A mean field game model for the evolution of cities

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Abstract

We propose a (toy) MFG model for the evolution of residents and firms densities, coupled both by labour market equilibrium conditions and competition for land use (congestion). This results in a system of two Hamilton-Jacobi-Bellman and two Fokker-Planck equations with a new form of coupling related to optimal transport. This MFG has a convex potential which enables us to find weak solutions by a variational approach. In the case of quadratic Hamiltonians, the problem can be reformulated in Lagrangian terms and solved numerically by an IPFP/Sinkhorn-like scheme as in [4]. We present numerical results based on this approach, these simulations exhibit different behaviours with either agglomeration or segregation dominating depending on the initial conditions and parameters.

Keywords: Mean field games, convex duality, optimal transport, labour market equilibrium, Iterative Proportional Fitting procedure (IPFP).

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1 Introduction

Economic equilibria frequently unfold over different time scales – locally in time, some parameters are taken as constant in the determination of a short-term or instaneous equilibrium, while in the long run they become state

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variables. Dynamics of cities and spatial labour markets are a prominent example: in the short run, when choosing where to work or who to hire, inhabitants and firms can be viewed as taking their geographical position as given (they cannot instantly relocate). In that way, labour market equilibrium can be approached as a static problem. In the long run (years or decades), however, people and firms can move, reshaping the city structure. Crucially, the properties of instaneous equilibrium are contingent on given densities which are themselves contingent on agents' motions which reflect their expectations. This creates strategic interactions over the dynamics of instaneous equilibria. On top of this, cities dynamics are subject to congestion that can be expressed through rents and also affect strategic moving decisions.

We propose a stylized model for the previous dynamics using the mean field game approach, pioneered by P.-L. Lions and the third author [21], [22], [23] to analyze equilibria in differential games with infinitely many players. Mean field games involving several populations have been studied recently by Cirant [13] and Achdou, Bardi and Cirant [1]. We consider a two-populations mean field game (e.g. workers and firms); the main mathematical novelty resides in coupling the running cost functions for the two populations by the potentials of an optimal transport problem between their respective distributions. With the square euclidean distance as ground cost, this amounts to couple the MFG system with Monge-Ampère equations at each time. This comes from (classically) interpreting equilibrium on the labour market for given densities as an optimal transport problem – the Kantorovich potentials of the dual problem then represent equilibrium payoffs for each population, see [11].

The market for land is modeled using absentee landlords, hence rents are simply given by an inverse demand function of the total mass of workers and firms at any point in space. This gives rise to a congestion-type running cost in the MFG problem. The full MFG system we consider is of the form :

$$\begin{cases} -\partial_t \phi_i - \nu_i \Delta \phi_i + H_i(x, \nabla \phi_i) = f(m_1 + m_2) + \alpha_i, \quad \phi_i(T, .) = 0, \\ \partial_t m_i - \nu_i \Delta m_i - \operatorname{div}(m_i \nabla_p H_i(x, \nabla \phi_i)) = 0, \quad m_i(0, .) = m_i^0, \end{cases} \tag{1.1}$$

for $i = 1, 2$; where ϕ_i are the value functions (solving the HJB equation in (1.1)), m_i are the densities (solving the Fokker-Planck equations in (1.1)) and $\alpha_i(t, .)$ are the potentials of an optimal transport problem between $m_1(t, .)$ and $m_2(t, .)$. For instance, in the case of a quadratic commuting cost, the potentials $\alpha_i(t, .)$ are related to the densities $m_i(t, .)$ by the Monge-Ampère equations:

$$\det(I - D^2\alpha_1) m_2(x - \nabla \alpha_1(x)) = m_1, \ \det(I - D^2\alpha_2) m_1(x - \nabla \alpha_2(x)) = m_2.$$

The paper is organized as follows : section 2 introduces the model in more details by building up from rent and labour market interactions to workers and firms optimal control problems in order to derive the MFG system (1.1). Section 3 recasts the MFG system as a variational problem and adopts a convex duality approach to prove existence of weak solutions. Section 4 specializes to the case of quadratic Hamiltonians (where the system admits classical solutions, see [10]) and the problem is recasted in a Lagrangian formulation as an entropy minimization problem on the space of measures on paths. Section 5 builds on the entropy-minimization interpretation of the system to construct an efficient IPFP-like algorithm to obtain numerical solutions (generalizing the celebrated Sinkhorn algorithm in optimal transport); numerical simulations are presented at the end of the section, displaying several examples of dynamics and exploring sensitivity to model parameters. Section 6 concludes.

2 The model

We consider a continuous time model with a finite horizon T. For the sake of simplicity, we will work in the periodic space setting and denote by $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$ the flat d-dimensional torus. The main unknowns of the model will be the time-dependent densities of workers and firms which we denote respectively $m_1(t,.)$, $m_2(t,.)$, where $t \in [0,T]$. The initial distributions m_1^0 and m_2^0 are given, and assumed to be everywhere positive, Lipschitz, and normalized to have unit total mass:

$$m_i^0 \in W^{1,\infty}(\mathbb{T}^d),\ \min_{\mathbb{T}^d} m_i^0 > 0,\ \int_{\mathbb{T}^d} m_i^0(x)\mathrm{d}x = 1,\ i = 1,2. \tag{2.1}$$

The structure of the MFG model, detailed in the next paragraphs, is the following:

- at each given time t, workers and firms interact in two ways: i) they compete for land use which results in a common rent which is an increasing function of the total density $m_1(t,.) + m_2(t,.)$, this is a local interaction which has a standard congestion effect, ii) they interact through the labour market, wages paid by firms should be such that this market is at equilibrium, this is a non-local interaction since workers choose where to work so as to maximize their revenue i.e. the wage they get net of some commuting cost,
- workers/firms located at $x \in \mathbb{T}^d$ at time $t \in [0,T]$, taking $m_1(s,.), m_2(s,.)$ for $s \geq t$ as a given prior, solve their individual stochastic control

problem taking into account the previous interactions (as well as an individual mobility cost), this resuts in two Hamilton-Jacobi-Bellman equations,

- from the optimal feedback laws resulting form these HJB's equations, one deduces the actual evolution of densities by solving a pair of Fokker-Planck equations,
- the MFG equilibrium system then expresses that this evolution coincides with the initial priors.

Interactions: rents and the labour market

Rents: Given $(t, x) \in (0, T) \times \mathbb{T}^d$ and $m_1(t, x)$, $m_2(t, x)$ the densities of workers/firms at x at time t, the rent $R(t, x)$ should be such that the total demand $m_1(t, x) + m_2(t, x)$ on the estate market equals the supply which is given by an exogenously given increasing supply function S of the rent. Thus, the rent R at (t, x) should be such that supply and demand concide i.e. $m_1(t, x) + m_2(t, x) = S(R(t, x))$. Inverting this monotone relation i.e. formally setting $f := S^{-1}$, we get

$$R(t, x) = f(m_1(t, x) + m_2(t, x)) \tag{2.2}$$

so that the rent is a local and increasing function of the total density. The coupling induced by the estate market simply is a joint congestion effect.

Labour market: Given $t \in [0, T]$ and $m_1(t, .)$ and $m_2(t, .)$ the spatial distributions at time t of workers and firms, which are probability measures on $\mathbb{T}^d$, firms and workers also interact through the labour market which we will assume to be at equilibrium. Firms located at y, propose a wage $w(t, y)$ to workers. There is a monetary commuting cost $c(x, y)$ for workers commuting from their residence location x to a job location y. The commuting cost is assumed to be continuous and nonnegative

$$c \in C(\mathbb{T}^d \times \mathbb{T}^d),\ c(x, y) \geq 0,\ \forall (x, y) \in \mathbb{T}^d \times \mathbb{T}^d. \tag{2.3}$$

Since workers are rational they choose their job location so as to maximize wage net of commuting cost which gives the following form for the revenue $r(t, x)$ of workers living at $x \in \mathbb{T}^d$:

$$r(t, x) := \max_{y \in \mathbb{T}^d} \{w(t, y) - c(x, y)\}. \tag{2.4}$$

By construction for every x and y one has

$$w(t,y) - r(t,x) \leq c(x,y)$$

and agents living at x work only at locations where the previous inequality is an equality. The pair of wage and revenue functions $y \mapsto w(t,y)$, $x \mapsto r(t,x)$ induces an equilibrium on the labour market if it is continuous and there exists a probability measure γ on $\mathbb{T}^d \times \mathbb{T}^d$ (where $\gamma(A \times B)$ represents the proportion of workers living in A and working in B) such that

- γ is a transport plan between $m_1(t,.)$ and $m_2(t,.)$ i.e. γ has marginals $m_1(t,.)$ and $m_2(t,.)$[1], which means that γ is consistent with the distributions of workers and firms,
- $w(t,y) - r(t,x) = c(x,y)$ on $\mathrm{spt}(\gamma)$, the support of γ, which means that γ is consistent with the rationality of workers.

The equilibrium conditions above are well-known to be related to the primal-dual optimality conditions for the Monge-Kantorovich mass transport problem (see [33], [32] [30]). More precisely, given the probability measures m_1 and m_2, consider

$$C(m_1, m_2) := \inf_{\gamma \in \Pi(m_1,m_2)} \int_{\mathbb{T}^d \times \mathbb{T}^d} c(x,y) \mathrm{d}\gamma(x,y) \tag{2.5}$$

where $\Pi(m_1, m_2)$ denotes the set of transport plans between m_1 and m_2, Kantorovich duality expresses that $C(m_1, m_2)$ can be expressed by the dual formula

$$C(m_1, m_2) := \sup_{(\alpha_1,\alpha_2) \in C(\mathbb{T}^d) \times C(\mathbb{T}^d)} \left\{ \int_{\mathbb{T}^d} \alpha_1 m_1 + \int_{\mathbb{T}^d} \alpha_2 m_2 \ : \ \alpha_1 \oplus \alpha_2 \leq c \right\} \tag{2.6}$$

where $\alpha_1 \oplus \alpha_2$ denotes the separable function

$$\alpha_1 \oplus \alpha_2(x,y) := \alpha_1(x) + \alpha_2(y), \ (x,y) \in \mathbb{T}^d \times \mathbb{T}^d.$$

The dual formulation (2.6) admits solutions α_1, α_2 (called Kantorovich potentials) which may be assumed to satisfy:

$$\alpha_1(x) = \min_{y \in \mathbb{T}^d} \{c(x,y) - \alpha_2(y)\}, \ \forall x \in \mathbb{T}^d$$

and $\gamma \in \Pi(m_1, m_2)$ solves (2.5) exactly when $\alpha_1 \oplus \alpha_2 = c$ on $\mathrm{spt}(\gamma)$. In other words, equilibrium on the labour market at time t, is equivalent to the fact

[1] which means that $\gamma(A \times \mathbb{T}^d) = m_1(t, A)$ and $\gamma(\mathbb{T}^d \times B) = m_2(t, B)$, for every Borel subsets A and B of $\mathbb{T}^d$.

that the pair $(\alpha_1(t,.), \alpha_2(t,.)) = (w(t,.), -r(t,.))$ satisfies $\alpha_1(t,.) \oplus \alpha_2(t,.) \leq c$ and is optimal for the Kantorovich dual. A concise way to rewrite the equilibrium condition therefore reads as

$$\alpha_1(t,.)\oplus\alpha_2(t,.) \leq c,\ C(m_1(t,.), m_2(t,.)) = \int_{\mathbb{T}^d} \alpha_1(t,.)m_1(t,.)+\int_{\mathbb{T}^d} \alpha_2(t,.)m_2(t,.). \tag{2.7}$$

Remark 2.1. We will rather use the symmetric notation (α_1, α_2) instead of $(w, -r)$. Let us point out that the labour-market equilibrium condition (2.7) never determines wages and revenues uniquely since adding a constant to α_1 and substracting it from α_2 does not affect this condition.

Remark 2.2. In the quadratic cost case i.e. when

$$c(x,y) = \frac{1}{2}\,\mathrm{dist}^2_{\mathbb{T}^d}(x,y),\ \text{ where } \mathrm{dist}_{\mathbb{T}^d}(x,y) := \min_{k\in\mathbb{Z}^d} |x+k-y|$$

and m_1 and m_2 are smooth positive densities, it is well-known (see [14], [5]) that the optimal α_1, α_2 in (2.5) is unique (up to an additive constant) and characterized by the condition

$$x \in \mathbb{R}^d \mapsto \frac{1}{2}|x|^2 - \alpha_i(x) \text{ convex}$$

for $i = 1, 2$ and the Monge-Ampère equations

$$\det(I - D^2\alpha_1)m_2(x - \nabla\alpha_1(x)) = m_1, \tag{2.8}$$

$$\det(I - D^2\alpha_2)m_1(x - \nabla\alpha_2(x)) = m_2. \tag{2.9}$$

Remark 2.3. Instead of assuming that workers choose to work at locations that maximize exactly wage net of transport cost, one can consider a regularization of (2.5) with a certain noise parameter $\sigma > 0$:

$$C_\sigma(m_1, m_2) := \inf_{\gamma\in\Pi(m_1,m_2)} \int_{\mathbb{T}^d\times\mathbb{T}^d} (c(x,y) + \sigma\log(\gamma(x,y))\gamma(x,y)\mathrm{d}x\mathrm{d}y \tag{2.10}$$

which (provided m_1 and m_2 have a finite entropy) has an almost closed-form solution $\gamma(x,y) = a_1(x)a_2(y)e^{-\frac{c(x,y)}{\sigma}}$ where a_1 and a_2 are such that the marginal constraints are met. This entropic regularization is very popular in numerical optimal transport because it can be solved iteratively by the Sinkhorn/IPFP algorithm (see [15], [28] and section 5).

Workers and firms optimal control problems

Given the rent $R = f(m_1 + m_2)$ and α_1 (negative of revenue), workers living at x at time t, seek to minimize the expected cost

$$\mathbb{E}\Big(\int_t^T [L_1(X_s, u_s) + R(s, X_s) + \alpha_1(s, X_s)]ds\Big) \tag{2.11}$$

where L_1 is a Lagrangian that captures the cost of motion for workers, u_s is an adapted control process, the SDE governing the evolution of the workers' position is

$$\mathrm{d}X_s = u_s\mathrm{d}s + \sqrt{2\nu_1}\mathrm{d}B_s,\ X_t = x, \tag{2.12}$$

where $\nu_1 > 0$ is the diffusivity parameter of the workers and B a standard brownian motion.

Similarly, given the rent $R = f(m_1 + m_2)$ and the wage α_2 paid to the workers, firms settled at y at time t, minimize

$$\mathbb{E}\Big(\int_t^T [L_2(Y_s, v_s) + R(s, Y_s) + \alpha_2(s, Y_s)]ds\Big) \tag{2.13}$$

where the Lagrangian L_2 models the mobility cost of firms, v_s is an adapted control process and

$$\mathrm{d}Y_s = v_s\mathrm{d}s + \sqrt{2\nu_2}\mathrm{d}B_s,\ Y_t = y, \tag{2.14}$$

where $\nu_2 > 0$ is the firms' diffusivity parameter.

The MFG system

The Hamiltonians H_1, H_2 associated with the Lagrangian of workers and firms respectively are given by $H_i(x, .) = L_i^*(x, -.)$, i.e.

$$H_i(x,p) := \sup_{v\in\mathbb{R}^d}\{-p\cdot v - L_i(x,v)\},\ x\in\mathbb{T}^d,\ p\in\mathbb{R}^d,\ i = 1, 2.$$

The MFG equilibrium system consists of two HJB equations for the value functions ϕ_1, ϕ_2 of agents and firms, coupled by rents and wages/revenues that clear the labour market, together with two Fokker-Planck equations for m_1 and m_2 with respective drifts $-\nabla_p H_1(x, \nabla\phi_1(x))$ and $-\nabla_p H_2(x, \nabla\phi_2(x))$ which are the optimal feedbacks for their respective control problems. More precisely, we look for functions

$$(t, x) \in (0, T) \times \mathbb{T}^d \mapsto (\phi_i(t, x), m_i(t, x), \alpha_i(t, x))_{i=1,2}$$

such that for $i = 1, 2$

$$-\partial_t \phi_i - \nu_i \Delta \phi_i + H_i(x, \nabla \phi_i) = f(m_1 + m_2) + \alpha_i, \ \ \phi_i(T, .) = 0, \tag{2.15}$$

$$\partial_t m_i - \nu_i \Delta m_i - \operatorname{div}(m_i \nabla_p H_i(x, \nabla \phi_i)) = 0, \ \ m_i(0, .) = m_i^0, \tag{2.16}$$

as well as for every $t \in (0, T)$:

$$\alpha_1(t, .) \oplus \alpha_2(t, .) \le c, \tag{2.17}$$

and

$$C(m_1(t, .), m_2(t, .)) = \int_{\mathbb{T}^d} \alpha_1(t, .) m_1(t, .) + \int_{\mathbb{T}^d} \alpha_2(t, .) m_2(t, .). \tag{2.18}$$

The last conditions (2.17)-(2.18) express that wages (α_2) and revenues ($-\alpha_1$) clear the labour market at every time (and, as explained in remark 2.2, take the form of Monge-Ampère equations in the special case where c is the squared distance).

3 A variational approach

3.1 Two problems in duality

At least formally, the right hand sides $(f(m_1 + m_2) + \alpha_1, f(m_1 + m_2) + \alpha_2)$ of the HJB equations (2.15), together with (2.17)-(2.18) can be seen as the derivative of the (convex) functional $(m_1, m_2) \mapsto C(m_1, m_2) + \int_{\mathbb{T}^d} F(m_1 + m_2) \mathrm{d}x$ with $F(m) = \int_0^m f(\alpha) \mathrm{d}\alpha$. Following the seminal work of Lasry and Lions [22], we see that the MFG system (2.15)-(2.16)-(2.17)-(2.18) has a convex potential structure and can therefore be seen as the optimality system for two convex minimization problems in duality.

From now on, we shall assume the following. For $i = 1, 2$, the Lagrangian L_i is continuous on $\mathbb{T}^d \times \mathbb{R}^d$, strictly convex, differentiable in its second argument with $\nabla_v L_i$ continuous and such that for some $M > 1$ and $s_i \in (1 + \infty)$, there holds

$$\frac{|v|^{s_i}}{M} - M \le L_i(x, v) \le M(|v|^{s_i} + 1), \ \forall (x, v) \in \mathbb{T}^d \times \mathbb{R}^d. \tag{3.1}$$

This of course implies that H_i is also continuous on $\mathbb{T}^d \times \mathbb{R}^d$, strictly convex, differentiable in its second argument with $\nabla_p H_i$ continuous and that for some $M > 1$ and for $r_i := \frac{s_i}{s_i - 1}$, the conjugate exponent of s_i, one has

$$\frac{|p|^{r_i}}{M} - M \le H_i(x, p) \le M(|p|^{r_i} + 1), \ \forall (x, p) \in \mathbb{T}^d \times \mathbb{R}^d \tag{3.2}$$

(possibly with a different positive constant M). As for the congestion term f: $\mathbb{R}_+ \to \mathbb{R}_+$ we assume that it is continuous, nondecreasing satisfies $f(0) = 0$ and that for some $M > 1$ and some $s \in (1, +\infty)$, it satisfies

$$\frac{m^{s-1}}{M} - M \leq f(m) \leq M(m^{s-1} + 1), \ \forall m \in \mathbb{R}_+. \tag{3.3}$$

We then set $F(m) := \int_0^m f(\alpha) \mathrm{d}\alpha$ for every $m \geq 0$ and note that F is convex with s-power growth. Finally, we define the functional $(m_1, m_2) \in L^1(\mathbb{T}^d) \times L^1(\mathbb{T}^d) \mapsto \mathcal{F}(m_1, m_2)$ by

$$\mathcal{F}(m_1, m_2) := \begin{cases} \int_{\mathbb{T}^d} F(m_1(x) + m_2(x)) dx, \text{ if } m_1 \geq 0, \ m_2 \geq 0, \\ +\infty \text{ otherwise.} \end{cases} . \tag{3.4}$$

We also define $G := (F + \chi_{\mathbb{R}_+})^*$, i.e.

$$G(\beta) := \sup_{m \geq 0} \{m\beta - F(m)\}. \tag{3.5}$$

Defining $r := \frac{s}{s-1}$ the conjugate exponent of s, note that our assumptions on f, imply that G is nondecreasing

$$G(\beta) = 0, \text{ for all } \beta \leq 0, \ \frac{\beta_+^r}{M} - M \leq G(\beta) \leq M(\beta_+^r + 1), \ \forall \beta \in \mathbb{R}. \tag{3.6}$$

(possibly with a different positive constant M). Throughout, this section, we will asume that (2.1), (2.3), (3.1), (3.3) are in force. To abbreviate notations, let $C := C([0, T] \times \mathbb{T}^d, \mathbb{R})$, $C^{1,2} = C^{1,2}([0, T] \times \mathbb{T}^d, \mathbb{R})$ denote the space of functions defined on $[0, T] \times \mathbb{T}^d$ which admit continuous first derivative in time and second derivatives in space, and define $C_T^{1,2}$ by:

$$C_T^{1,2} := \{\phi \in C^{1,2} \ : \ \phi(T, x) = 0, \ \forall x \in \mathbb{T}^d\}.$$

For $\phi = (\phi_1, \phi_2) \in C^{1,2} \times C^{1,2}$, and $\alpha = (\alpha_1, \alpha_2) \in C \times C$, let

$$\mathcal{G}(\phi, \alpha) := \int_0^T \int_{\mathbb{T}^d} G\Big(\max_{i=1,2}(-\partial_t \phi_i - \nu_i \Delta \phi_i + H_i(., \nabla \phi_i) - \alpha_i)\Big) - \sum_{i=1}^2 \int_{\mathbb{T}^d} \phi_i(0) m_i^0$$

and consider the variational problem

$$\inf \left\{\mathcal{G}(\phi, \alpha), \ \phi \in (C_T^{1,2})^2, \ \alpha \in C \times C, \ \alpha_1(t, .) \oplus \alpha_2(t, .) \leq c, \ \forall t\right\}. \tag{3.7}$$

Note that (3.7) is convex since $H_i(x, .)$ is convex and G is convex and nondecreasing. It can be rewritten in Fenchel-Rockafellar form as

$$\inf_{(\phi, \alpha) \in (C^{1,2} \times C)^2} I(\phi, \alpha) + J(\Lambda(\phi, \alpha)) \tag{3.8}$$

where $\Lambda(\phi,\alpha) = (\Lambda_1(\phi_1,\alpha_1), \Lambda_2(\phi_2,\alpha_2))$ and Λ_i is the continuous linear operator $C^{1,2} \times C \to C([0,T]\times\mathbb{T}^d)^{d+1}$

$$\Lambda_i(\phi_i,\alpha_i) := (-\partial_t\phi_i - \nu_i\Delta\phi_i - \alpha_i, -\nabla\phi_i),$$

and I and J are given respectively by:

$$I(\phi,\alpha) := \begin{cases} -\sum_{i=1}^2 \int_{\mathbb{T}^d}\phi_i(0)m_i^0 \text{ if } \alpha_1\oplus\alpha_2 \le c \text{ and } \phi_1(T,.) = \phi_2(T,.) = 0 \\ +\infty \text{ otherwise.} \end{cases}$$

and for $a = (a_1,a_2)$ in $C\times C$ and $b = (b_1,b_2)$ in $(C([0,T]\times\mathbb{T}^d,\mathbb{R}^d))^2$

$$J(a,b) := \int_0^T\int_{\mathbb{T}^d} G\Big(\max_{i=1,2}(a_i(t,x) + H_i(-b_i(t,x))\Big)\mathrm{d}x\mathrm{d}t$$

since J is continuous everywhere and I has a nonempty domain, we deduce from the Fenchel-Rockafellar Theorem that

$$\inf(3.7) + \min_{(m_i,w_i)_{i=1,2}\in(\mathcal{M}\times\mathcal{M}^d)^2}\{I^*(-\Lambda^*(m,w)) + J^*(m,w)\} = 0, \qquad (3.9)$$

where $\mathcal{M}$ denotes the space of Borel measures on $[0,T]\times\mathbb{R}^d$. A direct computation and the Kantorovich duality formula, give the following expression for $I^*(-\Lambda^*(m,w))$:

$$\begin{cases} \int_0^T C(m_1(t,.),m_2(t,.)), \text{ if } m_i\ge 0,\ \partial_t m_i - \nu_i\Delta m_i + \mathrm{div}(w_i) = 0,\ m_i(0,.) = m_i^0 \\ +\infty \text{ otherwise.} \end{cases}$$

As for J^*, using [29] as well as the definition of $\mathcal{F}$ and G (see for instance [20] for details in a similar variational MFG context), we have

$$J^*(m,w) = \sum_{i=1}^2\int_0^T\int_{\mathbb{T}^d} L_i(x,\frac{w_i}{m_i})m_i + \int_0^T \mathcal{F}(m_1(t,.),m_2(t,.))\mathrm{d}t$$

where $m_iL(x,\frac{w_i}{m_i})$ is a slight abuse of notations for the (convex lsc and 1-homogeneous) function

$$(m_i,w_i)\mapsto\begin{cases} m_iL(x,\frac{w_i}{m_i}) \text{ if } m_i>0, \\ 0 \text{ if } m_i = 0 \text{ and } w_i = 0, \\ +\infty \text{ otherwise.} \end{cases}$$

Observing that $J^*(m,w) < +\infty$ requires $m_i\in L^s$ and $w_i = m_iv_i$ with $m_i|v_i|^{s_i}\in L^1$, this implies that $w_i = m_i^{\frac{1}{r_i}}m_i^{\frac{1}{s_i}}v_i \in L^{\lambda_i}((0,T)\times\mathbb{T}^d)$ with

$$\lambda_i = \frac{rr_i}{rr_i - 1}. \qquad (3.10)$$

The dual of (3.7) thus reads as

$$\inf_{(m,w)\in\mathcal{K}} \mathcal{E}(m,w) := \sum_{i=1}^{2} \int_0^T \int_{\mathbb{T}^d} L_i(x, \frac{w_i}{m_i}) m_i + \int_0^T (C(m_1(t,.), m_2(t,.)) + \mathcal{F}(m_1(t,.), m_2(t,.))\mathrm{d}t \quad (3.11)$$

where $\mathcal{K}$ consists of all $(m,w) = (m_1, m_2, w_1, w_2)$ with $m_i \geq 0$, $m_i \in L^s((0,T)\times\mathbb{T}^d)$, $w_i \in L^{\lambda_i}((0,T)\times\mathbb{T}^d, \mathbb{R}^d)$ such that

$$\partial_t m_i - \nu_i \Delta m_i + \operatorname{div}(w_i) = 0 \text{ in } (0,T)\times\mathbb{T}^d,\ m_i(0,.) = m_i^0,$$

in the sense of distributions for $i = 1, 2$. Applying the Fenchel-Rockafellar Theorem thus gives

$$\inf(3.7) + \min_{(m,w)\in\mathcal{K}} \mathcal{E}(m,w) = 0. \quad (3.12)$$

and in particular the infimum is attained in (3.11).

3.2 Relaxed primal and weak solutions of the MFG system

Following [9], [25] (also see [8], [7], [19], [6], for the case of first-order variational MFG or transport problems), we will find weak solutions of the MFG system by considering a suitable relaxation of (3.7). Given $\alpha_i \in L^\infty((0,T)\times\mathbb{T}^d)$ and $\beta \in L^r((0,T)\times\mathbb{T}^d)$, we shall say that $\phi_i \in L^{r_i}((0,T), W^{1,r_i}(\mathbb{T}^d))$ is a weak subsolution of

$$-\partial_t\phi_i - \nu_i\Delta\phi_i + H_i(., \nabla\phi_i) \leq \alpha_i + \beta,\ \phi_i(T,.) \leq 0 \quad (3.13)$$

if for every $\eta \in C_c^\infty((0,T]\times\mathbb{T}^d)$ with $\eta \geq 0$, there holds

$$\int_0^T \int_{\mathbb{T}^d} (\partial_t\eta - \nu_i\Delta\eta)\phi_i + \int_0^T \int_{\mathbb{T}^d} H_i(., \nabla\phi_i)\eta \leq \int_0^T \int_{\mathbb{T}^d} (\alpha_i + \beta)\eta. \quad (3.14)$$

A crucial estimate for such weak subsolutions is provided by Theorem 3.3 of [9] which establishes that if ϕ_i is a bounded from below weak solution of (3.13), then

$$\|\phi_i\|_{L^\infty((0,T),L^{\eta_i}(\mathbb{T}^d))} + \|\phi_i\|_{L^{\gamma_i}((0,T)\times\mathbb{T}^d))} \leq C(\|\phi_{i-}\|_{L^\infty((0,T)\times\mathbb{T}^d)}, \|\alpha_i + \beta\|_{L^r((0,T)\times\mathbb{T}^d})) \quad (3.15)$$

where the exponents η_i and γ_i are given by

$$\eta_i = \begin{cases} \frac{d(r_i(r-1)+1)}{d-r_i(r-1)} \text{ if } 1+\frac{d}{r_i} > r, \\ \text{any exponent in } (1,+\infty) \text{ if } 1+\frac{d}{r_i} = r, \\ +\infty \text{ if } 1+\frac{d}{r_i} < r \end{cases} \tag{3.16}$$

and

$$\gamma_i = \begin{cases} \frac{(1+d)r_i r}{d-r_i(r-1)} \text{ if } 1+\frac{d}{r_i} > r, \\ \text{any exponent in } (1,+\infty) \text{ if } 1+\frac{d}{r_i} = r, \\ +\infty \text{ if } 1+\frac{d}{r_i} < r. \end{cases} \tag{3.17}$$

Note that γ_i and η_i can always be chosen such that $\gamma_i > \eta_i > r$ and $\gamma_i \geq r_i$, in particular if $m_i \in L^s$ and $\phi_i \in L^{\gamma_i}$ then $m_i\phi_i \in L^1$. We then define $\widetilde{\mathcal{A}}$ as the set of collections $(\phi_1, \phi_2, \alpha_1, \alpha_2, \beta)$ such that:

- $\phi_i \in L^{r_i}((0,T), W^{1,r_i}(\mathbb{T}^d)) \cap L^{\gamma_i}((0,T)\times\mathbb{T}^d)$,
- $\alpha_i \in L^\infty((0,T)\times\mathbb{T}^d)$ and $\alpha_1(t,.) \oplus \alpha_2(t,.) \leq c$ for a.e. $t \in (0,T)$,
- $\beta \in L^r((0,T)\times\mathbb{T}^d)$ and $\beta \geq 0$[2],
- for $i = 1, 2$, (3.13) holds in the weak sense of (3.14).

We shall say that a sequence $(\phi^n, \alpha^n, \beta^n)$ in $\widetilde{\mathcal{A}}$ converges weakly to (ϕ, α, β) which we will simply denote $(\phi^n, \alpha^n, \beta^n) \rightharpoonup (\phi, \alpha, \beta)$ if

$$\phi_i^n \rightharpoonup \phi_i \text{ in } L^{\gamma_i}((0,T)\times\mathbb{T}^d), \nabla\phi_i^n \rightharpoonup \phi_i \text{ in } L^{r_i}((0,T)\times\mathbb{T}^d), \tag{3.18}$$

and

$$\alpha_i^n \overset{*}{\rightharpoonup} \alpha_i \text{ in } L^\infty((0,T)\times\mathbb{T}^d),\ \beta^n \rightharpoonup \beta \text{ in } L^r((0,T)\times\mathbb{T}^d). \tag{3.19}$$

Since $H_1(x,.)$ and $H_2(x,.)$ are convex and satisfy (3.2), one immediately checks that $\widetilde{\mathcal{A}}$ is closed with respect to this weak convergence.

Let $(\phi, \alpha, \beta) = (\phi_1, \phi_2, \alpha_1, \alpha_2, \beta) \in \widetilde{\mathcal{A}}$, let $\eta \in W^{1,\infty}(\mathbb{T}^d)$ with $\min_{\mathbb{T}^d} \eta > 0$ and define for $i = 1, 2$ and $t \in (0,T)$:

$$\overline{\phi}_{i,\eta}(t) := \int_{\mathbb{T}^d} \phi_i(t,x)\eta(x)\mathrm{d}x,\ \overline{\psi}_{\alpha_i,\beta,\eta}(t) := \int_{\mathbb{T}^d} (\alpha_i(t,x) + \beta(t,x))\eta(x)\mathrm{d}x. \tag{3.20}$$

[2]Imposing $\beta \geq 0$ is not really a restriction since $G(\beta_+) = G(\beta)$ and (3.13) still holds when changing β into β_+.

It easily follows from (3.13) and the superlinearity of H_i that for some constant M_η one has

$$\frac{d}{dt}\overline{\phi}_{i,\eta}(t) + M_\eta + \overline{\psi}_{\alpha_i,\beta,\eta}(t) \geq 0, \tag{3.21}$$

and since $\overline{\psi}_{\alpha_i,\beta,\eta} \in L^r((0,T))$, we deduce that $\overline{\phi}_{i,\eta}$ is BV hence has a right (resp. left) limit $\overline{\phi}_{i,\eta}(t^+)$ (resp. $\overline{\phi}_{i,\eta}(t^-)$) at each $t \in [0,T)$ (resp. $t \in (0,T]$). Actually, more is true, indeed defining the $W^{1,r}((0,T))$ function

$$\overline{\Psi}_{\alpha_i,\beta,\eta}(t) := \int_0^t \overline{\psi}_{\alpha_i,\beta,\eta}(s)\mathrm{d}s = \int_0^t \int_{\mathbb{T}^d} (\alpha_i(s,x) + \beta(s,x))\eta(x)\mathrm{d}x\mathrm{d}s$$

since

$$t \mapsto \overline{\phi}_{i,\eta}(t) + M_\eta t + \overline{\Psi}_{\alpha_i,\beta,\eta}(t) \text{ is nondecreasing}$$

then for every $t \in [0,T)$ one has

$$\overline{\phi}_{i,\eta}(t^+) + M_\eta t + \overline{\Psi}_{\alpha_i,\beta,\eta}(t) = \inf_{\delta\in(0,T-t)} \left\{ \frac{1}{\delta} \int_t^{t+\delta} \int_{\mathbb{T}^d} (\phi_i + M_\eta s + \overline{\Psi}_{\alpha_i,\beta,\eta}(s))\mathrm{d}s\mathrm{d}x \right\}$$

from which we deduce that whenever a sequence $(\phi^n, \alpha^n, \beta^n)$ in $\widetilde{\mathcal{A}}$ converges weakly to some (ϕ, α, β), then for every $t \in [0,T)$, one has

$$\limsup_n \int_{\mathbb{T}^d} \phi_i^n(t^+, x)\eta(x)\mathrm{d}x \leq \overline{\phi}_{i,\eta}(t^+). \tag{3.22}$$

In particular, if we set

$$\int_{\mathbb{T}^d} \phi_i(0,x) m_i^0(x)\mathrm{d}x := \overline{\phi}_{i,m_i^0}(0^+), \tag{3.23}$$

we see that the functional

$$\widetilde{\mathcal{G}}(\phi, \alpha, \beta) := \int_0^T \int_{\mathbb{T}^d} G(\beta(t,x))\mathrm{d}x\mathrm{d}t - \sum_{i=1}^2 \int_{\mathbb{T}^d} \phi_i(0,x) m_i^0(x)\mathrm{d}x$$

is well-defined on $\widetilde{\mathcal{A}}$, convex and lsc for weak convergence. The relaxed formulation of (3.7) then reads:

$$\inf_{(\phi,\alpha,\beta)\in\widetilde{\mathcal{A}}} \widetilde{\mathcal{G}}(\phi, \alpha, \beta) \tag{3.24}$$

Remark 3.1. It is worth at this point remarking that (3.24) (as well as the unrelaxed problem (3.7)) has the following invariance property. If $(\phi, \alpha, \beta) \in \widetilde{\mathcal{A}}$ and $\mu \in L^\infty((0,T))$, setting $\widetilde{\alpha}_1(t,x) := \alpha_1(t,x)+\mu(t)$, $\widetilde{\alpha}_2(t,y) := \alpha_2(t,y)-\mu(t)$ and

$$\widetilde{\phi}_1(t,x) := \phi_1(t,x) + \int_t^T \mu(s)\mathrm{d}s, \widetilde{\phi}_2(t,x) := \phi_2(t,x) - \int_t^T \mu(s)\mathrm{d}s, \quad (3.25)$$

then $(\widetilde{\phi}, \widetilde{\alpha}, \beta) = (\widetilde{\phi}_1, \widetilde{\phi}_2, \widetilde{\alpha}_1, \widetilde{\alpha}_2, \beta) \in \widetilde{\mathcal{A}}$ and $\widetilde{\mathcal{G}}(\widetilde{\phi}, \widetilde{\alpha}, \beta) = (\phi, \alpha; \beta)$ (the fact that the boundary term remains unchanged follows from m_1^0 and m_2^0 having the same mass).

Obviously, $\inf(3.24) \leq \inf(3.7)$. The converse inequality follows from Lemma 5.3 in [9] which says that whenever $(m, w) \in \mathcal{K}$ is such that $\mathcal{E}(m, w) < +\infty$ and $(\phi, \alpha, \beta) \in \widetilde{\mathcal{A}}$ then for $i = 1, 2$,

$$-\int_{\mathbb{T}^d} \phi_i(0,x) m_i^0(x)\mathrm{d}x + \int_0^T \int_{\mathbb{T}^d} m_i\Big(\alpha_i + \beta + L_i(x, \frac{w_i}{m_i})\Big) \geq 0, \quad (3.26)$$

with an equality only if

$$w_i = -m_i \nabla_p H_i(., \nabla\phi_i)). \quad (3.27)$$

We thus have

Proposition 3.2. *The relaxed problem (3.24) satisfies the duality relation*

$$0 = \min_{(m,w)\in\mathcal{K}} \mathcal{E}(m,w) + \inf_{(\phi,\alpha,\beta)\in\widetilde{\mathcal{A}}} \widetilde{\mathcal{G}}(\phi, \alpha, \beta). \quad (3.28)$$

Proof. Let $(m, w) \in \mathcal{K}$ be such that $\mathcal{E}(m, w) < +\infty$ and $(\phi, \alpha, \beta) \in \widetilde{\mathcal{A}}$. Young's inequality gives

$$\int_0^T \int_{\mathbb{T}^d} G(\beta) + \int_0^T \mathcal{F}(m_1(t,.), m_2(t,.)\mathrm{d}t \geq \int_0^T \int_{\mathbb{T}^d} \beta(m_1 + m_2) \quad (3.29)$$

likewise,

$$\int_0^T C(m_1(t,.), m_2(t,.)\mathrm{d}t \geq \sum_{i=1}^2 \int_0^T \int_{\mathbb{T}^d} \alpha_i m_i. \quad (3.30)$$

Summing (3.29)-(3.30) with (3.26), exactly gives

$$\mathcal{E}(m,w) + \widetilde{\mathcal{G}}(\phi, \alpha, \beta) \geq 0$$

so that $\inf(3.24) \geq -\min(3.11)$ but since $\inf(3.24) \leq \inf(3.7)$, (3.28) follows from (3.12). $\square$

Corollary 3.3. *If $(m, w) \in \mathcal{K}$ solves (3.11) and $(\phi, \alpha, \beta) \in \widetilde{\mathcal{A}}$ solves (3.24), then (m, w, ϕ, α) is a weak solution of the MFG system (2.15)-(2.16)-(2.17)-(2.18), in the sense that:*

- *$\beta = f(m_1 + m_2)$ so that ϕ_i is a weak subsolution of (2.15),*
- *for $i = 1, 2$, $w_i = -m_i \nabla_p H_i(., \nabla \phi_i))$ so that (2.16) holds in the sense of distributions,*
- *for $i = 1, 2$, one has*

$$-\int_{\mathbb{T}^d} \phi_i(0, x) m_i^0(x) dx + \int_0^T \int_{\mathbb{T}^d} m_i \Big(\alpha_i + f(m_1 + m_2) + L_i(x, \frac{w_i}{m_i}) \Big) = 0$$

- *(2.17)-(2.18) hold for a.e. $t \in (0, T)$.*

Proof. If $(m, w) \in \mathcal{K}$ solves (3.11) and $(\phi, \alpha, \beta) \in \widetilde{\mathcal{A}}$ solves (3.24), one should have $\mathcal{E}(m, w) + \widetilde{\mathcal{G}}(\phi, \alpha, \beta) = 0$ so that (3.26), (3.29) and (3.30) should all be equalities implying $w_i = -m_i \nabla_p H_i(., \nabla \phi_i))$, $\beta = f(m_1 + m_2)$ and that (2.17)-(2.18) hold for a.e. t.

□

As for the existence of a solution to the relaxed problem (3.24), again following closely [9], we get:

Theorem 3.4. *The relaxed problem (3.24) admits at least one solution. In particular, the MFG system (2.15)-(2.16)-(2.17)-(2.18) admits a weak solution in the sense of corollary 3.3*

Proof. In what follows, M will denote a positive constant which may vary from one line to another. Let us start with (ϕ^n, α^n), a minimizing sequence for the unrelaxed problem (3.7). Set then

$$\beta^n := \max(0, \max_{i=1,2}(-\partial_t \phi_i^n - \nu_i \Delta \phi_i^n + H_i(x, \nabla \phi_i^n) - \alpha_i^n))$$

so that $(\phi^n, \alpha_n, \beta^n)$ is minimizing for (3.24). Let us then define

$$\widetilde{\alpha}_2^n(y) := \min_{x \in \mathbb{T}^d} \{c(x, y) - \alpha_1^n(y)\}, \ \widetilde{\alpha}_1^n(x) := \min_{y \in \mathbb{T}^d} \{c(x, y) - \widetilde{\alpha}_2^n(y)\} \tag{3.31}$$

it is easy to see that $\widetilde{\alpha}_i^n \geq \alpha_i^n$ and $\widetilde{\alpha}_1^n \oplus \widetilde{\alpha}_2^n \leq c$ so that $(\phi^n, \widetilde{\alpha}^n, \beta^n)$ is admissible for (3.24), but the advantage of using $\widetilde{\alpha}_i^n$ instead of α_i^n is that these functions are uniformly continuous in space:

$$|\widetilde{\alpha}_i^n(t,x) - \widetilde{\alpha}_i^n(t,y)| \leq \omega_c(\mathrm{dist}_{\mathbb{T}^d}(x,y)), \ \forall (t,x,y) \tag{3.32}$$

where ω_c is a modulus of continuity of c. Now thanks to the invariance property of remark 3.1, we can normalize $\widetilde{\alpha}_1^n$ in such a way that $-\mu^n(t) := \int_{\mathbb{T}^d} \widetilde{\alpha}_1^n(t,x)\mathrm{d}t = 0$, (up to replacing $\widetilde{\alpha}_2^n$ by $\widetilde{\alpha}_2^n + \mu^n$ and modifying ϕ_i^n accordingly, see (3.25)). Since $\widetilde{\alpha}_1^n$ now has zero spatial mean, (3.32) gives a uniform bound on $\widetilde{\alpha}_1^n$, but also on $\widetilde{\alpha}_2^n$ thanks to (3.31):

$$\|\widetilde{\alpha}_i^n\|_{L^\infty((0,T)\times\mathbb{T}^d)} \leq M. \tag{3.33}$$

Now, since $\widetilde{\alpha}_i^n$ and β^n are continuous, let $\widetilde{\phi}_n$ be the viscosity solution of

$$-\partial_t \widetilde{\phi}_i^n - \nu_i \Delta \widetilde{\phi}_i^n + H_i(x, \nabla \widetilde{\phi}_i^n) = \widetilde{\alpha}_i^n + \beta^n, \ \widetilde{\phi}_i^n(T,.) = 0 \tag{3.34}$$

by comparison $\widetilde{\phi}_i^n \geq \phi_i^n$ and thanks to $\beta^n + \widetilde{\alpha}_i^n \geq -M$, we also have by comparison that $\widetilde{\phi}_i^n$ is uniformly bounded from below, $\widetilde{\phi}_i^n \geq -M$. Moreover since $H_i(x,.)$ is convex, (3.34) also holds in the weak sense, $(\widetilde{\phi}^n, \widetilde{\alpha}^n, \beta^n) \in \widetilde{\mathcal{A}}$ and it is also minimizing since $\widetilde{\phi}_i^n(0,.) \geq \phi_i^n(0,.)$. In particular, we have

$$\int_0^T \int_{\mathbb{T}^d} G(\beta^n) - \sum_{i=0}^2 \int_{\mathbb{T}^d} \widetilde{\phi}_i^n(0,x) m_i^0(x)\mathrm{d}x \leq M. \tag{3.35}$$

But multiplying (3.34) by m_i^0, and using (3.2), (3.33) and (2.1), we get

$$\frac{1}{M}\int_0^T \int_{\mathbb{T}^d} |\nabla \widetilde{\phi}_i^n|^{r_i} + \int_{\mathbb{T}^d} \widetilde{\phi}_i^n(0,x) m_i^0(x)\mathrm{d}x \leq M(1 + \int_{\mathbb{T}^d} \beta^n) \tag{3.36}$$

which together with (3.35) gives

$$\|\beta^n\|_{L^r((0,T)\times\mathbb{T}^d)} + \|\nabla \widetilde{\phi}_i^n\|_{L^{r_i}((0,T)\times\mathbb{T}^d)} \leq M. \tag{3.37}$$

Thanks to the fact that $\widetilde{\phi}_i^n$ is uniformly bounded from below, (3.15) also gives an L^{γ_i} bound for $\widetilde{\phi}_i^n$. Passing to a subsequence if necessary, we may therefore assume that the minimizing sequence $(\widetilde{\phi}^n, \widetilde{\alpha}^n, \beta^n)$ weakly converges (in the sense of (3.18)-(3.19)), we have already observed that $\widetilde{\mathcal{A}}$ is sequentially weakly closed and that $\widetilde{\mathcal{G}}$ is sequentially weakly lsc (see (3.22)) which enables us to conclude that (3.24) admits at least one solution. $\square$

4 Quadratic Hamiltonians

We now specialize to the quadratic Hamiltonian case:

$$L_i(x,v) = \frac{\theta_i}{2}|v|^2, \; H_i(x,p) = \frac{1}{2\theta_i}|p|^2, \; \forall (x,v,p) \in \mathbb{T}^d \times \mathbb{R}^d \times \mathbb{R}^d, \tag{4.1}$$

where $\theta_i > 0$ captures the (inverse) mobility of workers and firms (one can think that $\theta_2 > \theta_1$). In this case, thanks to the Hopf-Cole transform, one can use the arguments of section 4 of [10] to obtain a priori bounds and construct classical solutions of the system (2.15)-(2.16)-(2.17)-(2.18). Another special feature of the quadratic case is that it can be reformulated as an entropy minimization problem at the level of the path space and this formulation will be the starting point of our numerical scheme in section 5. This Lagrangian viewpoint was already used in the MFG setting in [4].

4.1 A Lagrangian formulation

Let us start with the Eulerian problem (3.11) which in the quadratic Hamiltonian setting, takes the form

$$\inf_{(m,w)\in\mathcal{K}} \sum_{i=1}^{2} \frac{\theta_i}{2} \int_0^T \int_{\mathbb{T}^d} \frac{|w_i|^2}{m_i} + \int_0^T (C+\mathcal{F})(m_1(t,.), m_2(t,.))\mathrm{d}t. \tag{4.2}$$

The Lagrangian formulation of this problem, relies on the following result of Dawson and Gärtner [16] (also see section II.1.4 in Föllmer [17]). Given $\nu > 0$ a diffusivity parameter let R_ν be the reversible Wiener measure, i.e. the Borel probability measure on the path space $\Omega := C([0,T], \mathbb{T}^d)$

$$R_\nu := \int_{\mathbb{T}^d} \mathrm{Law}(x + \sqrt{2\nu}B)\mathrm{d}x$$

where B is the standard Brownian motion (on $\frac{1}{\sqrt{2\nu}}\mathbb{T}^d$) starting at 0. Given $Q \in \mathcal{P}(\Omega)$ another Borel probability measure on Ω, we denote by $H(Q|R_\nu)$ the relative entropy of Q with respect to R_ν:

$$\mathrm{H}(Q|R_\nu) := \begin{cases} \int_\Omega \log\left(\frac{dQ}{dR_\nu}\right)\mathrm{d}Q \text{ if } Q \ll R_\nu \\ +\infty \text{ otherwise} \end{cases}$$

where $\frac{dQ}{dR_\nu}$ stands for the Radon-Nikodym derivative of Q with respect to R_ν. For $t \in [0,T]$, we denote by e_t the evaluation at time t i.e. $e_t(\omega) := \omega(t)$ for every $\omega \in \Omega$, hence for $Q \in \mathcal{P}(\Omega)$ and $t \in [0,T]$, $Q^t := e_{t\#}Q$ is the marginal of

Q at time t. Given a flow of marginals $t \in [0,T] \mapsto m(t,.) \in \mathcal{P}(\mathbb{T}^d)$, Dawson and Gärtner [16], in connection with large deviation principles, established the following

$$\begin{aligned} &\inf_w \left\{ \frac{1}{2}\int_0^T \int_{\mathbb{T}^d} \frac{|w(t,x)|^2}{m(t,x)} \mathrm{d}x \ \mathrm{d}t \ : \partial_t m - \nu \Delta m + \mathrm{div}(w) = 0 \right\} \\ &= \inf_{Q \in \mathcal{P}(\Omega)} \left\{ H(Q|R_\nu), \ \ e_{t\#}Q = m_t, \ \forall t \in [0,T] \right\} - H(m_0|R_\nu^0). \end{aligned} \tag{4.3}$$

Setting $R_i := R_{\nu_i}$, this enables us to reformulate (4.2) as

$$\begin{aligned} \inf_{(Q_1,Q_2)\in \mathcal{P}(\Omega)^2 \,:\, e_{0\#}Q_i = m_i^0} \mathcal{H}(Q_1,Q_2) &:= \theta_1 H(Q_1|R_1) + \theta_2 H(Q_2|R_2) \\ &+ \int_0^T (C+\mathcal{F})(e_{t\#}Q_1, e_{t\#}Q_2)\mathrm{d}t. \end{aligned} \tag{4.4}$$

Note that (4.4) being a strictly convex minimization problem, its solution is unique hence so is the solution of (4.2) since the optimal measures $m_i(t,.)$ for (4.2) are the time-marginals of the optimal measures on paths Q_i in (4.4).

4.2 Beyond the potential case

One strong limitation of the variational approach for MFG involving several population of players is that it imposes symmetric interactions. However, it is reasonable to assume that there are non-symmetric externalities (the disutility of living close to a polluting factory for instance). A more general situation allowing for such non symmetric externalities, is to assume that these are given by two potentials V_i: $(m_1,m_2) \in \mathcal{P}(\mathbb{T}^d) \mapsto V_i[m_1,m_2] \in C(\mathbb{T}^d)$ which are regular in the sense that

$$V_i \in C((\mathcal{P}(\mathbb{T}^d), W_1)^2, C(\mathbb{T}^d)) \tag{4.5}$$

where W_1 denotes the 1-Wasserstein metric on $\mathcal{P}(\mathbb{T}^d)$. This leads to the MFG system

$$-\partial_t \phi_i - \nu_i \Delta \phi_i + \frac{1}{2\theta_i}|\nabla \phi_i|^2 = f(m_1+m_2) + \alpha_i + V_i[m_1,m_2], \ \ \phi_i(T,.) = 0, \tag{4.6}$$

$$\partial_t m_i - \nu_i \Delta m_i - \mathrm{div}(m_i \nabla \phi_i) = 0, \ \ m_i(0,.) = m_i^0, \tag{4.7}$$

supplemented by conditions (2.17)-(2.18) relating α_1, α_2 to m_1, m_2. Thanks to (4.3) this can easily be reformulated as a fixed-point problem at the

Lagrangian level. Let us equip $\mathcal{P}(\Omega)$ with the narrow topology. Given $(\widetilde{Q}_1, \widetilde{Q}_2) \in \mathcal{P}(\Omega)^2$ let

$$T(\widetilde{Q}_1, \widetilde{Q}_2) := \operatorname{argmin}\left\{\mathcal{H}_{\widetilde{Q}_1,\widetilde{Q}_2}(Q_1, Q_2) \ : \ (Q_1, Q_2) \in \mathcal{P}(\Omega)^2 \ : \ e_{0\#}Q_i = m_i^0\right\} \tag{4.8}$$

where

$$\mathcal{H}_{\widetilde{Q}_1,\widetilde{Q}_2}(Q_1, Q_2) := \mathcal{H}(Q_1, Q_2) + \sum_{i=1}^{2} \int_0^T \int_{\mathbb{T}^d} V_i[e_{t\#}\widetilde{Q}_1, e_{t\#}\widetilde{Q}_2] \mathrm{d}e_{t\#}Q_i(x)\mathrm{d}t$$

and $\mathcal{H}$ is as in (4.4). It is easy to see that T is well-defined and by construction

$$T(\mathcal{P}(\Omega)^2) \subset \{(Q_1, Q_2) \in \mathcal{P}(\Omega)^2 \ : \ \mathcal{H}_{\widetilde{Q}_1,\widetilde{Q}_2}(Q_1, Q_2) \leq \mathcal{H}_{\widetilde{Q}_1,\widetilde{Q}_2}(R_1, R_2)\}$$

hence for some $M > 0$

$$T(\mathcal{P}(\Omega)^2) \subset K_M := \left\{(Q_1, Q_2) \in \mathcal{P}(\Omega)^2 \ : \ H(Q_i|R_i) \leq M, \ e_{0\#}Q_i = m_i^0\right\}$$

since K_M is a uniformly integrable subset of $L^1(R_1) \times L^1(R_2)$ it is tight hence $T(\mathcal{P}(\Omega)^2)$ is relatively compact for the narrow topology. Moreover, the weak lower-semi-continuity of $\mathcal{H}$ together with (4.5) implies that T is narrowly continuous. It therefore follows from Schauder's fixed point Theorem that T admits at least a fixed point (Q_1, Q_2). The flow of marginals $m_i(t, .) := e_{t\#}Q_i$ therefore (at least formally, or in the weak sense) solves (4.6) for some ϕ_i and α_i such that (4.7)-(2.17)-(2.18) hold.

5 An IPFP scheme

The Lagrangian formulation of the problem is amenable to a numerical strategy that is a direct generalization of the famous Sinkhorn algorithm [31], also known as Iterative Proportional Fitting Procedure, which has received a lot of interest in recent years for its many applications related to optimal transport – see notably [15], [28], [3], [26], [12], [4]. The algorithm presented here is an extension of the algorithm used in [4], which itself can be viewed as a variant of the general form presented in [12].

5.1 Discretization, optimality system

To simplify exposition, we discretize (4.4) both in time and space. As explained in [27], a good way to view the IPFP procedure is to understand it as alternate maximization on the dual. Hence, we shall first derive a convenient expression for the discretized dual problem and explain how each block optimization is performed in practice. This will allow us to give a simple form for Sinkhorn iterations; the primal-dual conditions also give a very compact form for the optimal measures as diagonal scalings of a kernel.

We denote by S the discretized space grid – throughout we will only refer to indices $i \in S$ but it is to be understood that this corresponds to points x_i on a grid. The time interval $[0, T]$ is discretized with $N+1$ steps ; similarly we will refer to the time indices $k \in \{0, 1, ..., N\}$ where it is to be understood that they map to the discretized time grid $\{0, \frac{T}{N}, ..., T\}$. Denote $dt := \frac{T}{N}$ the size of one step on the time grid.

We denote by R_i^N the discretization of the reversible Wiener measure on S^{N+1} (path measures becoming tensors in this framework) with viscosity parameter ν_i. Similary the path measures Q_i become discrete probability measures in $\mathcal{P}(S^{N+1})$. Throughout this section we adopt the convention that Q_1^k denotes the k-th marginal of Q_1 in the canonical projection. Similarly for other variables, we will generally reserve subscripts to denote population $\{1, 2\}$, use superscripts for the time dimension, and write the space dimension as an input. For example $Q_1^k(i)$ denotes the mass of population 1 on grid point i at time k ; $Q_2(i_0, ..., i_N)$ denotes the mass of population 2 moving from i_0 to i_1 ... to i_N along the dynamics. Define the discrete analog of the relative entropy (or Kullback-Leibler divergence) :

$$H(p|q) := \sum_i \left(p_i \left(\log \left(\frac{p_i}{q_i} \right) - 1 \right) + q_i \right),$$

where, with a slight abuse of notation, we do not specify the underlying space of integration/summation.

We also regularize the instantaneous optimal transport (representing labour market equilibrium) problem by introducing an entropy term – this has several advantages. First, it makes the computation of the transport cost also amenable to a Sinkhorn-like approach, hence it allows us to rewrite the whole problem as a nested entropy minimization problem and perform all iterations jointly. Second, it is a well-known result that for a small regularization parameter the solution of the regularized problem tends to the classical optimal transport solution, hence we can recover the solution of (4.4). Third, it is also well-known that entropic regularization of optimal transport can be interpreted as adding noise in the coupling – which can be a heuristically desirable feature, as it can be interpreted as e.g. resulting from random preference shocks, which is common in the economics literature on matching models (see e.g. [18]). The discrete regularized optimal transport problem for given measures m_1 and m_2 writes as:

$$C^\sigma(m_1, m_2) := \inf_{\gamma \in \Pi(m_1,m_2)} \sum_{(i,j)\in S^2} \gamma(i,j)c(i,j) + \sigma \sum_{(i,j)\in S^2} \gamma(i,j)(\log(\gamma(i,j)) - 1),$$

where c is the ground cost and σ the regularization parameter and we write everything in grid coordinates. This rewrites (up to a constant), as an entropy minimization problem :

$$C^\sigma(m_1, m_2) := \sigma \inf_{\gamma \in \Pi(m_1,m_2)} H(\gamma|\xi),$$

where ξ is the Gibbs kernel ξ:

$$\xi(i,j) := e^{-c(i,j)/\sigma}.$$

We also rewrite the running cost functions $\mathcal{F}$ and F in a discretized equivalent of (3.4)

$$\mathcal{F}(m_1, m_2) = \begin{cases} \sum_{i \in S} F(m_1(i) + m_2(i)) & \text{if } m_1, m_2 \geq 0 \\ +\infty & \text{otherwise.} \end{cases}$$

The Lagrangian problem (4.4) thus rewrites in discretized form as :

$$\min_{\substack{Q_1, Q_2 \in \mathcal{P}(S^{N+1}) \\ Q_1^0 = m_1^0 \\ Q_2^0 = m_2^0}} \theta_1 H(Q_1|R_1^N) + \theta_2 H(Q_2|R_2^N) + dt \sum_{k=0}^{N} C^\sigma(Q_1^k, Q_2^k) + dt \sum_{k=1}^{N} \mathcal{F}(Q_1^k, Q_2^k) \tag{5.1}$$

which is a strictly convex finite dimensional problem. For convenience in writing the dual problem, furthermore define $\mathcal{F}_0 := \chi_{\{m_1^0\}\times\{m_2^0\}}$ the indicator (in the convex analysis sense $\mathcal{F}_0(m_1, m_2) = 0$ if $m_1 = m_1^0$ and $m_2 = m_2^0$ and $+\infty$ otherwise) for the initial condition. By a standard Lagrangian duality argument (note that (5.1) is a convex minimization problem with finitely many linear marginal constraints), we arrive at the following dual formulation which will be essential for the algorithm.

Proposition 5.1. *The dual problem of (5.1) is given by :*

$$\begin{aligned}
\sup_{\substack{u_1^k,u_2^k,v_1^k,v_2^k\in\mathbb{R}^S\\ k=0,\dots,N}} \quad & -\theta_1 \sum_{(i_0,\dots,i_N)\in S^{N+1}} \exp\left(\sum_{k=0}^{N}(v_1^k(i_k)-\frac{dt\sigma}{\theta_1}u_1^k(i_k))\right) R_1^N(i_0,\dots,i_N)\\
& -\theta_2 \sum_{(i_0,\dots,i_N)\in S^{N+1}} \exp\left(\sum_{k=0}^{N}(v_2^k(i_k)-\frac{dt\sigma}{\theta_2}u_2^k(i_k))\right) R_2^N(i_0,\dots,i_N)\\
& -dt\sigma\sum_{k=0}^{N}\sum_{(i,j)\in S^2} e^{u_1^k(i)}e^{u_2^k(j)}\xi(i,j)\\
& -dt\sum_{k=1}^{N}\mathcal{F}^*\left(-\frac{\theta_1}{dt}v_1^k,-\frac{\theta_2}{dt}v_2^k\right)\\
& -\mathcal{F}_0^*(-\theta_1 v_1^0,-\theta_2 v_2^0).
\end{aligned} \tag{5.2}$$

Moreover, strong duality holds in the sense that the (attained) value of (5.2) coincides with the minimum in (5.1).

We have used the same convention for potentials in the dual problem that superscripts denote time – e.g. u_1^k is the vector (or tensor) over the space grid for potential u_1 at time k. The expression above may seem daunting and notationally cumbersome; it can however be decomposed in rather familiar terms and is directly analogous to e.g. Theorem 3.2. in [12] or Proposition 5.1. in [4]. The dual potentials u_1^k and u_2^k are similar to Kantorovitch potentials in the optimal transport problem (instantaneous matching) and represent Lagrange multipliers for the first and second marginal constraints on the coupling γ^k. The usual primal-dual relations give us that the optimal transport plan γ^k between Q_1^k and Q_2^k can be expressed as a diagonal scaling of the Gibbs kernel ξ:

$$\gamma^k(i,j) := e^{u_1^k(i)}e^{u_2^k(j)}\xi(i,j). \tag{5.3}$$

Hence the third line in the dual problem simply corresponds to integration of the plans γ^k over space and time. Similarly, the potential v_1^k correspond to an indirect marginal constraint on Q_1^k the k-th marginal of Q_1, where optimality conditions on the marginals are captured in the Legendre conjugate $\mathcal{F}^*$. The optimal measure Q_1 is also expressed as a diagonal scaling of the associated Wiener measure R_1^N by the tensor products of exponential potentials – primal-dual conditions give :

$$Q_1(i_0, ..., i_N) = \exp\left(\sum_{k=0}^{N} v_1^k(i_k) - \frac{dt\sigma}{\theta_1} u_1^k(i_k)\right) R_1^N(i_0, ..., i_N) \tag{5.4}$$

$$Q_2(i_0, ..., i_N) = \exp\left(\sum_{k=0}^{N} v_2^k(i_k) - \frac{dt\sigma}{\theta_2} u_2^k(i_k)\right) R_2^N(i_0, ..., i_N). \tag{5.5}$$

Notice that the optimal measure depend on both potentials – this is intuitive, because the u potentials capture the transport cost, whereas the v capture the congestion cost and both combine to define the optimal scaling. It is customary in the literature to simplify notations by using direct sums or tensor products – in our setup where different dimensions are at play, it seemed that explicit notations, although cumbersome at first glance, ultimately improve expositional clarity.

5.2 IPFP scheme

The originality of the problem above with respect to related problems in [12] or [4] is twofold – because we are looking at a two-populations problem, the set of potentials is doubled once; because we introduced an extra coupling in the cost function, the set of potentials is doubled a second time. The overall structure of the problem, however, remains similar and lends itself directly to application of an Iterative Proportional Fitting Procedure scheme (or generalized Sinkhorn algorithm, or Dykstra algorithm) by block optimization for the potentials. The scheme we propose is nothing but alternate maximization (coordinate ascent) on the dual problem (5.2). For general convergence results for block coordinate optimization, a recent and complete reference is Beck and Tetruashvili [2]. We now proceed to detail the iterations.

Optimization with respect to u_1^k or u_2^k: Updates of u_1^k, u_2^k correspond to traditional Sinkhorn iterations. For instance, u_1^k is determined by the marginal constraint that the first marginal of γ^k is Q_1^k. Solving the first-order conditions with respect to u_1^k in the dual problem indeed gives the Sinkhorn-like scaling update :

$$e^{u_1^k(i)} = \left(\frac{e^{v_1^k(i)}\kappa_1^k(i)}{\sum_{j\in S} e^{u_2^k(j)}\xi(i,j)}\right)^{\frac{\theta_1}{\theta_1 + dt\sigma}}, \tag{5.6}$$

where :

$$\kappa_1^k(i) = \sum_{(i_0,...,i_{k-1},i_{k+1},...,i_N)\in S^N} \exp\left(\sum_{\substack{l=0\\l\neq k}}^{N}(v_1^l(i_l) - \frac{dt\sigma}{\theta_1}u_1^l(i_l))\right) \times R_1^N(i_0,...,i_{k-1},i,i_{k+1},...,i_N)$$

is a (rescaled) projection of Q_1. This is a pure scaling step which can easily be vectorized and is in fact computationally inexpensive. The only numerical difficulty resides with computations of integrals against the Wiener measure; however, using the decomposition of the Wiener measure, this step can be reduced to successive convolutions against the heat kernel. By using this trick, we can avoid having to store the whole Wiener measure tensor and operate instead only on the heat kernel over S. Update steps for u_2^k are fully identical.

Updating the v potentials: To simplify notations, take the exponential transform of the potentials :

$$a_1^k(i) := e^{u_1^k(i)} \quad a_2^k(i) := e^{u_2^k(i)}.$$

The optimal measures are given with this notation by :

$$\gamma^k(i,j) = a_1^k(i)\xi(i,j)a_2^k(j) \tag{5.7}$$

$$Q_1(i_0,...,i_N) = \left(\bigotimes_{k=0}^{N} e^{v_1^k}(a_1^k)^{-\frac{dt\sigma}{\theta_1}}\right)(i_0,...,i_N) \times R_1^N(i_0,...,i_N) \tag{5.8}$$

$$Q_2(i_0,...,i_N) = \left(\bigotimes_{k=0}^{N} e^{v_2^k}(a_2^k)^{-\frac{dt\sigma}{\theta_2}}\right)(i_0,...,i_N) \times R_2^N(i_0,...,i_N), \tag{5.9}$$

where $\bigotimes$ denotes the tensor product : $\left(\bigotimes_{k=0}^N a_1^k\right)(i_0,...,i_N) = a_1^0(i_0) \times a_1^1(i_1) \times ... \times a_1^N(i_N)$. Note that it will be convenient *not* to take the exponential transform for the v potentials.

Optimization with respect to v_1^0 or v_2^0: The potentials v_1^0 and v_2^0 also have direct scaling updates given by the marginal constraint on initial densities. It is straightforward to see that the update on v_1^0 is given by:

$$e^{v_1^0} := \frac{m_1^0}{\alpha_1^0}$$

where the division and multiplication of vectors is understood pointwise and we define :

$$\alpha_1^k(i) := a_1^k(i)^{-\frac{dt\sigma}{\theta_1}} \sum_{(i_0,\dots,i_{k-1},i_{k+1},\dots,i_N)\in S^N} \left(\bigotimes_{l\neq k} e^{v_1^l}(a_1^l)^{-\frac{dt\sigma}{\theta_1}} \right)(i_0,...i_{k-1},i_{k+1},...,i_N) \times R_1^N(i_0,...i_{k-1},i,i_{k+1},...,i_N)$$

which plays a similar role as κ before, being some scaled projection of Q_1 given by integration of potentials over all paths that pass through grid point i at time k. We define the update on v_2^0 and $\alpha_2^k(i)$ symmetrically.

Optimization with respect to v_1^k, v_2^k: For $k \geq 1$, we can perform the updates on v_1^k and v_2^k as one block. Indeed, observe that this update amounts to solving :

$$\begin{aligned}\sup_{v_1^k,v_2^k\in\mathbb{R}^S} & -\theta_1 \sum_{(i_0,\dots,i_N)\in S^{N+1}} \left(\bigotimes_{k=0}^{N} e^{v_1^k}(a_1^k)^{-\frac{dt\sigma}{\theta_1}} \right)(i_0,...,i_N) \times R_1^N(i_0,...,i_N) \\ & -\theta_2 \sum_{(i_0,\dots,i_N)\in S^{N+1}} \left(\bigotimes_{k=0}^{N} e^{v_2^k}(a_2^k)^{-\frac{dt\sigma}{\theta_2}} \right)(i_0,...,i_N) \times R_2^N(i_0,...,i_N) \\ & - dt\mathcal{F}^*\left(-\frac{\theta_1}{dt}v_1^k, -\frac{\theta_2}{dt}v_2^k\right).\end{aligned}$$

Defining $G := (F + \chi_{\mathbb{R}_+})^*$ as before, we have :

$$\mathcal{F}^*\left(-\frac{\theta_1}{dt}v_1^k, -\frac{\theta_2}{dt}v_2^k\right) = \sum_{i\in S} G\left(\max\{-\frac{\theta_1}{dt}v_1^k(i), -\frac{\theta_2}{dt}v_2^k(i)\}\right).$$

Although it might be suitable numerically to solve for the whole vectorized problem at once, the problem fully decouples according to each coordinate due to the local nature of the cost – for expositional purposes, it is clearer to study the update for $v_1^k(i), v_2^k(i)$ for given k, i :

$$\sup_{v_1^k(i),v_2^k(i)} -\theta_1\alpha_1^k(i)e^{v_1^k(i)} - \theta_2\alpha_2^k(i)e^{v_2^k(i)} - dtG\left(\max\{-\frac{\theta_1}{dt}v_1^k(i), -\frac{\theta_2}{dt}v_2^k(i)\}\right). \tag{5.10}$$

It is straightforward to see that problem (5.10) actually boils down to a one-dimensional problem:

$$\min_{\beta\geq 0} \theta_1\alpha_1^k(i)e^{-\frac{dt}{\theta_1}\beta} + \theta_2\alpha_2^k(i)e^{-\frac{dt}{\theta_2}\beta} + dtG(\beta) \tag{5.11}$$

from which we recover the potentials using β the solution of this problem :

$$v_1^k(i) = -\frac{dt}{\theta_1}\beta$$
$$v_2^k(i) = -\frac{dt}{\theta_2}\beta.$$

This gives a very tractable form to these updates which is easily amenable to various numerical solution methods – in particular, taking F to be e.g. a power function we have an explicit form for G. Furthermore, as previously mentioned, this minimization step can be vectorized to obtain the vectors v_1^k, v_2^k in a single step. The steps of the algorithm are summarized in compact form below in Algorithm 1.

5.3 Numerical results

We now present numerical results[3] obtained using the previously introduced algorithm. Throughout, we use the following conventions for model parameters:

- space S is the discretized one-dimensional torus (the circle),
- time horizon is T and the number of time steps is $N+1$ (discrete time is indexed by $k = 0, ..., N$),
- θ_1, θ_2 are the mobility parameters (higher θ means higher movement cost),
- σ is the regularization parameter for labour market equilibrium OT problem,
- ν_1, ν_2 are the respective diffusivity parameters for residents and firms,
- the congestion/rent function is given by :

  $$F(x) = \frac{ax^p}{p},$$

  hence the Legendre transform $(F + \chi_{\mathbb{R}_+})$ is simply :

  $$G(\beta) = \left(\frac{1}{a}\right)^{\frac{1}{p-1}} \left(\frac{p-1}{p}\right) \beta_+^{\frac{p}{p-1}}$$

[3]The Matlab code for the simulations presented in this section is available at `https://github.com/CesarBarilla/MFG-Cities_Code`. Animated GIFs of the simulations and additional cases are also available online at `https://cesarbarilla.github.io/research/mfg-cities`.

Algorithm 1 Iterative proportionnal fitting algorithm

Input : Initial potentials a_1^k, a_2^k, v_1^k, v_2^k.

for all $l \geq 0$ **do**

  for $k = 0, ..., N$ **do**

$$a_1^k = \left(\frac{e^{v_1^k} \kappa_1^k}{\sum_{j \in S} a_2^k \xi(\cdot, j)} \right)^{\frac{\theta_1}{\theta_1 + dt\sigma}}$$

$$a_2^k = \left(\frac{e^{v_2^k} \kappa_2^k}{\sum_{i \in S} a_1^k \xi(i, \cdot)} \right)^{\frac{\theta_2}{\theta_2 + dt\sigma}}$$

  end for

$$e^{v_1^0} := \frac{m_1^0}{\alpha_1^0}$$

$$e^{v_2^0} := \frac{m_2^0}{\alpha_2^0}$$

  for $k = 1, ..., N$ **do**

    Solve for all $i \in S$:

$$\min_{\beta \geq 0} \theta_1 \alpha_1^k(i) e^{-\frac{dt}{\theta_1}\beta} + \theta_2 \alpha_2^k(i) e^{-\frac{dt}{\theta_2}\beta} + dtG(\beta)$$

    Set :

$$v_1^k(i) = -\frac{dt}{\theta_1}\beta$$
$$v_2^k(i) = -\frac{dt}{\theta_2}\beta$$

  end for

  $Q_1 = \left(\bigotimes_{k=0}^N e^{v_1^k} (a_1^k)^{-\frac{dt\sigma}{\theta_1}} \right) \times R_1^N$

  $Q_2 = \left(\bigotimes_{k=0}^N e^{v_2^k} (a_2^k)^{-\frac{dt\sigma}{\theta_2}} \right) \times R_2^N$

end for

higher a and p mean stronger congestion,

- the ground cost is taken to be either the geodesic distance (labeled as *linear*), either its square root (labeled *sqrt*) or its square distance (labeled *quadratic*).

The simulations below only vary in the parameters $(T, N, \theta_1, \theta_2, \sigma, \nu_1, \nu_2, p, a)$ and the ground cost – all of which are specified for each corresponding plot at the bottom of the figure.

For interpreting graphs, recall that we take population 1 (blue/solid curves) as inhabitants and 2 (red/dashed curves) as firms. Generally, we study cases with $\theta_1 < \theta_2$ i.e it is *more costly for firms to relocate* and $\nu_1 > \nu_2$ i.e *inhabitants diffuse more.* This is chosen for consistency with intuition.

Several interesting empirical observations emerge from the simulations:

- **Ground cost matters:** Transport cost between the densities acts as an agglomeration force in the model – indeed, total commuting cost is minimized at zero when the two densities completely overlap. It is no surprise hence that, all else being equal, a stronger transport cost will generate more agglomeration effects and overlapping densities in the long run while a weaker transport cost can let the congestion effect dominate in equilibrium. Figures 1, 2, 3 show three simulations illustrating this, where all parameters are kept identical except for the ground cost (respectively square root, linear, quadratic). All three simulations start from the same initial condition: a single peaked compactly supported distribution for each population. Clearly, 1 showcases segregation while 3 generates overlapping densities in the long run – 2 is an intermediate case. Intuitively, a linear or concave commuting cost makes economic sense: the incremental difference in commuting cost has no particular reason to be increasing with distance to the workplace; on the contrary, it seems reasonable that marginal cost of commute would be decreasing in distance (once you live in the suburbs and far enough from the city center, living just a little further will hardly make a difference in commuting time). It's intuitive why such settings would lead to more segregated city patterns with a centre/suburbs configuration.

- **Sensitivity to initial conditions:** The kind of equilibrium and city configurations that arises in the long run appears to be highly sensitive to initial conditions. As an example, consider figures 4a and 4b. Both have identical parameters but the initial distributions have been swapped ; the initial densities are mixtures of gaussians such that one population is centered around one peak while the other has three main

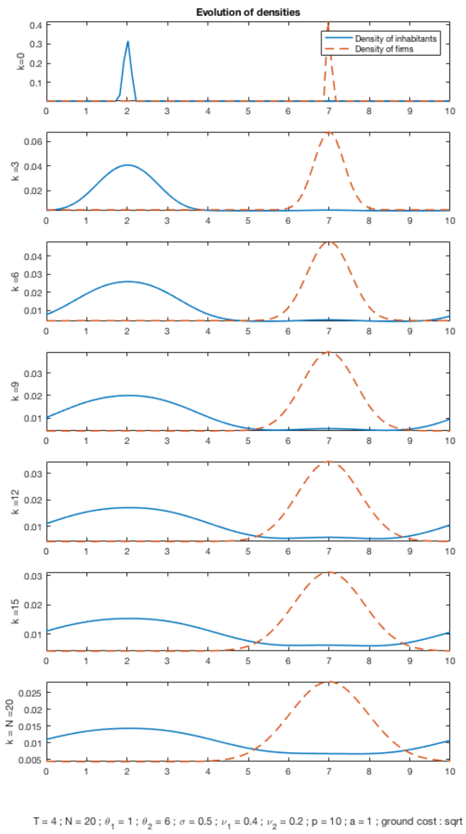

Evolution of densities
Density of inhabitants
Density of firms
k=0
k =3
k =6
k =9
k =12
k =15
k = N =20
T = 4 ; N = 20 ; θ_1 = 1 ; θ_2 = 6 ; σ = 0.5 ; ν_1 = 0.4 ; ν_2 = 0.2 ; p = 10 ; a = 1 ; ground cost : sqrt

Figure 1

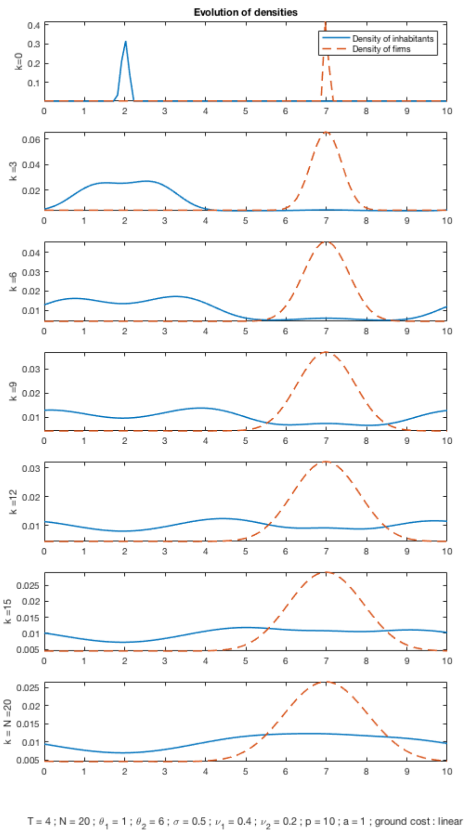

Evolution of densities
Density of inhabitants
Density of firms
k=0
k =3
k =6
k =9
k =12
k =15
k = N =20
T = 4 ; N = 20 ; θ_1 = 1 ; θ_2 = 6 ; σ = 0.5 ; ν_1 = 0.4 ; ν_2 = 0.2 ; p = 10 ; a = 1 ; ground cost : linear

Figure 2

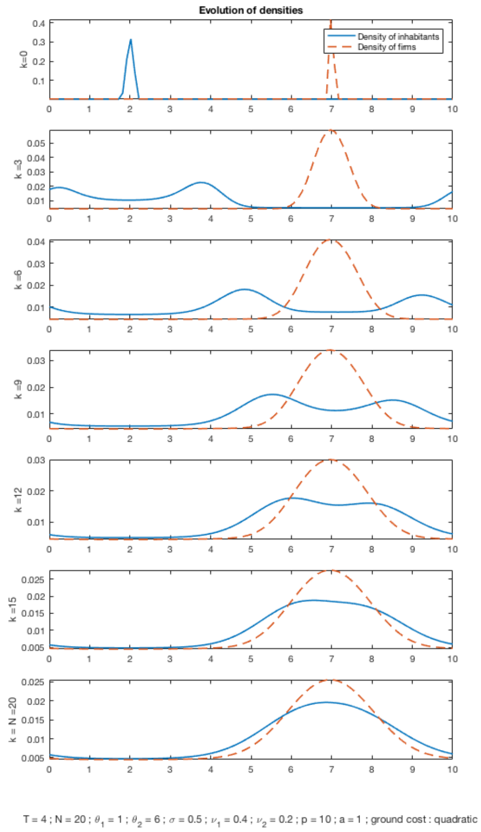
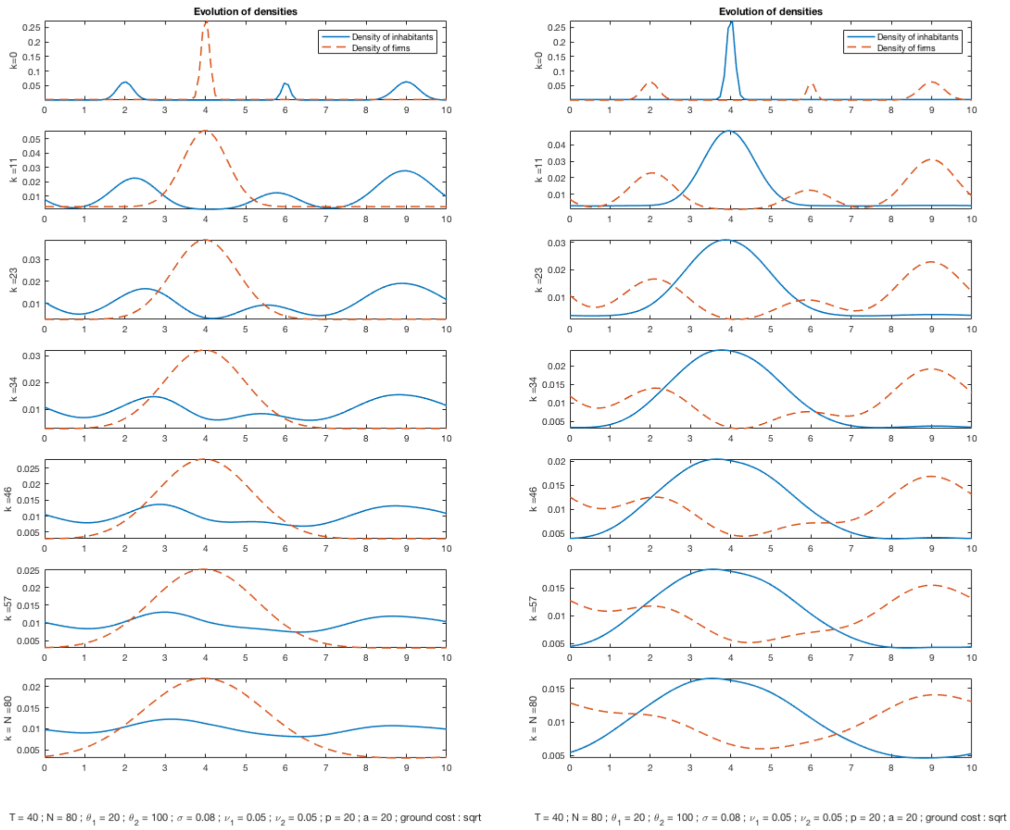

Evolution of densities
Density of inhabitants
Density of firms
k=0
k =3
k =6
k =9
k =12
k =15
k = N =20
T = 4 ; N = 20 ; θ_1 = 1 ; θ_2 = 6 ; σ = 0.5 ; ν_1 = 0.4 ; ν_2 = 0.2 ; p = 10 ; a = 1 ; ground cost : quadratic

Figure 3

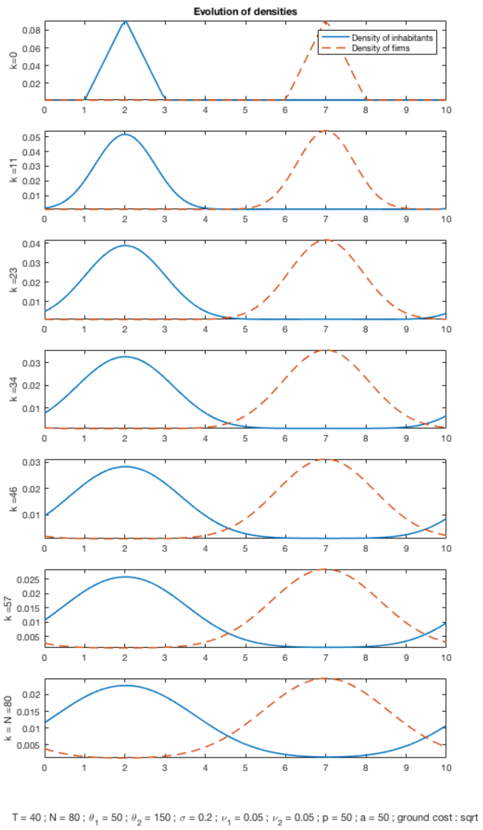


(a) (b)

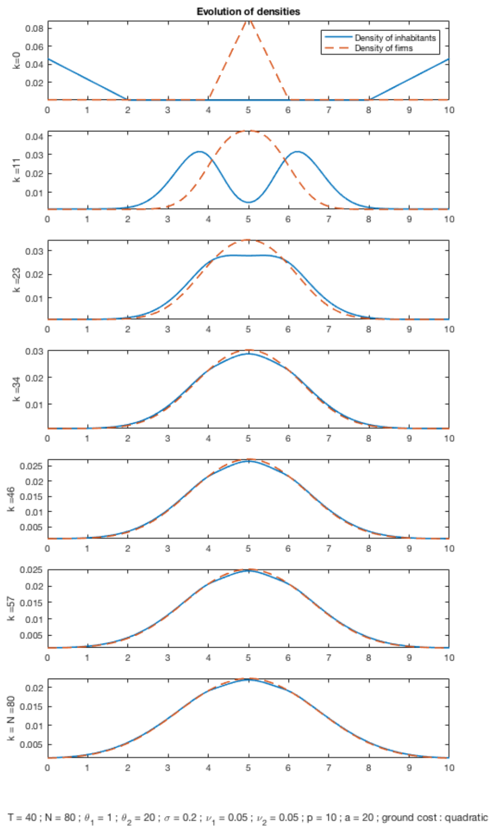
Figure 4

peaks. This could for instance be interpreted as an initial situation with one industrial center and three small villages (or vice-versa). In both cases, we see the emergence of a center/periphery structure with the population that was initially more concentrated in the center. Observe that the asymmetry in the population characteristics nonetheless plays a part in shaping the solution: although a similar behavior is observed in terms of formation of a center, when the more mobile population is not in the center its density becomes relatively closer to a uniform distribution 4a. In the opposite case, we see a pattern closer to a bimodal city with two centers emerging.

- **Convergence to matching distributions and speed:** In general, if the aggregating effects are high enough relative to segregating effects



(a)



(b)

Figure 5

(high transport cost, high diffusion, low motion cost, low congestion cost), then the densities will tend to converge quite quickly towards each other to completely overlap. The speed of that process and the shape of the final density is determined quite intuitively by the parameters – the density with least motion cost will tend to converge towards the other faster, lower overall motion costs speeds up convergence. See notably Figure 5b and 3 or see online for additional figures and examples[4]. On the contrary, figure 5a provides an example where congestion dominates and the initial segregation is perpetuated in a smoothed fashion in equilibrium.

- **Segregation patterns:** One of the main interest of the figures pre-

[4]https://cesarbarilla.github.io/research/mfg-cities

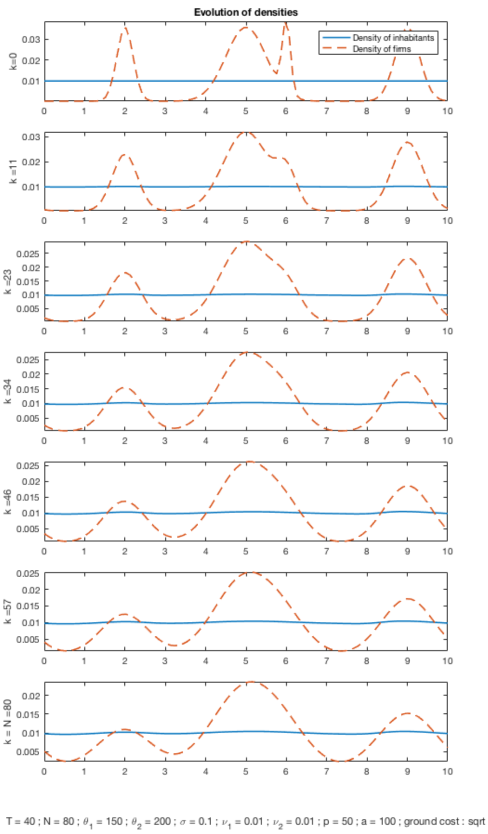


Figure 6

sented is not so much this or that particular configuration or example but rather the fact that the model, with a very sparse and stylized set of explanatory factors is able to generate quite a rich variety of segregation patterns through the interplay of model parameters. With a simple geographical labour market and congestion, several usual city patterns can be obtained in equilibrium – the American-style city with a business center and residential suburbs in 4a, its inverted ("European") form in 4b, a bimodal city in 5a, a near-uniform city with several industrial centers in 6. High sensitivity to parameters and strong dependency to initial parameters display complex dynamics: for instance, when a segregated city appears with a clear center/periphery structure, it can be but is not necessarily centered around the less mobile population – it tends to be if population are initially spread out enough but it can also center around the population that was initially more concentrated.

6 Conclusion

We have proposed a two-populations mean field game model for the evolution of cities with a coupling related to optimal transport (or its entropically regularized variant) so as to capture equilibrium on the labour market at each time. In the case of quadratic Hamiltonians, taking advantage of an entropy minimization formulation of the problem, we have proposed a numerical scheme in the spirit of the celebrated IPFP/Sinkhorn algorithm.

The variety of patterns that appear in simulations highlights that for all its sparsity, the model can provide intuitive and somehow *realistic* city dynamics for a rich array of configurations. This underlines the relevance of the mean-field game approach to model this kind of dynamics. It should be noted in addition that the algorithm and solution method provided is efficient and scalable. Lastly, note that although we have focused on a geographical approach modeling the dynamics of cities, this model could easily be reinterpreted for any situation that has similar essential ingredients : two populations constrained to be in an instaneous equilibrium in which some characteristic is taken as given locally in time, but such that this characteristic can be continuously altered (at a cost) in time. One prominent example would be to reinterpret the model in a skill space instead of geographical space: assume that workers have skill measured on some arbitrary space and that each firm needs a specific type of worker such that output is decreasing in the skill-space distance between its *ideal worker* and the worker actually hired. Taking skills as given, this is a matching problem that can be viewed as an optimal transport problem and is formally analogous to our geographical labour market. Now assume that workers can obtain training to change their skill and firms can shift their activity and corresponding skill demand – both of which at a cost. Instantaneously, there still needs to be an equilibrium but dynamically the distributions of both workers and firms in the skill space are altered through this process. Furthermore, assume there is an exogenous demand constraint on each skill – if too many firms/workers are producing the same good, profits are reduced (equivalently they pay some cost in profits loss). This corresponds formally to some congestion as introduced in the previous model. Hence the MFG system can almost directly be reinterpreted in this framework and the results and methods presented here translated to that framework. More generally, it seems that the multiscale intertwined equilibria framework presented here could prove relevant as a blueprint to model an array of economic interactions – which we did not exhaustively explore.

The contributions of this paper reside mainly in introducing a mathematical and numerical framework which is a combination of mean-field games with several populations and optimal transport. We proved results guaranteeing that the model is well-posed, proposed a numerical solution method, and explored one possible application – leaving space for further developments. Let us briefly discuss two possible developments in cities modeling. The observed concentration of activities in cities reflects the existence of strong production externalities. Positive externalities or spillover effects (absent from our model) act as a major agglomeration force and thus play a key role in urban economics. They should therefore be taken into account in more realistic equilibrium models as in Lucas and Rossi-Hansberg [24]. Another challenging extension is to add a major player to our model so as to analyze or design urban public policies.

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