Supremal representation of L^∞ functionals

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Abstract

We characterize the maps $F = F(u, A)$ defined for $u \in W^{1,\infty}$ and A open, which can be written as supremal functionals of the form $F(u, A) = \operatorname{ess\,sup}_{x\in A} f(x, u(x), Du(x))$.

Mathematics Subject Classification (2000): 49J45, 28A20.

Keywords: supremal functionals, Calculus of Variations in L^∞.

1 Introduction

The classical problems in Calculus of Variations are formulated through the introduction of an integral functional. This viewpoint had brought to refer as "'variational functional"' any functional $F(u, A)$ defined on a space of functions X and on a class $\mathcal{A}$ of open sets such that $F(u, \cdot)$ is a regular measure with respect to A (see [21], [26]).

In this way, an integral functional belongs to this class and, thanks to the representation results shown in [14] and in [15], all the variational functionals which satisfy some lower continuity properties with respect to u, can be written in the integral form

$$G(u, A) = \int_A g(x, u, Du)dx.$$

In several applications, an integral functional is not suitable to describe certain phenomena or to express an intrinsic property of some body. In fact, in many variational problems one can be interested to control the maximum of the pointwise values of a some quantity instead of its mean value. As consequence, in the last years a new class of functionals has been introduced and studied. These functionals have been called L^∞-*functionals* since the natural setting in which they are defined is the space L^∞ or $W^{1,\infty}$ and the natural form of representing them is the so called *supremal form*:

$$F(u, A) = \operatorname*{ess\,sup}_{x\in A} f(x, u(x), Du(x)). \tag{1.1}$$

For this reason they are also referred as *supremal functionals* (see [1]). Some concrete examples in which it is necessary to consider these new variational functionals can be found in [10], in [8] and in [22] where the problem of modeling the first dielectric breakdown of a composite conductor is studied. The increasing number of variational problems formulated through supremal functionals has raised a lot of questions: the identification of qualitative conditions which imply the lower semicontinuity of these functionals w.r.t. the weak* convergence of $W^{1,\infty}$ (see the results stated in [8], [1], [18]), the problem of the relaxation and more generally of the stability of this class w.r.t. Γ-convergence (see [10], [11], [25]), the particular case of the the homogenization (see [2] for the 1-dimensional case, [17] and [12] for the general case), the problem of the existence of absolute minimizers for a supremal functionals (see [3], [4], [5], [6], [9], [18]) the problem of characterizing the class of the functionals which can be written in the supremal form (1.1). In the resolution of these questions one encounters several difficulties. Indeed the techniques that have been developed for solving this kind of problems in the integral case strongly exploit the peculiar properties of the integral functional and do not work in the supremal case because of intrinsic differences between supremal functionals and integral ones. First, while an integral functional is absolutely continuous w.r.t. the Lebesgue measure, for a supremal functional sets of arbitrary small measure can be relevant. Moreover the additivity property, characteristic for the integral functionals, is replaced in the supremal case by the so called *countable supremality*, i.e.

$$F\left(u, \bigcup_{i=1}^{\infty} A_i\right) = \bigvee_{i=1}^{\infty} F(u, A_i).$$

This property remains the most delicate condition that one has to verify in order to obtain the representation in a supremal form of a functional. In this paper we are concerned with the following problem: fixed a bounded open set of $\mathbf{R}^N$ and given a functional $F(u, A)$ defined on the Lipschitz continuous functions $u \in W^{1,\infty}(\Omega)$ and on the open subsets A of $\Omega \subset \mathbf{R}^N$ we want to establish abstract conditions under which it is possible to represent the functional F in the supremal form (1.1). This problem follows very naturally from the paper [1] where, given a complete measure space $(\Omega, \mathcal{F}, \mu)$, the authors show a complete characterization of the class of all l.s.c. functionals $F : L^\infty_\mu \times \mathcal{F} \to \overline{\mathbf{R}}$ which can be represented in the supremal form

$$F(u, B) = \mu\text{-}\operatorname*{ess\,sup}_{x \in B} f(x, u(x)).$$

Similar representation problems, proved in the case of integral functional in [23], [24],[13], [14], can find some interesting applications in the study of many problems of calculus of variation, as Γ convergence, homogenization and relaxation. The representation in integral form for additive functionals on $W^{1,p}(\Omega)$ makes use of the Radon-Nikodym theorem for measure: after obtaining a representation formula on the piecewise affine functions, by continuity one can extend it on $W^{1,p}(\Omega)$ thanks to a density argument. In the supremal case, there is a Radon-Nikodym theorem (see [7]) but it requires the definition of the functional on all the Borel sets, which is much too rigid for applications.

The main result of this paper is the abstract representation result stated by Theorem 2.2. The argument of its proof are new and based on some fine properties of the Lipschitz-continuous functions. The formula (3.5) which identifies the right function f which represents the supremal functional is different from the one which gives the integrand for an integral functional. In fact, for a supremal functional F given by (1.1) in order to deduce $f(x, u, \xi)$ it is necessary to minimize F in the class of all Lipschitz continuous functions u which are differentiable in x with $u(x) = u$ and $Du(x) = \xi$. In the integral case in order to compute the integrand g it is sufficient to know the values of the functional on the affine functions (see Lemma 2.7 in [14] and Theorem in [13]). This difference comes from the fact that a supremal functional is not uniquely defined by its values on the smooth functions. In the next section, we give an example of two different l.s.c. supremal functionals which coincide on the smooth functions.

The paper is organized as follows: We state and discuss the representation result in a first part (section 2). The second part (section 3) is devoted to proofs.

2 Statement of the supremal representation

In this section we state the main result of this paper which is the supremal representation theorem for w^* lower semicontinuous functionals on $W^{1,\infty}(\Omega)$. Let us first introduce some notations. Throughout this paper, we assume that Ω is some open bounded domain of $\mathbf{R}^N$. We denote by $\mathcal{A}$ the family of open subsets of Ω, and by $\mathcal{B}_N$ the Borel σ-field of $\mathbf{R}^N$ (when $N = 1$ we will write simply $\mathcal{B}$). Moreover we denote by $|\cdot|$ the euclidean norm on $\mathbf{R}$ and by $||\cdot||$ the euclidean norm on $\mathbf{R}^N$, by $B_r(x)$ the open ball $\{y \in \Omega \ : \ ||x-y|| < r\}$ and by $\mathcal{L}$ the Lebesgue measure on $\mathbf{R}^N$. In the sequel we will work with *supremal functionals* on $W^{1,\infty}(\Omega)$, which means with functionals of the form

$$F(u) = \operatorname*{ess\,sup}_{x\in\Omega} f(x, u(x), Du(x)).$$

We refer to the function f, which represents the functional, as *supremand.* We give the following precise definitions.

Definition 2.1 *A function* $f : \Omega \times \mathbf{R} \times \mathbf{R}^N \to \overline{\mathbf{R}}$ *is said to be*

(a) *a* Caratheodory supremand *if:*

(i) for every $(t, \xi) \in \mathbf{R} \times \mathbf{R}^N$ *the function* $x \mapsto f(x, t, \xi)$ *is measurable in* Ω

(ii) for a. a. $x \in \Omega$ *the function* $(t, \xi) \mapsto f(x, t, \xi)$ *is continuous in* $\mathbf{R} \times R^N$

(b) *a* level convex Caratheodory supremand *if* f *is a Caratheodory supremand and* $f(x, u, \cdot)$ *is level convex on* $\mathbf{R}^N$ *for almost every* $x \in \Omega$ *and for every* $u \in \mathbf{R}$*, i.e. for every* $t \in \mathbf{R}$ *the level set* $\{\xi \in \mathbf{R}^d \ : \ f(x, u, \cdot) \le t\}$ *is convex;*

In this paper we show that any mapping $F : W^{1,\infty}(\Omega) \times \mathcal{A} \to \overline{\mathbf{R}}$ satisfying a certain set of assumptions may actually be written as a supremal functional for a suitable normal supremand f.

Theorem 2.2 *Let* $F : W^{1,\infty}(\Omega) \times \mathcal{A} \to \overline{\mathbf{R}}$ *be a functional. Assume that* F *satisfies the following properties:*

(i) *(locality)* $F(u, A) = F(v, B)$ *for every* $u, v \in W^{1,\infty}(\Omega)$ *such that* $u(x) = v(x)$ *for any* $x \in A \cup B$ *and for every* $A, B \in \mathcal{A}$ *with* $\mathcal{L}(A \triangle B) = 0$

(ii) *(countable supremality) for every* $A_i \in \mathcal{A}$ *and* $u \in W^{1,\infty}(\Omega)$

$$F\left(u, \bigcup_{i=1}^{\infty} A_i\right) = \bigvee_{i=1}^{\infty} F(u, A_i) \tag{2.2}$$

(iii) *(strong continuity) for any* $M > 0$ *there exists some modulus* ω_M *such that*

$$|F(u, A) - F(v, A)| \le \omega_M(||u - v||_{W^{1,\infty}(A)})$$

for every $A \in \mathcal{A}$*, for every* $u, v \in W^{1,\infty}(\Omega)$ *s.t.* $||u||_{W^{1,\infty}(\Omega)}, ||v||_{W^{1,\infty}(\Omega)} \le M$;

(iv) *(w* lower semicontinuity):* $F(\cdot, A)$ *is weakly* lower semicontinuous for every* $A \in \mathcal{A}$*;*

(v) *(coercivity): there exists an increasing continuous function* $\alpha : \mathbf{R}^+ \to \mathbf{R}^+$ *such that* $\lim_{t\to+\infty} \alpha(t) = +\infty$ *and for every* $A \in \mathcal{A}$*,* $F(\cdot, A) \ge \alpha(||\cdot||_{W^{1,\infty}(A)})$*.*

Then there exists a Caratheodory supremand $f : \Omega \times \mathbf{R} \times \mathbf{R}^d \to \mathbf{R}$ *such that*

$$F(u, A) = \operatorname*{ess\,sup}_{A} f(x, u(x), Du(x)) \tag{2.3}$$

for any $u \in W^{1,\infty}(\Omega)$ *and any* $A \in \mathcal{A}$*.*

The proof of the result is given in the next section. Now some comments of the result are in order.

Let us first recall that in the 1-dimensional case, thanks to Theorem 4.1 in [1], the supremand f which represents F in the supremal form (2.3) is level convex in the third variable. In the N-dimensional case, if F satisfies a continuity property w.r.t. x we can also prove the same property for f :

Proposition 2.3 *Let* F *be as in Theorem 2.2 and suppose that* F *satisfies the additional assumption:*

(vi) *(continuity) for any $M > 0$ there exists some modulus ρ_M such that*

$$|F(u, B_r(x_0)) - F(u_{x_o - y_o}, B_r(y_0))| \leq \rho_M(|x_0 - y_0|)$$

where $u_{x_0-y_0}(z) = u(z + x_0 - y_0)$ and $||u||_{W^{1,\infty}(\Omega)} \leq M$.

Then the function f which represents F in the supremal form (2.3) is continuous in x and level convex with respect to the last variable.

The proof is given at the end of the next section. In the general case, i.e., without assumption (vi), it is an open problem which are the qualitative property of f w.r.t. ξ in order to have the weak* lower semicontinuity of the functional.

Second let us point out that the assumptions of the Theorem are very natural. Indeed, if $f : \Omega \times \mathbf{R} \times \mathbf{R}^N \to \mathbf{R}$ is a normal level convex supremand such that

(a) $|f(x, u, \xi) - f(x, v, \eta)| \leq \omega_M(|\xi - \eta| + |u - v|)$ for a.e. $x \in \Omega$ and for every $\xi, \eta \in B_M(0)$ and $|u|, |v| \leq M$;

(b) $f(x, u, \cdot) \geq \alpha(|\cdot|)$ for a.e. $x \in \Omega$ and for every $u \in \mathbf{R}$,

then the functional F defined by $F(u, A) = \operatorname{ess\,sup}_A f(x, u(x), Du(x))$ for all $u \in W^{1,\infty}(\Omega)$ and for all $A \in \mathcal{A}$, satisfies the assumptions (i), (ii), (iii), (vi), (v) of Theorem 2.2.

We now point out the differences between our result and the symmetric one in the integral case. Recall first that the analogous representation result in the integral case (see Theorem 1.1 in [15]) only requires with the following condition:

(i)' $F(u, A) = F(v, A)$ for every u, v such that $u(x) = v(x)$ for a.e. $x \in A$

which is weaker than (i). Indeed for an integral functional the set locality property

(i)" $F(u, A) = F(u, B)$ for every u and $A, B \in \mathcal{A}$ s.t. $\mathcal{L}(A \triangle B) = 0$

follows by the condition that $F(u, \cdot)$ is a measure which is absolutely continuous w.r.t. the Lebesgue measure. On the contrary, in the supremal case we can not weaken the assumption (i) with the assumption (i)'. In fact, a functional F can satisfy (i)' and the countable supremality (iii) and not verify the set locality property (i)". This is for instance the case of of the functional

$$F(u, (a, b)) := \begin{cases} \operatorname*{ess\,sup}_{x \in (a,b)} |u'(x)| & \text{if } b \leq 1 \text{ or } a \geq 1 \\ \operatorname*{ess\,sup}_{x \in (a,b)} |u'(x)| \vee 1 & \text{if } a < 1 < b \end{cases}$$

Note that $F(0, (\frac{1}{2}, \frac{3}{2})) = 1$ and $F(0, (\frac{1}{2}, 1) \cup (1, \frac{3}{2})) = 0$ even if $|(\frac{1}{2}, \frac{3}{2}) \setminus (\frac{1}{2}, 1) \cup (1, \frac{3}{2})| = 0$.

Finally we want to underline another stricking difference between the integral and supremal case. In analogy with the integral case (see the paper [14] of Buttazzo and Dal Maso), one could think of choosing as supremand for the functional F the function $\overline{f}$ defined as

$$\overline{f}(x, u, \xi) := \inf\{F(\varphi_{x,u,\xi}, A) : \ x \in A, A \in \mathcal{A}\} \tag{2.4}$$

where $\varphi_{x,u,\xi}(y) := u + \xi \cdot (y - x)$. If F satisfies the properties (i), (ii), (iii), (iv), one can show that $\overline{f}$ represents the functional F on the affine function and then on the smooth functions proceeding exactly as in the integral case (see Lemma 2.6 in [15]). But, while an integral functional is completely determined by its values on the affine functions and on the open sets, in the supremal case, two functionals satisfying properties (i), (ii), (iii), (iv) and coinciding on the smooth functions, can be different on $W^{1,\infty}(\Omega)$. More precisely, while the following inequality always holds true:

$$F(u, A) \leq \operatorname*{ess\,sup}_A \overline{f}(x, u(x), Du(x)) \qquad \forall u \in W^{1,\infty}(\Omega)$$

the reverse inequality can be false. We illustrate this phenomenon by the following example: Let $E \subset (0,1)$ be a dense open set such that $|E| > 0$ and $|E^c| > 0$ (where $E^c = (0,1) \setminus E$) and let us define

$$F(u) := \operatorname*{ess\,sup}_{(0,1)}(1 + \mathbf{1}_E(x))|u'(x)| \qquad \forall u \in W^{1,\infty}(0,1)$$

where

$$\mathbf{1}_E(x) := \begin{cases} 1 & \text{if } x \in E \\ 0 & \text{otherwise.} \end{cases}$$

Then formula (2.4) gives

$$\overline{f}(x,s) = |s| \lim_{r\to 0+} \sup_{t\in(x-r,r+x)} (1 + \mathbf{1}_E(t)) = 2|s| \ .$$

This function does not represent the functional F on $W^{1,\infty}(0,1)$. Indeed, if we choose $u(x) := \int_0^x \mathbf{1}_{E^c}(s)ds$, then $u \in W^{1,\infty}(0,1) \setminus C^1(0,1)$ and

$$F(u) = \|u'\|_\infty = 1 \ < \ 2 \operatorname*{ess\,sup}_{(0,1)} \overline{f}(x,u(x)) = 2 \ .$$

Note however that $\overline{f}$ represents the supremal functional F on $\mathcal{C}^1((0,1))$.

3 Proof of the main result

The proof of Theorem 2.2 is quite intricated and shall be achieved through several intermediate steps. We will give the proof of the result in the case $\alpha(t) = t$, since it is always possible to reduce the problem to this case.

Before starting the proof of the Theorem, we need some preliminary results.

Lemma 3.1 *Let u, v belong to $W^{1,\infty}(\Omega)$, $x \in \Omega$ be a point of differentiability of u and v, and suppose that $u(x) = v(x)$ and $Du(x) = Dv(x)$. Then, for any $\epsilon > 0$, for any $r > 0$, there is some $r' \in (0,r)$, some open set $A \in \mathcal{A}$, with $B_{r'/2}(x) \subset A \subset B_{r'}(x)$ and $|\partial A| = 0$, and some $\alpha \in (0,\epsilon)$, $\beta \in (0,\epsilon)$ such that*

$$u(y) = v(y) + \alpha - \beta|y-x| \quad \forall y \in \partial A \quad \text{and} \quad u(y) < v(y) + \alpha - \beta|y-x| \quad \forall y \in A \ .$$

Proof. Without loss of generality, we assume that $x = 0$, $u(x) = v(x) = 0$ and $Du(x) = Dv(x) = 0$. We can find $r' \in (0,r)$, with $r' < 1$, such that

$$|u(y)| \leq \frac{\epsilon}{8}|y| \quad \text{and} \quad |v(y)| \leq \frac{\epsilon}{8}|y| \qquad \forall y \in B_{r'}(x) \ .$$

Let $\alpha \in (\frac{9r'\epsilon}{16}, \frac{5r'\epsilon}{8})$ and $\beta = \frac{7\epsilon}{8}$. Then, for all $y \in \partial B_{r'}(0)$,

$$v(y) + \alpha - \beta|y| \leq \frac{\epsilon}{8}r' + \alpha - \beta r' \leq -\frac{r'\epsilon}{8} \leq u(y) \ .$$

Moreover, for all $y \in B_{r'/2}(0)$,

$$v(y) + \alpha - \beta|y| \geq -\frac{\epsilon}{8}\frac{r'}{2} + \alpha - \beta\frac{r'}{2} \ > \ \frac{r'\epsilon}{16} \geq u(y) \ .$$

Then the open set $A_\alpha = \{y \in B_{r'}(x) \mid u(y) < v(y) + \alpha - \beta|y-x|\}$ satisfies our requirements provided it has a zero measure. Since u and v are lipschitz continuous, this is the case for almost every α thanks to the Coarea Formula. □

We are going to show the representation formula (2.3) for the functional F holds with the map f defined as follows:

$$f(x,t,\xi) := \inf\left\{F(u,B_r(x)) \mid \ r>0,\ u \in W^{1,\infty}(\Omega) \text{ s.t. } x \in \widehat{u}, \text{ with } u(x)=t,\ Du(x)=\xi \ \right\} \tag{3.5}$$

where

$$\widehat{u} := \{x \in \Omega : x \text{ is a Lebesgue point of } Du \text{ and a differentiable point of } u\}\,.$$

(Notice that, thanks to the Radamacher Theorem a.e. $x \in \Omega$ belongs to $\widehat{u}$.)

Lemma 3.2 *Under the assumptions of Theorem 2.2, f is bounded on bounded sets.*

Proof. Let $M>0$ and $(t,\xi) \in \mathbf{R}\times\mathbf{R}^N$ such that $|t|+\|\xi\| \le M$. Fix $x_0 \in \Omega$ and $r>0$ such that $B_{2r}(x_0) \subset\subset \Omega$. Let $\varphi_{x,t,\xi}(y) := t+\xi\cdot(y-x)$ and note that $\varphi_{x,t,\xi} \in W^{1,\infty}(\Omega)$ since Ω is bounded. From the definition of f and the continuity assumption on F, for any $(x,t,\xi) \in B_r(x_0)\times\mathbf{R}\times\mathbf{R}^N$ we have

$$\begin{aligned} 0 \le f(x,t,\xi) \le F(\varphi_{x,t,\xi}, B_r(x_0)) &= F(\varphi_{x,t,\xi}, B_r(x_0)) \\ &\le F(0,\Omega) + \omega_m(\|\varphi_{x,t,\xi}\|_{W^{1,\infty}(B_r(x_0))}) \\ &\le F(0,\Omega) + \omega_m(|t| + \|\xi\|(r+1)) \end{aligned}$$

where $m = |t| + \|\xi\|(diam(\Omega)+1)$. Hence f is bounded on bounded sets. □

Next we show that minimizers in (3.5) are uniformly bounded.

Lemma 3.3 *Let $M>0$ be fixed. Then there is some constant $K=K(M)$ such that, for any $(x,t,\xi) \in \Omega\times\mathbf{R}\times\mathbf{R}^N$ with $|t|+\|\xi\| \le M$, for any $r>0$ with $B_r(x)\subset\Omega$, for any $v \in W^{1,\infty}(\Omega)$,*

$$[\ v(x)=t \text{ and } F(v,B_r(x)) \le f(x,t,\xi)+1\ ] \quad \Rightarrow \quad \|v\|_{W^{1,\infty}(B_r(x))} \le K\ .$$

Proof. Since f is bounded on bounded sets, there exists a positive constant $M_1 = M_1(M)$ such that $f(x,t,\xi) \le M_1$ for any $(x,t,\xi) \in \Omega\times\mathbf{R}\times\mathbf{R}^N$ with $|t|+\|\xi\| \le M$. Now fix $(x,t,\xi) \in \Omega\times\mathbf{R}\times\mathbf{R}^N$ with $|t|+\|\xi\| \le M$. If $F(v,B_r(x)) \le f(x,t,\xi)+1$ with $B_r(x)\subset\Omega$ and $v \in W^{1,\infty}(\Omega)$ such that $v(x)=t$, then from the coercivity condition on F and the upper bound on f, we have

$$\|Dv\|_{L^\infty(B_r(x))} \le F(v,B_r(x)) \le f(x,t,\xi)+1 \le M_1+1\ .$$

Since $v(x)=t$ and $\|Dv\|_{L^\infty(B_r(x))} \le M_1+1$, this leads to

$$\|v\|_{L^\infty(B_r(x))} \le M + \|Dv\|_{L^\infty(B_r(x))} r \le M + (M_1+1)diam(\Omega)\ .$$

Therefore we have proved that $\|v\|_{W^{1,\infty}(B_r(x)}$ is bounded by some constant K depending only on M. □

Next we show some regularity properties of f.

Lemma 3.4 *Under the assumptions of Theorem 2.2, f is a Caratheodory supremand.*

Proof. In order to show (i) of definition (2.1), let $(t,\xi) \in \mathbf{R}\times\mathbf{R}^N$ and $\lambda \in \mathbf{R}$ be fixed. Define the sets

$$A(x) := \{u \in W^{1,\infty}(\Omega)\ :\ x \in \widehat{u} \text{ with } u(x)=t \text{ and } Du(x)=\xi,\},$$

and

$$\begin{aligned} K_\lambda :&= \{x\in\Omega\ :\ \forall u \in A(x)\ \forall r>0 \text{ s.t. } B_r(x)\subset\Omega\ F(u,B_r(x)) \ge \lambda\} \\ &= \{x \in \Omega\ :\ f(x,u,\xi) \ge \lambda\}. \end{aligned}$$

If we prove that K_λ is measurable for every $\lambda \in \mathbf{R}$, then $f(\cdot,u,\xi)$ is measurable. We adapt the proof of Lemma 2.3 in [7]. Suppose that K_λ is not measurable. Then there is a set C with $K_\lambda \subset C$ s.t. C is measurable and of minimal measure. Let $x_0 \in \widehat{C}\setminus K_\lambda$ where $\widehat{C}$ the set of the Lebesgue points of density of C. From the definition of K_λ, there is an open set $A_0 \in \mathcal{A}$ and some $u \in A(x_0)$ such that $x_0 \in A_0$ and $F(u, A_0) < \lambda$. From the strong continuity of $F(\cdot, A)$, one can find $\epsilon > 0$ such that

$$\|v - u\|_{W^{1,\infty}(A_0)} \leq \epsilon \quad \Rightarrow \quad F(v, A_0) < \lambda \,. \tag{3.6}$$

Let

$$A_1 = \left\{ \; x \in A_0 \mid x \in \widehat{u},\, |u(x) - t| \leq \epsilon/2,\, |Du(x) - \xi| \leq \tfrac{\epsilon}{2diam(A)} \; \right\}$$

Note that, A_1 is measurable and since x_0 is a Lebesgue point of Du, then $|A_1| > 0$.

We claim that $A_1 \cap K_\lambda = \emptyset$. In fact, if $x \in A_1$, then the function $v_x \in W^{1,\infty}(\Omega)$ defined by

$$v_x(y) := u(y) + (u(x_0) - u(x)) + \langle Du(x_0) - Du(x), y - x\rangle$$

belongs to $A(x)$. Since $\|v_x - u\|_{W^{1,\infty}(A_0)} \leq \epsilon$, from (3.6), we have that $F(v_x, A_0) < \lambda$. So $x \notin K_\lambda$. In particular $K_\lambda \subset C \setminus A_1$. Moreover the set $C\backslash A_1$ is still measurable, with a measure smaller than the measure of C, and since $K \subset (C\backslash A_1)$, we have contradicted the minimality of K.

To show (ii) of definition (2.1), let us fix $x \in \Omega$, $(t, \xi) \in \mathbf{R} \times \mathbf{R}^N$, $(t', \xi') \in \mathbf{R} \times \mathbf{R}^N$ and $\epsilon > 0$. From the definition of f we can find some $r > 0$ and some $u \in W^{1,\infty}(\Omega)$ such that $u(x) = u$, $Du(x) = \xi$ and $f(x, u, \xi) \leq F(u, B_r(x)) + \epsilon$. Then

$$f(x, t', \xi') \leq F(u + \varphi_{x,t',\xi'} - \varphi_{x,t,\xi}, B_r(x)) \leq F(u, B_r(x)) + \omega_M(\|\varphi_{x,t',\xi'} - \varphi_{x,u,\xi}\|_{W^{1,\infty}(B_r(x))}) \,,$$

where $\varphi_{x,t,\xi}(y) := t + \xi \cdot (y - x)$ and where $M = \max\{\|u\|_{W^{1,\infty}(B_r(x)}, |t|, |t'|, ||\xi||, ||\xi'||\}$. Note that M actually only depends on $|t|, |t'|, ||\xi||, ||\xi'||$ thanks to Lemma 3.3. Moreover,

$$\|\varphi_{x,t',\xi'} - \varphi_{x,u,\xi})\|_{W^{1,\infty}(B_r(x))} \leq |t' - t| + ||\xi' - \xi||(1 + r) \,.$$

Hence

$$f(x, t', \xi') \leq f(x, t, \xi) + \epsilon + \omega_M(|t' - t| + ||\xi' - \xi||(1 + r)) \,,$$

which leads to the desired result as $\epsilon \to 0^+$. $\square$

Proof of Theorem 2.2. Let $u \in W^{1,\infty}(\Omega)$. Our aim is to prove (2.3). For this, let us denote by $L(u)$ the set of points which are at the same time of points of differentiability of u, Lebesgue points of Du and Lebesgue points of $f(x, u(x), Du(x))$. For any $A \in \mathcal{A}$, for any $x \in L(u) \cap A$, for any $r > 0$ with $B_r(x) \subset A$, we have: $f(x, u(x), Du(x)) \leq F(u, B_r(x)) \leq F(u, A)$. Hence

$$F(u, A) \geq \operatorname*{ess\,sup}_x f(x, u(x), Du(x)) \,.$$

In order to prove the reverse inequality, we first notice that there is some constant $M = M(\|u\|_{W^{1,\infty}(\Omega)})$ such that, for any $x \in L(u)$, for any $r > 0$, such that $B_r(x) \subset \Omega$, for any $v \in W^{1,\infty}(\Omega)$,

$$[\; v(x) = u(x) \text{ and } F(v, B_r(x)) \leq f(x, u, Du(x)) + 1 \;] \quad \Rightarrow \quad \|v\|_{W^{1,\infty}(B_r(x))} \leq M \,. \tag{3.7}$$

This is a straightfoward application of Lemma 3.3 combined with the fact that f is bounded on bounded sets (Lemma 3.2).

Let us now fix $\varepsilon > 0$. From the definition of f, for any $x \in L(u) \cap A$ there is some $r_x \in (0, \epsilon)$ with $B_{r_x}(x) \subset A$, some $v_x \in W^{1,\infty}(\Omega)$ with $v_x(x) = u(x)$, $Dv_x(x) = Du(x)$ and

$$f(x, u(x), Du(x)) \geq F(v_x, B_{r_x}(x)) - \varepsilon \,. \tag{3.8}$$

According to Lemma 3.1 we can find some $r'_x \in (0, r_x)$, some open set A_x with $B_{r'_x/2}(x) \subset A_x \subset B_{r'_x}(x)$ and $|\partial A_x| = 0$, and some constants $\alpha_x \in (0, \varepsilon)$, $\beta_x \in (0, \varepsilon)$ with

$$u(y) = v_x(y) + \alpha_x - \beta_x|y - x| \quad \text{on } \partial A_x \quad \text{and} \quad u(y) < v_x(y) + \alpha_x - \beta_x|y - x| \quad \text{in } A_x \,.$$

From Vitali covering Theorem (see for instance Corollary 10.6 of [19]), we can find a sequence (x_n) such that the disjoint family $(A_{x_n})_{n\in\mathbf{N}}$ covers A: $|A\setminus\bigcup_n A_{x_n}|=0$. Let us now set

$$w_\varepsilon(x)=v_{x_n}(x)+\alpha_{x_n}-\beta_{x_n}|x-x_n| \quad \text{if } x \text{ belongs to some } A_{x_n} \text{ and } \quad w_\varepsilon(x)=u(x) \quad \text{otherwise.} \tag{3.9}$$

We claim that w_ε belongs to $W^{1,\infty}(\Omega)$, with a Lipschitz constant independant of ε, that w_ε converges to u in L^∞ and that

$$\operatorname*{ess\,sup}_A f(x,u(x),Du(x)) \geq F(w_\varepsilon,A)-\omega_M(2\varepsilon)-\varepsilon\ , \tag{3.10}$$

for some M.

Note that this statement completes the proof of the representation formula (2.3) because, from lower semicontinuity of F, letting $\varepsilon\to 0^+$ in (3.10) gives

$$\operatorname*{ess\,sup}_A f(x,u(x),Du(x)) \geq \liminf_{\varepsilon\to 0^+} F(w_\varepsilon,A) \geq F(u,A)\ .$$

Let us first show that the w_ε belongs to $W^{1,\infty}(\Omega)$. For this we note that w_ε is the pointwise limit of the Lipschitz maps v_n defined inductively by $v_0=u$, and

$$v_{n+1}(x)=v_{x_{n+1}}(x)+\alpha_{x_{n+1}}-\beta_{x_{n+1}}|x-x_{n+1}| \text{ if } x \text{ belongs to } A_{x_{n+1}} \text{ and } v_{n+1}(x)=v_n(x) \text{ otherwise.}$$

The maps v_n are equilipschitz continuous, with a Lipschitz constant independant of ε, because the v_{x_n} are equilipschitz continuous in A_{x_n} from (3.7) and because $\alpha_{x_n}\in(0,\epsilon)$ and $\beta_{x_n}\in(0,\epsilon)$. Hence the w_ε belong to $W^{1,\infty}(\Omega)$ with a Lipschitz norm which does not depend on ε. Let us denote by k a constant such that $\|w_\varepsilon\|_{W^{1,\infty}(\Omega)}\leq k$ for any $\varepsilon>0$. Now we show that $(w_\varepsilon)_\varepsilon$ converges to u in L^∞ as $\varepsilon\to 0^+$. If $x\in A_{x_n}$ for some n, then there is some $y\in\partial A_{x_n}$ with $|x-y|\leq 2r_{x_n}\leq 2\epsilon$ because $A_{x_n}\subset B_{r_{x_n}}(x_n)$ and $r_{x_n}\leq\epsilon$. Hence

$$|w_\varepsilon(x)-u(x)|\leq |w_\varepsilon(x)-w_\varepsilon(y)|+|w_\varepsilon(y)-u(y)|+|u(y)-u(x)|\leq 2k\varepsilon+0+2k\varepsilon$$

because $w_\varepsilon(y)=u(y)$. So $|w_\varepsilon(x)-u(x)|\leq 4k\varepsilon$ for any $x\in\bigcup_n A_{x_n}$. Since the family (A_{x_n}) covers A, then $|w_\varepsilon(x)-u(x)|\leq 4k\varepsilon$ for any $x\in A$. Finally $|w_\varepsilon(x)-u(x)|\leq 4k\varepsilon$ for any $x\in\Omega$ because $w_\varepsilon=u$ in $\Omega\setminus A$. So we have proved that w_ε converges uniformly to u.

It remains to show (3.10). Since $\alpha_{x_n}\in(0,\varepsilon)$, $\beta_{x_n}\in(0,\varepsilon)$ and $A_{x_n}\subset B_\varepsilon(x_n)$, then, by (3.9), we have that

$$\|v_{x_n}-w_\varepsilon\|_{L^\infty(A_{x_n})}=\sup_{A_{x_n}}|\alpha_{x_n}-\beta_{x_n}|x-x_n||\leq 2\varepsilon \quad \text{and} \quad \|Dv_{x_n}-Dw_\varepsilon\|_{L^\infty(A_{x_n})}\leq\varepsilon.$$

Thus, by using (3.8) and the continuity assumption of F, we get

$$f(x_n,u(x_n),Du(x_n))\geq F(v_{x_n},A_{x_n})-\varepsilon\geq F(w_\varepsilon,A_{x_n})-\omega_{M'}(\|v_{x_n}-w_\varepsilon\|_{W^{1,\infty}(A_{x_n})})-\varepsilon$$

where $M'=k+3\epsilon$. In particular we have that

$$f(x_n,u(x_n),Du(x_n))\geq F(w_\varepsilon,A_{x_n})-\omega_M(3\varepsilon)-\varepsilon\ .$$

Since x_n belongs to $L(u)\cap A$, the last inequality leads to

$$\operatorname*{ess\,sup}_A f(x,u(x),Du(x))\geq\sup_n f(x_n,u(x_n),Du(x_n))\geq\sup_n F(w_\varepsilon,A_{x_n})-\omega_M(3\varepsilon)-\varepsilon\geq F(w_\varepsilon,A)-\omega_M(3\varepsilon)-\varepsilon$$

because of the countable supremality of F and the fact that $|A\setminus\bigcup_n A_{x_n}|=0$. So (3.10) is established and the proof of the representation formula is complete. □

Proof of Proposition 2.3 : We first prove that for every $M>0$, and for all $(t,\xi)\in\mathbf{R}\times\mathbf{R}^N$ such that $|t|+|\xi|<M$ there exists $K=K(M)$ such that $|f(x,t,\xi)-f(y,t,\xi)|\leq\rho_K(|x-y|)$. In fact fix $M>0$, $(t,\xi)\in\mathbf{R}\times\mathbf{R}^N$ such that $|t|+|\xi|<M$ and $x,y\in\Omega$. Then for every $\varepsilon>0$ there exists $B_r(y)\subset\Omega$ and $u\in W^{1,\infty}(\Omega)$ such that $y\in\hat{u}$, $u(y)=t$, $Du(y)=\xi$ and $F(u,B_r(y))\leq f(y,u,\xi)+\epsilon$. By lemma 3.3 there is some constant $K=K(M)$ such that $\|u\|_{W^{1,\infty}(B_r(x))}\leq K$. Then

$$f(x,t,\xi)\leq F(u_{y-x},B_r(x))\leq F(u,B_r(y))+\rho_K(|x-y|)\leq f(y,t,\xi)+\varepsilon+\rho_K(|x-y|)$$

which leads to the desired result as $\epsilon\to 0^+$. By applying Theorem 2.7 in [8] f is level convex in the third variable. □

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