On the energy of a flow arising in shape optimization

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Abstract :   In [8] we have defined a viscosity solution for the gradient flow of the exterior Bernoulli free boundary problem. We prove here that the associated energy is non decreasing with the flow. For this we build a discrete gradient flow in the flavour of Almgren, Taylor and Wang [2] and prove its convergence to the viscosity solution.

1   Introduction

In this paper we continue our investigation of a gradient flow for the Bernoulli free boundary problem initiated in [8]. The exterior Bernoulli free boundary problem amounts to minimize the capacity of a set under volume constraints. Using a Lagrange multiplier $\lambda > 0$, this problem can be recasted into the minimization with respect to the set Ω of the functional

$$\mathcal{E}_\lambda(\Omega) = \mathrm{cap}(\Omega) + \lambda|\Omega| \, ,$$

where $\mathrm{cap}(\Omega)$ denotes the capacity of the set Ω with respect to some fixed set S and $|\Omega|$ denotes the volume of Ω. The set Ω is constrained to satisfy the inclusion $S \subset\subset \Omega$. Notice that there is a "competition" between the two terms in the minimization: the capacity is nondecreasing with respect to inclusion whereas the volume is nondecreasing.

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Such a problem has quite a long history and we refer to the survey paper [12] for references and interpretations in physics. Our study is motivated by several papers in numerical analysis where discrete gradient flows are built via a level-set approach in order to solve free boundary and shape optimization problems: see [1] and the references therein for the recent advances in this area. In this framework, the exterior Bernoulli free boundary problem appears as a model problem in order to better understand this numerical approach.

Let us now go further into the description of the gradient flow for $\mathcal{E} := \mathcal{E}_1$ (we work here in the case $\lambda = 1$ for simplicity of notations). The energy $\mathcal{E}$ being defined on sets, a gradient flow for $\mathcal{E}$ is a family of sets $(\Omega(t))_{t \geq 0}$ evolving with a normal velocity which decreases instantaneously the most the energy. For the Bernoulli problem, the corresponding evolution law is given by:

$$V_{t,x} = h(x, \Omega(t)) := -1 + \bar{h}(x, \Omega(t)) \qquad \text{for all } t \geq 0, \ x \in \partial\Omega(t) . \qquad (1)$$

In the above equation, $V_{t,x}$ is the normal velocity of the set $\Omega(t)$ at the point x at time t and $\bar{h}(x, \Omega)$ is a non local term of Hele-Shaw type given, for any set Ω with smooth boundary, by

$$\bar{h}(x, \Omega) = |\nabla u(x)|^2 , \qquad (2)$$

where $u : \Omega \to \mathbb{R}$ is the capacity potential of Ω with respect to S, i.e., the solution of the following partial differential equation

$$\begin{cases} -\Delta u = 0 & \text{in } \Omega \backslash S, \\ u = 1 & \text{on } \partial S, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \qquad (3)$$

The set S is a fixed source and we always assume above that S is smooth and $S \subset\subset \Omega(t)$. Let us underline that $h(x, \Omega)$ is well defined as soon as Ω has a "smooth" (say for instance $\mathcal{C}^2$) boundary and that $S \subset\subset \Omega$.

The reason why a smooth solution $(\Omega(t))$ of the geometric equation (1) can be considered as a gradient flow of the energy

$$\mathcal{E}(\Omega) = |\Omega| + \text{cap}(\Omega) \qquad (4)$$

is the following: from Hadamard formula we have

$$\frac{d}{dt}\mathcal{E}(\Omega(t)) = \int_{\partial\Omega(t)} \left(1 - |\nabla u|^2\right) V_{t,x} = -\int_{\partial\Omega(t)} \left(-1 + |\nabla u|^2\right)^2 \leq 0 .$$

Hence the choice of $V_{t,x} = h(x, \Omega(t))$ in (1) appears to be the one which decreases instantaneously the most the energy $\mathcal{E}$. In order to minimize the energy $\mathcal{E}$, it is therefore very natural to follow the gradient flow (1). This is precisely what is done numerically in [1].

In general the geometric flow (1) does not have classical solutions. In order to define the flow after the onset of singularities, we have introduced in [8] a notion of generalized (viscosity) solution and investigated its existence as well as its uniqueness. The aim of this paper is to prove that the energy $\mathcal{E}$ is non increasing along the generalized flow. This question is certainly essential to better explain the numerical schemes of [1]. This also fully justifies the terminology of "gradient flow" for our generalized solutions.

Such energy estimates are difficult to derive from the notion of viscosity solutions. Indeed this latter notion is defined through a comparison principle, which has very little to do with the energy associated to the flow. To the best of our knowledge, such a question has only be settled for the mean curvature motion (MCM in short), which corresponds to the gradient flow of the perimeter. There are two proofs of the fact that the perimeter of the viscosity solution to the mean curvature flow decreases: the first one is due to Evans and Spruck in their seminal papers [10, 11]; it is based on a regularized version of the level set formulation for the flow and is probably specific to local evolution equations. The other proof is due to Chambolle [9]. Its starting point is the fondamental construction of Almgrem, Taylor and Wang [2] who built generalized solutions of the MCM in a variational way as limits of "discrete gradient flow" for the perimeter (the so-called minimizing movements. See also Ambrosio [5]). The key argument of Chambolle's paper [9] is that Almgren, Taylor and Wang's generalized solutions coincide with the viscosity solutions, at least for a large class of initial sets. Hence the energy estimate available from [2]—which allows to compare the energy of the evolving set with the energy of the initial position—can also be applied to the viscosity solution. Since the viscosity solution enjoys a semi-group property, one can conclude that the energy is decreasing along the flow.

For proving that the energy $\mathcal{E}$ is decreasing along our viscosity solutions of (1), we borrow several ideas from Almgren, Taylor and Wang [2] and Chambolle [9]. As in [2] for the MCM, we start with a construction of discrete gradient flow (Ω_n^h) for the energy $\mathcal{E}$: namely Ω_{n+1}^h is obtained from Ω_n^h as a minimizer of a functional $J_h(\Omega_n^h, \cdot)$ which is equal to $\mathcal{E}$ plus a penalizing term. The penalizing term—which depends on the time-step h—prevents

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the minimizing set Ω_{n+1}^h from being too far from Ω_n^h. Then, as in Chambolle [9], we prove that the limits of these discrete gradient flows converge to the viscosity solution of our equation (1) as the time-step h goes to 0. In [9], this convergence is proved by using the convexity of the equivalent of our functional $J_h(\Omega_h^n, \cdot)$ for the MCM. We use instead here directly a weak form of the Euler equation for minimizers of $J_h(\Omega_h^n, \cdot)$ as described by Alt and Caffarelli [3] for the Bernoulli problem. We then conclude that the energy of the flow is non increasing.

The paper is organized in the following way. In Section 2 we recall the construction of [8] for the viscosity solutions of (1). Section 3 is devoted to suitable generalizations of the capacity and capacity potential needed for our estimates. In Section 4 we introduce the functional J_h and build the discrete motions, the limits of which are discussed in section 5. The fact that the energy is decreasing along the flow is finally proved in Section 6.

2 Definitions and notations for the generalized flow

Let us first fix some basic notations: if A, B are subsets of $\mathbb{R}^N$, then $A \subset\subset B$ means that the closure $\overline{A}$ of A is a compact subset which satisfies $\overline{A} \subset \text{int}(B)$, where $\text{int}(B)$ is the interior of B. We set

$$\mathcal{D} = \{K \subset\subset \mathbb{R}^N \ : \ S \subset\subset K\}.$$

Throughout the paper $|\cdot|$ denotes the euclidean norm (of $\mathbb{R}^N$ or $\mathbb{R}^{N+1}$, depending on the context) and $B(x, R)$ denotes the open ball centered at x and of radius R. If E is a measurable subset of $\mathbb{R}^N$, we also denote by $|E|$ the Lebesgue measure of E. If K is a subset of $\mathbb{R}^N$ and $x \in \mathbb{R}^N$, then $d_K(x)$ denotes the usual distance from x to K: $d_K(x) = \inf_{y \in K} |y - x|$. The signed distance d_K^s to K is defined by

$$d_K^s(x) = \begin{cases} d_K(x) & \text{if } x \notin K, \\ -d_{\partial K}(x) & \text{if } x \in K, \end{cases} \tag{5}$$

where $\partial K = \overline{K} \backslash \text{int}(K)$ is the boundary of K. Let Ω be an open bounded subset of $\mathbb{R}^N$. We denote by $\mathcal{C}_c^\infty(\Omega)$ the set of smooth functions with com-

pact support in Ω, and by $H_0^1(\Omega)$ its closure for the H^1 norm.

Here and throughout the paper, we assume that

S is the closure of an open, nonempty, bounded subset of $\mathbb{R}^N$ with a $\mathcal{C}^2$ boundary. $\qquad$ (6)

The generalized solution of the front propagation problem (1) is defined though their graph: if $(\Omega(t))_{t\geq 0}$ is the familly of evolving sets, then its graph is the subset of $\mathbb{R}^+ \times \mathbb{R}^N$ defined by

$$\mathcal{K} = \{(t,x) \in \mathbb{R}^+ \times \mathbb{R}^N \ : \ x \in \Omega(t)\} \ .$$

We denote by (t,x) an element of such a set, where $t \in \mathbb{R}^+$ denotes the time and $x \in \mathbb{R}^N$ denotes the space. We set

$$\mathcal{K}(t) \ = \ \{x \in \mathbb{R}^N \mid (t,x) \in \mathcal{K}\} \ .$$

The closure of the set $\mathcal{K}$ in $\mathbb{R}^{N+1}$ is denoted by $\overline{\mathcal{K}}$. The closure of the complementary of $\mathcal{K}$ is denoted $\widehat{\mathcal{K}}$:

$$\widehat{\mathcal{K}} = \overline{(\mathbb{R}^+ \times \mathbb{R}^N) \backslash \mathcal{K}}$$

and we set

$$\widehat{\mathcal{K}}(t) = \{x \in \mathbb{R}^N \mid (t,x) \in \widehat{\mathcal{K}}\} \ .$$

We use here repetitively the terminology of [6, 7, 8]:

- A *tube* $\mathcal{K}$ is a subset of $\mathbb{R}^+ \times \mathbb{R}^N$, such that $\overline{\mathcal{K}} \cap ([0,t] \times \mathbb{R}^N)$ is a compact subset of $\mathbb{R}^{N+1}$ for any $t \geq 0$.

- A tube $\mathcal{K}$ is *left lower semi-continuous* if

  $\forall t > 0, \ \forall x \in \mathcal{K}(t), \ \text{if } t_n \to t^-, \ \exists x_n \in \mathcal{K}(t_n) \text{ such that } x_n \to x \ .$

- If $s = 1, 2$ or $(1,1)$, then a $\mathcal{C}^s$ tube $\mathcal{K}$ is a tube whose boundary $\partial \mathcal{K}$ has at least $\mathcal{C}^s$ regularity.

- A *regular tube* $\mathcal{K}_r$ is a tube with a non empty interior and whose boundary has at least $\mathcal{C}^1$ regularity, such that at any point $(t,x) \in \mathcal{K}_r$ the outward normal (ν_t, ν_x) to $\mathcal{K}_r$ at (t,x) satisfies $\nu_x \neq 0$. In this case, its *normal velocity* $V_{(t,x)}^{\mathcal{K}_r}$ at the point $(t,x) \in \partial \mathcal{K}_r$ is defined by

  $$V_{(t,x)}^{\mathcal{K}_r} = -\frac{\nu_t}{|\nu_x|},$$

  where (ν_t, ν_x) is the outward normal to $\mathcal{K}_r$ at (t,x).

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- A $\mathcal{C}^1$ regular tube $\mathcal{K}_r$ is *externally tangent* to a tube $\mathcal{K}$ at $(t, x) \in \mathcal{K}$ if

$$\mathcal{K} \subset \mathcal{K}_r \text{ and } (t, x) \in \partial \mathcal{K}_r .$$

  It is *internally tangent* to $\mathcal{K}$ at $(t, x) \in \widehat{\mathcal{K}}$ if

$$\mathcal{K}_r \subset \mathcal{K} \text{ and } (t, x) \in \partial \mathcal{K}_r .$$

- We say that a sequence of $\mathcal{C}^{1,1}$ tubes $(\mathcal{K}_n)$ converges to some $\mathcal{C}^{1,1}$ tube $\mathcal{K}$ *in the $\mathcal{C}^{1,\mathrm{b}}$ sense* if $(\mathcal{K}_n)$ converges to $\mathcal{K}$ and $(\partial \mathcal{K}_n)$ converges to $\partial \mathcal{K}$ for the Hausdorff distance, and if there is an open neighborhood $\mathcal{O}$ of $\partial \mathcal{K}$ such that, if $d^s_{\mathcal{K}}$ (respectively $d^s_{\mathcal{K}_n}$) is the signed distance to $\mathcal{K}$ (respectively to $\mathcal{K}_n$), then $(d^s_{\mathcal{K}_n})$ and $(\nabla d^s_{\mathcal{K}_n})$ converge uniformly to $d^s_{\mathcal{K}}$ and $D\mathbf{d}_{\mathcal{K}}$ on $\mathcal{O}$ and $\|D^2 d^s_{\mathcal{K}_n}\|_\infty$ are uniformly bounded on $\mathcal{O}$.

We are now ready to define the generalized solutions of (1):

Definition 2.1 *Let $\mathcal{K}$ be a tube and $K_0 \in \mathcal{D}$ be an initial set.*

1. *$\mathcal{K}$ is a viscosity subsolution to the front propagation problem (1) if $\mathcal{K}$ is left lower semi-continuous and $\mathcal{K}(t) \in \mathcal{D}$ for any t, and if, for any $\mathcal{C}^2$ regular tube $\mathcal{K}_r$ externally tangent to $\mathcal{K}$ at some point (t, x), with $\mathcal{K}_r(t) \in \mathcal{D}$ and $t > 0$, we have*

$$V^{\mathcal{K}_r}_{(t,x)} \leq h(x, \mathcal{K}_r(t))$$

   where $V^{\mathcal{K}_r}_{(t,x)}$ is the normal velocity of $\mathcal{K}_r$ at (t, x).

   We say that $\mathcal{K}$ is a subsolution to the front propagation problem with initial position K_0 if $\mathcal{K}$ is a subsolution and if $\overline{\mathcal{K}}(0) \subset \overline{K_0}$.

2. *$\mathcal{K}$ is a viscosity supersolution to the front propagation problem if $\widehat{\mathcal{K}}$ left lower semi-continuous, and $\mathcal{K}(t) \subset \mathcal{D}$ for any t, and if, for any $\mathcal{C}^2$ regular tube $\mathcal{K}_r$ internally tangent to $\mathcal{K}$ at some point (t, x), with $\mathcal{K}_r(t) \in \mathcal{D}$ and $t > 0$, we have*

$$V^{\mathcal{K}_r}_{(t,x)} \geq h(x, \mathcal{K}_r(t)) .$$

   We say that $\mathcal{K}$ is a supersolution to the front propagation problem with initial position K_0 if $\mathcal{K}$ is a supersolution and if $\widehat{\mathcal{K}}(0) \subset \overline{\mathbb{R}^N \backslash K_0}$.

3. *Finally, we say that a tube $\mathcal{K}$ is a viscosity solution to the front propagation problem (with initial position K_0) if $\mathcal{K}$ is a sub- and a supersolution to the front propagation problem (with initial position K_0).*

In [8] we have proved that for any initial position there is a maximal solution, with a closed graph, which contains any subsolution of the problem, as well as a minimal solution, which has an open graph, and is contained in any supersolution of the problem.

3   Capacity and capacity potential

Let S be as in (6). For an open bounded subset Ω of $\mathbb{R}^N$ such that $S \subset\subset \Omega$, the capacity of Ω with respect to S is defined by

$$\text{cap}(\Omega) = \inf \left\{ \int_{\Omega\backslash S} |\nabla\phi|^2 \ : \ \phi \in \mathcal{C}_c^\infty(\Omega), \ \phi = 1 \text{ on } S \right\} .$$

Obviously $\text{cap}(\Omega)$ is non increasing with respect to the set Ω (for inclusion). For a general reference on the subject, see for instance [14].

Remark 3.1 (Classical capacity potential) If Ω is any bounded open subset of $\mathbb{R}^N$, then

$$\text{cap}(\Omega) = \inf \left\{ \int_{\Omega\backslash S} |\nabla v|^2 \ : \ v \in H_0^1(\Omega), \ v = 1 \text{ on } S \right\}$$

and the infimum is achieved for a unique $u \in H_0^1(\Omega)$, called the capacity potential of Ω with respect to S, such that $u = 1$ on S, u is harmonic in $\Omega\backslash S$ and $|\{u > 0\}\backslash\Omega| = 0$ (namely, $u = 0$ a.e. in $\mathbb{R}^N\backslash\Omega$). If Ω has a $\mathcal{C}^{1,1}$ boundary, then it is known that the infimum is achieved by a function $u \in \mathcal{C}^2(\Omega\backslash S) \cap \mathcal{C}^1(\overline{\Omega\backslash S})$ which is a classical solution to (3).

For any set E (not necessarily open) such that $S \subset\subset E$, we define a generalized capacity by

$$\text{cap}(\overline{E}) = \sup \left\{ \text{cap}(\Omega) \mid E \subset\subset \Omega, \ \Omega \text{ open and bounded} \right\} .$$

With this definition, $\text{cap}(\overline{E})$ is non increasing with respect to the set E. Notice that this notion of capacity does not take into account "thin closed sets" in the sense that, if $\overline{F} = E$, then $\text{cap}(\overline{E}) = \text{cap}(\overline{F})$ even when $|E\backslash F| \neq 0$. By construction, if E is open, then we have

$$\text{cap}(\overline{E}) \leq \text{cap}(E)$$

but equality does not hold in general. Nevertheless, there is equality if the boundary of the set is regular enough:

Lemma 3.1 *If Ω is an open bounded subset of $\mathbb{R}^N$, with $S \subset\subset \Omega$ and with a $\mathcal{C}^{1,1}$ boundary, we have $\mathrm{cap}(\Omega) = \mathrm{cap}(\overline{\Omega})$.*

Proof of Lemma 3.1. We have to prove that $\mathrm{cap}(\overline{\Omega}) \geq \mathrm{cap}(\Omega)$. It is enough to show that, if

$$\Omega_n = \left\{ y \in \mathbb{R}^N \ : \ d_\Omega(y) < 1/n \right\} ,$$

then $\mathrm{cap}(\Omega_n) \to \mathrm{cap}(\Omega)$ as $n \to +\infty$. Indeed, for n large enough, Ω_n has also a $\mathcal{C}^{1,1}$ boundary. Then from classical regularity arguments, the harmonic potential u_n to Ω_n converges to the capacity potential u of Ω for the $\mathcal{C}^{1,\alpha}$ norm, where $\alpha \in (0,1)$. Whence the result.

<div align="right">QED</div>

Lemma 3.2 *Let E_n be a bounded sequence of subsets of $\mathbb{R}^N$, for which there exists some $r > 0$ with $S_r \subset E_n$ for any n, where*

$$S_r = \{ y \in \mathbb{R}^N \ : \ d_S(y) \leq r \} . \tag{7}$$

Let us denote by K the Kuratowski upper limit of the (E_n), namely

$$K = \left\{ x \in \mathbb{R}^N \ : \ \liminf_n d_{E_n}(x) = 0 \right\} .$$

Then

$$\liminf_n \mathrm{cap}(\overline{E_n}) \geq \mathrm{cap}(\overline{K}) .$$

Proof of Lemma 3.2. Let Ω be any open bounded set such that $K \subset\subset \Omega$. Since (E_n) is bounded and has for upper-limit K, the inclusion $E_n \subset \Omega$ holds for n large enough. Hence $\mathrm{cap}(\overline{E_n}) \geq \mathrm{cap}(\Omega)$ for every n. Therefore

$$\liminf_n \mathrm{cap}(\overline{E_n}) \geq \mathrm{cap}(\Omega) .$$

The open set Ω being arbitrary, the desired conclusion holds.

<div align="right">QED</div>

Let Ω be an open bounded subset of $\mathbb{R}^N$, with $S \subset\subset \Omega$. We denote by $H_0^1(\overline{\Omega})$ the intersection sequence of the spaces $H_0^1(\Omega_n)$ where (Ω_n) is a decreasing sequence of open bounded sets, such that $\Omega \subset\subset \Omega_n$ and $\overline{\Omega} = \cap_n \Omega_n$. One easily checks that $H_0^1(\overline{\Omega})$ does not depend on the sequence (Ω_n).

Lemma 3.3 *Assume that $|\partial\Omega| = 0$. Then the following equality holds:*

$$\mathrm{cap}(\overline{\Omega}) = \inf \left\{ \int_{\Omega\backslash S} |\nabla v|^2 \; : \; v \in H_0^1(\overline{\Omega}), \; v = 1 \text{ on } S \right\} \; ,$$

and there is a unique $u \in H_0^1(\overline{\Omega})$ such that

$$u = 1 \text{ on } S \quad and \quad \int_{\mathbb{R}^N\backslash S} |\nabla u|^2 = \int_{\Omega\backslash S} |\nabla u|^2 = \mathrm{cap}(\overline{\Omega}) \; .$$

Moreover u is harmonic in $\Omega\backslash S$ and $|\{u > 0\}\backslash\Omega| = 0$.

Definition 3.4 *Such a function u is called the capacity potential of $\overline{\Omega}$ with respect to S.*

Remark 3.2
1. If $\partial\Omega$ is $\mathcal{C}^{1,1}$, then the capacity potential u of $\overline{\Omega}$ with respect to S is the (classical) solution of (3) and is equal to the (classical) capacity potential of Ω (see Remark 3.1).
2. In what follows, we study the energy of subsets $\Omega \supset\supset S$ which is defined as the sum of the capacity and the volume of Ω with respect to S (see (4)). This energy is well-defined for bounded sets $\Omega \supset\supset S$. It is why we assumed all the sets to be bounded. But let us mention that all classical results of this section hold replacing Ω, S bounded by $\Omega\backslash S$ bounded. We need this generalization in the proof of Lemma 4.5.

Proof of Lemma 3.3. The proof is easily obtained by approximation. By construction of $\mathrm{cap}(\overline{\Omega})$, we can find a decreasing sequence of open bounded sets Ω_n such that

$$\Omega \subset\subset \Omega_n \; , \; \bigcap_n \Omega_n = \overline{\Omega} \text{ and } \mathrm{cap}(\overline{\Omega}) = \lim_n \mathrm{cap}(\Omega_n) \; .$$

Let u_n be the (classical) capacity potential of Ω_n. From the maximum principle, the sequence (u_n) is decreasing, and converges to some u which is nonnegative with a support in $\overline{\Omega}$ and equals 1 on S. In particular, $\{u > 0\} \subset \Omega$ a.e. since $|\partial\Omega| = 0$. Furthermore, by classical stability result, u is harmonic in Ω because so are the u_n. Since we can find a smooth function ϕ with compact support in Ω such that $\phi = 1$ on S, we have

$$\int_{\Omega_n\backslash S} |\nabla u_n|^2 \leq \int_{\Omega_n\backslash S} |\nabla \phi|^2 = \int_{\Omega\backslash S} |\nabla \phi|^2,$$

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which proves that (u_n) is bounded in $H^1(\mathbb{R}^N)$. Thus the limit u belongs to $H^1(\mathbb{R}^N)$. Since $u_n \in H_0^1(\Omega_n)$ with $H_0^1(\Omega_{n+1}) \subset H_0^1(\Omega_n)$, u belongs to $H_0^1(\Omega_n)$ for any n. Therefore $u \in H_0^1(\overline{\Omega})$. In particular, the support of u lies in $\overline{\Omega} = \Omega$ a.e.. So we have,

$$
\begin{aligned}
\mathrm{cap}(\overline{\Omega}) &= \lim_n \mathrm{cap}(\Omega_n) = \lim_n \int_{\Omega_n \setminus S} |\nabla u_n|^2 \\
&= \liminf_n \int_{\mathbb{R}^N \setminus S} |\nabla u_n|^2 \geq \int_{\mathbb{R}^N \setminus S} |\nabla u|^2 = \int_{\Omega \setminus S} |\nabla u|^2. \quad (8)
\end{aligned}
$$

For every n,

$$
\begin{aligned}
\mathrm{cap}(\Omega_n) = \int_{\Omega_n \setminus S} |\nabla u_n|^2 &= \inf \left\{ \int_{\Omega_n \setminus S} |\nabla v|^2 \ : \ v \in H_0^1(\Omega_n), \ v = 1 \text{ on } S \right\} \\
&\leq \inf \left\{ \int_{\Omega \setminus S} |\nabla v|^2 \ : \ v \in H_0^1(\overline{\Omega}), \ v = 1 \text{ on } S \right\},
\end{aligned}
$$

since $H_0^1(\overline{\Omega}) \subset H_0^1(\Omega_n)$. Letting n go to infinity, we obtain

$$
\mathrm{cap}(\overline{\Omega}) \leq \inf \left\{ \int_{\Omega \setminus S} |\nabla v|^2 \ : \ v \in H_0^1(\overline{\Omega}), \ v = 1 \text{ on } S \right\}.
$$

From (8), we get the equality in the above inequality and the fact that u is optimal. Uniqueness of u comes from the strict convexity of the criterium.

<div align="right">QED</div>

4 The discrete motions

Let us fix $h > 0$ which has to be understood as a time step. Let us recall that S is the closure of an open bounded subset of $\mathbb{R}^N$ with $\mathcal{C}^2$ boundary. We introduce the functional space

$$
E(S) := \{ u \in H^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N) \ : \ u = 1 \text{ on } S \text{ and } u \text{ has a compact support} \}.
$$

If S and S' are two compact subsets of $\mathbb{R}^N$ with $\mathcal{C}^2$ boundary such that $S \subset S'$, then we note that $E(S') \subset E(S)$.

For any bounded open subset Ω of $\mathbb{R}^N$ with $S \subset\subset \Omega$ we define the functional $J_h : E(S) \to \mathbb{R}$ by setting

$$
J_h^S(\Omega, u) = \int_{\mathbb{R}^N \setminus S} |\nabla u|^2 + \mathbf{1}_{\{u>0\}} \left(1 + \frac{1}{h} d_\Omega^s \right)_+.
$$

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where d_Ω^s is the signed distance to Ω defined by (5), $\mathbf{1}_A$ denotes the indicator function of any set $A \subset \mathbb{R}^N$ and $r_+ = r \vee 0$ for any $r \in \mathbb{R}$. We write $J_h(\Omega, u)$ if there is no ambiguity on S.

Let us recall some existence and regularity results given in [3]:

Proposition 4.1 (Alt and Caffarelli [3]) *Let Ω be an open subset of $\mathbb{R}^N$ such that $\Omega \backslash S$ is bounded and with $S \subset\subset \Omega$. Then there is at least a minimizer $u \in E(S)$ to $J_h(\Omega, \cdot)$. Moreover u is Lipschitz continuous and is harmonic in $\{u > 0\} \backslash S$. Finally $\mathcal{H}^{N-1}(\partial\{u > 0\}) < +\infty$.*

Remark 4.1 We note that $S \subset\subset \{u > 0\}$ because u is Lipschitz continuous with $u = 1$ in S.

The existence of u and its Lipschitz continuity come from Theorem 1.3 and Corollary 3.3 of [3]. The fact that u has a compact support is established in Lemma 2.8, and its harmonicity in Lemma 2.4. The finiteness of $\mathcal{H}^{N-1}(\partial\{u > 0\})$ is given in Theorem 4.5.

We are now ready to define the discrete motions.

Let $\Omega_0 \supset\supset S$ be a fixed initial condition. We define by induction the sequence (Ω_n^h) of open bounded subsets of $\mathbb{R}^N$ with $\Omega_n \supset\supset S$ by setting

$$\Omega_0^h := \Omega_0 \quad \text{and} \quad \Omega_{n+1}^h := \{u_n > 0\} \cup \{x \in \Omega_n^h \ : \ d_{\partial\Omega_n^h}(x) > h\},$$

where

$$u_n \in \operatorname*{argmin}_{v \in E(S)} J_h^S(\Omega_n^h, v).$$

We call *discrete motion* such a family of open sets. Of course, the discrete motion is defined in order that it converges to a solution of the front propagation problem (1) (see Theorem 5.2 and Remark 4.2).

In order to investigate the behavior of discrete motions, we need some properties on the minimizers of J_h.

Lemma 4.2 *Let Ω and u be as in Proposition 4.1. Let $\Omega' = \{u > 0\} \cup \hat{\Omega}_h$ where*

$$\hat{\Omega}_h := \{y \in \Omega \ : \ d_{\partial\Omega}(y) > h\} = \{y \in \mathbb{R}^N \ : \ d_\Omega^s(y) < -h\}. \qquad (9)$$

Then $|\partial\Omega'| = 0$ and u is the capacity potential to $\overline{\Omega'}$

Remark 4.2 We do not claim that u is positive in Ω'. For instance, consider a set Ω with two connected components Ω_1 and Ω_2 such that $S \subset\subset \Omega_1$. In this case, $u \equiv 0$ in Ω_2. Notice that it explains why we define $\Omega_{n+1}^h := \{u_n > 0\} \cup \{x \in \Omega_n^h : d_{\partial\Omega_n^h}(x) > h\}$. Adding the set $\{x \in \Omega_n^h : d_{\partial\Omega_n^h}(x) > h\}$ prevents the discrete motion from the sudden disappearance of a connected component. Indeed, the discrete motion is built in order to approach a solution of the front propagation problem (1) and a connected component which does not contain any part of the source is expected to move with a constant normal velocity -1.

Proof of Lemma 4.2. Let us first notice that $|\partial\Omega'| = 0$. Indeed we already know that $|\partial\{u > 0\}| = 0$ (because its $\mathcal{H}^{N-1}$−measure is finite from Proposition 4.1). On the other hand $\partial\hat{\Omega}_h \subset \{y \in \Omega : d_{\partial\Omega'}(y) = h\}$ has also a finite $\mathcal{H}^{N-1}$−measure thanks to [4, Lemma 2.4].

Let now $\epsilon > 0$ be fixed and set, for any $\alpha > 0$, $\Omega_\alpha = \{y \in \mathbb{R}^N : d_{\Omega'}(y) < \alpha\}$. The set Ω_α is open, bounded and satisfies $\Omega' \subset\subset \Omega_\alpha$. Moreover, since $\mathbf{1}_{\Omega_\alpha} \to \mathbf{1}_{\overline{\Omega'}}$ and Ω' is bounded with $|\partial\Omega'| = 0$, for $\alpha > 0$ enough small, we have

$$\int_{\Omega_\alpha \setminus \Omega'} \left(1 + \frac{1}{h}d_{\partial\Omega}^s\right)_+ \leq \epsilon . \tag{10}$$

Let v be the capacity potential of Ω_α and set

$$v_k(x) = v(x) + \frac{1}{k}d_{\mathbb{R}^N \setminus \Omega_\alpha}(x) \qquad \forall x \in \mathbb{R}^N .$$

Then (v_k) converges to v in $H^1(\mathbb{R}^N)$ and $|\Omega_\alpha \setminus \{v_k > 0\}| = 0$. Therefore

$$J_h(\Omega, v_k) = \int_{\mathbb{R}^N \setminus S} |\nabla v_k|^2 + \mathbf{1}_{\{v_k > 0\}} \left(1 + \frac{1}{h}d_\Omega^s\right)_+$$

$$\xrightarrow[k]{} \operatorname{cap}(\Omega_\alpha) + \int_{\mathbb{R}^N \setminus S} \mathbf{1}_{\Omega_\alpha} \left(1 + \frac{1}{h}d_\Omega^s\right)_+ .$$

Since $J_h(\Omega, v_k) \geq J_h(\Omega, u)$, we get from (10)

$$\begin{aligned} \operatorname{cap}(\overline{\Omega'}) \quad &\geq \quad \operatorname{cap}(\Omega_\alpha) \\ &\geq \quad \lim_k J_h(\Omega, v_k) - \int_{\mathbb{R}^N \setminus S} \mathbf{1}_{\Omega_\alpha} \left(1 + \frac{1}{h}d_\Omega^s\right)_+ \\ &\geq \quad J_h(\Omega, u) - \int_{\mathbb{R}^N \setminus S} \mathbf{1}_{\Omega_\alpha} \left(1 + \frac{1}{h}d_\Omega^s\right)_+ \\ &\geq \quad \int_{\mathbb{R}^N \setminus S} |\nabla u|^2 - \mathbf{1}_{\Omega_\alpha \setminus \{u > 0\}} \left(1 + \frac{1}{h}d_\Omega^s\right)_+ \end{aligned}$$

$$\geq \int_{\mathbb{R}^N \setminus S} |\nabla u|^2 - \mathbf{1}_{\Omega_\alpha \setminus \Omega'} \left(1 + \frac{1}{h} d_\Omega^s\right)_+$$

$$\geq \int_{\mathbb{R}^N \setminus S} |\nabla u|^2 - \epsilon$$

Thus $\int_{\mathbb{R}^N \setminus S} |\nabla u|^2 \leq \text{cap}(\overline{\Omega'})$, which proves from Lemma 3.3 that u is the capacity potential of $\overline{\Omega'}$.

<div align="right">QED</div>

Next we need to compare solutions to $J_h(\Omega, \cdot)$ for different S and Ω.

Proposition 4.3 *Let S_1 and S_2 be the closure of two open bounded subsets of $\mathbb{R}^N$ with $\mathcal{C}^2$ boundary, Ω_1 and Ω_2 be open bounded subsets of $\mathbb{R}^N$ such that $S_1 \subset\subset \Omega_1$ and $S_2 \subset\subset \Omega_2$. Let u_1 and u_2 be, respectively, minimizers of $J_h^{S_1}(\Omega_1, \cdot)$ and $J_h^{S_2}(\Omega_2, \cdot)$. If $S_1 \subset S_2$ and $\Omega_1 \subset \Omega_2$, then $u_1 \wedge u_2$ and $u_1 \vee u_2$ are, respectively, minimizers of $J_h^{S_1}(\Omega_1, \cdot)$ and $J_h^{S_2}(\Omega_2, \cdot)$.*

Remark 4.3
1. In particular, if $J_h^{S_2}(\Omega_2, \cdot)$ has a unique minimizer u_2, then $\{u_1 > 0\} \subset \{u_2 > 0\}$.
2. This Proposition still holds true if we replace, for $i = 1, 2$, Ω_i, S_i bounded by $\Omega_i \setminus S_i$ bounded; see Remark 3.2 and Lemma 4.5.

Proof of Proposition 4.3. We have

$$J_h^{S_1}(\Omega_1, u_1 \wedge u_2) + J_h^{S_2}(\Omega_2, u_1 \vee u_2)$$
$$= J_h^{S_1}(\Omega_1, u_1) + J_h^{S_2}(\Omega_2, u_2)$$
$$+ \int_{\mathbb{R}^N \setminus S_1} (|\nabla(u_1 \wedge u_2)|^2 - |\nabla u_1|^2) + (\mathbf{1}_{\{u_1 \wedge u_2 > 0\}} - \mathbf{1}_{\{u_1 > 0\}}) \left(1 + \frac{1}{h} d_{\Omega_1}^s\right)_+$$
$$+ \int_{\mathbb{R}^N \setminus S_2} (|\nabla(u_1 \vee u_2)|^2 - |\nabla u_2|^2) + (\mathbf{1}_{\{u_1 \vee u_2 > 0\}} - \mathbf{1}_{\{u_2 > 0\}}) \left(1 + \frac{1}{h} d_{\Omega_2}^s\right)_+.$$

Since $\Omega_1 \subset \Omega_2$ we have $d_{\Omega_2}^s \leq d_{\Omega_1}^s$ in $\mathbb{R}^N$. Hence, a straightforward computation leads to

$$\mathbf{1}_{\{u_1 \wedge u_2 > 0\}} \left(1 + \frac{1}{h} d_{\Omega_1}^s\right)_+ + \mathbf{1}_{\{u_1 \vee u_2 > 0\}} \left(1 + \frac{1}{h} d_{\Omega_2}^s\right)_+$$
$$\leq \mathbf{1}_{\{u_1 > 0\}} \left(1 + \frac{1}{h} d_{\Omega_1}^s\right)_+ + \mathbf{1}_{\{u_2 > 0\}} \left(1 + \frac{1}{h} d_{\Omega_2}^s\right)_+.$$

<div align="center">13</div>

Moreover, by classical results,

$$|\nabla(u_1 \wedge u_2)|^2 + |\nabla(u_1 \vee u_2)|^2 = |\nabla u_1|^2 + |\nabla u_2|^2 \quad \text{a.e. in } \mathbb{R}^N.$$

It follows

$$
\begin{aligned}
& J_h^{S_1}(\Omega_1, u_1 \wedge u_2) + J_h^{S_2}(\Omega_2, u_1 \vee u_2) \\
\leq \; & J_h^{S_1}(\Omega_1, u_1) + J_h^{S_2}(\Omega_2, u_2) \\
& + \int_{S_1 \setminus S_2} (|\nabla(u_1 \wedge u_2)|^2 - |\nabla u_1|^2) + (\mathbf{1}_{\{u_1 \wedge u_2 > 0\}} - \mathbf{1}_{\{u_1 > 0\}}) \left(1 + \frac{1}{h} d_{\Omega_1}^s \right)_+ .
\end{aligned}
$$

But $u_1 \wedge u_2 = u_1$ on S_1 which gives

$$J_h^{S_1}(\Omega_1, u_1 \wedge u_2) + J_h^{S_2}(\Omega_2, u_1 \vee u_2) \leq J_h^{S_1}(\Omega_1, u_1) + J_h^{S_2}(\Omega_2, u_2). \quad (11)$$

Since u_1 and u_2 are minimizers we have

$$J_h^{S_1}(\Omega_1, u_1) \leq J_h^{S_1}(\Omega_1, u_1 \wedge u_2) \quad \text{and} \quad J_h^{S_2}(\Omega_2, u_2) \leq J_h^{S_2}(\Omega_2, u_1 \vee u_2). \quad (12)$$

The inequalities in (11) and (12) are therefore equalities. Hence $u_1 \wedge u_2$ and $u_1 \vee u_2$ are respectively minimizers of $J_h^{S_1}(\Omega_1, \cdot)$ and $J_h^{S_2}(\Omega_2, \cdot)$.

$$\text{QED}$$

We define the energy $\mathcal{E}(\overline{\Omega})$ by

$$\mathcal{E}(\overline{\Omega}) = |\Omega| + \operatorname{cap}(\overline{\Omega}).$$

(compare with (4)).

Lemma 4.4 *Let (Ω_n^h) be a discrete motion with $|\partial \Omega_0^h| = 0$. Then the energy $\mathcal{E}(\overline{\Omega_n^h})$ is non increasing with respect to n. More precisely,*

$$\mathcal{E}(\overline{\Omega_{n+1}^h}) - \mathcal{E}(\overline{\Omega_n^h}) \leq \int_{\mathbb{R}^N} \left(\mathbf{1}_{\Omega_n^h \setminus \{d_{\partial \Omega_n^h}^s < -h\}} - \mathbf{1}_{\{u_n > 0\} \setminus \{d_{\partial \Omega_n^h}^s < -h\}} \right) \frac{1}{h} d_{\Omega_n^h}^s \leq 0,$$

where u_n is a minimizer for $J_h(\Omega_n^h, \cdot)$.

Proof of Lemma 4.4. Let us fix n. In order to simplify the notations, let us set

$$\Omega := \Omega_n^h, \qquad \hat{\Omega}_h := \{x \in \Omega \; : \; d_\Omega^s(x) < -h\} = \{x \in \Omega \; : \; d_{\partial\Omega}(x) > h\} .$$

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Let u_0 be the capacity potential of $\overline{\Omega}$ and u be a minimizer to $J_h(\Omega, \cdot)$. We finally set $\Omega' := \Omega^h_{n+1} = \{u > 0\} \cup \hat{\Omega}_h$. Recall that $\Omega' \in \mathcal{D}$ and that $|\partial\Omega'| = 0$: indeed this is true for $n = 0$ from the assumption and by Lemma 4.2 for $n \geq 1$. With these notations we have to prove that

$$\mathcal{E}(\overline{\Omega'}) \leq \mathcal{E}(\overline{\Omega}) .$$

For this we introduce for any $k \geq 1$ the function u_k defined by

$$u_k(x) = \begin{cases} u_0(x) + \frac{1}{k}d_{\partial\Omega}(x) & \text{if } x \in \Omega, \\ u_0(x) & \text{otherwise.} \end{cases}$$

Then (u_k) converges to u_0 in $H^1(\mathbb{R}^N)$ and $\{u_k > 0\} = \Omega$ a.e. because $\{u_0 > 0\} \subset \overline{\Omega}$ and $|\partial\Omega| = 0$. Hence

$$
\begin{aligned}
\lim_k J_h(\Omega, u_k) &= \lim_k \int_{\mathbb{R}^N \setminus S} |\nabla u_k|^2 + \mathbf{1}_{\{u_k > 0\}}(1 + \frac{1}{h}d_\Omega^s)_+ \\
&= \text{cap}(\overline{\Omega}) + \int_{\mathbb{R}^N \setminus S} \mathbf{1}_\Omega (1 + \frac{1}{h}d_\Omega^s)_+ \\
&= \mathcal{E}(\overline{\Omega}) - |\Omega| + \int_{\Omega \setminus \hat{\Omega}_h} (1 + \frac{1}{h}d_\Omega^s) \\
&= \mathcal{E}(\overline{\Omega}) + \int_{\Omega \setminus \hat{\Omega}_h} \frac{1}{h}d_\Omega^s - |\hat{\Omega}_h|.
\end{aligned}
$$

On the other hand, since $\text{cap}(\overline{\Omega'}) = \int_{\mathbb{R}^N \setminus S} |\nabla u|^2$ from Lemma 4.2 and since $|\overline{\Omega'}| = |\Omega'|$, we also have

$$
\begin{aligned}
J_h(\Omega, u) &= \int_{\mathbb{R}^N \setminus S} |\nabla u|^2 + \mathbf{1}_{\{u > 0\}}(1 + \frac{1}{h}d_\Omega^s)_+ \\
&= \mathcal{E}(\overline{\Omega'}) - |\Omega'| + \int_{\{u > 0\} \setminus \hat{\Omega}_h} (1 + \frac{1}{h}d_\Omega^s) \\
&= \mathcal{E}(\overline{\Omega'}) + \int_{\{u > 0\} \setminus \hat{\Omega}_h} \frac{1}{h}d_\Omega^s - |\hat{\Omega}_h|.
\end{aligned}
$$

Writing that $J_h(\Omega, u) \leq J_h(\Omega, u_k)$, we get the desired claim.

<div style="text-align: right">QED</div>

Next we show that the solution does not blow up when h becomes small.

Lemma 4.5 *Let $R > 0$ and $r_0 \in (0, R/2^{1/(N-2)})$ be fixed. Let us also fix M such that $\sqrt{1+M} \geq 4(N-2)/r_0$. Then there is some $h_0 = h_0(N, r_0, R, M)$ such that, for any $h \in (0, h_0)$ and $r \in (r_0, R/2^{1/(N-2)})$, for any $\Omega \in \mathcal{D}$ open bounded, for any $x \notin \overline{\Omega}$ with $r \leq d_\Omega(x)$, $R \leq d_S(x)$ and for any u minimizer to $J_h(\Omega, \cdot)$, we have*

$$d_{\{u>0\} \cup \hat{\Omega}_h}(x) \geq r - Mh,$$

where $\hat{\Omega}_h$ is defined by (9).

Proof of Lemma 4.5. The idea is to compare the solution with radial ones. For simplicity we assume that $N \geq 3$, the computation in the case $N = 2$ being similar. We also suppose without loss of generality that $x = 0$.

Let us first investigate the problem of minimizing $J_h^{B_R^c}(B_r^c, \cdot)$, where $B_r = B(0, r)$ and $B_R = B(0, R)$. Notice that neither the source B_R^c, nor the subset B_r^c is bounded but $B_r^c \backslash B_R^c = B_R \backslash B_r$ is bounded so the previous results on the minimization problem apply (see Remark 3.2). Standard
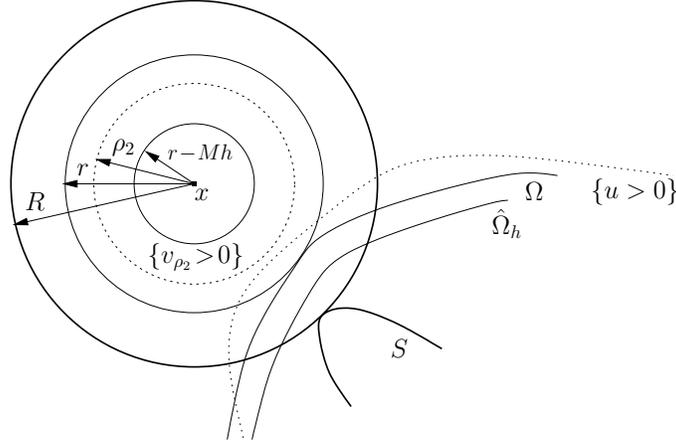

Figure 1: *Illustration of the proof of Lemma 4.5.*

symmetrization arguments show that a minimizer v to $J_h^{B_R^c}(B_r^c, \cdot)$ must be radially symmetric. For $\rho \in (0, R)$, let us denote by v_ρ the (radial) harmonic function which vanishes on ∂B_ρ and is equal to 1 on ∂B_R. We also set $J_h(\rho) := J_h^{B_R^c}(B_r^c, v_\rho)$. Notice that a minimizer of $J_h^{B_R^c}(B_r^c, \cdot)$ has to be either of the form v_ρ with ρ minimizer of $J_h(\cdot)$, or constant equal to $v_0 := 1$. Let us fix h_0 enough small in order that $r + h < R$ for $h \in (0, h_0)$. We have

$$J_h(0^+) = J_h^{B_R^c}(B_r^c, v_0) = \frac{\alpha_{N-1}(r+h)^{N+1}}{hN(N+1)},$$

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where α_{N-1} is the volume of the unit sphere of $\mathbb{R}^N$. For $J_h(\rho)$ with $\rho > 0$, we distinguish two cases. If $r + h < \rho < R$, then

$$J_h(\rho) = \frac{\alpha_{N-1}(N-2)}{\rho^{2-N} - R^{2-N}}.$$

If $0 < \rho \le r + h$, then

$$\frac{J_h(\rho)}{\alpha_{N-1}} = \frac{N-2}{\rho^{2-N} - R^{2-N}} + \frac{1}{h}\left(\frac{(r+h)^{N+1}}{N(N+1)} + \frac{\rho^{N+1}}{N+1} - \frac{(r+h)\rho^N}{N}\right).$$

We show that v_0 cannot be a minimizer by comparing $J_h(0^+)$ with $J_h(\rho)$ for $0 < \rho \le r + h$. Choosing $\rho = \beta\sqrt{h}$ with $\beta > 0$, we have

$$\frac{1}{\alpha_{N-1}}(J_h(\rho) - J_h(0))$$

$$= \rho^{N-2}\left(\frac{N-2}{1 - (\rho/R)^{N-2}} + \frac{\rho^3}{h(N+1)} - \frac{(r+h)\rho^2}{hN}\right)$$

$$\le \rho^{N-2}\left(\frac{N-2}{1 - \beta^{N-2}h^{(N-2)/2}/R^{N-2}} + \frac{\beta^3 h^{1/2}}{N+1} - \frac{r\beta^2}{N}\right). \tag{13}$$

Recalling that $r_0 \in (0, R/2^{1/(N-2)})$ is fixed, we choose

$$\beta > \frac{N(2(N-2)+1)}{r_0} \tag{14}$$

and then $h_0 = h_0(N, \beta, r_0, R) > 0$ enough small such that

$$1 - \frac{\beta^{N-2}h_0^{(N-2)/2}}{R^{N-2}} > \frac{1}{2} \quad \text{and} \quad \frac{\beta^3 h_0^{1/2}}{N+1} < 1. \tag{15}$$

For all $h \in (0, h_0)$, we obtain that (13) is negative, which proves that v_0 is not a minimizer.

Therefore minimizers have to be of the form v_ρ for some $\rho \in (0, R)$. On $(r + h, R)$, $J_h(\rho)$ is increasing. For $\rho \in (0, r + h)$, we have

$$\frac{J_h'(\rho)}{\alpha_{N-1}} = \frac{(N-2)^2 \rho^{1-N}}{(\rho^{2-N} - R^{2-N})^2} + \frac{\rho^N}{h} - \frac{(r+h)\rho^{N-1}}{h}.$$

The stationary points of J_h on $(0, r + h]$ satisfy

$$f(\rho) := \frac{(N-2)^2}{[\rho(1 - (\rho/R)^{N-2})]^2} - \frac{1}{h}(r + h - \rho) = 0. \tag{16}$$

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Notice that $\rho \mapsto f(\rho)$ is convex on $(0, r + h]$ and tends to $+\infty$ as $\rho \to 0^+$ and as $\rho \to R^-$. If we find some value ρ for which $f(\rho)$ is negative, then there are exactly two solutions to (16).

For this, let us choose $\rho = \beta\sqrt{h}$ with $\beta > 0$. Then

$$h\, f(\beta\sqrt{h}) = \frac{(N-2)^2}{\beta\left(1 - (\beta h^{1/2}/R)^{N-2}\right)^{N-2})^2} - r - h + \beta h^{1/2}.$$

Choosing $\beta > 0$ satisfying (14) and

$$\beta > \frac{4(N-2)}{r_0^{1/2}}$$

and h_0 satisfying (15) and

$$\beta h_0^{1/2} < \frac{r_0}{2}, \tag{17}$$

we obtain that $f(\beta\sqrt{h}) < 0$ for $h \in (0, h_0)$.

Let us fix $h \in (0, h_0)$ and let ρ_1 and ρ_2 be respectively the smallest and largest solutions to (16). With the arguments just developed above, we know that $\rho_1 \leq \beta\sqrt{h} \leq \rho_2$, where β is defined as above. Since $J_h'(\rho) = \alpha_{N-1}\rho^{N-1}f(\rho)$, we have

$$
\begin{aligned}
J_h''(\rho_1) &= \alpha_{N-1}\rho_1^{N-1}f'(\rho_1) \\
&= \alpha_{N-1}\rho_1^{N-1}\left[\frac{-2(N-2)^2(1 - (N-1)(\rho_1/R)^{N-2})}{(\rho_1(1 - (\rho_1/R)^{N-2}))^3} + \frac{1}{h}\right] \\
&\leq \alpha_{N-1}\rho_1^{N-1}\left[-\frac{2(N-2)^2}{\rho_1^3}\left(1 - (N-1)\left(\frac{\rho_1}{R}\right)^{N-2}\right) + \frac{\beta^2}{\rho_1^2}\right].
\end{aligned}
$$

If we choose $h_0 > 0$ satisfying (15), (17) and furthermore

$$(N-1)\left(\frac{\beta h_0^{1/2}}{R}\right)^{N-2} < \frac{1}{2} \quad \text{and} \quad h_0^{1/2} < \frac{N-2}{\beta^3}, \tag{18}$$

we obtain that $J_h''(\rho_1) < 0$ for $h \in (0, h_0)$ and ρ_1 is not a minimum to J_h. Therefore, J_h is increasing on $(0, \rho_1)$, decreasing on (ρ_1, ρ_2) and increasing on (ρ_2, R). The minimum is achieved at $\rho = \rho_2$.

Let us now estimate ρ_2. We suppose that h_0 satisfies (15), (17), (18) and

$$h_0 \leq \frac{r_0}{2M} \quad \text{where} \quad 1 + M \geq \frac{16(N-2)^2}{r_0^2}.$$

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Then, for all $h \in (0, h_0)$ and $r \in (r_0, R/2^{1/(N-2)})$, we have $r - Mh \geq r_0/2 > 0$ and we compute

$$
\begin{aligned}
f(r - Mh) &= \frac{(N-2)^2}{(r-Mh)^2(1-((r-Mh)/R)^{N-2})^2} - (1+M) \\
&\leq \frac{4(N-2)^2}{(r-Mh)^2} - (1+M) \\
&\leq 0.
\end{aligned}
$$

Therefore $\rho_2 \geq r - Mh$.

To summerize, we know that, setting $h_0 = h_0(N, r_0, R, M)$ small enough, for all $h \in (0, h_0)$ and $r \in (r_0, R/2^{1/(N-2)})$, the problem consisting of minimizing $J_h^{B_R^c}(B_r^c, \cdot)$ has a unique solution v_{ρ_2}, which is radially symmetric and such that $\rho_2 \geq r - Mh$.

Let now $\Omega \in \mathcal{D}$, $x \notin \overline{\Omega}$ with $R \leq d_S(x)$, $r \leq d_\Omega(x)$ and let u be a minimizer to $J_h(\Omega, \cdot)$. Since $S \subset B_R^c(x)$ and $\Omega \subset B_r^c(x)$, Proposition 4.3 states that $\{u > 0\} \subset \{v_{\rho_2} > 0\} \subset B_{r-Mh}^c(x)$ (see Figure 1 for a picture). Since $\hat{\Omega}_h \subset \Omega \subset B_r^c$, finally, we have $d_{\{u>0\} \cup \hat{\Omega}_h}(x) \geq r - Mh$.

$$\text{QED}$$

Finally we explain that the set $\{u > 0\}$ satisfies some inequalities in a viscosity sense. Here again the regularity results of Alt and Caffarelli [3] play a crucial role. Let Σ be an open set with $\mathcal{C}^{1,1}$ boundary such that $S \subset\subset \Sigma$ and $\Sigma \setminus S$ is bounded. We denote by u_S^Σ the (classical) solution to (3) (replacing Ω by Σ), i.e., the capacity potential of Σ with respect to S.

Lemma 4.6 *Let Ω be a bounded open subset of $\mathbb{R}^N$ with $S \subset\subset \Omega$ and u be a minimizer to $J_h(\Omega, \cdot)$. We set*

$$
\hat{\Omega}_h = \{x \in \Omega \mid d_{\partial\Omega}(x) > h\} \qquad and \qquad \Omega' = \{u > 0\} \cup \Omega_h \, .
$$

Let Σ is an open bounded subset of $\mathbb{R}^N$ with $\mathcal{C}^{1,1}$ boundary.

1. [**Outward estimate**] *Suppose that Σ is such that*

$$
\{u > 0\} \subset \Sigma \qquad and \qquad \exists x \in \partial\Sigma \cap \partial\{u > 0\} \, .
$$

Then

$$
|\nabla u_S^\Sigma(x)| \geq \left(1 + \frac{1}{h} d_\Omega^s(x)\right)_+^{1/2} \, .
$$

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2. [**Inward estimate**] *Let us now assume that Σ is such that*

$$S \subset\subset \Sigma, \qquad \Sigma \subset \Omega' \qquad and \qquad \exists x \in \partial\Sigma \cap \partial\Omega' \ .$$

Then

$$|\nabla u_S^\Sigma(x)| \le \left(1 + \frac{1}{h} d_\Omega^s(x)\right)_+^{1/2} \ .$$

Proof of Lemma 4.6. Let us set $g_\Omega(x) = (1 + d_\Omega^s(x)/h)_+$. We first prove the outward estimate. From [3, Lemma 4.10] we have

$$\limsup_{\substack{x' \to x \\ x' \in \{u > 0\}}} \frac{u(x')}{d_B(x')} \ge \sqrt{g_\Omega(x)}$$

for any ball B contained in $\{u = 0\}$ and tangent to $\{u > 0\}$ at x. Let ν be the outward unit normal to Σ at x and $r > 0$ be such that the ball $B := B(x + r\nu, r)$ is tangent to Σ at x. Then B is also tangent to $\{u > 0\}$ at x. Since by the maximum principle, $u \le u_S^\Sigma$, we have

$$
\begin{aligned}
|\nabla u_S^\Sigma(x)| &= \limsup_{x' \to x, \ x' \in \{u > 0\}} \frac{u_S^\Sigma(x')}{d_B(x')} \\
&\ge \limsup_{x' \to x, \ x' \in \{u > 0\}} \frac{u(x')}{d_B(x')} \\
&\ge \sqrt{g_\Omega(x)}.
\end{aligned}
$$

We now turn to the proof of the inward estimate. We first prove that $u_S^\Sigma \le u$ in $\{u_S^\Sigma > 0\}$. Indeed from Lemma 4.2, u is the capacity potential of $\overline{\Omega'}$. In particular u is harmonic in $\Omega'\backslash S \supset \Sigma\backslash S$, $u = u_S^\Sigma$ on ∂S and $0 = u_S^\Sigma \le u$ on $\partial\{u_S^\Sigma > 0\}$. Hence $u_S^\Sigma \le u$ in $\{u_S^\Sigma > 0\}$. Let us note that $u = 0$ on $\partial\Omega'$. Therefore $u(x) = u_S^\Sigma(x) = 0$.

We now consider two cases. If $x \notin \partial\{u > 0\}$, then $x \in \partial\hat\Omega_h$; thus $d_\Omega^s(x) = -h$ and $g_\Omega(x) = 0$. But $0 \le u_S^\Sigma \le u = 0$ in a neighborhood of x so that $\nabla u_S^\Sigma(x) = 0$. Therefore

$$|\nabla u_S^\Sigma(x)| = 0 = g_\Omega(x).$$

Let us now consider the case $x \in \partial\{u > 0\}$. Then [3, Theorem 6.3] states that

$$\sup_{B(x,r)} |\nabla u| \le \sqrt{g_\Omega(x)} + m(r),$$

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where $m(r) \to 0$ as $r \to 0^+$. Since we want to prove that $|\nabla u_S^\Sigma(x)| \le \sqrt{g_\Omega(x)}$, we can assume without loss of generality that $\nabla u_S^\Sigma(x) \ne 0$. Let ν be the outward unit normal to Σ at x. Since $\nu = -\nabla u_S^\Sigma(x)/|\nabla u_S^\Sigma(x)|$, for $r > 0$ sufficiently small, the segment $]x, x - r\nu[$ is contained in Σ and in $\{u_S^\Sigma > 0\}$, and thus in $\{u > 0\}$. So u is smooth at each point of this segment. Since moreover $u \ge u_S^\Sigma$, we have, for some $\xi \in (x, x - r\nu)$,

$$
\begin{aligned}
u_S^\Sigma(x - r\nu) &\le u(x - r\nu) = u(x) + \langle \nabla u(\xi), -r\nu \rangle \\
&\le r\left(\sqrt{g_\Omega(x)} + m(r) \right).
\end{aligned}
$$

Therefore

$$
|\nabla u_S^\Sigma(x)| = \lim_{r \to 0^+} \frac{u_S^\Sigma(x - r\nu)}{r} \le \sqrt{g_\Omega(x)} .
$$

<div align="right">QED</div>

5 Discrete motions and viscosity solutions

Let us fix Ω_0 open and bounded such that $S \subset\subset \Omega_0$. Let $(\Omega_n^h)_n$ be a discrete motion with $\Omega_0^h = \Omega_0$.

Let us now introduce a lower and upper envelope for the sequences $(\Omega_n^h)_n$ as the time-step h tends to 0^+: the upper envelope $\mathcal{K}^*$ is

$$
\mathcal{K}^*(t) := \left\{ x \in \mathbb{R}^N \ : \ \begin{array}{l} \exists h_k \to 0^+,\ n_k \to +\infty,\ x_k \in \Omega_{n_k}^{h_k}, \\ \text{with } x_k \to x \text{ and } h_k n_k \to t \end{array} \right\}, \tag{19}
$$

while the lower envelope $\mathcal{K}_*$ is defined by its complementary:

$$
\mathbb{R}^N \backslash \mathcal{K}_*(t) = \left\{ x \in \mathbb{R}^N \ : \ \begin{array}{l} \exists h_k \to 0^+,\ n_k \to +\infty,\ x_k \notin \Omega_{n_k}^{h_k}, \\ \text{with } x_k \to x \text{ and } h_k n_k \to t \end{array} \right\}. \tag{20}
$$

Lemma 5.1 *The set $\mathcal{K}^*$ is closed while $\mathcal{K}_*$ is open. Moreover the maps $t \to \mathcal{K}^*(t)$ and $t \to \widehat{\mathcal{K}_*}(t)$ are left lower-semicontinuous on $(0, +\infty)$.*

Proof of Lemma 5.1. The fact that set $\mathcal{K}^*$ is closed comes from its construction since the upper limit of sets is always closed. The argument works in a symmetric way for $\mathcal{K}_*$.

We now prove that $t \to \mathcal{K}^*(t)$ is left lower-semicontinuous on $(0, +\infty)$ (see Section 2 for a definition). We proceed by contradiction assuming there exist $t > 0$, $x \in \mathcal{K}^*(t)$, $\rho > 0$ and a sequence $t_p \to t^-$ such that $B(x, \rho) \cap \mathcal{K}^*(t_p) = \emptyset$. Therefore $d_{\mathcal{K}^*(t_p)}(x) \ge \rho > 0$ for all p.

<div align="center">21</div>

Set $R = d_S(x)$, $r_0 < \min\{\rho, R/2^{1/(N-2)}\}$ and $M > 0$ such that $\sqrt{1+M} \geq 8(N-2)/r_0$. Then Lemma 4.5 states that there is some $h_0 = h_0(N, r_0, R, M)$ with the following property: for any $r \in (r_0/2, R/2^{1/N-2})$ and $h \in (0, h_0)$, for any Ω with $r \leq d_\Omega(x)$ and for any u minimizer to $J_h(\Omega, \cdot)$, we have $d_{\{u>0\}\cup\hat{\Omega}_h}(x) \geq r - Mh$, where $\hat{\Omega}_h = \{d_\Omega^s < -h\}$.

For $h \in (0, h_0)$, let $n_h = [t_p/h]$ be the integer part of t_p/h. From the definition of $\mathcal{K}^*(t_p)$ and r_0, we can find some $h_1 \in (0, h_0)$ such that $d_{\Omega_{n_h}^h}(x) \geq r_0$ for any $h \in (0, h_1)$. We are going to prove by induction that

$$d_{\Omega_{n_h+kh}^h}(x) \geq r_0 - Mkh \qquad \text{for all } k \in \{0, \ldots, k_0^h\}, \tag{21}$$

where $k_0^h = [r_0/(2Mh)]$. Indeed inequality (21) holds for $k = 0$. Assume that it holds for some $k < k_0^h$. Let u be a minimizer for $J_h(\Omega_{n_h+kh}^h, \cdot)$ and define

$$\Omega_{n_h+(k+1)h}^h = \{u > 0\} \cup \{y \in \Omega_{n_h+kh}^h \ : \ d_{\Omega_{n_h+kh}^h}(y) > h\} \ .$$

Then since $r_0 - Mkh \geq r_0/2$ and $r_0 - Mkh \leq r_0 \leq R/2^{1/(N-2)}$, we have from Lemma 4.5 recalled above that

$$d_{\Omega_{n_h+(k+1)h}^h}(x) \geq r_0 - Mkh - Mh \ .$$

So (21) is proved.

Let us set $\tau = r_0/(4M)$ and fix $s \in (0, \tau)$. Let (k_h) be such that $k_h h \to s$ as $h \to 0^+$. We notice that $k_h \in \{0, \ldots, k_0^h\}$ for h sufficiently small. Letting $h \to 0^+$ in inequality (21) for any such (k_h) implies that

$$d_{\mathcal{K}^*(t_p+s)}(x) \geq r_0 - Ms \geq r_0/2 > 0. \tag{22}$$

Since τ does not depend on x and t_p and since $t_p \to t^-$, for p large enough, we have $s = t - t_p \leq \tau$. Therefore, from (22), we obtain $d_{\mathcal{K}^*(t)}(x) = d_{\mathcal{K}^*(t_p+s)}(x) \geq r_0/2 > 0$ which is a contradiction with the assumption $x \in \mathcal{K}^*(t)$.

The proof of the left lower semicontinuity of $\widehat{\mathcal{K}_*}$ is simpler. As above, we proceed by contradiction assuming that there exists $x \in \widehat{\mathcal{K}_*}(t)$ for $t > 0$ and a sequence $t_p \to t^-$ such that $d_{\widehat{\mathcal{K}_*}(t_p)}(x) \geq \rho > 0$ for all p. From the definition of $\Omega_{n_h+1}^h$, for (n_h) such that $n_h h \to t_p$ and h sufficiently small, we have $B_{\rho/2}(x) \subset \Omega_{n_h}^h$. From the definition of $\Omega_{n_h+1}^h$, we have therefore

$$B_{\rho/2-h}(x) \subset \{y \in \Omega_{n_h}^h \ : \ d_{\partial\Omega_{n_h}^h}(x) > h\} \subset \Omega_{n_h+1}^h \ .$$

By induction we prove in a similar way that, for any $k \leq \rho/(4h)$,

$$B_{\rho/2-kh}(x) \subset \{y \in \Omega^h_{n_h+(k-1)h} \; : \; d_{\partial\Omega^h_{n_h+(k-1)h}}(x) > h\} \subset \Omega^h_{n_h+k} \; .$$

Letting now $h \to 0^+$ we get at the limit:

$$B_{\rho/4}(x) \cap \widehat{\mathcal{K}_*}(t_p + s) = \emptyset \qquad \text{for all } s \in [0, \rho/4] \; .$$

Since ρ is independent of p, we get a contradiction by taking p big enough such that $t - t_p = s \leq \rho/4$.

<div align="right">QED</div>

Theorem 5.2 *The tube $\mathcal{K}^*$ (respectively $\mathcal{K}_*$) is a viscosity subsolution (respectively supersolution) to the front propagation problem $V = h(x, \Omega)$, where*

$$h(x, \Omega) = -1 + \bar{h}(x, \Omega)$$

and $\bar{h}$ is defined by (2).

Proof of Theorem 5.2. Let us set $\Omega^h := \bigcup_n \{nh\} \times \Omega^h_n$. Let $(t_0, x_0) \in \mathcal{K}^*$ with $t_0 > 0$, be such that there is a smooth regular tube $\mathcal{K}_r$ with $\mathcal{K}^* \subset \mathcal{K}_r$ and $x_0 \in \partial\mathcal{K}_r(t_0)$. Without loss of generality we can assume that $\mathcal{K}^* \cap \partial\mathcal{K}_r = \{(t_0, x_0)\}$. Then by standard stability arguments (see [7]), one can find a sequence of smooth regular tubes $\mathcal{K}_r^k$ converging to $\mathcal{K}_r$ in the $\mathcal{C}^{1,b}$ sense (see Section 2 for a definition), and sequences $h_k \to 0$ and $n_k \to +\infty$ such that $\Omega^{h_k} \subset \mathcal{K}_r^k$, $(n_k h_k, x_k) \to (t_0, x_0)$, $x_k \in \partial\Omega^{h_k}_{n_k}$ and such that $x_k \in \partial\mathcal{K}_r^k(n_k h_k)$.

Let u be a minimizer to $J_{h_k}(\Omega^{h_k}_{n_k-1}, \cdot)$. By definition of the discrete motion, we have

$$\Omega^{h_k}_{n_k} = \{u > 0\} \cup \{y \in \Omega^{h_k}_{n_k-1} \; : \; d^s_{\Omega^{h_k}_{n_k-1}}(y) < -h_k\} \; . \tag{23}$$

Let $v_k := u_S^{\mathcal{K}_r^k(n_k h_k)}$ be the capacity potential of $\mathcal{K}_r^k(n_k h_k)$.

Let us first assume that $x_k \in \partial\{u > 0\}$ for some subsequence of (x_k) (still denoted by (x_k)). The case $x_k \in \text{int}\{u = 0\}$ for any k is treated later. From the discrete viscosity condition in Lemma 4.6 and the inclusion $\Omega^{h_k}_{n_k-1} \subset \mathcal{K}_r^k((n_k - 1)h_k)$, we know that

$$|\nabla v_k(x_k)| \geq \left(1 + \frac{1}{h_k} d^s_{\Omega^{h_k}_{n_k-1}}(x_k)\right)^{1/2}_+ \geq \left(1 + \frac{1}{h_k} d^s_{\mathcal{K}_r^k((n_k-1)h_k)}(x_k)\right)^{1/2}_+ \; .$$

<div align="center">23</div>

Hence

$$\frac{1}{h_k}d^s_{\mathcal{K}^k_r((n_k-1)h_k)}(x_k) \le -1 + |\nabla v_k(x_k)|^2 \ . \tag{24}$$

Let us now recall that the normal velocity of $\mathcal{K}^k_r$ at a point $(t,x) \in \partial\mathcal{K}^k_r$ is given by $-\frac{\partial}{\partial t}d^s_{\mathcal{K}^k_r(t)}(x)$. Since $x_k \in \partial\mathcal{K}^k_r(n_kh_k)$, $(n_kh_k,x_k) \to (t_0,x_0)$ and since $\mathcal{K}^k_r$ converges to $\mathcal{K}_r$, we have therefore that

$$\begin{aligned}
d^s_{\mathcal{K}^k_r((n_k-1)h_k)}(x_k) &= d^s_{\mathcal{K}^k_r(n_kh_k)}(x_k) - h_k\frac{\partial}{\partial t}d^s_{\mathcal{K}^k_r(n_kh_k)}(x_k) + h_k\epsilon(k) \\
&= h_k V^{\mathcal{K}_r}_{(t_0,x_0)} + h_k\epsilon(k) \ ,
\end{aligned}$$

where $\epsilon(k) \to 0$ as $k \to +\infty$ and $V^{\mathcal{K}_r}_{(t_0,x_0)}$ is the normal velocity of $\mathcal{K}_r$ at (t_0,x_0). From (24) we get for k large enough,

$$h(x_k,\mathcal{K}^k_r(n_kh_k)) = -1 + |\nabla v_k(x_k)|^2 \ge V^{\mathcal{K}_r}_{(t_0,x_0)} + \epsilon(k) \ .$$

Letting $k \to +\infty$, we obtain

$$h(x_0,\mathcal{K}_r(t_0)) = \lim_k h(x_k,\mathcal{K}^k_r(n_kh_k)) \ge V^{\mathcal{K}_r}_{(t_0,x_0)}.$$

The above equality is a straightforward application of [13, Theorem 8.33] since $\mathcal{K}^k_r$ converges to $\mathcal{K}_r$ in the $\mathcal{C}^{1,\mathrm{b}}$ sense (see Section 2 for a definition).

We now assume that $x_k \in \mathrm{int}\{u=0\}$ for any k. Then we have from (23) that

$$d^s_{\Omega^{h_k}_{n_k-1}}(x_k) = -d_{\partial\Omega^{h_k}_{n_k-1}}(x_k) = -h_k \ .$$

Arguing as above we get

$$-h_k = d^s_{\Omega^{h_k}_{n_k-1}}(x_k) \ge d^s_{\mathcal{K}^k_r((n_k-1)h_k)}(x_k) = h_k V^{\mathcal{K}_r}_{(t_0,x_0)} + h_k\epsilon(k) \ ,$$

where $\epsilon(k) \to 0$. Dividing by h_k and letting $k \to +\infty$ gives

$$V^{\mathcal{K}_r}_{(t_0,x_0)} \le -1 \le -1 + |\nabla u^{\mathcal{K}_r(t_0)}_S(x_0)|^2 = h(x_0,\mathcal{K}_r(t_0)) \ .$$

So we have finally proved that $\mathcal{K}^*$ is a subsolution.

We now show that $\mathcal{K}_*$ is a supersolution. The proof starts exactly as above: if there is a smooth regular tube $\mathcal{K}_r$ with $\mathcal{K}_r \subset \mathcal{K}_*$ and some $(t_0,x_0) \in \partial\mathcal{K}_*$ with $t_0 > 0$ and $x_0 \in \partial\mathcal{K}_r(t_0)$, then one can find a sequence of smooth regular tubes $\mathcal{K}^k_r$ converging to $\mathcal{K}_r$ in the $\mathcal{C}^{1,\mathrm{b}}$ sense and sequences $h_k \to 0$ and $n_k \to +\infty$ such that $\mathcal{K}^k_r(nh_k) \subset \Omega^{h_k}_n$ for any n, $(n_kh_k,x_k) \to (t_0,x_0)$,

24

$x_k \in \partial\Omega^{h_k}(n_k h_k) \cap \partial\mathcal{K}_r^k(n_k h_k)$. Let u be a minimizer to $J_{h_k}(\Omega_{n_k-1}^{h_k}, \cdot)$. Then (23) holds for $\Omega_{n_k}^{h_k}$.

Then using Lemma 4.6 we get

$$|\nabla v_k(x_k)| \leq \left(1 + \frac{1}{h_k} d_{\Omega_{n_k-1}^{h_k}}^s(x_k)\right)_+^{1/2} \leq \left(1 + \frac{1}{h_k} d_{\mathcal{K}_r^k((n_k-1)h_k)}(x_k)\right)_+^{1/2}, \tag{25}$$

where $v_k := u_S^{\mathcal{K}_r^k(n_k h_k)}$ is the capacity potential of $\mathcal{K}_r^k(n_k h_k)$ with respect to S. Since $x_k \in \partial\Omega^{h_k}(n_k h_k)$, we have from (23) that $d_{\Omega^{h_k}(n_k h_k)}^s(x_k) \geq -h_k$. Therefore inequality (25) can also be written as

$$\frac{1}{h_k} d_{\mathcal{K}_r^k((n_k-1)h_k)}(x_k) \geq -1 + |\nabla v_k(x_k)|^2 .$$

As before we have

$$d_{\mathcal{K}_r^k((n_k-1)h_k)}^s(x_k) = h_k V_{(t_0,x_0)}^{\mathcal{K}_r} + h_k \epsilon(k) .$$

Hence

$$h(x_k, \mathcal{K}_r^k(n_k h_k)) = -1 + |\nabla v_k(x_k)|^2 \leq V_{(t_0,x_0)}^{\mathcal{K}_r} + \epsilon(k) \to V_{(t_0,x_0)}^{\mathcal{K}_r} .$$

Then we can complete the proof as above to get the required condition:

$$h(x_0, \mathcal{K}_r(t_0)) \leq V_{(t_0,x_0)}^{\mathcal{K}_r} .$$

<div align="right">QED</div>

In particular we get immediately the following Theorem:

Theorem 5.3 *Let Ω_0 be an open bounded subset of $\mathbb{R}^N$ such that $S \subset\subset \Omega_0$. Let $\mathcal{K}^+$ and $\mathcal{K}^-$ be, respectively, the largest and smallest viscosity solutions to the front propagation problem (1) with initial position Ω_0. Then*

$$\mathcal{K}^- \subset \mathcal{K}_* \subset \mathcal{K}^* \subset \mathcal{K}^+ .$$

In particular, if the problem has a unique solution, i.e., $\overline{\mathcal{K}^-} = \mathcal{K}^+$, then

$$\overline{\mathcal{K}^-} = \overline{\mathcal{K}_*} = \mathcal{K}^* = \mathcal{K}^+ .$$

Proof of Theorem 5.3. Since $\mathcal{K}^+$ contains any subsolution and $\mathcal{K}^-$ is contained in any supersolution (see [8]), we have $\mathcal{K}^* \subset \mathcal{K}^+$ and $\mathcal{K}^- \subset \mathcal{K}_*$. Inclusion $\mathcal{K}_* \subset \mathcal{K}^*$ holds by construction. Whence the result.

<div align="right">QED</div>

6   The energy is decreasing along the flow

Let Ω_0 be a bounded open subset of $\mathbb{R}^N$. Let we assume that the front propagation problem (1) with initial position Ω_0 has a unique solution and, furthermore that

$$|\partial\Omega_0| = 0 \qquad \text{and} \qquad |\mathcal{K}^+\backslash\mathcal{K}^-| = 0, \qquad\qquad (26)$$

where $\mathcal{K}^+$ and $\mathcal{K}^-$ denote the maximal and minimal solutions respectively.

Theorem 6.1 *Under assumption (26), there is a set $\mathcal{T} \subset [0, +\infty)$ of full measure such that*

$$\mathcal{E}\left(\mathcal{K}^+(t)\right) \leq \mathcal{E}\left(\mathcal{K}^+(s)\right) \qquad \text{for all } s, t \in \mathcal{T}, \ s < t \ .$$

Remark 6.1 Assumption (26) is not too restrictive. Indeed, it is generic in the following sense: let $(\Omega_0^\lambda)_{\lambda > 0}$ be a strictly increasing family of bounded open initial positions containing the source, i.e.,

$$\text{for all } 0 < \lambda < \lambda', \quad S \subset\subset \Omega_0^\lambda \subset\subset \Omega_0^{\lambda'}.$$

If $\mathcal{K}_\lambda^+$ (respectively $\mathcal{K}_\lambda^-$) is the maximal (respectively minimal) viscosity solution to (1) with initial position Ω_0^λ, then (26) holds for all $\lambda > 0$ except for a countable subset. See [8] for details. For simplicity of notations, we have chosen to consider the case $\lambda = 1$ and to assume that (26) holds for the initial position Ω_0.

Proof of Theorem 6.1. Let (Ω_n^h) be a discrete motion starting from Ω_0. Recall for later use that, from Lemma 4.4,

$$\mathcal{E}(\overline{\Omega_n^h}) \leq \mathcal{E}(\overline{\Omega_0}) \qquad \forall n \geq 0, \ \forall h > 0 \ , \qquad\qquad (27)$$

because we have assumed that $|\partial\Omega_0| = 0$. Let $\mathcal{K}^*$ and $\mathcal{K}_*$ be the associated generalized evolutions defined by (19) and (20). We have

$$\mathcal{K}^- \subset \mathcal{K}_* \subset \mathcal{K}^* \subset \mathcal{K}^+ \ .$$

Let

$$\mathcal{T} := \left\{ t \in [0, +\infty) \ : \ \left|\mathcal{K}^+(t)\backslash\mathcal{K}^-(t)\right| = 0 \right\} \ .$$

From assumption (26) and Fubini Theorem, the set $\mathcal{T}$ is of full measure in $[0, +\infty)$.

We first prove that

$$\mathcal{E}(\mathcal{K}^+(t)) \leq \mathcal{E}(\overline{\Omega_0}) \qquad \forall t \in \mathcal{T} \, . \tag{28}$$

For this, let $t \in \mathcal{T}$, $h_k \to 0^+$ and $n_k \to +\infty$ such that $h_k n_k \to t$. For simplicity we set $\Omega_k := \Omega_{n_k}^{h_k}$. Since the Kuratowski upper limit of the (Ω_k) is contained in $\mathcal{K}^+(t)$, which is a compact subset, the sequence (Ω_k) is bounded. Since moreover the upper limit of $(\mathbb{R}^N \backslash \Omega_k)$ is contained in $\mathbb{R}^N \backslash \mathcal{K}^-(t)$, the latter with a boundary at a positive distance from S, there is some $r > 0$ such that $S_r \subset \Omega_k$ for any k sufficiently large (see (7) for a definition of S_r). Since finally the capacity is non increasing with respect to the inclusion, we get from Lemma 3.2:

$$\liminf_k \text{cap}(\overline{\Omega_k}) \geq \text{cap}(\mathcal{K}^+(t)) \, . \tag{29}$$

The next step towards (28) amounts to show that

$$|\mathcal{K}^+(t)| \leq \liminf |\Omega_k| \, . \tag{30}$$

Let $R > 0$ be sufficiently large so that $\mathcal{K}^+(t) \subset\subset B_R$, where $B_R = B(0, R)$. By definition of the Kuratowski upper limit and the construction of $\mathcal{K}_*$, we have

$$\mathbf{1}_{B_R \backslash \mathcal{K}_*(t)} \geq \limsup_k \mathbf{1}_{B_R \backslash \Omega_k} \, .$$

Fatou Lemma then states that

$$|B_R \backslash \mathcal{K}_*(t)| \geq \limsup |B_R \backslash \Omega_k| \, ,$$

whence (30) since $\mathcal{K}^-(t) \subset \mathcal{K}_*(t)$ and $|\mathcal{K}^+(t)| = |\mathcal{K}^-(t)|$ because $t \in \mathcal{T}$.

Combining (29), (30) and (27) finally gives

$$\mathcal{E}(\mathcal{K}^+(t)) \leq \liminf_k \mathcal{E}(\overline{\Omega_k}) \leq \mathcal{E}(\overline{\Omega_0}) \qquad \forall t \in \mathcal{T} \, .$$

This proves (28).

Let now $0 \leq s \leq t$ with $s, t \in \mathcal{T}$. From the uniqueness of the solution starting from K_0, the maximal solution to the front propagation problem starting at time s from $\mathcal{K}^+(s)$ is equal at time t to $\mathcal{K}^+(t)$. Since $|\partial \mathcal{K}^+(s)| = 0$, because $s \in \mathcal{T}$, inequality (28) states that

$$\mathcal{E}(\mathcal{K}^+(t)) \leq \mathcal{E}(\mathcal{K}^+(s)) \, ,$$

which is the desired result.

$$\text{QED}$$

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