it enlightens on the geometrical nature of condition (16). Indeed, if one could hope possible to find a purely analytical proof of Theorem 4.2, we have no idea how to prove uniqueness with initial data like the u_0 above by pdes methods: the solutions can have arbitrary growth, they have no expected properties, etc.

Proof of Theorem 4.2. From [3, Theorem 6.2], it is sufficient to prove that the generalized evolution Γ_t of $\Gamma_0 := \mathrm{Graph}(u_0)$ does not develop any interior. We proceed in two steps.

Step 1. To emphasize the main ideas without technicality, we suppose first that u_0 is convex in $I\!R^N$.
Up to translate Γ_0, we can assume that there exists $\rho > 0$ such that $\overline{B(0,\rho)} \subset \mathrm{Epi}(u_0)$. Consider the family $(\mathcal{A}_\varepsilon)_{\varepsilon>0} \subset \mathcal{T}$, defined by

$$\mathcal{A}_\varepsilon = (1+\varepsilon)Id, \quad 0 < \varepsilon < 1,$$

and set $\Gamma_0^\varepsilon = \mathcal{A}_\varepsilon(\Gamma_0)$.

We aim at applying Theorem (3.1). We claim there exists a sequence $(\eta_\varepsilon)_{\varepsilon>0}$ of positive numbers such that

$$d_e(\Gamma_0, \mathcal{A}_\varepsilon(\Gamma_0)) \geq \eta_\varepsilon. \tag{22}$$

We prove this claim. Let $z_0 = (x_0, y_0) \in \Gamma_0$ and $z_\varepsilon = \mathcal{A}_\varepsilon(z_0) = (1+\varepsilon)z_0$. For ξ in the convex subdifferential $\partial u_0(x_0)$ of u_0 at x_0, consider the support hyperplane

$$H_0 = \{z = (x, y) \in \mathbb{R}^{N+1} : y = \langle \xi, x - x_0 \rangle + y_0\},$$

to $\mathrm{Epi}(u_0)$ passing through z_0 (see Figure 2). Since u_0 is convex, $\mathrm{Epi}(u_0)$ lies in the half-space $\{y \geq \langle \xi, x - x_0 \rangle + y_0\}$; it follows

$$\mathrm{d}(z_\varepsilon, \Gamma_0) \geq \mathrm{d}(z_\varepsilon, H_0) = \mathrm{d}(H_\varepsilon, H_0),$$

where $H_\varepsilon = \{z = (x, y) : y = \langle \xi, x - (1+\varepsilon)x_0 \rangle + (1+\varepsilon)y_0\}$ is the parallel hyperplane to H_0 passing through z_ε. Noticing that $H_\varepsilon = \mathcal{A}_\varepsilon(H_0)$, we obtain

$$\mathrm{d}(H_\varepsilon, H_0) \geq \varepsilon\, \mathrm{d}(0, H_0) \geq \rho\varepsilon.$$

Finally, for every $z_\varepsilon \in \Gamma_0^\varepsilon$, $\mathrm{d}(z_\varepsilon, \Gamma_0) \geq \rho\varepsilon$; therefore (22) holds with $\eta_\varepsilon = \rho\varepsilon$. It follows that assumption (16) of Theorem (3.1) holds and we obtain the desired conclusion.

Note that Step 1 provides a new demonstration of [3, Theorem 10.1].

Step 2. The general case.
From now on, we assume that the map u_0 is convex at infinity. Let $R_0 > 0$ be some constant such that u_0 is convex on any convex subset of $\mathbb{R}^N - B(0, R_0)$. For $x \in \mathbb{R}^N$, with $|x| > R_0$, we define the subdifferential $\partial u_0(x)$ as the subdifferential at x of the restriction of u_0 to any convex neighbourhood of x contained in $\mathbb{R}^N - B(0, R_0)$. Since the notion of subdifferential is local, $\partial u_0(x)$ is well defined. Let us point out that $\partial u_0(x)$ enjoys the following property: if $p \in \partial u_0(x)$, then

$$\forall y \in \mathbb{R}^N \text{ with } [x, y] \cap \overline{B(0, R_0)} = \emptyset,\ u_0(y) \geq u_0(x) + \langle p, y - x \rangle\ .$$

and that $\partial u_0(x)$ is non empty as soon as $|x| > R_0$.

With this in mind, let us state a preliminary (and technical) lemma, proved below.

Lemma 4.1 *Assume that u_0 is convex at infinity. Then there is some radius $R > 0$ and some constant $c \in \mathbb{R}$ such that*
(i) u_0 is convex on any convex subset of $\mathbb{R}^N - B(0, R)$,
(ii) for any $x \notin B(0, R+1)$, for any $p \in \partial u_0(x)$,

$$u_0(x) - \langle p, x \rangle \leq -|p| + c\ . \tag{23}$$

Remark 4.1 Lemma 4.1 has the following geometric interpretation: Let $\mathcal{C}$ be any open convex subset of $\mathbb{R}^N - B(0, R+1)$. Let $u_0^{\mathcal{C}}$ be the smallest convex function which coincides with u_0 on $\mathcal{C}$, namely

$$\forall x \in \mathbb{R}^N,\ u_0^{\mathcal{C}}(x) = \sup\{u_0(z) + \langle p, x - z \rangle\ :\ z \in \mathcal{C} \text{ and } p \in \partial u_0(z)\}\ .$$

Then inequality (23) states that $u_0^{\mathcal{C}}$ is bounded from above by the constant c on the ball $B(0, 1)$.

For any $\varepsilon > 0$ and any $\lambda \in (0,1)$, let us set

$$u_{\varepsilon,\lambda}(x) = (1-\lambda)\left[u_0\left(\frac{x}{1-\lambda}\right) + \varepsilon\right] .$$

Let us underline that the graph of $u_{\varepsilon,\lambda}$ is the image of the graph of u_0 by the similitude $\mathcal{A}_{\varepsilon,\lambda}$ defined by

$$\mathcal{A}_{\varepsilon,\lambda}(z) = (1-\lambda)(z + (0,\varepsilon)).$$

Note that $\mathcal{A}_{\varepsilon,\lambda} \to Id$ as $\varepsilon, \lambda \to 0$.

To conclude applying Theorem 3.1, it is sufficient to prove the following claim: *for any $\varepsilon > 0$, there is some $\lambda_\varepsilon \in (0,1)$ such that, for any $\lambda \in (0, \lambda_\varepsilon)$,*

$$\mathrm{d}\left(\mathrm{Graph}(u_0), \mathrm{Graph}(u_{\varepsilon,\lambda})\right) \; > \; 0 \; .$$

Let R and c be as in Lemma 4.1. Without loss of generality, up to some translation, we can assume that $c = -1$. Thus we have

$$\forall x \notin B(0, R+1),\ \forall p \in \partial u_0(x),\ u_0(x) - \langle p, x\rangle \leq -(1 + |p|) \; . \tag{24}$$

Since u_0 is continuous, we have

$$\forall z \in I\!R^N,\ \mathrm{d}((z, u_0(z)), \mathrm{Graph}(u_0) + (0,\varepsilon)) > 0 \; .$$

Therefore, $\gamma_\varepsilon = \min_{|z|\leq R+1} \mathrm{d}((z, u_0(z)), \mathrm{Graph}(u_0) + (0,\varepsilon))$ is positive.

Let $x, y \in I\!R^N$. We want to estimate from below $|(x, u_0(x)) - (y, u_{\varepsilon,\lambda}(y))|$ by some constant independant of x and y. For this, let us first assume that $x \in B(0, R+1)$. We can also suppose that $|x - y| \leq 1$. Then

$$|(x,u_0(x))-(y,u_{\varepsilon,\lambda}(y))| \geq |(x,u_0(x))-(y,u_0(y)+\varepsilon)|-|(u_0(y)+\varepsilon)-(1-\lambda)(u_0(\frac{y}{1-\lambda})+\varepsilon)| \; .$$

Since $(y, u_0(y) + \varepsilon)$ belongs to $\mathrm{Graph}(u_0) + (0, \varepsilon)$, we have

$$|(x, u_0(x)) - (y, u_{\varepsilon,\lambda}(y))| \geq \gamma_\varepsilon - \left[\lambda\varepsilon + \lambda|u_0(\frac{y}{1-\lambda})| + |u_0(y) - u_0(\frac{y}{1-\lambda})|\right] \; .$$

We can choose $\lambda_\varepsilon > 0$ small enough such that, for every $\lambda \in (0, \lambda_\varepsilon)$,

$$\forall y \in B(0, R+2),\ \ \lambda\varepsilon + \lambda|u_0(\frac{y}{1-\lambda})| + |u_0(y) - u_0(\frac{y}{1-\lambda})| \leq \gamma_\varepsilon/2 \; .$$

This leads to

$$|(x, u_0(x)) - (y, u_{\varepsilon,\lambda}(y))| \geq \gamma_\varepsilon/2 \; .$$

Let us now assume that $x \notin B(0, R+1)$. Let us choose some $p \in \partial u_0(x)$. Since $|y - x| \leq 1$ and $|x| \geq R+1$, the segment $[x, y/(1-\lambda)]$ is a subset of $I\!R^N - \overline{B(0,R)}$. Thus,

$$u_0(\frac{y}{1-\lambda}) \geq u_0(x) + \langle p, \frac{y}{1-\lambda} - x\rangle \; ,$$

which implies that

$$u_{\varepsilon,\lambda}(y) \geq (1-\lambda)[u_0(x) - \langle p, x\rangle + \varepsilon] + \langle p, y\rangle \ .$$

Let us define

$$\forall z \in I\!\!R^N,\ \pi(z) = (1-\lambda)[u_0(x) - \langle p, x\rangle + \varepsilon] + \langle p, z\rangle \ .$$

Let us notice that, on the one hand $\pi(x) \geq u_0(x)$ because of (24) and, on the another hand $u_{\varepsilon,\lambda}(y) \geq \pi(y)$. Therefore

$$|(x, u_0(x)) - (y, u_{\varepsilon,\lambda}(y))| \geq d((x, u_0(x)), \mathrm{Graph}(\pi)) = \gamma[1+|p|^2]^{-\frac{1}{2}} \ ,$$

where $\gamma = \lambda(\langle p, x\rangle - u_0(x)) + (1-\lambda)\varepsilon$. Therefore, using (24), we get

$$|(x, u_0(x)) - (y, u_{\varepsilon,\lambda}(y))| \geq \lambda \frac{1+|p|}{(1+|p|^2)^{\frac{1}{2}}} \geq \frac{\lambda}{2} \ .$$

In conclusion, we have proved that, for any $\varepsilon > 0$ and any $\lambda \in (0, \lambda_\varepsilon)$,

$$\mathrm{d}\,(\mathrm{Graph}(u_0), \mathrm{Graph}(u_{\varepsilon,\lambda})) \ \geq \ \min\{\gamma_\varepsilon/2, \lambda/2\} \ > 0 \ ,$$

which completes the proof of Theorem 4.2. □

Proof of Lemma 4.1. Let $R_0 > 0$ be some constant such that u_0 is convex on any convex subset of $I\!\!R^N - B(0, R_0)$. Let us fix $z \in I\!\!R^N$ with $|z| \leq 1$ and let us set $u_z(\cdot) = u(\cdot + z)$. Then u_z is convex on any convex subset of $I\!\!R^N - B(0, R_0 + 1)$.

We claim that, for any $x \in I\!\!R^N$ with $|x| > R_0 + 2$, for any $p \in \partial u_z(x)$, for any $q \in \partial u_z(y)$ where $y = (R_0 + 2)x/|x|$, we have

$$u_z(x) - \langle p, x\rangle \ \leq \ u_z(y) - \langle q, y\rangle \ . \tag{25}$$

Indeed, since the segment $[x, y]$ has an empty intersection with $B(0, R_0 + 1)$, and since u_z is convex on any convex subset of $I\!\!R^N - B(0, R_0 + 1)$, we have

$$u_z(y) \geq u_z(x) + \langle p, y - x\rangle \ . \tag{26}$$

Moreover, from the convexity of u_z on some convex neighbourhood of the segment $[x, y]$, we have $\langle p - q, x - y\rangle \geq 0$. Since $x - y = (|x|/(R_0 + 2) - 1)y$, with $|x| > R_0 + 2$, this implies that $\langle p, y\rangle \ \geq \ \langle q, y\rangle$. From this inequality and from (26) we deduce that

$$u_z(y) \geq u_z(x) - \langle p, x\rangle + \langle q, y\rangle \ ,$$

which proves our claim.

Since u_0 is convex on any convex subset of $I\!\!R^N - B(0, R_0)$, u_0 is locally Lipschitz continuous on this set. Let L be some Lipschitz constant of u_0 on the set $B(0, R_0 + 3) -$

$B(0, R_0+1)$. In particular, for any $y \in B(0, R_0+3) - B(0, R_0+1)$ and any $q \in \partial u_0(y)$, we have $|q| \leq L$.

Let us fix $z \in B(0,1)$ and $x \in I\!R^N$ with $|x| \geq R_0+3$. We apply (25) to z and to $x-z$: (25) states that

$$\forall p \in \partial u_0(x),\ u_0(x) - \langle p, x-z \rangle \ \leq\ u_0(y+z) - \langle q, y \rangle$$

where $y = (R_0+2)(x-z)/|x-z|$ and $q \in \partial u_0(y+z)$. Let us notice that $|q| \leq L$ since $y+z \in B(0, R_0+3) - B(0, R_0+1)$. Therefore

$$\forall p \in \partial u_0(x),\ u_0(x) - \langle p, x-z \rangle \ \leq\ \|u_0\|_{L^\infty(B(0,R_0+3))} + L(R_0+2)\ .$$

Since this inequality holds true for any z with $|z| \leq 1$, we finally deduce that

$$\forall p \in \partial u_0(x),\ u_0(x) - \langle p, x \rangle \leq c - |p|\ ,$$

with $c = \|u_0\|_{L^\infty(B(0,R_0+3))} + L(R_0+2)$. This is the desired result if we set $R = R_0+3$. $\square$

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