On the selection of maximal Cheeger sets

G. Buttazzo, G. Carlier, M. Comte

March 20, 2007

Abstract

Given a bounded open subset Ω of $\mathbb{R}^d$ and two positive weight functions f and g, the Cheeger sets of Ω are the subdomains C of finite perimeter of $\overline{\Omega}$ that maximize the ratio $\int_C f(x)\,dx \,/ \int_{\partial^* C} g(x)\,d\mathcal{H}^{d-1}$. Existence of Cheeger sets is a well-known fact. Uniqueness is a more delicate issue and is not true in general (although it holds when Ω is convex and $f \equiv g \equiv 1$ as recently proved in [4]). However, there always exists a unique maximal (in the sense of inclusion) Cheeger set and this paper addresses the issue of how to determine this maximal set. We show that in general the approximation by the p-Laplacian does not provide, as $p \to 1$, a selection criterion for determining the maximal Cheeger set. On the contrary, a different perturbation scheme, based on the constrained maximization of $\int_\Omega f\big(u - \varepsilon\Phi(u)\big)\,dx$ for a strictly convex function Φ, gives, as $\varepsilon \to 0$, the desired maximal set.

Keywords: Cheeger sets, p-Laplacian approximation, concave penalization, 1-Laplacian type operators.

MSC 2000: 49J45, 49Q10, 49R50, 35J20, 35P99

1 Introduction

Given a bounded open Lipschitz subset Ω of $\mathbb{R}^d$ Cheeger sets are defined as the subsets C of Ω which maximize the ratio $|C|/\operatorname{Per}(C)$ where $\operatorname{Per}(C)$ is the perimeter of C and $|C|$ denotes the Lebesgue measure of C. By the direct methods of the calculus of variations and in particular by the De Giorgi theory of perimeters and BV spaces (see for instance [2]) the existence of a Cheeger set follows straightforwardly.

We consider here a slightly more general situation, where the Lebesgue measure and the perimeter are weighted by two weight functions f and g; more precisely, we consider the problem

$$\mu_1 := \sup\left\{ \frac{\int_C f\,dx}{\int_{\partial^* C} g\,d\mathcal{H}^{d-1}} \;:\; C \subset \overline{\Omega} \right\} \tag{1.1}$$

where $\partial^* C$ is the reduced boundary of C and $\mathcal{H}^{d-1}$ is the Hausdorff $d-1$ dimensional measure. Again, under the assumptions

- $f \in L^\infty(\Omega)$, $f > 0$ a.e.,
- g is continuous on $\overline{\Omega}$ and $\inf g > 0$,

the direct methods of the calculus of variations apply and provide the existence of a Cheeger set.

An important fact is that the shape optimization problem (1.1) is tightly related to the variational problem:

$$\mu_1 = \sup\left\{\int_\Omega f u\, dx \ : \ u \in BV_0(\Omega),\ \int_{\mathbb{R}^d} g\, d|Du| \le 1\right\}, \tag{1.2}$$

where $BV_0(\Omega)$ denotes the class of functions in $BV(\mathbb{R}^d)$ that vanish outside $\overline{\Omega}$. Note that for $u \in BV_0(\Omega)$, one has:

$$\int_{\mathbb{R}^d} g\, d|Du| = \int_\Omega g\, d|Du| + \int_{\partial\Omega} g|u|\, d\mathcal{H}^{d-1}.$$

We remark that μ_1 coincides with the inverse of the first eigenvalue λ_1 of an operator of 1-Laplacian type. More precisely,

$$\lambda_1 = \frac{1}{\mu_1} = \inf\left\{R(u) \ : \ u \in BV_0(\Omega),\ u \ne 0\right\} \tag{1.3}$$

where $R(u)$ is the Rayleigh quotient

$$R(u) := \frac{\int_{\mathbb{R}^d} g\, d|Du|}{\int_\Omega f|u|\, dx}.$$

In the sequel, we will denote by χ_C the characteristic function of the set $C \subset \mathbb{R}^d$, and define the set of solutions of (1.2):

$$Q := \left\{u \in BV_0(\Omega) \ : \ \int_{\mathbb{R}^d} g\, d|Du| \le 1,\ \int_\Omega f u\, dx = \mu_1\right\} \tag{1.4}$$

as well as the family of Cheeger sets:

$$\mathcal{C} := \left\{C \subset \overline{\Omega} \ : \ \chi_C \in BV_0(\Omega),\ \int_C f\, dx = \mu_1 \int_{\partial^* C} g\, d\mathcal{H}^{d-1}\right\}. \tag{1.5}$$

Of course χ_C is a solution of (1.3) whenever $C \in \mathcal{C}$. There is however a more precise relationship between Cheeger sets and solutions of (1.3): u solves (1.3) if and only if all its level sets are Cheeger sets. We refer to Theorem 2 of [3] for a proof, the case $f = g = 1$ being well-known.

Except under special additional assumptions (for instance when $f = g = 1$ and Ω is convex, see [4]), one cannot expect Cheeger sets to be unique and examples are known where they are actually infinitely many (see for instance [8, 9] and the examples of Section 2). On the other hand, the family of Cheeger sets $\mathcal{C}$ is stable by countable union (see Theorem 3 of [3]). This implies that $\mathcal{C}$ possesses a maximal element in the sense of inclusion, the *maximal Cheeger set* of Ω:

Proposition 1.1. *There exists a unique maximal Cheeger set, i.e. a unique $C_0 \in \mathcal{C}$ such that for every $C \in \mathcal{C}$, C is included in C_0 up to a Lebesgue negligible set.*

Proof. Let us consider the problem of maximizing the Lebesgue measure of C among all Cheeger sets $C \in \mathcal{C}$. Since $\{\chi_C \ : \ C \in \mathcal{C}\}$ is compact in $L^1(\Omega)$, we obtain the existence of some maximizer C_0. If $C \in \mathcal{C}$, since $C \cup C_0 \in \mathcal{C}$, we have $C \subset C_0$ up to a Lebesgue negligible set, therefore C_0 is a maximal Cheeger set and it is obviously the only one (up to a Lebesgue negligible set again). □

The question of determining the maximal Cheeger set then becomes an interesting issue. The main focus of the present paper is to investigate whether natural approximation schemes select at the limit the maximal Cheeger set.

A possibility, that has been investigated for instance in [8, 10], is to consider the solutions u_p of a PDE involving the p-Laplace operator and to let p tend to 1. The hope is that the limit function u is a characteristic function of the form $u = \alpha\chi_C$ with C the maximal Cheeger set, or that at least the support of u is the maximal Cheeger set. The arguments in favour of this approach are that it works when $f = g = 1$ and Ω is convex (see [8]) and that in any case all level sets of the limit function u are Cheeger sets (see [3]).

We show in Section 2 that the procedure above cannot be expected to work in general. We consider the p-approximation of problem (1.2)

$$\mu_p := \sup\Big\{ \int_\Omega f u \, dx \ : \ \int_\Omega g|Du|^p \, dx \le 1, \ u \in W_0^{1,p}(\Omega)\Big\}. \tag{1.6}$$

The unique (nonnegative) maximizer u_p of (1.6) is of course the solution of the PDE

$$-\operatorname{div}\big(g|Du|^{p-2}Du\big) = \lambda_p f, \ u \in W_0^{1,p}(\Omega), \ \text{with } \lambda_p := \frac{1}{\mu_p}. \tag{1.7}$$

As $p \to 1$, the maximal values μ_p in (1.6) tend to the maximal Cheeger value μ_1 in (1.1) (see Proposition 2.1) and, denoting by u_p the unique solution of the PDE (1.7), we have convergence of some subsequence to some solution u of (1.2). However, we show by some onedimensional examples, that neither the limit of u_p nor its support identify in general the maximal Cheeger set.

In Section 3 we consider a different kind of approximation:

$$\sup\Big\{ \int_\Omega f\big(u - \varepsilon\Phi(u)\big) \, dx \ : \ \int_{\mathbb{R}^d} g \, d|Du| \le 1, \ u \in BV_0(\Omega)\Big\} \tag{1.8}$$

We show that if the function Φ is strictly convex and $\Phi(0) = 0$ then the optimal solutions u_ε of problem (1.8) tend as $\varepsilon \to 0$ to a characteristic function $u = \alpha\chi_C$ where C is the maximal Cheeger set in Ω.

2 The p-Laplacian approximation

2.1 Convergence of the p-Laplacian approximation

By classical arguments, we know that there exists a unique solution, denoted by u_p, of the variational problem (1.6). In addition $u_p > 0$ in Ω (see [14]) and by standard elliptic regularity theory (see [13] or [6]), $u_p \in C^{1,\alpha}(\Omega)$ whenever g is of class C^1 (an assumption that we won't need here). Moreover, u_p is the unique solution of the p-Laplace equation

$$-\operatorname{div}\left(g|Du|^{p-2}Du\right) = \lambda_p f, \qquad u \in W_0^{1,p}(\Omega) \tag{2.1}$$

where $\lambda_p = 1/\mu_p$ and μ_p is the maximal value of (1.6). By construction, one has:

$$\mu_p = \int_\Omega f u_p \, dx \qquad \text{and} \qquad \left(\int_\Omega g|Du_p|^p \, dx\right)^{1/p} = 1. \tag{2.2}$$

This section is devoted to the convergence of μ_p to μ_1 and to the convergence (up to a subsequence) of the maximizers u_p to some maximizer of (1.2).

Proposition 2.1. *As $p \to 1$ the maximal values μ_p in* (1.6) *tend to the maximal Cheeger value μ_1 in* (1.1).

Proof. Using Hölder's inequality, we have

$$\int_\Omega g|Du_p| \, dx \leq \left(\int_\Omega g \, dx\right)^{(p-1)/p} \left(\int_\Omega g|Du_p|^p \, dx\right)^{1/p} = \left(\int_\Omega g \, dx\right)^{(p-1)/p}. \tag{2.3}$$

We thus deduce

$$\mu_1 \geq \frac{\int_\Omega f u_p \, dx}{\int_\Omega g|Du_p| \, dx} \geq \mu_p \left(\int_\Omega g \, dx\right)^{(1-p)/p}, \tag{2.4}$$

hence

$$\mu_1 \geq \limsup_{p \to 1} \mu_p \left(\int_\Omega g \, dx\right)^{(p-1)/p} = \limsup_{p \to 1} \mu_p. \tag{2.5}$$

Let $\delta > 0$; by standard approximation results (see in particular Remark 2.12 in [7] and Proposition 3.15 in [2]), there exists a nonnegative function $v \in C^\infty(\mathbb{R}^d)$ with $v \equiv 0$ on $\mathbb{R}^d \setminus \Omega$ such that

$$\int_\Omega g|Dv| \, dx = 1 \qquad \text{and} \qquad \int_\Omega f v \, dx \geq \mu_1 - \delta. \tag{2.6}$$

We then have

$$\mu_p \geq \frac{\int_\Omega f v \, dx}{\left(\int_\Omega g|Dv|^p \, dx\right)^{1/p}}$$

so that

$$\liminf_{p\to1}\mu_p \geq \liminf_{p\to1}\frac{\int_\Omega fv\,dx}{\left(\int_\Omega g|Dv|^p\,dx\right)^{1/p}} \geq \mu_1-\delta. \tag{2.7}$$

Since $\delta>0$ is arbitrary in (2.7), together with (2.5) we finally get $\mu_1=\lim\mu_p$. □

Proposition 2.2. *Up to a subsequence, $(u_p)_p$ converges in $L^1(\Omega)$, as $p\to1$, to a solution u of* (1.2).

Proof. Combining Hölder's inequality with the fact that $\int_\Omega g|Du_p|^p\,dx\leq1$ we obtain $\int_\Omega|Du_p|\,dx\leq M$ for a suitable constant $M\geq0$. The sequence $(u_p)_p$ is therefore bounded in $BV_0(\Omega)$, and thus precompact in $L^1(\Omega)$. Hence (u_p) converges, up to a subsequence (still denoted (u_p)) to some u in $L^1(\Omega)$. Applying standard lower-semi continuity results (see for instance Corollary 1 of [3]), we then get

$$\int_{\mathbb{R}^d} g\,d|Du| \leq \liminf_{p\to1}\int_\Omega g|Du_p|\,dx \leq \liminf_{p\to1}\Big(\int_\Omega g|Du_p|^p\,dx\Big)^{1/p} \tag{2.8}$$

where the second inequality follows from (2.3). Therefore $\int_{\mathbb{R}^d} g\,d|Du|\leq1$. On the other hand

$$\lim_{p\to1}\int_\Omega fu_p\,dx=\int_\Omega fu\,dx. \tag{2.9}$$

Finally we get

$$\frac{\int_\Omega fu\,dx}{\int_\Omega g\,d|Du|} \geq \limsup_{p\to1}\frac{\int_\Omega fu_p\,dx}{(\int_\Omega g|Du_p|^p\,dx)^{1/p}}=\mu_1 \tag{2.10}$$

which concludes the proof. □

Getting back to the main purpose of the present paper, namely the selection of the maximal Cheeger set, at this point, two questions naturally arise:

- is the limit function u (up to a multiplicative constant) the characteristic function of the maximal Cheeger set?
- in case of a negative answer to the previous question, does the support of u identify the maximal Cheeger set?

As we shall see in the next section, by means of simple one-dimensional counter-examples, the answer is actually negative to both questions.

2.2 The one-dimensional case

In this section, we consider problem (1.6) (equivalently equation (2.1)) in dimension one. In this case the differential equation (2.1) can be explicitly integrated and this will enable us to analyze the limit of maximizers of problem (1.6) as $p \to 1$.

We take $\Omega := (-1, 1)$ and f, g two even functions (that satisfy the general assumptions of the paper); then, it is easy to see that the solution of (1.6) is even too. Setting $n = \frac{1}{p-1}$ (so that $n \to +\infty$ as $p \to 1$) we thus consider the maximization problem

$$\sup \Big\{ \int_{-1}^{1} f u \, dx \ : \ \int_{-1}^{1} g |u'|^{1+1/n} \, dx \leq 1, \ u \in W_0^{1,1+1/n}(-1,1) \Big\} \tag{2.11}$$

and denote by w_n the solution of (2.11). We also set

$$F(x) := \int_0^x f(t) \, dt, \qquad h(x) := F(x)/g(x).$$

Proposition 2.3. *The (even) solution of* (2.11) *is given by*

$$w_n(x) = \frac{\int_x^1 h^n \, dt}{\Big(2 \int_0^1 g h^{n+1} \, dt \Big)^{n/(n+1)}} \qquad \forall x \in [0, 1]. \tag{2.12}$$

Proof. Obviously, w_n is proportional to u_n that solves

$$g(x) |u_n'(x)|^{-1+1/n} u_n'(x) = -F(x). \tag{2.13}$$

Thus u_n is decreasing on $[0, 1]$ and by (2.13) $-u_n'(x) = h^n(x)$. Integrating once more and using the fact that $u_n(1) = 0$ leads to $u_n(x) = \int_x^1 h^n(t) \, dt$. Now we set $w_n = C_n u_n$ with C_n such that $2 \int_0^1 g |w_n'|^{1+1/n} \, dt = 1$ which proves the result. □

Proposition 2.4. *There exist Cheeger sets of* $(-1, 1)$ *which are symmetric intervals.*

Proof. From Proposition 2.2, we know that u_n converges in L^1, up to a subsequence, to a solution u of (1.2). Since u_n is even and nonincreasing on $(0, 1]$, the same holds for u. From Theorem 2 of [3] the level sets of u are Cheeger sets. Therefore there exists a symmetric interval which is a Cheeger set. □

Determining Cheeger sets of the form $[-a, a]$ amounts to solve

$$\sup \big\{ h(a) \ : \ a \in [0, 1] \big\}.$$

Proposition 2.5. *If $h(x) \leq h(1)$ for every $x \in [0,1]$ then the maximal Cheeger set coincides with the whole interval $[-1,1]$.*

Proof. Indeed in this case $a = 1$ is a maximizer of h. $\square$

We now study the behaviour of the functions w_n as $n \to +\infty$. The function $h(x)$ is bounded and, since the expression of w_n is homogeneous of degree zero in h, we may assume that $\max h = 1$.

Proposition 2.6. *Assume that in an interval $[a,b]$ with $0 \leq a < b < 1$ we have $h(x) = 1$ and that $h(x) < 1$ in an open interval $(\alpha, 1)$. Then $w_n(x) \to 0$ for every $x \in (\alpha, 1)$.*

Proof. We have

$$2\int_0^1 gh^{n+1}\,dt \geq 2\int_a^b gh^{n+1}\,dt = 2\int_a^b g\,dt \tag{2.14}$$

therefore

$$\Big(2\int_0^1 gh^{n+1}\,dt\Big)^{n/(n+1)} \geq \Big(2\int_a^b g\,dt\Big)^{n/(n+1)} \tag{2.15}$$

and then for n large enough

$$\Big(2\int_0^1 gh^{n+1}\,dt\Big)^{n/(n+1)} \geq \int_a^b g\,dt. \tag{2.16}$$

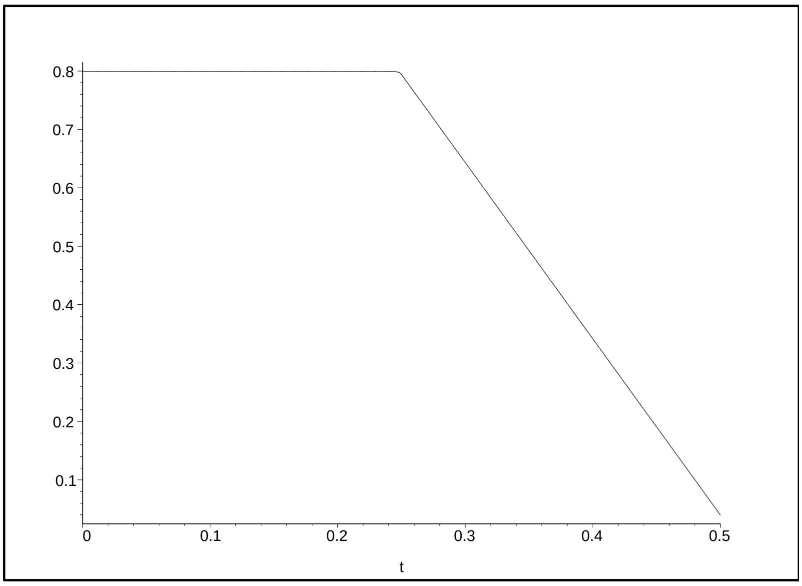
On the other hand, since $h(x) < 1$ in $(\alpha, 1)$, $h^n(x) \to 0$ as $n \to \infty$ in $(\alpha, 1)$. Taking into account the expression of w_n given by (2.12), this gives the result. $\square$

Putting together Propositions 2.5 and 2.6 we can easily construct functions f and g such that the maximal Cheeger set is the whole interval $[-1,1]$ whereas the limit function $\lim_n w_n(x)$ vanishes in a neighbourhood of 1.

Example 2.7. Taking for instance $f \equiv 1$ and

$$g(x) = \begin{cases} 1/4 & \text{if } x \in [0, 1/4] \\ x & \text{if } x \in [1/4, 1/2] \\ \dfrac{4x}{3 + |4x-3|} & \text{if } x \in [1/2, 1] \end{cases}$$

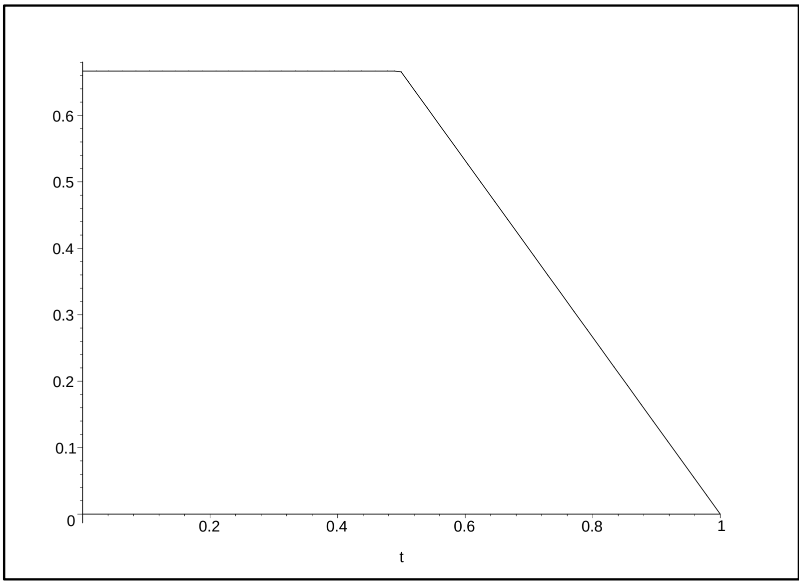
provides the desired counterexample. Indeed in this case, by Proposition 2.5, the maximal Cheeger set is the full interval (and all the intervals $[-a, a]$ with $a \in [1/4, 1/2]$ are Cheeger sets) whereas the limit of the w_n's vanishes on $[1/2, 1]$. We have plotted the graph of $w = \lim_n w_n$ in the next figure.



Example 2.8. Take $f \equiv 1$ and

$$g(x) = \begin{cases} 1/2 & \text{if } x \in [0, 1/2] \\ x & \text{if } x \in [1/2, 1]. \end{cases}$$

All the intervals $[-a, a]$ with $a \in [1/2, 1]$ are Cheeger sets and w_n again converges to some function plotted below which is not a characteristic function but whose support is the maximal Cheeger set.



Remark 2.9. We notice that in both examples, $\lim_n w_n$ is a solution of (1.2) which is continuous and nonconstant. Of course, such solutions can exist only if there is a continuum of Cheeger sets as in the previous examples.

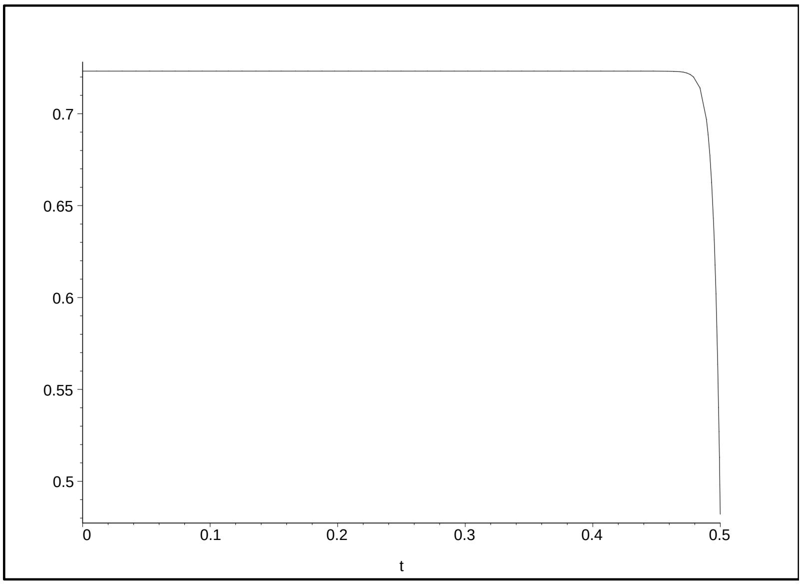
Example 2.10. We now consider a case where h achieves its maximum only at 1 and 1/2, for instance:

$$h(x) = \begin{cases} 2x & \text{if } x \in [0,1/2] \\ 2(1-x) & \text{if } x \in [1/2,3/4] \\ 2x-1 & \text{if } x \in [3/4,1] \end{cases}$$

which is obtained by taking $f \equiv 1$ and

$$g(x) = \begin{cases} 1/2 & \text{if } x \in [0,1/2] \\ x/(2-2x) & \text{if } x \in [1/2,3/4] \\ x/(2x-1) & \text{if } x \in [3/4,1]. \end{cases}$$

In this case, w_n still converges to a multiple of the characteristic of $[-1/2,1/2]$. The next graph represents w_{100}.



3 Concave penalizations select maximal Cheeger sets

In this section, we approximate the maximization problem

$$\sup\left\{ \int_\Omega f u \, dx \ : \ \int_{\mathbb{R}^d} g \, d|Du| \le 1,\ u \in BV_0(\Omega) \right\} \tag{3.1}$$

by the strictly concave penalization

$$\sup\left\{ \int_\Omega f\big(u - \varepsilon\Phi(u)\big) \, dx \ : \ \int_{\mathbb{R}^d} g \, d|Du| \le 1,\ u \in BV_0(\Omega) \right\} \tag{3.2}$$

where $\varepsilon > 0$ is a perturbation parameter and Φ is a strictly convex nonnegative function that satisfies:

$$\Phi(0) = 0, \qquad 0 \leq \Phi(t) < +\infty \quad \forall t \in \mathbb{R}^+. \tag{3.3}$$

Again, we denote by μ_1 the optimal value of (3.1). We recall that, from Theorem 4 of [3], the set Q of solutions of (3.1) is in fact included in $L^\infty(\Omega)$.

Theorem 3.1. *Let u_ε be the solution of* (3.2)*; then the following holds:*

- *$(u_\varepsilon)_\varepsilon$ converges in $L^1(\Omega)$, as $\varepsilon \to 0^+$, to the solution $\overline{u}$ of*

$$\inf\left\{\int_\Omega f\Phi(u)\,dx \;:\; u \in Q\right\}, \tag{3.4}$$

- *$\overline{u} = \alpha\chi_{C_0}$ for some $\alpha > 0$ and $C_0 \subset \overline{\Omega}$,*
- *C_0 is the maximal Cheeger set, i.e. $C_0 \in \mathcal{C}$ and C_0 contains every other Cheeger set (up to a Lebesgue negligible set).*

Proof. Since $(u_\varepsilon)_\varepsilon$ is bounded in $BV(\Omega)$, it admits a subsequence (not relabeled) that converges in $L^1(\Omega)$ to some $\overline{u} \in BV_0(\Omega)$ and

$$\overline{u} \geq 0, \qquad \int_{\mathbb{R}^d} g\,d|D\overline{u}| \leq 1.$$

Let $v \in Q$; for every $\varepsilon > 0$ we have

$$0 \geq \int_\Omega f(u_\varepsilon - v)\,dx \geq \varepsilon \int_\Omega f\big(\Phi(u_\varepsilon) - \Phi(v)\big)\,dx. \tag{3.5}$$

Letting $\varepsilon \to 0^+$ in (3.5) and using the facts that $\Phi \geq 0$ and $\Phi(v)$ is bounded since $v \in L^\infty(\Omega)$ (by Theorem 4 of [3]), we then get

$$\int_\Omega f\overline{u}\,dx = \int_\Omega fv\,dx = \mu_1$$

hence $\overline{u} \in Q$. Dividing by ε in (3.5), thanks to Fatou's Lemma, we get

$$\int_\Omega f\Phi(\overline{u})\,dx \leq \liminf_{\varepsilon\to 0^+} \int_\Omega f\Phi(u_\varepsilon)\,dx \leq \int_\Omega f\Phi(v)\,dx$$

so that $\overline{u}$ solves (3.4). By the strict convexity of Φ, the minimization problem (3.4) admits $\overline{u}$ as unique solution and the whole family $(u_\varepsilon)_\varepsilon$ converges to $\overline{u}$.

Let us now prove the second assertion. Assume by contradiction that $\overline{u}$ is not of the form $\alpha\chi_{C_0}$ with $\alpha > 0$ and $C_0 \subset \overline{\Omega}$; then $\overline{u} \neq \overline{w}$ with

$$\overline{w} := \frac{\int_\Omega f\overline{u}\,dx}{\int_{\{\overline{u}>0\}} f\,dx}\chi_{\{\overline{u}>0\}}. \tag{3.6}$$

From Theorem 3 of [3] it follows that the set $C = \{\overline{w} > 0\} = \{\overline{u} > 0\}$ is a Cheeger set and that $\overline{w} \in Q$. Now using $\Phi(0) = 0$, the fact that $\overline{u} \neq \overline{w}$ and Jensen's inequality, we get

$$\int_\Omega f\Phi(\overline{u})\,dx = \int_{\{\overline{u}>0\}} f\Phi(\overline{u})\,dx > \left(\int_{\{\overline{u}>0\}} f\,dx\right)\Phi\left(\frac{\int_\Omega f\overline{u}\,dx}{\int_{\{\overline{u}>0\}} f\,dx}\right) = \int_\Omega f\Phi(\overline{w})\,dx$$

contradicting the fact that $\overline{u}$ solves (3.4). This proves that $\overline{u} = \alpha\chi_{C_0}$ with

$$C_0 = \{\overline{u} > 0\} \qquad \text{and} \qquad \alpha = \frac{\int_\Omega f\overline{u}\,dx}{\int_{\{\overline{u}>0\}} f\,dx}.$$

It remains to prove that C_0 is the maximal Cheeger set. Let us remark that for every $C \in \mathcal{C}$, one has

$$\frac{\chi_C}{\int_{\partial^* C} g\,d\mathcal{H}^{d-1}} = \frac{\mu_1\chi_C}{\int_C f\,dx} \in Q$$

so that by (3.4)

$$\left(\int_C f\,dx\right)\Phi\left(\frac{\mu_1}{\int_C f\,dx}\right) \geq \left(\int_{C_0} f\,dx\right)\Phi\left(\frac{\mu_1}{\int_{C_0} f\,dx}\right) \qquad \forall C \in \mathcal{C}. \tag{3.7}$$

Moreover, the function $t \mapsto t\Phi(\frac{\mu_1}{t})$ is decreasing, and thus (3.7) implies

$$\int_{C_0} f\,dx \geq \int_C f\,dx \qquad \forall C \in \mathcal{C}. \tag{3.8}$$

Since $C_0 \cup C \in \mathcal{C}$ for every $C \in \mathcal{C}$, we then have

$$\int_{C\setminus C_0} f\,dx = 0 \qquad \forall C \in \mathcal{C}$$

and since $f > 0$ this proves that $C \subset C_0$ (up to a negligible set). □

4 Concluding remarks and related problems

This paper has focused on the selection of the maximal Cheeger set and we have given elementary examples for which there are several (even infinitely many) Cheeger sets. In such nonuniqueness cases, we have shown that the natural p-Laplacian approximation does not always select the maximal Cheeger set (but the concave penalization scheme of Section 3 does). However, when there is a unique Cheeger set (equivalently when (1.2) possesses a unique solution), Propositions 2.1 and 2.2 of course imply the convergence of the p-Laplacian approximations u_p to (a multiple of) the characteristic function of the unique Cheeger set. Of course, when there is such uniqueness, the selection of the maximal Cheeger set is not a relevant issue. In fact, nonuniqueness is rather rare as the following genericity result shows:

Proposition 4.1. *Let $g \in C^0(\overline{\Omega})$ with $g \geq g_0$ for a positive constant g_0. Then there exists a G_δ dense subset X of $C^0(\overline{\Omega}, \mathbb{R}^+)$ such that for every $f \in X$,* (1.2) *admits a unique solution (equivalently $\mathcal{C}$ is a singleton).*

Proof. For every $f \in C^0(\overline{\Omega})$ (not necessarily nonnegative) we define

$$V(f) := \sup \left\{ \int_\Omega f u \, dx \ : \ u \in BV_0(\Omega), \ u \geq 0, \ \int_{\mathbb{R}^d} g \, d|Du| \leq 1 \right\},$$

then V is a convex continuous (even Lipschitz) functional on $C^0(\overline{\Omega})$. Since $C^0(\overline{\Omega})$ is separable and complete, when equipped with the sup norm, it follows from a theorem of Mazur (see [11] or Theorem 1.20 in [12]) that V is Gâteaux differentiable on a G_δ dense subset of $C^0(\overline{\Omega})$. Generic uniqueness then follows at once from the fact that the subgradient of V at $f \in C^0(\overline{\Omega}, \mathbb{R}^+)$ is exactly the set of solutions of (1.2). □

Remark 4.2. The previous proof works in the same way when the weight f is taken in any separable Banach space naturally related to the problem (e.g. $L^q(\Omega)$ with $q \in [d, +\infty)$). A similar proof also works for fixed f and a generic g.

Also, in the present paper, we have only considered the stationary case, although another related interesting issue is the asymptotic behaviour of the (motion by mean curvature-like) evolution equation

$$\partial_t u - \operatorname{div} \left(g \frac{Du}{|Du|} \right) = f.$$

Acknowledgements. This research has been conceived during a visit of the first author to CEREMADE of Université de Paris Dauphine; he wishes to thank this institution for the warm and friendly atmosphere provided during all the visit.

References

[1] F. Alter, V. Caselles, A. Chambolle: *A characterization of convex calibrable sets in* $\mathbb{R}^N$*.* Math. Ann., **332** (2) (2005), 329–366.

[2] L. Ambrosio, N. Fusco, D. Pallara: *Functions of Bounded Variation and Free Discontinuity Problems.* Oxford Mathematical Monographs, Oxford University Press, New York (2000).

[3] G. Carlier and M. Comte: *On a weighted total variation minimization problem.* Preprint Laboratoire Jacques-Louis Lions, Université Pierre et Marie Curie, Paris (2006).

[4] V. Caselles, A. Chambolle and M. Novaga: *Uniqueness of the Cheeger set of a convex body.* Preprint Università di Pisa, Pisa (2006) (to appear on Pacific J. Math.), available at `http://cvgmt.sns.it`.

[5] F. Demengel and R. Temam: *Convex functions of a measure and application.* Indiana Univ. Math. J., **33** (1984), 673–709.

[6] E. DiBenedetto: $C^{1,\alpha}$ *local regularity of weak solutions of degenerate elliptic equations*, Nonlinear Analysis: Theory, Method and Applications, **7** (1983), 827-850.

[7] E. Giusti: *Minimal Surfaces and Functions of Bounded Variation.* Monographs in Mathematics **80**, Birkhäuser, Basel (1984).

[8] B. Kawohl, V. Fridman: *Isoperimetric estimates for the first eigenvalue of the p-Laplace operator and the Cheeger constant.* Comment. Math. Univ. Carolinae, **44** (4) (2003), 659–667.

[9] B. Kawohl, T. Lachand-Robert: *Characterization of Cheeger sets for convex subsets of the plane.* Pacific J. Math., **225** (2006), 103–118.

[10] B. Kawohl, M. Novaga: *The p-Laplace eigenvalue problem as* $p \to 1$ *and Cheeger sets in a Finsler metric.* Preprint Università di Pisa, Pisa (2006) (to appear in J. Convex Anal.), available at `http://cvgmt.sns.it`.

[11] S. Mazur: *Über konvexe Mengen in linearen normierten Räumen.* Studia Math., **4** (1933), 70–84.

[12] R. R. Phelps: *Convex Functions, Monotone Operators and Differentiability.* Lect. Notes Math. **1364**, Springer-Verlag, Berlin (1993).

[13] J. Serrin: *Local behaviour of solutions of quasi-linear elliptic equations*, Acta Math. **111** (1964), 247–302.

[14] J. L. Vazquez: *A Strong Maximum Principle for some Quasilinear Elliptic Equations*, Appl. Math. and Optimization, **12** (1984), 191-202.

Giuseppe Buttazzo
Dipartimento di Matematica
Università di Pisa
Largo B. Pontecorvo, 5
56127 Pisa - ITALY
buttazzo@dm.unipi.it

Guillaume Carlier
CEREMADE
Université de Paris-Dauphine
Place du Maréchal De Lattre De Tassigny
75775 Paris Cedex 16 - FRANCE
carlier@ceremade.dauphine.fr

Myriam Comte
Laboratoire Jacques-Louis Lions
Université Pierre et Marie Curie
Boîte courrier 187
75252 Paris Cedex 05 - FRANCE
comte@ann.jussieu.fr