On the strict concavity of the harmonic radius in dimension $N \geq 3$

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Abstract : In this paper a result on the strict concavity of the harmonic radius in open convex domains of $\mathbb{R}^N$, for $N \geq 3$, is given. It implies the strict convexity of the Robin function and the uniqueness of the harmonic center in bounded convex domains.

Résumé : Dans cet article est énoncé un résultat de stricte concavité du rayon harmonique dans des ouverts convexes de $\mathbb{R}^N$, pour $N \geq 3$. Ceci implique la stricte convexité de la fonction de Robin et l'unicité du centre harmonique dans les ouverts convexes bornés.

1 Introduction

The main goal of this paper is to prove the strict concavity of the *harmonic radius* in convex bounded domains of $\mathbb{R}^N$, for $N \geq 3$, and the uniqueness of the harmonic center. Let us briefly recall what the harmonic radius and the harmonic center are: Let Ω be an open subset of $\mathbb{R}^N$ and let us denote by $G_x(\cdot)$ (for $x \in \Omega$) the Green function of the Laplace equation with Dirichlet boundary condition, i.e, the solution of

$$\left\{ \begin{array}{ll} -\Delta G_x(\cdot) = \delta_x \ , & \text{in } \Omega \\ G_x = 0 \ , & \text{on } \partial\Omega \end{array} \right.$$

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where δ_x denotes the Dirac measure at the point x. The function $G_x(\cdot)$ can be decomposed in two parts:

$$G_x(\cdot) = F_x(\cdot) - H_x(\cdot) \ .$$

The singular part F_x is given by

$$\forall y \in \Omega, \ F_x(y) = F(|y-x|) \text{ with } F(t) = \gamma t^{2-N}$$

(if $N \geq 3$), where $\gamma = \Gamma(N/2)/[2(N-2)\pi^{N/2}]$. The regular part H_x is the unique solution of the following equation:

$$\begin{cases} -\Delta H_x(\cdot) = 0 \ , & \text{in } \Omega \\ H_x = F_x & \text{on } \partial\Omega \end{cases}$$

The *Robin function* is the function $t(\cdot) : \Omega \to I\!\!R$ defined by

$$\forall x \in \Omega, \ t(x) = H_x(x)$$

The *harmonic radius* $r(\cdot) : \Omega \to I\!\!R$ is defined by

$$\forall x \in \Omega, \ r(x) = F^{-1}(t(x)) \ .$$

A *harmonic center* of Ω is a point of Ω where $r(\cdot)$ attains its maximum (or, equivalently, a point of Ω where $t(\cdot)$ attains its minimum). In dimension $N = 2$, the harmonic radius is usually known as the conformal radius. For $N \geq 3$, the notion of harmonic radius was introduced by Hersch [19].

Robin function, harmonic radius and harmonic center play a main role in several areas of mathematics and physics: See for instance the survey paper [2] and its bibliography. In particular, the harmonic center appears when studying elliptic boundary value problems involving concentration of energy (see in particular [2], [8], [9], [10], [11]). For this problem, the question of uniqueness of the harmonic center is crucial.

In dimension $N = 2$, the convexity of the Robin function has been proved by several authors [23], [20], [5], [15]. The concavity of the conformal radius has been established in [16], [21]. Moreover, it is known that the conformal radius is strictly concave if Ω is bounded. The key point in the proofs of the above mentioned authors is the fact that the Robin function is a solution of a well-known p.d.e.: the Liouville equation. This is no longer true in higher dimensions in general (for $N \geq 3$): The counter-example of the strip in dimension $N = 3$ is detailed in [2].

We prove in this paper that the harmonic radius is strictly concave when $N \geq 3$ unless Ω is a cylinder, a cone or the translate of a cone. To the best of our knowledge, even the question of the concavity of the harmonic radius, in dimension $N \geq 3$, was open up to now. Our result implies in particular that the harmonic center of a convex bounded set is unique and that the Robin function is strictly convex. Our proof is completely different from the case $N = 2$. We devote the rest of the introduction to explain its main ideas.

Surprisingly, the main difficulty is *not* the concavity of the harmonic radius, but its strict concavity. Let us briefly explain how the convexity of the Robin function - which is a somewhat simpler - can be proved. It is a consequence of two old results. The first one apparently goes back to Grötzsch in dimension $N = 2$ and can be found in [2] for $N \geq 2$. It states that the capacity of small balls is strongly related with the Robin function by:

$$cap_\Omega(B_\rho(x)) = \frac{1}{F(\rho) - t(x) + \mathcal{O}(\rho)} , \tag{1}$$

where $B_\rho(x)$ stands for the ball of radius $\rho > 0$ and of center $x \in \Omega$, $t(x)$ is the Robin function of Ω at the point x and $cap_\Omega(B_\rho(x))$ denotes the capacity of the ball $B_\rho(x)$ in the set Ω. Let us recall that the capacity of a set $D \subset\subset \Omega$ in Ω is defined by

$$cap_\Omega(D) = \inf\{\int_\Omega |\nabla u|^2 \mid u \in H_0^1(\Omega),\ u \geq 1 \text{ in } D\} .$$

Let us now fix two points x_0 and x_1 in Ω, $s \in [0,1]$ and let us set $x_s = (1-s)x_0 + sx_1$. In order to compare $(1-s)t(x_0) + st(x_1)$ with $t(x_s)$, a natural idea in view of (1) is to compare the capacity of the ball $B_\rho(x_s)$ and the convex combination $(1-s)cap_\Omega(B_\rho(x_0)) + scap_\Omega(B_\rho(x_s))$. This comparison derives from an inequality given by Borell in [3]: Let us denote by u_D^Ω the capacity potential of the subdomain D of Ω in Ω, i.e., the solution to

$$\begin{cases} -\Delta u = 0 & \text{in } \Omega \backslash \overline{D} , \\ u = 0 & \text{on } \partial\Omega , \\ u = 1 & \text{on } \partial D . \end{cases}$$

Let us recall that

$$cap_\Omega(D) = \int_{\Omega\setminus D} |\nabla u_D^\Omega|^2 = \int_{\partial D} |\nabla u_D^\Omega| \ .$$

Let Ω_0 and Ω_1 be two convex, open domains of $I\!R^N$, D_0 and D_1 be two convex, bounded, open subdomains of Ω_0 and Ω_1 respectively. Let us set

$$\forall s \in [0,1],\ \Omega_s = (1-s)\Omega_0 + s\Omega_1 \text{ and } D_s = (1-s)D_0 + sD_1$$

where the sum in the above equalities has to be understand in the Minkowski sense (see section 2 below). Borell's inequality states that

$$(2) \qquad \forall y \in \Omega_s\setminus D_s,\ u_{\Omega_s}^{D_s}(y) \geq \sup_{y_0,y_1} \min\{u_{\Omega_0}^{D_0}(y_0), u_{\Omega_1}^{D_1}(y_1)\}\ ,$$

where the supremum is taken over any pair $(y_0, y_1) \in (\Omega\setminus D_0) \times (\Omega\setminus D_1)$ such that $y_s = (1-s)y_0 + sy_1$. In particular, when $\Omega_0 = \Omega_1 = \Omega$ and $D_0 = B_\rho(x_0)$, $D_1 = B_\rho(x_1)$, then $D_s = B_\rho(x_s)$ and a simple consequence of Borell's inequality is that: $\forall w \in \partial B_1(0)$,

$$|\nabla u_\Omega^{B_\rho(x_s)}(x_s+\rho w)| \leq (1-s)|\nabla u_\Omega^{B_\rho(x_0)}(x_0+\rho w)| + s|\nabla u_\Omega^{B_\rho(x_1)}(x_1+\rho w)|$$

(see Proposition 2.4 below). Integrating the previous inequality over $\partial B_1(0)$ then gives the following inequality on the capacities:

$$cap_\Omega(B_\rho(x_s)) \leq (1-s)\, cap_\Omega(B_\rho(x_0)) + s\, cap_\Omega(B_\rho(x_1))\ .$$

Putting the above inequality into equality (1) gives the convexity of the Robin function when one lets $\rho \to 0^+$. The concavity of the harmonic radius can be obtained with a similar method, by choosing in a suitable way different radii for the ball centered at x_0 and for the ball centered at x_1.

The strict concavity of the harmonic radius (or the strict convexity of the Robin function) is much harder to establish and much more technical. For proving it, one of key points amounts to study closely the cases of equality in Borell's inequality (2). Our main result (Theorem 2.3) states that there is an equality in (2) if and only if there are $a > 0$ and $b \in I\!R^N$ such that

$$D_1 = aD_0 + b \text{ and } \Omega_1 = a\Omega_0 + b\ .$$

The cases of equality in inequality (2) - or, more precisely, in a Brunn-Minkowski inequality for capacity derived from (2) in [3] - have already been studied when $\Omega_i = I\!R^N$ in [6]. The proof in [6] is done in two steps. In the first step, the result is proved for sets D_0 and D_1 sufficiently close to balls by computing the first and second derivative of the capacity with respect to the domain. In the second step, the authors use the fact that the level sets of the capacity potential in $I\!R^N$ are asymptotically close to balls as well as a continuation method to complete the proof. The proof of Theorem 2.3 involves completely different arguments since, in our situation, i.e., when the sets Ω_i are bounded, the level sets of the functions $u_{D_1}^{\Omega_i}$ are never close to balls.

The paper is organized in the following way. In section 2, we study the cases of equality in Borell's inequality (2). The strict concavity of the harmonic radius is established in section 3. Some strong maximum principles for viscosity solutions needed in section 3 are recalled in the appendix.

Let us finally recall that the main results of this paper were announced in the note [7].

2 Conditions of equality in Borell's inequality

Let Ω be a convex, open domain of $I\!R^N$, D be a bounded, convex, open subdomain of Ω. We denote by u_D^Ω the solution of

$$(3)\qquad \begin{cases} -\Delta u = 0 & \text{in } \Omega\backslash\overline{D} \\ u = 0 & \text{on } \partial\Omega \\ u = 1 & \text{on } \partial D \end{cases}$$

In [12, 13, 14] (see also [22]) Gabriel has proved that the level sets of the functions u_D^Ω are bounded, convex, with positive gaussian curvature and that $\nabla u \neq 0$ in $\Omega\backslash\overline{D}$.

Let us now consider Ω_0 and Ω_1 two convex, open domains of $I\!R^N$, D_0 and D_1 two bounded, convex, open subdomains of Ω_0 and Ω_1 respectively. For $s \in [0,1]$, we denote by Ω_s and D_s the following sets:

$$\Omega_s = (1-s)\Omega_0 + s\Omega_1 = \{x \in I\!R^N,\ \exists x_0 \in \Omega_0,\ \exists x_1 \in \Omega_1 \text{ with } x = (1-s)x_0 + sx_1\}$$

and

$$D_s = (1-s)D_0 + sD_1 = \{x \in I\!R^N,\ \exists x_0 \in D_0,\ \exists x_1 \in D_1 \text{ with } x = (1-s)x_0 + sx_1\}\ .$$

Then Ω_s is a convex, open domain of $I\!R^N$, and that D_s is a bounded, convex, open subdomain of Ω_s.

In [3], Borell obtained the following result by refining Lewis method: Let us denote by $\widetilde{u}_s$ the function

$$\forall x \in \Omega_s \backslash D_s, \quad \widetilde{u}_s(x) = \sup_{x_0, x_1} \min\{u_{D_0}^{\Omega_0}(x_0), u_{D_1}^{\Omega_1}(x_1)\}$$

where the supremum is taken over the $x_0 \in \Omega_0 \backslash D_0$ and $x_1 \in \Omega_1 \backslash D_1$ such that $x = (1-s)x_0 + sx_1$.

Theorem 2.1 (Borell) *For any $x \in \Omega_s \backslash D_s$, we have*

$$u_{D_s}^{\Omega_s}(x) \geq \widetilde{u}_s(x) \ .$$

In the sequel we need the following reformulation of Borell's result:

Lemma 2.2 *The function $\widetilde{u}_s$ is a viscosity subsolution of equation (3) for $\Omega = \Omega_s$ and $D = D_s$.*

The Lemma implies Borell's inequality thanks to standard comparison principles for viscosity solutions. The idea of using viscosity solutions in order to establish qualitative properties of solutions of p.d.e.s is not new: For instance, several results concerning the convexity of various p.d.e.s have been established in [1] by using viscosity methods.

The proof of Lemma 2.2 is given below although most arguments of this proof can be traced back to Lewis or Borell's papers. We only give it because several estimates are crucial for the sequel.

Our main Theorem states that $\widetilde{u}_s \neq u_{D_s}^{\Omega_s}$ unless some strong restrictions on the functions $u_{D_0}^{\Omega_0}$ and $u_{D_1}^{\Omega_1}$ hold. More precisely, we have the following result:

Theorem 2.3 *Let us assume that $\widetilde{u}_s$ is harmonic for some $s \in (0,1)$. Then there are $a > 0$ and $b \in \mathbb{R}^N$ such that*

$$\forall x \in \Omega_1 \backslash D_1, \quad u_0(x) = u_1(ax+b) \ .$$

In particular,

$$\Omega_1 = a\Omega_0 + b \ \text{ and } \ D_1 = aD_0 + b \ .$$

Remark : Let us point out that, if $\Omega_0 = \Omega_1$, equality $\Omega_1 = a\Omega_0 + b$ implies that $(a,b) = (1,0)$ and thus that $D_0 = D_1$, unless Ω_0 is a cylinder (if $a = 1$ and $b \neq 0$), a cone (if $a \neq 1$ and $b = 0$) or the translate of a cone (if $a \neq 1$ and $b \neq 1$).

From now on, we concentrate on the proof of this Theorem. It shall need several steps. In order to simplify the notation, we set $u_0 = u_{D_0}^{\Omega_0}$, $u_1 = u_{D_1}^{\Omega_1}$. Let us start with proving Lemma 2.2.

Proof of Lemma 2.2: In order to prove that $\widetilde{u}_s$ is a subsolution of (3), we show that $\widetilde{u}_s = 0$ on $\partial\Omega_s$, that $\widetilde{u}_s = 1$ on ∂D_s and, finally, that $\widetilde{u}_s$ is a (viscosity) solution of

$$-\Delta u \leq 0 \text{ in } \Omega_s \backslash \overline{D}_s \ . \tag{4}$$

1) $\widetilde{u}_s = 0$ **on** $\partial\Omega_s$**.** Indeed, if $x = (1-s)x_0 + sx_1$ belongs to $\partial\Omega_s$, then x_0 and x_1 necessarily belong to $\partial\Omega_0$ and $\partial\Omega_1$. Thus

$$\widetilde{u}_s(x) = \min\{0, 0\} = 0 \ .$$

2) $\widetilde{u}_s = 1$ **on** ∂D_s**.** If x belongs to ∂D_s, then there is some $x_0 \in \partial D_0$ and $x_1 \in \partial D_1$ such that $x = (1-s)x_0 + sx_1$. Thus

$$\widetilde{u}_s(x) \geq \min\{u_0(x_0), u_1(x_1)\} = \min\{1, 1\} = 1 \ .$$

Since $0 \leq u_0, u_1 \leq 1$, the converse inequality is obvious. Therefore $\widetilde{u}_s = 1$ on ∂D_s.

3) $\widetilde{u}_s$ **is a (viscosity) solution of (4).** Let ϕ be a smooth test function such that $\widetilde{u}_s \leq \phi$ in $\Omega_s \backslash D_s$ and $\widetilde{u}_s(\bar{x}) = \phi(\bar{x})$ for some $\bar{x} \in \Omega_s \backslash \overline{D}_s$. We want to prove that

$$\Delta\phi(\bar{x}) \geq 0 \ .$$

Since $\nabla u_0 \neq 0$ in $\Omega_0 \backslash D_0$ and $\nabla u_1 \neq 0$ in $\Omega_1 \backslash D_1$ (cf. [22]), one shows easily that there are $\bar{x}_0 \in \Omega_0 \backslash \overline{D_0}$ and $\bar{x}_1 \in \Omega_1 \backslash \overline{D_1}$ such that $\bar{x} = (1-s)\bar{x}_0 + s\bar{x}_1$ and that

$$\widetilde{u}_s(\bar{x}) = u_0(\bar{x}_0) = u_1(\bar{x}_1) \ .$$

We claim that

$$\frac{\nabla\phi(\bar{x})}{|\nabla\phi(\bar{x})|} = \frac{\nabla u_0(\bar{x}_0)}{|\nabla u_0(\bar{x}_0)|} = \frac{\nabla u_1(\bar{x}_1)}{|\nabla u_1(\bar{x}_1)|} \tag{5}$$

and that

$$|\nabla\phi(\bar{x})| = \frac{1}{\frac{1-s}{|\nabla u_0(\bar{x}_0|} + \frac{s}{|\nabla u_1(x_1)|}} \ . \tag{6}$$

Proof of the claims (5) and (6): Let us set

$$\text{for } i=0,1,\;\; d_i = |\nabla u_i(\bar{x}_i)| \qquad \text{and} \qquad z_i = \frac{\nabla u_i(\bar{x}_i)}{|\nabla u_i(\bar{x}_i)|}\ .$$

Then, for any $\xi \in I\!\!R^N$ and any $h>0$, we have

$$\begin{array}{rcl}\widetilde{u}_s(\bar{x}) & \geq & \min\{u_0(\bar{x}_0 + \frac{h\xi}{1-s)}), u_1(\bar{x}_1 - \frac{h\xi}{s})\\ & \geq & \widetilde{u}_s(\bar{x}) + h\min\{\frac{d_0}{1-s} < z_0, \xi >, -\frac{d_1}{s} < z_1, \xi >\} + o(h)\ ,\end{array}$$

using Taylor formula. Thus we obtain

$$\forall \xi \in I\!\!R^N,\ \min\{\frac{d_0}{1-s} < z_0, \xi >, -\frac{d_1}{s} < z_1, \xi >\} \leq 0\ ,$$

which means that $z_0 = z_1$. We set $z = z_0 = z_1$.

Let now $\xi \in I\!\!R^N$ be nonzero and fixed. Then, for any $h>0$,

$$\phi(\bar{x} + \frac{h\xi}{d}) \geq \widetilde{u}_s(\bar{x} + \frac{h\xi}{d}) \geq \min\{u_0(\bar{x}_0 + \frac{h\xi}{d_0}), u_1(\bar{x}_1 + \frac{h\xi}{d_1})\}\ ,$$

where

(7) $$d = 1/[(1-s)/d_0 + s/d_1]\ .$$

Since $\phi(\bar{x}) = \widetilde{u}_s(\bar{x}) = u_0(\bar{x}_0) = u_1(\bar{x}_1)$, these inequalities imply that

$$\forall \xi \in I\!\!R^N,\ < \nabla\phi(\bar{x}), \frac{\xi}{d} > \geq < z, \xi >\ ,$$

after using Taylor formula. Since this holds true for any ξ, we obtain

$$\frac{\nabla\phi(\bar{x})}{d} = z\ ,$$

which entails (5) and (6).

It now suffices to prove the following **claim**:

(8) $$\frac{\nabla^2\phi(\bar{x})}{d^3}\ \geq\ (1-s)\frac{\nabla^2 u_0(\bar{x}_0)}{d_0^3} + s\frac{\nabla^2 u_1(\bar{x}_1)}{d_1^3}$$

(in the sense of symmetric matrices), where we have set again, for $i=0,1$, $d_i = |\nabla u_i(\bar{x}_i)|$ and

(9) $$d = 1/[(1-s)/d_0 + s/d_1] = |\nabla\phi(\bar{x})|\ .$$

Indeed, taking the trace in (8), we obtain that

$$\Delta\phi(\bar{x}) \geq 0$$

because u_0 and u_1 are harmonic. Therefore $\widetilde{u}_s$ is a subsolution.

Proof of claim (8) : Let us fix some $\xi \in I\!R^N$ such that $< z, \xi > \neq 0$ where

$$z = \frac{\nabla\phi(\bar{x})}{|\nabla\phi(\bar{x})|} = \frac{\nabla u_0(\bar{x}_0)}{|\nabla u_0(\bar{x}_0)|} = \frac{\nabla u_1(\bar{x}_1)}{|\nabla u_1(\bar{x}_1)|} \ .$$

For simplicity, we introduce the following notations:

$$A_0 =< \nabla^2 u_0(\bar{x}_0)\xi, \xi > \ , \ A_1 =< \nabla^2 u_1(\bar{x}_1)\xi, \xi >$$

and

$$\sigma =< z, \xi >, \qquad A =< \nabla^2\phi(\bar{x})\xi, \xi > \ .$$

If we set, for $h > 0$,

$$x_0^h = \bar{x}_0 + h\frac{\xi}{d_0}$$

and

$$x_1^h = \bar{x}_1 + h\frac{\xi}{d_1} + \frac{h^2}{2\sigma d_1}\left(\frac{A_0}{d_0^2} - \frac{A_1}{d_1^2}\right)\xi \ ,$$

then one can check easily that

$$u_0(x_0^h) = u_1(x_1^h) + h^2\epsilon(h) = u_0(x_0) + h\sigma + \frac{h^2 A_0}{2d_0^2} + h^2\epsilon(h) \ ,$$

using Taylor formula. Therefore,

$$\phi((1-s)x_0^h + sx_1^h) \geq \min\{u_0(x_0^h), u_1(x_1^h)\} = \phi(\bar{x}) + h\sigma + \frac{h^2 A_0}{2d_0^2} + h^2\epsilon(h) \ .$$

Computing the term in h^2 gives, since the term in h is vanishing,

$$\frac{A}{2d^2} + s\frac{d}{2d_1}\left(\frac{A_0}{d_0^2} - \frac{A_1}{d_1^2}\right) \ \geq \ \frac{A_0}{2d_0^2} \ .$$

From this and from (7), we deduce that

$$\frac{A}{d^3} \ \geq \ \frac{A_0}{d_0^2}\left(\frac{1}{d} - \frac{s}{d_1}\right) + s\frac{A_1}{d_1^3} = (1-s)\frac{A_0}{d_0^3} + s\frac{A_1}{d_1^3} \ .$$

In conclusion, we have for any ξ such that $< z, \xi > \neq 0$,

$$\frac{< \nabla^2\phi(\bar{x})\xi, \xi >}{d^3} \geq (1-s)\frac{< \nabla^2 u_0(\bar{x}_0)\xi, \xi >}{d_0^3} + s\frac{< \nabla^2 u_1(\bar{x}_1)\xi, \xi >}{d_1^3} ,$$

which implies the desired result for any ξ by density.

This completes the proof of Lemma 2.2.

$\square$

As the consequence of the previous proof, we have the following:

Proposition 2.4

1. *Let (x_0, x_1) belong to $(\Omega_0\backslash\overline{D_0}) \times (\Omega_1\backslash\overline{D_1})$ and be such that $u_0(x_0) = u_1(x_1)$ and*
$$\frac{\nabla u_0(x_0)}{|\nabla u_0(x_0)|} = \frac{\nabla u_1(x_1)}{|\nabla u_1(x_1)|} .$$
Then
$$\widetilde{u}_s((1-s)x_0 + sx_1) = u_0(x_0) = u_1(x_1) .$$

2. *Let us assume that D_0 and D_1 are smooth. Then we have, for any $x \in \partial D_s$,*
$$\liminf_{h\to 0^+} \frac{\widetilde{u}_s(x+hn) - 1}{h} \geq -\left[\frac{(1-s)}{|\nabla u_0(x_0)|} + \frac{s}{|\nabla u_1(x_1)|}\right]^{-1} ,$$
where n is a unit normal to D_s at x, for some $(x_0, x_1) \in \partial D_0 \times \partial D_1$ such that $x = (1-s)x_0 + sx_1$.

Remark 2.5 *If $\widetilde{u}_s$ is smooth, then (5), (6) and (8) hold true for $\phi = \widetilde{u}_s$.*

Proof of Proposition 2.4 :

1) Let us set $\lambda = u_0(x_0) = u_1(x_1) \in (0,1)$. We have

$$\{\widetilde{u}_s \geq \lambda\} = (1-s)\{u_0 \geq \lambda\} + s\{u_1 \geq \lambda\} ,$$

where, for simplicity, we denote by $\{\widetilde{u}_s \geq \lambda\}$ the set $\{x \in \Omega_s\backslash D_s \mid \widetilde{u}_s(x) \geq \lambda\}$, etc... Let x belong to $\partial\{\widetilde{u}_s \geq \lambda\}$ and be such that $n = -\frac{\nabla u_0(x_0)}{|\nabla u_0(x_0)|}$ is a normal to $\{\widetilde{u}_s \geq \lambda\}$ at x. Then a standard result in convex analysis states that, for any $(x_0', x_1') \in \{u_0 \geq \lambda\} \times \{u_1 \geq \lambda\}$ such that $x = (1-s)x_0' + sx_1'$, we have in fact: For $i = 0, 1$, $x_i' \in \partial\{u_i \geq \lambda\}$ and n is a normal to $\{u_i \geq \lambda\}$ at

x_i'. Accordingly, we have $u_i(x_i') = \lambda$. Moreover, since the level sets $\{u_i \geq \lambda\}$ are convex and smooth, with a positive gaussian curvature (cf. Lewis [22]), the fact that $u_i(x_i') = u_i(x_i) = \lambda$ and that n is normal to $\{u_i \geq \lambda\}$ at x_i and x_i' implies that $x_i = x_i'$ for $i = 0, 1$.

2) Since D_0 and D_1 are smooth, so is D_s. Following [3], it is easy to prove that $\widetilde{u}_s$ is derivable on $\Omega_s \backslash \overline{D_s}$. Moreover, because of the strict convexity of the level sets of u_1 and u_2, for any $x \in \Omega_s \backslash D_s$ there is a unique couple (x_0, x_1) in $(\Omega_0 \backslash D_0) \times (\Omega_1 \backslash D_1)$ such that $\widetilde{u}_s(x) = u_0(x_0) = u_1(x_1)$ and $x = (1-s)x_0 + sx_1$. For such a couple, we have, by using (5) and (6) with $\phi = \widetilde{u}_s$,

$$\frac{\nabla \widetilde{u}_s(x)}{|\nabla \widetilde{u}_s(x)|} = \frac{\nabla u_0(x_0)}{|\nabla u_0(x_0)|} = \frac{\nabla u_1(x_1)}{|\nabla u_1(x_1)|}$$

and

$$\frac{1}{|\nabla \widetilde{u}_s(x)|} = \frac{(1-s)}{|\nabla u_0(x_0)|} + \frac{s}{|\nabla u_1(x_1)|} \ .$$

We now choose any $x \in \partial D_s$. We denote by ν the outward normal to D_s at x. Then we have

$$\widetilde{u}_s(x + h\nu) = 1 + h < \nabla \widetilde{u}_s(x_h), \nu > \tag{10}$$

for some $x_h \in (x, x + h\nu)$. Let (x_0^h, x_1^h) be the unique couple such that $\widetilde{u}_s(x_h) = u_0(x_0^h) = u_1(x_1^h)$ and $x_h = (1-s)x_0^h + sx_1^h$. As h goes to 0^+, x_0^h and x_1^h converge, up to a subsequence, to some x_0 and x_1 belonging respectively to ∂D_0 and ∂D_1 and such that $x = (1-s)x_0 + sx_1$. Since the unit vector $\nu_0^h = -\frac{\nabla u_0(x_0^h)}{|\nabla u_0(x_0^h)|}$ is a normal to $\{u_0 \geq u_0(x_0^h)\}$ at x_0^h, the limit of ν_0^h when $h \to 0^+$ is also a normal to D_0 at x_0. Since D_0 is smooth and since, moreover, we have $x = (1-s)x_0 + sx_1$, one can show as in the proof of part 1 that this limit is necessarily equal to ν. Therefore we have

$$\lim_{h \to 0^+} \frac{\nabla \widetilde{u}_s(x_h)}{|\nabla \widetilde{u}_s(x_h)|} = \lim_{h \to 0^+} \frac{\nabla u_0(x_0^h)}{|\nabla u_0(x_0^h)|} = -\nu \tag{11}$$

and

$$\begin{aligned} \lim_{h\to 0^+} \tfrac{1}{|\nabla \widetilde{u}_s(x_h)|} &= \lim_{h\to 0^+} \left[\tfrac{(1-s)}{|\nabla u_0(x_0^h)|} + \tfrac{s}{|\nabla u_1(x_1^h)|} \right] \\ &= \tfrac{(1-s)}{|\nabla u_0(x_0)|} + \tfrac{s}{|\nabla u_1(x_1)|} \ . \end{aligned} \tag{12}$$

From (10), (11) and (12), we deduce that

$$\liminf_{h \to 0^+} \frac{\widetilde{u}_s(x + h\nu) - 1}{h} \ \geq \ -\left[\frac{(1-s)}{|\nabla u_0(x_0)|} + \frac{s}{|\nabla u_1(x_1)|} \right]^{-1} \ .$$

$\square$

For the proof of Lemma 2.7 below, we need the following well-known result (see for instance [1]):

Proposition 2.6 *Let $\mathcal{S}_N^-$ the set of symmetric definite negative matrices. Then*

1. *The map $\Phi : \mathcal{S}_N^- \to I\!R$ defined by*

$$\forall S \in \mathcal{S}_N^-,\ \Phi(S) = 1/Tr(S^{-1})$$

   is convex on $\mathcal{S}_N^-$.

2. *Moreover, for any symmetric negative matrices S_0 and S_1 and any $s \in (0,1)$, the following equality holds true:*

$$\Phi((1-s)S_0 + sS_1) = (1-s)\Phi(S_0) + s\Phi(S_1)$$

   if and only if $S_1 = \gamma S_0$, with $\gamma = Tr(S_1)/Tr(S_0)$.

Lemma 2.7 *Let us assume that $\widetilde{u}_s$ is harmonic for some $s \in (0,1)$. Let (x_0, x_1) belong to $(\Omega_0 \backslash D_0) \times (\Omega_1 \backslash D_1)$ and be such that*

1. *$u_0(x_0) = u_1(x_1)$ and*
2. *$\dfrac{\nabla u_0(x_0)}{|\nabla u_0(x_0)|} = \dfrac{\nabla u_1(x_1)}{|\nabla u_1(x_1)|}$.*

Then we have

$$\frac{\nabla^2 u_0(x_0)}{|\nabla u_0(x_0)|^2} = \frac{\nabla^2 u_1(x_1)}{|\nabla u_1(x_1)|^2} \ . \tag{13}$$

Proof of Lemma 2.7: The proof is rather long and technical. Let x_0 and x_1 be as in the Lemma. Let us set

$$n = \frac{\nabla u_0(x_0)}{|\nabla u_0(x_0)|} = \frac{\nabla u_1(x_1)}{|\nabla u_1(x_1)|} \ .$$

Using Proposition 2.4, part 1, we have

$$\widetilde{u}_s((1-s)x_0 + sx_1) = u_0(x_0) = u_1(x_1) \ . \tag{14}$$

Let us set

$$x_s = (1-s)x_0 + sx_1 \ .$$

Since $\widetilde{u}_s$ is harmonic, $\widetilde{u}_s$ is smooth. Therefore, Remark 2.5 states that one can replace ϕ by $\widetilde{u}_s$ into (5), (6) and (8). Namely, we have

$$(15)\quad \frac{1}{|\nabla \widetilde{u}_s(x_s)|} = \frac{1-s}{|\nabla u_0(x_0)|} + \frac{s}{|\nabla u_1(x_1)|} \qquad \text{and} \qquad \frac{\nabla \widetilde{u}_s(x_s)}{|\nabla \widetilde{u}_s(x_s)|} = n \, .$$

Moreover, we have, thanks to (8),

$$\frac{\nabla^2 \widetilde{u}_s(x_s)}{|\nabla \widetilde{u}_s(x_s)|^3} \geq (1-s)\frac{\nabla^2 u_0(x_0)}{|\nabla u_0(x_0)|^3} + s\frac{\nabla^2 u_1(x_1)}{|\nabla u_1(x_1)|^3} \, .$$

Since u_0, u_1 and $\widetilde{u}_s$ are harmonic, the matrix on the left-hand side has the same trace than the matrix on the right-hand side, which implies their equality:

$$(16)\qquad \frac{\nabla^2 \widetilde{u}_s(x_s)}{|\nabla \widetilde{u}_s(x_s)|^3} = (1-s)\frac{\nabla^2 u_0(x_0)}{|\nabla u_0(x_0)|^3} + s\frac{\nabla^2 u_1(x_1)}{|\nabla u_1(x_1)|^3} \, .$$

From now on, we set, for $i = 0, 1$,

$$(17)\quad A_i = \nabla^2 u_i(x_i),\ d_i = |\nabla u_i(x_i)|,\ \widetilde{A}_s = \nabla^2 \widetilde{u}_s(x_s) \ \text{ and } \ \widetilde{d}_s = |\nabla \widetilde{u}_s(x_s)| \, ,$$

$$(18)\quad B_i = \nabla^2 u_i(x_i)_{|_{n^\perp}},\ \theta_i = <\nabla^2 u_i(x_i)n, n> \ \text{ and } \ \xi_i = (\nabla^2 u_i(x_i)n)_{|_{n^\perp}}$$

and

$$(19)\quad \widetilde{B}_s = \nabla^2 \widetilde{u}_s(x_s)_{|_{n^\perp}},\ \widetilde{\theta}_s = <\nabla^2 \widetilde{u}_s(x_s)n, n> \ \text{ and } \ \widetilde{\xi}_s = (\nabla^2 \widetilde{u}_s(x_s)n)_{|_{n^\perp}}$$

where, for a symmetric matrix S, $S_{|_{n^\perp}}$ means the restriction of S to the orthogonal space of n, while, for a vector v, $v_{|_{n^\perp}}$ means the projection of v onto the orthogonal space of n. Let us recall that the matrices B_1 and B_2 are negative definite, because the level sets $\{u_i > t\}$ are convex with positive (Gaussian) curvature. Moreover, (16) implies that $\widetilde{B}_s$ is also negative definite.

We claim that

$$(20)\qquad \frac{\widetilde{B}_s}{\widetilde{d}_s} \geq \left((1-s)\left(\frac{B_0}{d_0}\right)^{-1} + s\left(\frac{B_1}{d_1}\right)^{-1} \right)^{-1} \, .$$

Proof of (20) : Let v_0 and v_1 be orthogonal to n, and let us set $v_s = (1-s)v_0 + sv_1$. For $i = 0, 1$ and $h > 0$, we have

$$u_i(x_i + hv_i - \frac{h^2}{2d_i} < B_i v_i, v_i > n) = u_i(x_i) + o(h^2) \, ,$$

because $< v_i, n >= 0$. From (14) and (17), we deduce that

$$\widetilde{u}_s(x_s + hv_s - \frac{h^2}{2}\left((1-s)\frac{< B_0 v_0, v_0 >}{d_0} + s\frac{< B_1 v_1, v_1 >}{d_1}\right) n) \geq \widetilde{u}_s(x_s) + o(h^2) ,$$

since $< v_s, n >= 0$. Using Taylor formula and (15), we get

$$\frac{< \widetilde{B}_s v_s, v_s >}{\widetilde{d}_s} \geq (1-s)\frac{< B_0 v_0, v_0 >}{d_0} + s\frac{< B_1 v_1, v_1 >}{d_1} . \tag{21}$$

Let us now fix v_s. The previous inequality holds true for any v_0, v_1 orthogonal to n such that $v_s = (1-s)v_0 + sv_1$. Take the maximum of the right-hand side of inequality (21) over such v_0, v_1 gives, after some computations (for similar computations, see [1]):

$$\frac{< \widetilde{B}_s v_s, v_s >}{\widetilde{d}_s} \geq < \left((1-s)\left(\frac{B_0}{d_0}\right)^{-1} + s\left(\frac{B_1}{d_1}\right)^{-1}\right)^{-1} v_s, v_s > ,$$

for any $v_s \perp n$. Hence our claim (20) is proved.

We now claim that

$$\frac{B_0}{d_0^2} = \frac{B_1}{d_1^2} \quad \text{and} \quad \frac{\theta_0}{d_0^2} = \frac{\theta_1}{d_1^2} \tag{22}$$

while

$$\frac{\widetilde{B}_s}{\widetilde{d}_s^2} = \frac{B_0}{d_0^2} \quad \text{and} \quad \frac{\widetilde{\theta}_s}{\widetilde{d}_s^2} = \frac{\theta_0}{d_0^2} \quad . \tag{23}$$

Proof of (22) and (23) : Let us consider the application $\Phi(S) = 1/Tr(S^{-1})$ defined of the set of symmetric negative matrices. Taking the trace in (20) and using the convexity of Φ (see Proposition 2.6) gives

$$\frac{\widetilde{d}_s}{Tr(\widetilde{B}_s)} \leq (1-s)\frac{d_0}{Tr(B_0)} + s\frac{d_1}{Tr(B_1)} . \tag{24}$$

Since u_0, u_1 and $\widetilde{u}_s$ are harmonic, inequality (24) can also be rewritten as

$$\frac{\widetilde{d}_s}{\widetilde{\theta}_s} \geq (1-s)\frac{d_0}{\theta_0} + s\frac{d_1}{\theta_1} , \tag{25}$$

where θ_0, θ_1 and $\widetilde{\theta}_s$ are defined in (18) and (19). From (16), we have

$$\frac{\widetilde{\theta}_s}{\widetilde{d}_s^3} = (1-s)\frac{\theta_0}{d_0^3} + s\frac{\theta_1}{d_1^3} . \tag{26}$$

We replace $\widetilde{\theta}_s$ and $\widetilde{d}_s$ in inequality (25) by using (15) and (26), and we obtain:

$$\frac{\theta_0 d_1^2}{\theta_1 d_0^2} + \frac{\theta_1 d_0^2}{\theta_0 d_1^2} \leq 2 \, . \tag{27}$$

But, for any $z > 0$, we have $z + 1/z \geq 2$ on $\mathbb{R}^+$, with an equality if and only if $z = 1$. This proves that in fact (27) is an equality and we have

$$\frac{\theta_0 d_1^2}{\theta_1 d_0^2} = 1 \, , \qquad \text{i.e.,} \quad \frac{\theta_0}{d_0^2} = \frac{\theta_1}{d_1^2} \, . \tag{28}$$

Moreover, the fact that (27) is an equality implies that inequality (24) is also an equality, which finally means an equality in (20). Then Proposition 2.6 states that

$$\frac{B_1}{d_1} = \gamma \frac{B_0}{d_0} \quad \text{where} \quad \gamma = \frac{d_0 Tr(B_1)}{d_1 Tr(B_0)} \, . \tag{29}$$

Taking the trace in (29) gives

$$\frac{\theta_1}{d_1} = \gamma \frac{\theta_0}{d_0} \, ,$$

because the u_i are harmonic. Using (28), we deduce that $\gamma = \frac{d_1}{d_0}$. Therefore, using (29) again, we can prove (22). Then the equalities in (23) follow from (15), (16) and (22) combined with the fact that $\widetilde{u}_s$ is harmonic. Hence our claims (22) and (23) are proved.

To complete the proof of the Lemma, it remains to show that we have:

$$\xi_1 = \gamma^2 \xi_0 \, . \tag{30}$$

with $\gamma = d_1/d_0$. This is rather technical, and we omit some details of the computation.

Proof of (30) : Let $\alpha \in \mathbb{R}$, and v_0, v_1 be orthogonal to n. Let us set

$$v_s = (1-s)v_0 + s v_1, \; w_i = \frac{\alpha}{d_i} n + v_i \text{ (for } i = 0,1) \, .$$

Then, if we set

$$w_s = (1-s) w_0 + s w_1 \, ,$$

we have, from (15),

$$w_s = \frac{\alpha}{\widetilde{d}_s} n + v_s \, .$$

Let us notice that we have, for $i = 0, 1$,

$$u_i(x_i + hw_i - \frac{h^2}{2d_i} < A_i w_i, w_i > n) = u_i(x_i) + h\alpha + o(h^2)$$

from which we deduce that

$$\widetilde{u}_s \left(x_s + hw_s - \frac{h^2}{2} \left((1-s) \frac{< A_0 w_0, w_0 >}{d_0} + s \frac{< A_1 w_1, w_1 >}{d_1} \right) n \right)$$
$$\geq \widetilde{u}_s(x_s) + h\alpha + o(h^2) \; .$$

Considering the Taylor expansion of the left-hand side of this inequality yields to

$$\text{(31)} \qquad \frac{< \widetilde{A}_s w_s, w_s >}{\widetilde{d}_s} \geq (1-s) \frac{< A_0 w_0, w_0 >}{d_0} + s \frac{< A_1 w_1, w_1 >}{d_1} \; .$$

Let us notice that, for $i = 0, 1$, we have

$$< A_i w_i, w_i > = < B_i v_i, v_i > + \frac{2\alpha}{d_i} < \xi_i, v_i > + \alpha^2 \theta_i / d_i^2$$

where the B_i, ξ_i and θ_i are defined in (18), while

$$< \widetilde{A}_s w_s, w_s > = < \widetilde{B}_s v_s, v_s > + \frac{2\alpha}{\widetilde{d}_s} < \widetilde{\xi}_s, v_s > + \alpha^2 \theta_s / \widetilde{d}_s^2 \; ,$$

where $\widetilde{B}_s$, $\widetilde{\xi}_s$ and $\widetilde{\theta}_s$ are defined in (19). From now on, to simplify the notations, we set

$$\gamma = \frac{d_1}{d_0} \qquad \text{and} \qquad \sigma = (1-s) + \frac{s}{\gamma} \; .$$

Let us notice that

$$\frac{\widetilde{d}_s}{d_0} = \frac{1}{\sigma} \; .$$

Eliminating $\widetilde{A}_s$, $\widetilde{d}_s$, A_1 and d_1 in (31) by using (22), (23) and the definition of γ and σ gives:

$$\begin{aligned} < \frac{B_0}{\sigma d_0} v_s, v_s > + \frac{2\alpha\sigma^2}{d_0^2} < \widetilde{\xi}_s, v_s > \; \geq \; & (1-s)(< \frac{B_0}{d_0} v_0, v_0 > + \frac{2\alpha}{d_0^2} < \xi_0, v_0 >) + \\ & s(\gamma < \frac{B_0}{d_0} v_1, v_1 > + \frac{2\alpha}{\gamma^2 d_0^2} < \xi_1, v_1 >) \end{aligned}$$

which holds true for any v_0 and v_1 orthogonal to n, such that $v_s = (1-s)v_0 + sv_1$. In order to exploit this inequality, we eliminate v_1 by setting $v_1 = (v_s - (1-s)v_0)/s$. Then the previous inequality becomes

$$\begin{aligned}(\frac{1}{\sigma}-\frac{\gamma}{s}) < \frac{B_0}{d_0}v_s, v_s > + \frac{2\alpha}{d_0^2} < \sigma^2\widetilde{\xi}_s - \frac{1}{\gamma^2}\xi_1, v_s > \ \geq \\ (1-s)\Big\{(1+\gamma(1-s)/s) < \frac{B_0}{d_0}v_0, v_0 > + \\ 2 < \left(\frac{\alpha}{d_0^2}(\xi_0 - \xi_1/\gamma^2) - \frac{\gamma}{sd_0}B_0 v_s\right), v_0 > \Big\}\end{aligned}$$

where

$$(1+\gamma(1-s)/s) = \gamma\sigma/s \ .$$

This inequality holds true for any v_0 orthogonal to n. Let us now recall that $B_0 < 0$. Hence, maximizing right-hand side of the previous inequality with respect to v_0 gives:

$$\begin{aligned}(\frac{1}{\sigma}-\frac{\gamma}{s}) < \frac{B_0}{d_0}v_s, v_s > + \frac{2\alpha}{d_0^2} < \sigma^2\widetilde{\xi}_s - \frac{1}{\gamma^2}\xi_1, v_s > \ \geq \\ -\frac{(1-s)sd_0}{\gamma\sigma} < B_0^{-1}\left(\frac{\alpha}{d_0^2}(\xi_0 - \xi_1/\gamma^2) - \frac{\gamma}{sd_0}B_0 v_s\right), \left(\frac{\alpha}{d_0^2}(\xi_0 - \xi_1/\gamma^2) - \frac{\gamma}{sd_0}B_0 v_s\right) >\end{aligned}$$

Since α is arbitrary and $B_0 < 0$, this last inequality when $\alpha \to +\infty$ implies that $\xi_0 - \xi_1/\gamma^2 = 0$, which is precisely (30).

□

In order to understand the meaning of equality (13) in Lemma 2.7, we need the following technical result:

Lemma 2.8 *Let us assume that u_0 and u_1 satisfy the conclusion of Lemma 2.7: Namely, we assume that, for any couple $(x_0, x_1) \in (\Omega_0 \backslash \overline{D_0}) \times (\Omega_1 \backslash \overline{D_1})$, such that*

$$u_0(x_0) = u_1(x_1) \qquad \text{and} \qquad \frac{\nabla u_0(x_0)}{|\nabla u_0(x_0)|} = \frac{\nabla u_1(x_1)}{|\nabla u_1(x_1)|} \tag{32}$$

we have

$$\frac{\nabla^2 u_0(x_0)}{|\nabla u_0(x_0)|^2} = \frac{\nabla^2 u_1(x_1)}{|\nabla u_1(x_1)|^2} \ . \tag{33}$$

Then, for any $w \in I\!R^N$, the set

$$K = \{(x_0, x_1) \text{ such that (32) holds true}\}$$

is locally invariant under the dynamical system

$$\frac{d}{dt}(x_0(t), x_1(t)) = \Phi(x_0(t), x_1(t)) \tag{34}$$

where

$$\Phi(x_0, x_1) = (\frac{w}{|\nabla u_0(x_0)|}, \frac{w}{|\nabla u_1(x_1)|}) \ .$$

Proof of Lemma 2.8: Let us first notice that K can be rewritten as

$$K = \left\{ (x_0, x_1) \mid u_0(x_0) \frac{\nabla u_0(x_0)}{|\nabla u_0(x_0)|} = u_1(x_1) \frac{\nabla u_1(x_1)}{|\nabla u_1(x_1)|} \right\} \ .$$

If we set

$$\phi(x_0, x_1) = u_0(x_0) \frac{\nabla u_0(x_0)}{|\nabla u_0(x_0)|} - u_1(x_1) \frac{\nabla u_1(x_1)}{|\nabla u_1(x_1)|} \ ,$$

then

$$D\phi(x_0, x_1) = \begin{pmatrix} \frac{\nabla u_0 \nabla u_0^*}{|\nabla u_0|} + \frac{u_0}{|\nabla u_0|}(\nabla^2 u_0 - \frac{\nabla u_0 \nabla u_0^* \nabla^2 u_0}{|\nabla u_0|^2}) \\ -\frac{\nabla u_1 \nabla u_1^*}{|\nabla u_1|} - \frac{u_1}{|\nabla u_1|}(\nabla^2 u_1 - \frac{\nabla u_1 \nabla u_1^* \nabla^2 u_1}{|\nabla u_1|^2}) \end{pmatrix}$$

where for simplicity, we have omitted x_0 and x_1 in the expression. Since the restriction of $\nabla^2 u_0$ to the orthogonal of ∇u_0 is negative definite, it is easy to check that $D\phi$ is surjective. Hence K is a smooth manifold. In order to prove that K is invariant with respect to Φ, it is enough to check that

$$\forall (x_0, x_1) \in K, \ D\phi(x_0, x_1)\Phi(x_0, x_1) = 0 \ .$$

This is a direct consequence of the fact that equality (33) holds true on K, and we leave this verification to the reader.

$\square$

Proof of Theorem 2.3: The idea is to prove the result locally, then to use a connectedness argument to obtain the global result. We proceed in three steps.

First step : We claim that, for any $x_0 \in \Omega_0 \backslash \overline{D_0}$, there exists a unique $x_1 \in \Omega_1 \backslash \overline{D_1}$, denoted $\Lambda(x_0)$, such that $(x_0, x_1) \in K$, where K is defined in Lemma 2.8.

For proving this claim, let $x_0 \in \Omega_0 \backslash \overline{D_0}$ and n be the outward normal to the smooth convex set $\{u_0 \geq u_0(x_0)\}$. Since the set $\{u_1 \geq u_0(x_0)\}$ is also smooth and convex, there is some x_1 such that $u_1(x_1) = u_0(x_0)$ and such that n is the normal to $\{u_1 \geq u_0(x_0)\}$ at x_1. Moreover, this point is unique since the set $\{u_1 \geq u_0(x_0)\}$ is strictly convex. Now it is clear that the point (x_0, x_1) belongs to K because, by construction,

$$u_1(x_1) = u_0(x_0) \qquad \text{and} \qquad -n = \frac{\nabla u_0(x_0)}{|\nabla u_0(x_0)|} = \frac{\nabla u_1(x_1)}{|\nabla u_1(x_1)|} \,.$$

Hence the claim is proved. Let us point out that, by definition, $\Lambda(\cdot)$ is invertible.

Second step : Let us now prove that, for any $\bar{x}_0 \in \Omega_0 \backslash \overline{D_0}$ and any $R > 0$ such that $B_R(\bar{x}_0) \subset\subset \Omega_0 \backslash \overline{D_0}$, there exists some constants $a > 0$ and $b \in \mathbb{R}^N$ such that

$$\forall x_0 \in B_R(\bar{x}_0),\ |\nabla u_0(x_0)| = a|\nabla u_1(x_1)| \tag{35}$$

with $x_1 = \Lambda(x_0)$, and

$$x_1 = \Lambda(x_0) = ax_0 + (\Lambda(\bar{x}_0) - a\bar{x}_0) = ax_0 + b \,. \tag{36}$$

Let us point out that the constants a and b depend a priori on $\bar{x}_0$ and R: $a = a_{\bar{x}_0}$ and $b = b_{\bar{x}_0}$.

Proof of (35) and (36) : Let $x_0 \in B_R(\bar{x}_0)$ with $x_0 \neq \bar{x}_0$. Let us set $x_1 = \Lambda(x_0)$ and $\bar{x}_1 = \Lambda(\bar{x}_0)$. Let us also set $w = x_0 - \bar{x}_0$. Let us consider $(x_0(\cdot), x_1(\cdot))$ the solution of the differential equation (34) starting from $(\bar{x}_0, \bar{x}_1)$. This solution is defined until either $x_0(\cdot)$ or $x_1(\cdot)$ reaches the boundary of $\Omega \backslash D_0$ or $\Omega \backslash D_1$, i.e., when $u_0(x_0(\cdot))$ or $u_1(x_1(\cdot))$ equals 0 or 1. Since K is invariant, we have $u_0(x_0(\cdot)) = u_1(x_1(\cdot))$. Therefore $x_0(\cdot)$ and $x_1(\cdot)$ reaches the boundary of $\Omega \backslash D_0$ and of $\Omega \backslash D_1$ in the same time, when $u_0(x_0(\cdot)) = u_1(x_1(\cdot))$ equals 0 or 1. In particular, the solution $(x_0(\cdot), x_1(\cdot))$ exists at least until the unique $\bar{t} > 0$ such that $x_0(\bar{t}) = x_0$. Since it stays in K, we also have

$$u_1(x_1(\bar{t})) = u_0(x_0(\bar{t})) = u_0(x_0)$$

and

$$\frac{\nabla u_1(x_1(\bar{t}))}{|\nabla u_1(x_1(\bar{t}))|} = \frac{\nabla u_0(x_0(\bar{t}))}{|\nabla u_0(x_0(\bar{t}))|} = \frac{\nabla u_0(x_0)}{|\nabla u_0(x_0)|} \,.$$

Therefore the point $(x_0, x_1(\bar{t}))$ belongs to K. Since (x_0, x_1) also belongs to K, we can conclude, from the first step, that $x_1(\bar{t}) = x_1 = \Lambda(x_0)$. Let us now compute the ratio $|\nabla u_0(x_0(t))|/|\nabla u_1(x_1(t))|$ on $[0,\bar{t}]$. We have

$$\frac{d}{dt}\left(\frac{|\nabla u_0|}{|\nabla u_1|}\right) = \frac{|\nabla u_0|}{|\nabla u_1|}\left(< \frac{\nabla^2 u_0}{|\nabla u_0|^2}w, \frac{\nabla u_0}{|\nabla u_0|} > - < \frac{\nabla^2 u_1}{|\nabla u_1|^2}w, \frac{\nabla u_1}{|\nabla u_1|} >\right)$$

where, to simplify the expressions, we have omitted the dependence of u_0 and u_1 with respect to $x_0(t)$ and $x_1(t)$. Using Lemma 2.7, we see that the right-hand side vanishes. Therefore, the ratio $a = |\nabla u_0(x_0(t))|/|\nabla u_1(x_1(t))|$ is constant along the trajectory and we have, in particular, for $t = 0$ and $t = \bar{t}$,

$$a = \frac{|\nabla u_0(x_0)|}{|\nabla u_1(x_1)|} = \frac{|\nabla u_0(\bar{x}_0)|}{|\nabla u_1(\bar{x}_1)|} .$$

This proves claim (35) in a neighborhood of $(\bar{x}_0, \bar{x}_1)$.

Since the ratio $a = |\nabla u_0(x_0(t))|/|\nabla u_1(x_1(t))|$ is constant along the trajectory $(x_0(\cdot), x_1(\cdot))$, we have from (34),

$$\forall t \in [0,\bar{t}], \qquad x_1'(t) = \frac{w}{|\nabla u_1(x_1(t))|} = \frac{aw}{|\nabla u_0(x_0(t))|} = a x_0'(t) .$$

Integrating this equality over $[0,\bar{t}]$ gives (36).

Third step : To complete the proof of the Theorem, it suffices to establish that the constants a and b defined in (35) and (36) do not depend on $\bar{x}_0$.

Let x_0 and y_0 belong to $\Omega_0\backslash\overline{D_0}$. We suppose that there are $R_{x_0} > 0$ and $R_{y_0} > 0$, denoted respectively $R_x = R_{x_0}$ and $R_y = R_{y_0}$ for simplicity, such that

$$B_{R_x}(x_0) \subset \Omega_0\backslash\overline{D_0},\ B_{R_y}(y_0) \subset \Omega_0\backslash\overline{D_0} \qquad \text{and} \qquad B_{R_x}(x_0) \cap B_{R_y}(y_0) \neq \emptyset .$$

From (35) we have

$$|\nabla u_0(z)| = a_{x_0}|\nabla u_1(\Lambda(z))| ,\ \forall z \in B_{R_x}(x_0) \cap B_{R_y}(y_0)$$

$$|\nabla u_0(z)| = a_{y_0}|\nabla u_1(\Lambda(z))| ,\ \forall z \in B_{R_x}(x_0) \cap B_{R_y}(y_0) .$$

Therefore $a_{x_0} = a_{y_0} = a$. Then, from (36), we have

$$\Lambda(z) = az + b_{x_0} \quad \forall z \in B_{R_x}(x_0) \cap B_{R_y}(y_0)$$

$$\Lambda(z) = az + b_{y_0} \quad \forall z \in B_{R_x}(x_0) \cap B_{R_y}(y_0) ,$$

from which we deduce that $b_{x_0} = b_{y_0} = b$.

If now x_0 and y_0 are any two points of $\Omega\backslash\overline{D_0}$, we can connect x_0 and y_0 by a finite number of balls $(B_i)_{i=1,\ldots,k}$, with $x_0 \in B_0$, $y_0 \in B_k$, and $B_i \cap B_{i+1} \neq \emptyset$, because $\Omega_0\backslash\overline{D_0}$ is connected. Then we can apply the previous construction to establish that there are constants $a > 0$ and $b \in I\!\!R^N$, such that

$$\forall z \in \Omega_0\backslash\overline{D_0},\ \Lambda(z) = az + b\ .$$

This implies, from the definition of the map $\Lambda(\cdot)$, that

$$\forall z \in \Omega_0\backslash\overline{D_0},\ u_1(az+b) = u_0(z)\ ,$$

because $(z, \Lambda(z)) \in K$.

$\square$

3 Strict concavity of the harmonic radius

Let us recall some notations of the introduction. If Ω is an open domain of $I\!\!R^N$, we denote by $t(\cdot)$ the Robin function of Ω and by $r(\cdot)$ the harmonic radius. These functions are related by the equality:

$$\forall y \in \Omega,\ t(y) = F(r(y))\ ,$$

where F is defined by

$$\forall t \geq 0,\ F(t) = \gamma t^{2-N}\ .$$

A harmonic center of Ω is a point where $r(\cdot)$ attains its maximum. In order to simplify the notations, we denote by u_D (instead of u_D^Ω) the solution to

$$\begin{cases} \nabla u = 0 \text{ in } \Omega\backslash\overline{D} \\ u = 0 \text{ on } \partial\Omega,\ u = 1 \text{ on } \partial D \end{cases}$$

where D is a subdomain of Ω.

Our aim is to prove the following result:

Theorem 3.1 *If Ω is an open subset of $I\!\!R^N$, the harmonic radius $r(\cdot)$ is strictly concave in Ω, unless Ω is a cylinder, a cone or the translate of a cone.*

As a consequence, we have

Corollary 3.2 *If Ω is an open convex bounded subset of $I\!R^N$, the Robin function $t(\cdot)$ is strictly convex in Ω, and there is a unique harmonic center in Ω.*

Proof : The strict convexity of the Robin function comes from the fact that $r(x) = F(t(x))$, with F strictly convex and decreasing and $t(\cdot)$ strictly concave. The uniqueness of the harmonic center is obvious.

□

To prove Theorem 3.1, we need several preliminary results, given through six Proposition.

As mentioned in the introduction, our starting point is the following estimate for the capacity of small balls (see for instance [2]):

Proposition 3.3 *Let $x \in \Omega$. Then, for any $\rho > 0$, we have*

$$cap_\Omega(B_\rho(x)) = \frac{1}{F(\rho) - t(x) + \mathcal{O}(\rho)} \ .$$

Proposition 3.4 *Let $x \in \Omega$. Then we have, for $\rho > 0$ small enough,*

$$F(\rho)u_{B_\rho(x)} \geq G_x \ \text{ in } \Omega\backslash B_\rho(x) \tag{37}$$

and the map $F(\rho)u_{B_\rho(x)}$ converges to the Green function G_x, uniformly on compacta of $\Omega\backslash\{x\}$, when $\rho \to 0^+$.

Proof of Proposition 3.4 : Let us recall that the Green function G_x is harmonic in $\Omega\backslash\{x\}$ and vanishes on $\partial\Omega$. For any $\rho > 0$ sufficiently small, the functions $G_x(\cdot)$ and $(F(\rho) - t(x))u_{B_\rho(x)}(\cdot)$ are two solutions of the Lapace equation in $\Omega\backslash\overline{B}_\rho(x)$ and are equal on $\partial\Omega$. On $\partial B_\rho(x)$, we have

$$G_x(\cdot) = F(\rho) - H_x(\cdot), \quad \text{ while } \quad (F(\rho) - t(x))u_{B_\rho(x)}(\cdot) = F(\rho) - t(x) \ .$$

Hence we have, from the maximum principle,

$$\|G_x(\cdot) - (F(\rho) - t(x))u_{B_\rho(x)}(\cdot)\|_{L^\infty(\Omega\backslash B_\rho(x))} = \|H_x(\cdot) - t(x)\|_{L^\infty(\partial B_\rho(x))} \ .$$

Since H_x is continuous in Ω, this proves the uniform convergence of $(F(\rho) - t(x))u_{B_\rho(x)}(\cdot)$ to $G_x(\cdot)$ on the compacta of $\Omega\backslash\{x\}$ when $\rho \to 0^+$. Let us now recall that $u_{B_\rho(x)}(\cdot)$ converges uniformly to 0^+ on the compacta of $\Omega\backslash\{x\}$ when $\rho \to 0^+$. Hence the map $F(\rho)u_{B_\rho(x)}$ converges to the Green function G_x, uniformly on compacta of $\Omega\backslash\{x\}$, when $\rho \to 0^+$.

Since $t(x)$ is positive and $H_x(\cdot)$ continuous at x, we have

$$\forall y \in \partial B_\rho(x),\ F(\rho) \geq G_x(y) + t(x) - \|H_x(\cdot) - t(x)\|_{L^\infty(\partial B_\rho(x))} \ \geq \ G_x(y)$$

for $\rho > 0$ small enough. Since $G_x(\cdot)$ and $F(\rho)u_{B_\rho(x)}(\cdot)$ are harmonic in $\Omega\backslash\overline{B_\rho(x)}$, coincide on $\partial\Omega$ and satisfy $G_x(\cdot) \leq F(\rho)u_{B_\rho(x)}(\cdot)$ on $\partial B_\rho(x)$, the maximum principle give inequality (37).

□

Let now x_0 and x_1 be two fixed points of Ω, with $x_0 \neq x_1$, $s \in (0,1)$, $x_s = (1-s)x_0 + sx_1$. We also fix $\rho > 0$, $\alpha_0 > 0$ and $\alpha_1 > 0$ such that

$$(1-s)\alpha_0 + s\alpha_1 = 1\ . \tag{38}$$

The real number $\rho > 0$ is going to tend to 0^+, while α_0 and α_1 are to be chosen later, in the proof of Theorem 3.1. Let us set

$$\rho_0 = \rho_0(\rho) = \alpha_0\rho \text{ and } \rho_1 = \rho_1(\rho) = \alpha_1\rho\ .$$

We clearly have $\rho = (1-s)\rho_0 + s\rho_1$. We always choose ρ sufficiently small such that

$$B_\rho(x_s) \subset\subset \Omega,\ B_{\rho_0}(x_0) \subset\subset \Omega \text{ and } B_{\rho_1}(x_1) \subset\subset \Omega\ .$$

We finally set, to simplify the notations, for $i = 0, 1$,

$$u_i^\rho = u_{B_{\rho_i}(x_i)}\ .$$

Let us define, as previously, $\widetilde{u}_s^\rho$ by

$$\widetilde{u}_s^\rho(y) = \sup_{y_0, y_1} \min\{u_0^\rho(y_0), u_1^\rho(y_1)\}$$

where the supremum is taken over the $y_0 \in \Omega\backslash B_{\rho_0}(x_0)$ and $y_1 \in \Omega\backslash B_{\rho_1}(x_1)$ such that $y = (1-s)y_0 + sy_1$. In the same way, we define

$$\widetilde{w}_s(y) = \sup_{y_0, y_1} \min\{\alpha_0^{N-2}G_{x_0}(y_0), \alpha_1^{N-2}G_{x_1}(y_1)\}$$

where the supremum is taken over the $y_0 \in \Omega\backslash\{x_0\}$ and $y_1 \in \Omega\backslash\{x_1\}$ such that $y = (1-s)y_0 + sy_1$.

We claim that

Proposition 3.5 *The function $F(\rho)\widetilde{u}_s^\rho$ converges to $\widetilde{w}_s$ uniformly on compacta of $\Omega\backslash\{x_s\}$ as $\rho \to 0^+$.*

Proof of Proposition 3.5 : Let (y^n) be a sequence of points converging to some $y \in \Omega\backslash\{x_s\}$ and $\rho^n \to 0^+$. We have to prove that the sequence $(F(\rho^n)\widetilde{u}_s^{\rho^n}(y^n))$ converges to $\widetilde{w}_s(y)$. Let y_0 and y_1 belong respectively to $\Omega\backslash\{x_0\}$ and $\Omega\backslash\{x_1\}$, and be such that $y = (1-s)y_0 + sy_1$. For n sufficiently large, the points $y_0^n = y_0 + (y^n - y)/2(1-s)$ and $y_1^n = y_1 + (y^n - y)/2s$ belong respectively to $\Omega\backslash B_{\rho_0^n}(x_0)$ and $\Omega\backslash B_{\rho_1^n}(x_1)$ and satisfy $y^n = (1-s)y_0^n + sy_1^n$. Therefore

$$F(\rho^n)\widetilde{u}_s^{\rho^n}(y^n) \geq \min\{F(\rho^n)u_0^{\rho^n}(y_0^n), F(\rho^n)u_1^{\rho^n}(y_1^n)\} \ .$$

Since the y_i^n converge to $y_i \neq x_i$, Proposition 3.4 and the fact that $F(\rho^n) = \alpha_i^{N-2}F(\rho_i^n)$ imply that

$$\liminf_n F(\rho^n)\widetilde{u}_s^{\rho^n}(y^n) \geq \min\{\alpha_0^{N-2}G_{x_0}(y_0), \alpha_1^{N-2}G_{x_1}(y_1)\}$$

i.e.,

$$\liminf_n F(\rho^n)\widetilde{u}_s^{\rho^n}(y^n) \geq \widetilde{w}_s(y) \ . \tag{39}$$

To complete the proof, it remains to show that

$$\widetilde{w}_s(y) \geq \limsup_n F(\rho^n)\widetilde{u}_s^{\rho^n}(y^n) \ . \tag{40}$$

For this, let y_0^n and y_1^n belong respectively to $\Omega\backslash B_{\rho_0^n}(x_0)$ and $\Omega\backslash B_{\rho_1^n}(x_1)$, satisfy $y^n = (1-s)y_0^n + sy_1^n$ and be such that

$$u_0^{\rho^n}(y_0^n) = u_1^{\rho^n}(y_1^n) = \widetilde{u}_s^{\rho^n}(y^n) \ . \tag{41}$$

Let y_0 and y_1 be the limit (up to a subsequence) of (y_0^n) and (y_1^n). We now prove that $y_0 \neq x_0$ and $y_1 \neq x_1$. Since we have $y = (1-s)y_0 + sy_1$ and since $y \neq x_s$, we cannot have simultaneously $y_0 = x_0$ and $y_1 = x_1$ because $x_s = (1-s)x_0 + sx_1$. Let us assume for instance that $y_0 \neq x_0$. Then $F(\rho^n)u_0^{\rho^n}(y_0^n)$ converges to $\alpha_0^{N-2}G_{x_0}(y_0)$ from Proposition 3.4. Therefore, from (41), we have

$$\lim F(\rho^n)u_1^{\rho^n}(y_1^n) = \lim F(\rho^n)u_0^{\rho^n}(y_0^n) = \alpha_0^{N-2}G_{x_0}(y_0) \ . \tag{42}$$

From inequality (37) in Proposition 3.4, we have

$$F(\rho^n)u_1^{\rho^n}(y_1^n) \geq \alpha_1^{N-2}G_{x_1}(y_1^n) \ .$$

Hence y_1^n cannot converge to x_1 since otherwise we would have

$$\lim_n F(\rho^n) u_1^{\rho^n}(y_1^n) \geq \lim_n \alpha_1^{N-2} G_{x_1}(y_1^n) = +\infty \ ,$$

which is in contradiction with (42). Therefore we have proved that $y_0 \neq x_0$ and $y_1 \neq x_1$. Using (41), we have

$$\begin{aligned} \lim_n F(\rho^n) \widetilde{u}_s^{\rho^n}(y^n) &= \lim_n F(\rho^n) u_0^{\rho^n}(y_0^n) = \alpha_0^{N-2} G_{x_0}(y_0) \\ &= \lim_n F(\rho^n) u_1^{\rho^n}(y_1^n) = \alpha_1^{N-2} G_{x_1}(y_1) \end{aligned}$$

from which we conclude that

$$\widetilde{w}_s(y) \geq \min\{\alpha_0^{N-2} G_{x_0}(y_0), \alpha_1^{N-2} G_{x_1}(y_1)\} = \lim_n F(\rho^n) \widetilde{u}_s^{\rho^n}(y^n) \ .$$

This proves inequality (40) and completes the proof of the Proposition.

□

Proposition 3.6 *If we set* $z_s = G_{x_s} - \widetilde{w}_s$*, then we have*

$$\liminf_{y \to x_s} z_s(y) \ > \ 0 \ .$$

Proof of Proposition 3.6 : Let us set $u_s^\rho = u_{\overline{B_\rho(x_s)}}$. From Lemma 2.2, $\widetilde{u}_s^\rho$ is a subsolution of Laplace equation in $\Omega \backslash \overline{B_\rho(x_s)}$. Hence $F(\rho)(u_s^\rho - \widetilde{u}_s^\rho)$ is a non negative super-solution of this equation. From Proposition 3.4 and Proposition 3.5, $F(\rho)(u_s^\rho - \widetilde{u}_s^\rho)$ converges to $z_s = G_{x_s} - \widetilde{w}_s$ uniformly on compacta of $\Omega \backslash \{x_s\}$ as $\rho \to 0^+$. Therefore, from standard stability results on viscosity solutions, z_s is also a non negative super-solution of Laplace equation in $\Omega \backslash \{x_s\}$.

Our aim is to use a strong maximum principle (Theorem 4.2 in the appendix), stating that, if z_s is a non negative super-solution of Laplace equation, such that $z_s \not\equiv 0$, then

$$\liminf_{y \to x_s} z_s(y) \ > \ 0 \ .$$

So it remains to prove that $z_s \not\equiv 0$. We argue by contradiction, by assuming that $z_s \equiv 0$, i.e., $G_{x_s} \equiv \widetilde{w}_s$. Then $\widetilde{w}_s$ is harmonic in $\Omega \backslash \{x_s\}$. Since $x_0 \neq x_1$ and since $\bigcap_{t>0} \{G_{x_i} > t\} = \{x_i\}$ for $i = 0, 1$, we can choose $t > 0$ sufficiently large so that

$$\{\alpha_0^{N-2} G_{x_0}(\cdot) > t\} \neq \{\alpha_1^{N-2} G_{x_1}(\cdot) > t\} \ .$$

We denote respectively by D_0 and D_1 these sets. Then the map $\alpha_0^{N-2}G_{x_0}(\cdot)/t$ is equal to the solution $u^\Omega_{D_0}$ of (3), while $\alpha_0^{N-2}G_{x_1}(\cdot)/t$ is equal to $u^\Omega_{D_1}$. Moreover, we clearly have

$$\forall y \in (1-s)D_0 + sD_1, \quad \frac{\widetilde{w}_s(y)}{t} = \sup_{y_0,y_1} \min\{u^\Omega_{D_0}(y_0), u^\Omega_{D_1}(y_1)\}$$

where the supremum is taken over the $y_0 \in \Omega\backslash D_0$ and $y_1 \in \Omega\backslash D_1$ such that $y = (1-s)y_0 + sy_1$. We now apply Theorem 2.3 to the sets $\Omega_0 = \Omega_1 = \Omega$ and D_0 and D_1: Since $\widetilde{w}_s(\cdot)/t$ is harmonic, there are $a > 0$ and $b \in I\!R^N$ such that $\Omega = a\Omega + b$ and $D_1 = aD_0 + b$. Since Ω is neither a cylinder, a cone or the translate of a cone, we necessarily have $a = 1$ and $b = 0$, which implies that $D_1 = D_0$. This is impossible from our assumption. Hence we have found a contradiction and we have proved that the non negative function z_s, which is a super-solution of Laplace equation in $\Omega\backslash\{x_s\}$, is not equal to zero. Using Theorem 4.2 completes the proof of the Proposition.

□

For any $\rho > 0$, we set $z^\rho_s = F(\rho)(u^\rho_s - \widetilde{u}^\rho_s)$, where $u^\rho_s = u_{B_\rho(x_s)}$. Let us recall that, according to Propositions 3.4 and 3.5, z^ρ_s converges to z_s uniformly on any compact of $\Omega\backslash\{x_s\}$ and that z^ρ_s is a super-solution of Laplace equation.

Proposition 3.7 *For any constant $k > 0$, there is some $\rho_k > 0$ such that, for any $\rho \in (0, \rho_k)$, there is a neighborhood U_ρ of $\partial B_\rho(x_s)$ with*

$$\forall y \in U_\rho \backslash B_\rho(x_s), \quad z^\rho_s(y) \geq k d_{\partial B_\rho(x_s)}(y)\ .$$

Proof of Proposition 3.7 : Let us fix $k > 0$. We have, from Proposition 3.6, that

$$m = \liminf_{y \to x_s} z_s(y) \ > \ 0\ .$$

Let $\bar{r} > 0$ be such that $B_{2\bar{r}}(x_s) \subset \Omega$ and such that

$$\forall y \in B_{2\bar{r}}(x_s),\ z_s(y) \geq m/2\ . \tag{43}$$

Let us fix in a first step $r \in (0, \bar{r})$, to be chosen later. For any $\rho \in (0, r/2)$ and any x such that $|x - x_s| = r$, we define the map ϕ_x by

$$\phi_x(y) = \alpha|y - x|^{2-N} + \beta$$

where α and β are such that

$$\left\{ \begin{array}{l} \phi_x(\cdot)_{|\partial B_{r/2}(x)} = m/4 = \alpha(r/2)^{2-N} + \beta \\ \phi_x(\cdot)_{|\partial B_{r-\rho}(x)} = 0 = \alpha(r-\rho)^{2-N} + \beta \end{array} \right. \tag{44}$$

Let us notice that, from (44), we can deduce that $\alpha > 0$ since $r - \rho > r/2$. Since z_s^ρ converges uniformly to z_s as $\rho \to 0^+$ on compacta of $\Omega \backslash \{x_s\}$, we can choose ρ sufficiently small (say $\rho \in (0, \bar{\rho}_k)$, for some $\bar{\rho}_k > 0$), such that

$$\left\{ \begin{array}{ll} \phi_x(\cdot) = m/4 < z_s^\rho(\cdot) & \text{on } \partial B_{r/2}(x) \\ \phi_x(\cdot) = 0 \leq z_s^\rho(\cdot) & \text{on } \partial B_{r-\rho}(x) \end{array} \right.$$

because of (43). Thus, since $\phi_x(\cdot)$ is harmonic, and $z_s^\rho(\cdot)$ is a super-solution of Laplace equation, we have

$$\forall y \in B_{r-\rho}(x) \backslash B_{r/2}(x)\ , \ \ \phi_x(y) \leq z_s^\rho(y)\ . \tag{45}$$

Let now $y \in B_{r/2}(x_s) \backslash B_\rho(x_s)$ be fixed. We consider the point x such that $|x - x_s| = r$ and such that y belongs to the segment $[x_s, x]$. We clearly have that $y \in B_{r-\rho}(x) \backslash B_{r/2}(x)$ and that

$$|x - y| = r - \rho - d_{\partial B_\rho(x_s)}(y)\ .$$

We now apply inequality (45) for this choice of x and y. This gives

$$z_s^\rho(y) \geq \alpha[r - \rho - d_{\partial B_\rho(x_s)}(y)]^{2-N} + \beta = \phi_x(y) \tag{46}$$

from the definition of $\phi_x(y)$. Since $\alpha > 0$, the map $a \to \alpha[r - \rho - a]^{2-N} + \beta$ is convex in $[0, r/2 - \rho]$. It vanishes at $a = 0$, from (44), and its derivative at that point is equal to $\alpha(N-2)(r-\rho)^{1-N}$. Thus (46) implies that

$$\forall y \in B_{r/2}(x_s) \backslash B_\rho(x_s),\ z_s^\rho(y) \geq \alpha(N-2)(r-\rho)^{1-N} d_{\partial B_\rho(x_s)}(y)\ .$$

Now it remains to show that $\alpha(N-2)(r-\rho)^{1-N}$ can be chosen arbitrarily large (i.e., larger than k). Indeed, after computing α from the equalities (44), we get

$$\alpha(N-2)(r-\rho)^{1-N} = \frac{m}{4r} \left[\frac{(1-\rho/r)^{1-N}}{(1/2)^{2-N} - (1-\rho/r)^{2-N}} \right] \geq \frac{m}{4r}\ .$$

Thus, if we set $\rho_k = \min\{\bar{\rho}_k, m/4k\}$ and $U_\rho = B_{r/2}(x_s)$, the proof of the Proposition is complete.

$\square$

Proposition 3.8 *For any constant $k > 0$, there is some $\rho_k > 0$ such that, for any $\rho \in (0, \rho_k)$, we have*

$$\frac{\rho^{N-1}}{cap_\Omega(B_\rho(x_s)) + k\rho^{N-1}\sigma_N/F(\rho)} \geq \frac{(1-s)\rho_0^{N-1}}{cap_\Omega(B_{\rho_0}(x_0))} + \frac{s\rho_1^{N-1}}{cap_\Omega(B_{\rho_1}(x_1))}$$

where $\sigma_N = |\partial B(0,1)|$.

Proof of Proposition 3.8 : Let us fix $k > 0$. From Proposition 3.7, there is a neighborhood U_ρ of $\partial B_\rho(x_s)$ with

$$\forall y \in U_\rho \backslash B_\rho(x_s), \;\; z_s^\rho(y) \geq k d_{\partial B_\rho(x_s)}(y) \; .$$

Hence we have, from the definition of z_s^ρ,

$$\forall y \in U_\rho \backslash B_\rho(x_s), \;\; u_s^\rho(y) \geq \tilde{u}_s^\rho(y) + k d_{\partial B_\rho(x_s)}(y)/F(\rho) \; . \tag{47}$$

Let us fix some $y = x_s + \rho w \in \partial B_\rho(x_s)$ (where $w \in \partial B_1(0)$). There is a unique pair $(y_0, y_1) \in \partial B_\rho(x_0) \times \partial B_\rho(x_1)$ such that $y = (1-s)y_0 + sy_1$: Namely $y_0 = x_0 + \rho w$ and $y_1 = x_1 + \rho w$. Then Proposition 2.4, part 2, states that

$$\liminf_{h \to 0^+} \frac{(\tilde{u}_s^\rho(y + hw) - 1)}{h} \geq -\left[\frac{1-s}{|\nabla u_0(y_0)|} + \frac{s}{|\nabla u_1(y_1)|}\right]^{-1} \; .$$

Using inequality (47), we have, for any $w \in \partial B_1(0)$:

$$|\nabla u_s^\rho(x_s + \rho w)| \leq \left[\frac{1-s}{|\nabla u_0(x_0 + \rho_0 w)|} + \frac{s}{|\nabla u_1(x_1 + \rho_1 w)|}\right]^{-1} - k/F(\rho) \; ,$$

since u_s^ρ is regular. After integration over $\partial B_1(0)$, we get

$$\frac{1}{\rho^{1-N} cap_\Omega(B_\rho(x_s)) + k\sigma_N/F(\rho)} \geq \left[\int_{\partial B_1(0)} \frac{1}{\frac{1-s}{|\nabla u_0(x_0+\rho_0 w)|} + \frac{s}{|\nabla u_1(x_1+\rho_1 w)|}}\right]^{-1} ,$$

because

$$cap_\Omega(B_\rho(x_s)) = \int_{\partial B_\rho(x_s)} |\nabla u_s^\rho| = \rho^{N-1} \int_{\partial B_1(0)} |\nabla u_s^\rho(x_s + \rho w)| \; .$$

Since the map $f \to 1/\int(1/f)$ is concave on the set of positive functions, we obtain:

$$\frac{\rho^{N-1}}{cap_\Omega(B_\rho(x_s)) + k\rho^{N-1}\sigma_N/F(\rho)} \geq \frac{(1-s)}{\rho_0^{1-N} cap_\Omega(B_{\rho_0}(x_0))} + \frac{s}{\rho_1^{1-N} cap_\Omega(B_{\rho_1}(x_1))}$$

which is the desired result.

$\square$

Proof of Theorem 3.1 : We now prove that the harmonic radius is strictly concave. We argue by contradiction. Let us suppose that there exists x_0 and x_1 belonging to Ω, with $x_0 \neq x_1$, and some $s \in (0,1)$ such that

$$r(x_s) \leq (1-s)r(x_0) + sr(x_1) \ ,$$

where we have set, as previously, $x_s = (1-s)x_0 + sx_1$. From the definition of the Robin function and of the harmonic radius, this inequality is equivalent to the following:

$$t(x_s) \geq \gamma[(1-s)r(x_0) + sr(x_1)]^{2-N} \tag{48}$$

because $a \to a^{2-N}$ is decreasing. From now on, we fix some $\rho > 0$, and we set, for $i = 0, 1$,

$$\alpha_i = \frac{r(x_i)}{(1-s)r(x_0) + sr(x_1)} \ . \tag{49}$$

Let us notice that the α_i satisfy (38). Let us recall that the ρ_i are defined by $\rho_i = \alpha_i \rho$

Using Proposition 3.3, Proposition 3.8 can be rewritten as

$$\frac{\rho^{N-1}}{\frac{1}{F(\rho) - t(x_s) + \mathcal{O}(\rho)} + k\rho^{N-1}\sigma_N/F(\rho)} \geq \tag{50}$$
$$(1-s)\rho_0^{N-1}(F(\rho_0) - t(x_0)) + s\rho_1^{N-1}(F(\rho_1) - t(x_1)) + \mathcal{O}(\rho^N)$$

because ρ_0 and ρ_1 are of the same order as ρ (cf. (49)). Let us first compute the right-hand side R of inequality (50). Using (49), R is equal to:

$$R = \gamma\rho - \rho^{N-1}\frac{(1-s)r(x_0)^{N-1}t(x_0) + sr(x_1)^{N-1}t(x_1)}{[(1-s)r(x_0) + sr(x_1)]^{N-1}} + \mathcal{O}(\rho^N)$$

because $F(a) = \gamma a^{2-N}$ and $(1-s)\rho_0 + s\rho_1 = \rho$. Since $t(x_i) = \gamma(r(x_i))^{2-N}$, we have

$$R = \gamma\rho - \gamma\rho^{N-1}((1-s)r(x_0) + sr(x_1))^{2-N} + \mathcal{O}(\rho^N) \ .$$

We now consider the left-hand side L of (50):

$$\begin{aligned}
L = & \frac{\rho^{N-1}(F(\rho)-t(x_s)+\mathcal{O}(\rho))}{1+k\rho^{N-1}\sigma_N+\mathcal{O}(\rho^{N-1}/F(\rho))} \\
= & (\gamma\rho-\rho^{N-1}t(x_s)+\mathcal{O}(\rho^N))(1-k\rho^{N-1}\sigma_N+\mathcal{O}(\rho^{N-1}/F(\rho))+\mathcal{O}_k(\rho^{2N-2})) \\
= & \gamma\rho-\rho^{N-1}t(x_s)-\gamma k\rho^N\sigma_N+\mathcal{O}(\rho^N)+\mathcal{O}_k(\rho^{2N-2})
\end{aligned}$$

where the $\mathcal{O}(\rho^N)$ does not depend on k. Combining the computation for the right-hand side with the computation for the left-hand side gives, after simplification:

$$-\rho^{N-1}t(x_s)-\gamma k\rho^N\sigma_N \;\geq\; -\gamma\rho^{N-1}((1-s)r(x_0)+sr(x_1))^{2-N}+\mathcal{O}(\rho^N)+\mathcal{O}_k(\rho^{2N-2})$$

Using assumption (48), this implies that

$$\gamma k\rho^N\sigma_N \;\leq\; \mathcal{O}(\rho^N)+\mathcal{O}_k(\rho^{2N-2}) \;,$$

i.e.,

$$\gamma k\sigma_N \;\leq\; \mathcal{O}(1)+\mathcal{O}_k(\rho^{N-2}) \;.$$

Since $N \geq 3$, letting $\rho \to 0^+$, we find that there is some constant M, independent of k, such that

$$\gamma k\sigma_N \;\leq\; M \;.$$

Hence we have found a contradiction since k is arbitrary.

□

4 Appendix: Some maximum principles

We recall here two results on the maximum principles for viscosity super-solutions of Laplace equation:

$$-\Delta u \;=\; 0 \;. \tag{51}$$

The first Theorem is very classical (see for instance Proposition 4.9 in [4]) and we omit its proof. We give the proof of the second one for sake of completeness.

Theorem 4.1 *Let u be a continuous non negative super-solution (in the viscosity sense) of (51) in some open connected set $\Omega \subset I\!R^N$. If $u \not\equiv 0$, then u is positive in Ω.*

Theorem 4.2 *Let Ω be some open connected subset of $I\!R^N$ and $x_0 \in \Omega$. Let u be a continuous non negative super-solution (in the viscosity sense) of (51) in $\Omega\backslash\{x_0\}$. If $u \not\equiv 0$, then*

$$\liminf_{x\to x_0} u(x) \ > \ 0 \ .$$

Proof of Theorem 4.2 : Let us assume that, contrary to our claim, we have:

(52) $$\liminf_{x\to x_0} u(x) \ = \ 0 \ .$$

Let us fix $\rho > 0$ such that $B_\rho(x_0) \subset\subset \Omega$, and let us set

(53) $$m = \min_{y\in\partial B_\rho(x_0)} u(y) \ .$$

Thanks to Theorem 4.1, we know that m is positive because u is positive in $\Omega\backslash\{x_0\}$. Let us define the function ϕ_γ by

$$\phi_\gamma(y) = \frac{m/2+1}{\rho^\gamma}|y-x_0|^\gamma - 1$$

where $\gamma \in (0,1)$ is to be defined later. Let us notice that we have

(54) $$\forall y \in \partial B_\rho(x_0),\ \phi_\gamma(y) = m/2 \ < \ u(y)$$

because of (53), and that

$$\forall y \neq x_0, \quad \lim_{\gamma\to 0^+} \phi_\gamma(y) = m/2 \ .$$

From our assumption (52), there is some $y_0 \in B_\rho(x_0)\backslash\{x_0\}$ with $u(y_0) < m/2$. Let us choose $\gamma > 0$ sufficiently small so that $u(y_0) < \phi_\gamma(y_0)$. For this choice of γ, we have, on the one hand, that there is some $y_0 \in B_\rho(x_0)\backslash\{x_0\}$ such that $u(y_0) < \phi_\gamma(y_0)$, and, on the other hand, $\phi_\gamma < u$ on $\partial B_\rho(x_0)$ thanks to (54) and

$$\phi_\gamma(x_0) = -1 < 0 \leq \liminf_{x\to x_0} u(x) \ .$$

So we can deduce that there is some $\bar{y}$ point of minimum of $u - \phi_\gamma$ in $B_\rho(x_0)\backslash\{x_0\}$. Since u is a viscosity super-solution of (51), we have

$$-\Delta\phi_\gamma(\bar{y}) \geq 0 \ .$$

However, we have

$$-\Delta\phi_\gamma(\bar{y}) = -\frac{m/2+1}{\rho^\gamma}|\bar{y}-x_0|^{\gamma-2}\gamma(\gamma-2+N) < 0 \ ,$$

since $N \geq 3$. So we have found a contradiction.

□

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