Computing the demand of agents with a law invariant utility

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Abstract

We consider a class of concave law invariant utilities which contains the Rank Dependent Expected Utility (RDU). For this class, we show that computation of the demand for a contingent claim, although not as simple as in the Expected Utility (EU) case, is still tractable. Specific attention is given to the RDU case. Numerous examples are fully solved.

1 Introduction

The axiomatic aspects of the Rank Dependent Expected Utility model (RDU from now on) have extensively been discussed and it is well known that the RDU accounts for a number of violations of Expected Utility as Allais' paradox. Necessary and sufficient conditions for RDU to be second order stochastic dominance (SSD from now on) preserving are well known and equivalent to its concavity as functional over random variables. Technical isues as differentiability of RDU as functional over lotteries (see for example Chew et al [7] or Wang [21] and the bibliographies listed therein) or over random variables (see Carlier and Dana [2]) are also well understood. However, RDU has not been used much for analyzing problems of economics of uncertainty, in particular for infinite state spaces except in the recent mathematical finance literature on risk measures (see for example [19] and [14]). This is due to the technical difficulty of solving maximization problems for such utilities. The aim of this paper is to provide tools to study the demand for state contingent claims of a RDU agent. We show that computations, although not as simple as for an Expected utility (EU from now on), are still tractable. The technique we use, applies to a larger set of utilities that we next introduce.

We consider concave utilities which are additively separable with respect to the quantile:

$$U(X) := \int_0^1 L(t, F_X^{-1}(t))dt + g(F_X^{-1}(0)). \tag{1}$$

In (1), X is a random variable on a non atomic space with distribution function F_X and F_X^{-1} is a version of the inverse of F_X or quantile of X. The term

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$g(F_X^{-1}(0)) = g(\text{essinf } X)$ accounts for a specific weight given to the minimal value of X. Let us remark that U is stochastic dominance preserving if and only if L is submodular and concave nondecreasing in its second argument.

The class of utilities of type (1) contains the set of SSD preserving RDU. Indeed, from [7], any SSD preserving RDU is the Choquet integral of $u(X)$, a concave function of X with respect to a convex distortion $f : [0,1] \to [0,1]$ continuous or discontinuous at one. When the distortion is continuous,

$$E_f(u(X)) = \int_0^1 f'(1-t)u(F_X^{-1}(t))dt \tag{2}$$

and is obtained from (1) by setting $g = 0$ and $L(t,x) = f'(1-t)u(x)$, while if f is discontinuous,

$$E_f(u(X)) = (1 - f(1^-))u(F_X^{-1}(0)) + \int_0^1 f'(1-t)u(F_X^{-1}(t))dt \tag{3}$$

defines a utility of the type (1) with $L(t,x) = f'(1-t)u(x)$ and $V(x) = (1 - f(1^-))u(x)$ and $f(1^-)) = \lim_{x\uparrow 1} f(x)$. When $u(x) = x$ in formula (3), one obtains Yaari's utilities.

In the case $g = 0$, utility functions of type (1) were first introduced by Green and Jullien [12]. In Epstein and Chew [6], these utilities are called *Rank linear utilities* (RLU from now on). Green and Jullien gave a representation theorem under an axiom over distribution functions weaker than the independence axiom, the *ordinal independence* axiom. While providing unifying foundations for non expected utilities theories, Epstein and Chew [6] discussed representations of type (1) and provided a representation theorem under an axiom of *rank separability.* Axiomatic foundations for (1) were finally given by Chew and Wakker [8] under a *comonotone sure thing principle.* Necessary and sufficient conditions for these utilities to be second order stochastic dominance preserving were given by Epstein and Chew [6] and Green and Jullien [12]. Ordering of risk aversions for utilities of type (1) was characterized by Green and Jullien [12].

In [19], Schied solved the Yaari and the expected shortfall demand problems and obtained discontinuous solutions. In this paper, we focus on the demand of RLU agents when $L(t,.)$ is strictly concave. We discuss demand for state contingent claims under the assumption that the pricing density ψ has a continuous distribution function F_ψ. For SSD preserving utilities, it is shown in [4] that a demand problem for contingent claims may be brought down to a *quantile demand problem.* The quantile of a random variable being a nondecreasing function, the quantile demand problem reduces to a variational problem with a monotonicity constraint.

Calculus of variations problems with a monotonicity constraint appeared in one dimensional *adverse selection theory* (see for example, Mirrlees [16], Mussa-Rosen [17], Spence [20], Guesnerie-Laffont [13], Rochet [18]). Bank and Riedel [1] also dealt with such problems to solve intertemporal utility maximization problems. In particular, Mussa and Rosen [17] developed a method called the *ironing procedure* to characterize solutions. They showed that there is a partition of the type space consisting of:

- sub-intervals on which the solution is constant: such intervals are called *bunches*,
- sub-intervals on which the solution is increasing and coincides with the maximizer without the monotonicity constraint.

Specific attention is then given to the RDU case, assuming the distortion either continuous or discontinuous. We first show that the demand of a RDU agent with a continuous distortion is decreasing in the pricing density ψ if and only if the rate of growth of the quantile of the pricing density is greater than the distortion index of the agent $f"/f'$. This condition is also equivalent to $\ln \psi$ being more disperse in the sense of Bickel-Lehman than $\ln f'$. Otherwise, there are ranges of the pricing density on which the demand is constant. We next show that a RDU agent behaves almost as if she was perceiving a perturbation of the pricing density and was an EU agent, the *perceived* pricing density depending only on the distortion and on the pricing density. In the RDU case, distortion of the probabilities and risk aversion have therefore separate effects on demand. The distortion determines the perceived pricing density. Risk aversion determines demand in function of the perceived pricing density almost as for a EU model. In particular if $f'(0) = 0$ ($f'(1) = \infty$), the perceived pricing density is constant for high (low) values of the true pricing density and therefore the demand function is constant for low (high) values of the pricing density. The previous analysis extends to the case of discontinuous distortions with the further property that cautiousness induces the perceived pricing density to be constant for low values of the pricing density and thus the demand to be constant for high values of the pricing density. We also prove that the RDU demand function is *strangled* either if $f'(0) = 0$ and the agent is averse to the worse state or if $f'(0) = 0$ and $f'(1) = \infty$.

The general techniques of the paper are illustrated by several examples for which we obtain closed-form solutions for both RDU and RLU cases. The RDU examples prove that the demand may have an arbitrary number of bunches. Another example suggests that the RLU model allows richer income effects than the RDU.

The paper is organized as follows: In section 2, standard definitions and properties are reviewed. Utilities of type (1) are introduced. Necessary and sufficient conditions for such utilities to be SSD preserving are provided. Section 3 is devoted to the reformulation of a RLU demand problem as a variational problem with a monotonicity constraint. Section 4 specializes on the RDU case. Section 5 is devoted to examples.

2 On rank linear utility functionals

2.1 Definitions

Given as primitive is a probability space $(\Omega, \mathcal{B}, P)$. Let X be a random variable and let $F_X(t) = P(X \leq t)$, $t \in \mathbb{R}$ denote its distribution function. The generalized inverse of F_X is defined by:

$$F_X^{-1}(0) = \text{essinf}\, X \text{ and } F_X^{-1}(t) = \inf\{z \in \mathbb{R} : F_X(z) \geq t\}, \text{ for all } t \in (0, 1]$$

Let us recall that the random variable X dominates Y in the sense of second order stochastic dominance (SSD), which will be denoted $X \succeq_2 Y$ if $E(u(X)) \geq E(u(Y))$, for every utility function $u : \mathbb{R} \to \mathbb{R}$ concave nondecreasing. We also recall the characterization:

Proposition 1 *The random variable X dominates Y in the sense of second order stochastic dominance if and only if*

$$\int_0^1 g(s)F_X^{-1}(s)ds \geq \int_0^1 g(s)F_Y^{-1}(s)ds, \ \forall g : \ [0,1] \to \mathbb{R}_+ \text{ nonincreasing.} \quad (4)$$

We also recall that X strictly dominates Y in the sense of SSD (notation $X \succ_2 Y$) if $E(u(X)) > E(u(Y))$ for every strictly concave nondecreasing utility function u. Equivalently, X strictly dominates Y if and only if the inequality is strict in (4), for every decreasing function g.

The fact that two random variables X on $(\Omega, \mathcal{B}, P)$ and Y on $(\Omega', \mathcal{B}', P')$ have the same probability law will be denoted $X \overset{d}{\sim} Y$. For a map $V : L^\infty(\Omega) \to \mathbb{R} \cup \{-\infty\}$, the domain of V is defined by
dom $V := \{X \in L^\infty(\Omega) \ : \ V(X) > -\infty\}$.

Definition 1 1. *A map $V : L^\infty(\Omega) \to \mathbb{R} \cup \{-\infty\}$ is (strictly) monotone if $X \geq Y$ a.e. implies $V(X) \geq V(Y)$ (resp. $V(X) > V(Y)$ whenever $X \geq Y$ a.e. and $P(X \neq Y) > 0$ and $Y \in$dom V).*

2. *A map $V : L^\infty(\Omega) \to \mathbb{R} \cup \{-\infty\}$ is law invariant if $V(X) = V(Y)$ whenever $X \overset{d}{\sim} Y$.*

3. *A map $V : L^\infty(\Omega) \to \mathbb{R} \cup \{-\infty\}$ (strictly) preserves SSD if $X \succeq_2 Y$ implies $V(X) \geq V(Y)$ (resp. $V(X) > V(Y)$ whenever $X \succ_2 Y$ and $Y \in$dom V).*

Since $X \overset{d}{\sim} Y$ is equivalent to $X \succeq_2 Y$ and $Y \succeq_2 X$, SSD preserving functions are law invariant. As $X + Y \succ_2 X$ for any $Y \geq 0, Y \neq 0$, SSD preserving functions (strictly SSD preserving functions) are monotone (strictly monotone). The converse is not true. For a counterexample, see Dana [9]. However, we have the following result proven in Dana [9]:

Proposition 2 *Let $(\Omega, \mathcal{B}, P)$ be non-atomic and let $V : L^\infty(\Omega) \to \mathbb{R} \cup \{-\infty\}$ be concave, $\sigma(L^\infty(\Omega), L^1(\Omega))$ upper semi-continuous. Then V is SSD preserving if and only if V is law invariant and monotone.*

In the remainder of the paper, we shall assume that $(\Omega, \mathcal{B}, P)$ is non-atomic that is there exists a random variable U on $(\Omega, \mathcal{B}, P)$ uniformly distributed on $[0, 1]$.

2.2 Stochastic dominance

In the paper, we pay special attention to quantile based-utilities defined by integrals. These utilities generalize the rank dependent expected utility (see next subsection) and are of the form:

$$V(X) := V_L(X) + g(F_X^{-1}(0)), \quad (5)$$

with

$$V_L(X) := \int_0^1 L(t, F_X^{-1}(t))dt, \quad \text{for all } X \in L^\infty(\Omega, \mathcal{B}, P). \tag{6}$$

Necessary and sufficient conditions for a utility V_L defined by (6) to be SSD preserving are next provided (see [5] for a proof). For the sake of simplicity, L is assumed to be smooth in what follows.

Proposition 3 *Let $(\Omega, \mathcal{B}, P)$ be non atomic and V_L be of type (6). Let $L \in C^2([0,1] \times \mathbb{R})$. The following are equivalent:*

1. *V_L is SSD preserving,*
2. *$\partial_x L \geq 0$, $\partial_{xx} L \leq 0$ and $\partial_{tx} L \leq 0$ on $[0,1] \times \mathbb{R}$*
3. *V_L is concave, monotone and $\sigma(L^\infty(\Omega), L^1(\Omega))$ upper semi-continuous.*

If, in addition to the assumptions of the previous proposition, we either assume that $L(t,.)$ is strictly concave for every $t \in [0,1]$, or that $\partial_{tx} L < 0$ then V_L is strictly SSD preserving. Indeed, assume first that $L(t,.)$ is strictly concave and let X and Y be in $L^\infty(\Omega, \mathcal{B}, P)$ with $X \succ_2 Y$. By strict concavity and since $F_X^{-1} \neq F_Y^{-1}$, we have:

$$V_L(Y) - V_L(X) < \int_0^1 \partial_x L(t, F_X^{-1}(t))(F_Y^{-1}(t) - F_X^{-1}(t))dt$$

Let $g(t) := \partial_x L(t, F_X^{-1}(t))$. By assumption, g is nonnegative. Since F_X^{-1} can be approximated by smooth nondecreasing functions in the a.e. convergence, we may assume that $F_X^{-1} := x$ is a smooth nondecreasing function. Hence

$$\frac{d}{dt}[\partial_x L(t, x(t))] = \partial_{tx} L(t, x(t)) + \partial_{xx} L(t, x(t))x'(t) \leq 0,$$

and g is nonincreasing. From (4), we deduce that $V_L(Y) < V_L(X)$, hence V_L is strictly SSD preserving. Finally, if $\partial_{tx} L < 0$ then g defined above is decreasing, hence if $X \succ_2 Y$, then

$$V_L(Y) - V_L(X) \leq \int_0^1 g(t)(F_Y^{-1}(t) - F_X^{-1}(t))dt < 0.$$

2.3 Choquet integral with respect to a convex distortion

A convex distortion is a convex increasing map $f : [0,1] \to [0,1]$ such that $f(0) = 0$, $f(1) = 1$. Since f is convex and $f(0) = 0$, f is continuous on $[0,1[$. The Choquet integral of $X \in L^\infty(\Omega)$ with respect to the capacity $f(P)$, denoted $E_f(X)$, is defined by

$$E_f(X) = \int_{-\infty}^0 (f(P(\{X > t\})) - 1)dt + \int_0^\infty f(P(\{X > t\}))dt$$

Since f is nondecreasing, convex and finite, it is differentiable a.e. and $f' \in L^1_+[0,1]$. When f is continuous, one has

$$E_f(X) = \int_0^1 f'(1-t)F_X^{-1}(t)dt.$$

When f is discontinuous at 1, let $f(1^-) = \lim_{x\uparrow 1} f(x)$ and $\tilde{f}(x) = \frac{f(x)}{f(1^-)}$ if $x < 1$ and $\tilde{f}(1) = 1$ (note that $\tilde{f}$ is continuous). Since E_f is translation invariant and $X + \|X\|_\infty \geq 0$, we may assume that $X \geq 0$. We then have:

$$\begin{aligned}
E_f(X) = & \int_0^{\text{essinf }(X)} f(P(X > t))dt + \int_{\text{essinf }(X)}^{\infty} f(P(X > t))dt \\
=&\text{essinf }(X) + f(1^-) \left[\int_0^{\infty} \tilde{f}(P(X > t))dt - \int_0^{\text{essinf }(X)} \tilde{f}(P(X > t))dt \right]. \\
=&(1 - f(1^-))\text{essinf }(X) + f(1^-)E_{\tilde{f}}(X) \\
=&(1 - f(1^-))F_X^{-1}(0) + \int_0^1 f'(1-t)F_X^{-1}(t)dt
\end{aligned}$$

Hence, $\quad E_f(u(X)) = (1 - f(1^-))u(F_X^{-1}(0)) + \int_0^1 f'(1-t)u(F_X^{-1}(t))dt$

and therefore a RDU is a utility of type (5) with $L(t, x) = f'(1-t)u(x)$ and $g(x) = (1 - f(1^-))u(x)$.

3 Reformulation of the RLU demand problem

Let $L_+^\infty(\Omega)$ be the set of bounded state contingent consumptions. Let $\psi \in L_+^1(\Omega)$ with $E(\psi) = 1$ be a pricing density. Consider an agent with utility $V : L^\infty(\Omega) \to \mathbb{R} \cup \{-\infty\}$ and income $w > 0$. The agent's demand for state contingent claims is the solution to :

$$(\mathcal{D}) \ \sup\{V(X) : E(\psi X) \leq w, \, X \geq 0\}. \tag{7}$$

3.1 Quantile-based reformulation

Let us assume that F_ψ is continuous (or equivalently that F_ψ^{-1} is strictly increasing). We recall that this implies that $F_\psi(\psi)$ is uniformly distributed. We further assume that V is strictly SSD preserving and concave. Let

$$\mathcal{A} := \{x : [0, 1] \to \mathbb{R}_+, \, x \text{ nondecreasing}\}$$

Intuition suggests that the demand problem may be restricted to the class of nonincreasing function of the price. Since F_ψ^{-1} is strictly increasing, any nonincreasing function of the price is of the form $x(1 - F_\psi(\psi))$ with $x \in \mathcal{A}$. Indeed if $X = f(\psi)$ with f nonincreasing, then $F_X^{-1}(t) := x(t) = f(F_\psi^{-1}(1-t))$. As F_ψ^{-1} is strictly increasing, we have $f(u) = x(1 - F_\psi(u))$. Hence (7) may be restricted to claims of the form $X = x(1 - F_\psi(\psi))$ with $x \in \mathcal{A}$.

Let U be a uniformly distributed random variable on $(\Omega, \mathcal{B}, P)$ and let us define $v(x) := V(x \circ U)$ for $x \in \mathcal{A}$. Since V is law invariant, v does not depend on the choice of U. The following proposition that is proved in details in [4] shows that the demand problem may be brought down to a quantile problem. Defining $q(t) := F_\psi^{-1}(1-t)$, q is decreasing and nonnegative.

Proposition 4 *$\overline{X}$ is a solution of $(\mathcal{D})$ iff $\overline{X} = \overline{x}(1 - F_\psi(\psi))$ and $\overline{x}$ is a solution of :*

$$(\tilde{\mathcal{D}})\ \sup\{v(x) : x \in \mathcal{A},\ x \text{ bounded and } \int_0^1 q(t)x(t)dt \leq w\}$$

Equivalently, there exists $\lambda > 0$ such that $\overline{x}_\lambda$ is a solution of

$$\sup_{y \in \mathcal{A}} v(y) - \lambda \int_0^1 q(t)y(t)dt \tag{8}$$

and $\int_0^1 q(t)\overline{x}_\lambda(t)dt = w$.

In proposition 4, we only required V to be strictly SSD preserving and concave. From now on, we assume that $V(X) := \int_0^1 L(t, F_X^{-1}(t))dt + g(F_X^{-1}(0))$. We then have:

$$v(x) = \int_0^1 L(t, x(t))dt + g(x(0)) \tag{9}$$

The demand problem in the RLU case then amounts to

$$\sup\{\int_0^1 L(t, x(t))dt + g(x(0)) : x \in \mathcal{A} \cap L^\infty,\ \int_0^1 qx \leq w\} \tag{10}$$

Equivalently $\overline{x}$ solves (10) if and only if there exists $\lambda > 0$ such that $\overline{x}_\lambda$ solves:

$$\sup_{y \in \mathcal{A} \cap L^\infty} v_\lambda(y) := \int_0^1 L(t, y(t))dt + g(y(0)) - \lambda \int_0^1 qy. \tag{11}$$

and $\overline{x}_\lambda$ satisfies the budget constraint. For a fixed λ, Program (11) is a variational problem subject to a monotonicity constraint. Problems with a similar mathematical structure have been studied in the theory of incentives.

3.2 Characterization of solutions

Throughout this section, we assume the following:

- $L \in C^1([0,1] \times \mathbb{R}_+^*, \mathbb{R})$, $g \in C^1(\mathbb{R}_+^*, \mathbb{R})$,
- for every $t \in [0,1]$, $L(t, .)$ is strictly concave increasing on $\mathbb{R}_+^*$, g is strictly concave increasing on $\mathbb{R}_+^*$, and:

$$\lim_{x \to +\infty} \sup_{t \in [0,1]} \frac{\max(L(t,x), 0)}{x} = 0 \tag{12}$$

- there exists $q_0 > 0$ such that $q \geq q_0$ on $[0,1]$, q is continuous on $[0,1]$,
- defining:

$$\tilde{x}_\lambda(t) := \text{argmax}_{x \in \mathbb{R}_+}(L(t,x) - \lambda q(t)x).$$

  $\tilde{x}_\lambda$ is Lipschitz continuous on $[0,1]$,

- **either**:

$$\lim_{x \to 0^+} g'(x) = +\infty, \text{ or} \tag{13}$$

$$\lim_{(t,x)\to(0^+,0^+)} \partial_x L(t,x) = \lim_{\varepsilon\to 0^+} \int_0^\delta \partial_x L(t,\varepsilon)dt = +\infty, \ \forall \delta \in (0,1). \tag{14}$$

The first three assumptions ensure the existence and uniqueness of a continuous solution to (10) and (11) for every $\lambda > 0$. Strict concavity of $L(t,.)$ is necessary to obtain the continuity of the solution: Schied [19] obtains discontinuous solutions in the linear case. Under the fourth assumption, the solutions are Lipschitz continuous, hence differentiable a.e.. When $g = 0$, the last assumption simplifies to (14), an Inada condition weaker than the assumption $\partial_x L(t, 0^+) = +\infty$ for all t that might seem more natural. In subsection 5.3, we discuss the case $L(t,x) := \ln(t+x)$ which satisfies (14) and $\partial_x L(t, 0^+) = t^{-1} \in \mathbb{R}$, for every $t > 0$. In the RDU case, $L(t,x) = f'(1-t)U(x)$, and (14) simplifies to the usual Inada condition $U'(0^+) = +\infty$. Conditions (13) or (14) ensure that solutions to (10), and (11) remain positive.

Existence, uniqueness and characterization results for (10) and (11) for every $\lambda > 0$ are next given.

Proposition 5 *Under the previous assumptions, one has:*

- *(10) admits a unique solution $\overline{x}$, and (11) admits a unique solution $\overline{x}_\lambda$ for every $\lambda > 0$,*
- *$\overline{x}_\lambda$ is Lipschitz continuous, hence differentiable a.e., for every $\lambda > 0$,*
- *let $\lambda > 0$ and $x \in \mathcal{A} \cap L^\infty$ and let Λ be defined for every $t \in [0,1]$ by:*

$$\Lambda'(t) := \partial_x L(t, x(t)) - \lambda q(t) \text{ and } \Lambda(1) = 0 \tag{15}$$

  then $x = \overline{x}_\lambda$ if and only if x is differentiable a.e. and:

$$\begin{aligned} &(i) \quad \Lambda \geq 0, \text{ and } \Lambda(t)x'(t) = 0 \text{ a.e.,} \\ &(ii) \quad x(0) > 0 \text{ and } \Lambda(0) = g'(x(0)). \end{aligned}$$

A detailed proof of proposition 5 may be found in [4]. Condition (i) is the usual complementary slackness condition associated to the monotonicity constraint : it means that a.e. either $\Lambda(t) = 0$ or $\overline{x}_\lambda{}'(t) = 0$. Let us remark that when $t \in (0,1)$ and $\Lambda(t) = 0 = \min \Lambda$, then $\Lambda'(t) = 0$ which implies $\overline{x}_\lambda(t) = \tilde{x}_\lambda(t)$. The optimality condition (i), then implies the simpler condition:

$$\overline{x}_\lambda{}'(t) \neq 0 \Rightarrow \overline{x}_\lambda(t) = \tilde{x}_\lambda(t). \tag{16}$$

To solve the demand problem (10) in practice, we shall proceed in two steps : we shall first compute, for a given λ, the solution $\overline{x}_\lambda$ of (11) by using proposition 5, then we shall find λ such that the budget constraint is satisfied by $\overline{x}_\lambda$.

If $g = 0$, we deduce from proposition 5 that if $\tilde{x}_\lambda$ is decreasing, then $\overline{x}_\lambda$ coincides with $\tilde{x}_\lambda$ at, at most one point, hence optimality condition (i) implies that $\overline{x}_\lambda$ is constant. In the polar case where $\tilde{x}_\lambda$ is nondecreasing, then $\overline{x}_\lambda = \tilde{x}_\lambda$.

When $\tilde{x}_\lambda$ is not monotone, finding the solution is more intricate. It first follows from proposition 5, that either $\overline{x}_\lambda$ is flat or $\overline{x}_\lambda = \tilde{x}_\lambda$ (the latter case can only occur on intervals where $\tilde{x}_\lambda$ is itself nondecreasing). Furthermore, the function Λ defined by (15) must remain nonnegative and $\Lambda = 0$ on intervals on which $\overline{x}_\lambda = \tilde{x}_\lambda$. By concavity of $L(t, .)$, Λ is nondecreasing whenever the graph of $\overline{x}_\lambda$ is below that of $\tilde{x}_\lambda$ and nonincreasing otherwise.

4 Applications to the RDU demand problem

4.1 The case of a continuous distortion

The RDU demand problem, with a continuous distortion and fixed multiplier $\lambda > 0$, which corresponds to the case $L(t, x) = f'(1-t)U(x)$ reads as:

$$\sup_{x \in \mathcal{A}} v_\lambda(x) := \int_0^1 (f'(1-t)U(x(t)) - \lambda q(t)x(t))dt \tag{17}$$

Assuming U' is differentiable, decreasing with $U'(0^+) = +\infty$, $U'(\infty) = 0$, let $I\ :]0, \infty[\to]0, \infty[$ denote the inverse of U'. The function $\tilde{x}_\lambda$ that maximizes pointwise the integrand in (17) is given by:

$$\tilde{x}_\lambda(t) = I\left(\frac{\lambda q(t)}{f'(1-t)}\right) = I\left(\frac{\lambda F_\psi^{-1}(1-t)}{f'(1-t)}\right). \tag{18}$$

Note that $\tilde{x}_\lambda$ has the same monotonicity as the ratio $f'(1-t)/F_\psi^{-1}(1-t)$ (independently of λ). In general, this ratio is not monotone increasing, but when it is, necessarily $\overline{x}_\lambda = \tilde{x}_\lambda$. Conversely, since either $\overline{x}_\lambda{}'(t) = 0$ or $\overline{x}_\lambda(t) = \tilde{x}_\lambda(t)$, $\overline{x}_\lambda$ is strictly increasing (meaning that the demand function is strictly decreasing) if and only if F_ψ^{-1}/f' is increasing. As a direct consequence we have:

Proposition 6 *The demand of an RDU agent with a continuous distortion is decreasing in the price if and only if F_ψ^{-1}/f' is increasing. Hence, if F_ψ^{-1}/f' is not monotone increasing, there are ranges of values of the pricing density for which the demand is constant.*

The condition F_ψ^{-1}/f' increasing means that F_ψ^{-1} grows faster than f'. This condition is equivalent to the distortion index f''/f' being smaller than the growth rate $(F_\psi^{-1})'/F_\psi^{-1}$. Hence if the distortion effect is high, then the demand function has some flat zones.

In general, F_ψ^{-1}/f' is not monotone. We may remark that the following function

$$h_\lambda := \frac{U'(\overline{x}_\lambda)}{\lambda}$$

is independent of the multiplier λ. Indeed, the optimality conditions may be expressed in terms of the (positive nonincreasing) function h_λ. They may be written as:

$$\frac{\Lambda(t)}{\lambda} = \int_0^t (f'(1-s)h_\lambda(s) - q(s))ds. \tag{19}$$

Hence h_λ is characterized by the conditions:

$$\int_0^t (f'(1-s)h_\lambda - q) \geq 0, \ \forall t \in [0,1] \ , \ h_\lambda \text{ positive nonincreasing,} \tag{20}$$

$$h'_\lambda(t) \int_0^t (f'(1-s)h_\lambda - q) = 0 \text{ a.e. and } \int_0^1 (f'(1-s)h_\lambda - q) = 0. \tag{21}$$

Since these conditions do not depend on λ and define h_λ in a unique way by proposition 5, h_λ is independent of λ hence will simply be denoted h. Once h is determined, $\overline{x}_\lambda$ is given by $\overline{x}_\lambda = I(\lambda h)$. The multiplier $\lambda > 0$ is determined by the budget constraint:

$$\int_0^1 \overline{x}_\lambda(s)q(s)ds = \int_0^1 I(\lambda h(s))q(s)ds = w. \tag{22}$$

Denoting by λ^* the root of (22) (existence and uniqueness follows from the strict monotonicity of I and $I(0) = \infty$, $I(\infty) = 0$), the optimal $\overline{x}$ is given by $\overline{x} = I(\lambda^* h)$.

To sum up, the demand $\overline{X}$ can be computed as follows:

- determine h by the conditions (20) and (21),
- determine $\lambda = \lambda^*$ by solving (22) ,
- the demand (as a function of the price ψ) is then given by

$$\overline{X}(\psi) = I(\lambda^* H(\psi)) \text{ with : } H(\psi) := h(1 - F_\psi(\psi)). \tag{23}$$

In practice, only the determination of h may be complicated. Let us remark though that there is a partition of $(0,1)$ consisting of subintervals on which h is constant and subintervals on which $h(t) = F_\psi^{-1}(1-t)/f'(1-t)$. We also wish to emphasize that h does not depend on the utility index U (only λ^* does). If we consider the EU case ($f(t) = t$) as a benchmark, it is easy to interpret formula (23) as a deformation of the EU demand. Indeed, in the EU case, one has $h = q$ and the demand is given by $X(\psi) = I(\lambda^* \psi)$ for some $\lambda^* > 0$. Comparing the previous EU formula with (23) : $\overline{X}(\psi) = I(\lambda^* H(\psi)) = I(\lambda^* h(1 - F_\psi(\psi))$, we may view the function H (which typically exhibits flat zones) as a deformation of the identity map due to uncertainty and dispersion of prices. Interpreting $H(\psi)$ as perceived price, we see that the demand of an RDU agent is almost (the value of the multiplier need not be the same) the same as if she was EU facing the perceived price $H(\psi)$. Let us point out that in general the perceived price is an atomic random variable.

Let us finally remark that, for fixed utility index, price distribution and distortion, λ^* (which is determined almost as for a EU model) is a decreasing function of revenue w. Hence the demand depends on w in an increasing way.

If the distortion of small or large probabilities is high, then there is an interval of prices on which the demand is constant:

Proposition 7 *If $f'(0) = 0$, the demand is constant for low prices. If $f'(1) = \infty$, the demand is constant for high prices.*

Proof. Let us assume that $f'(0) = 0$. Using (19) and (21), one has:

$$\frac{\Lambda(t)}{\lambda} = \int_t^1 (q(s) - f'(1-s)h(s))ds$$

hence $\Lambda(t) > 0$ and $\overline{x}'(t) = 0$ for t close to 1. Thus, the demand is constant for low prices. If $f'(1) = \infty$, one obtains in a similar way $\Lambda(t) > 0$ for small $t :$ the demand is constant for high prices. $\square$

Before going further, let us compute the demand as a function of h for standard utility index:

Example A

Assume that $U(x) = \ln(x)$. Then (22) reads as $\frac{1}{\lambda^*}\int_0^1 \frac{q}{h}dt = w$, therefore

$$\overline{x}(t) = \frac{w}{h(t)\int_0^1 \frac{q}{h}} \quad \text{hence} \quad \overline{X}(\psi) = \left(\frac{w}{\int_0^1 \frac{q}{h}dt}\right)\frac{1}{h(1-F_\psi(\psi))}.$$

In the EU case, $\overline{x} = \frac{w}{q}$ and $\overline{X} = \frac{w}{\psi}$.

Example B

Assume that $U(x) = \frac{x^{1-\gamma}}{1-\gamma}$, $\gamma < 1$. Then (22) reads as $\frac{1}{(\lambda^*)^{1/\gamma}}\int_0^1 \frac{q}{h^{1/\gamma}}dt = w$, therefore

$$\overline{x} = \frac{w}{h^{1/\gamma}\int_0^1 \frac{q}{h^{1/\gamma}}} \quad \text{hence} \quad \overline{X}(\psi) = \left(\frac{w}{\int_0^1 \frac{q}{h^{1/\gamma}}dt}\right)\frac{1}{h(1-F_\psi(\psi))^{1/\gamma}}$$

while in the EU case, $\overline{x} = \frac{w}{q^{1/\gamma}}\frac{1}{\int_0^1 q^{1-1/\gamma}dt}$ and $\overline{X} = \frac{w}{E(\psi^{1-1/\gamma})}\frac{1}{\psi^{1/\gamma}}$.

4.2 RDU with discontinuous distortion

In this paragraph, we extend the previous analysis to the case of discontinuous distortions. The RDU demand problem, with a discontinuous distortion and fixed multiplier $\lambda > 0$ reads as:

$$\sup_{x\in\mathcal{A}} v_\lambda(x) := \varepsilon U(x(0)) + (1-\varepsilon)\int_0^1 (f'(1-t)U(x(t)) - \lambda q(t)x(t))dt \tag{24}$$

with f differentiable and $\varepsilon > 0$. Let us define:

$$\tilde{x}_\lambda(t) = I\left(\frac{\lambda q(t)}{f'(1-t)(1-\varepsilon)}\right) = I\left(\frac{\lambda F_\psi^{-1}(1-t)}{f'(1-t)(1-\varepsilon)}\right). \tag{25}$$

To solve the demand problem in the present case, we introduce as in the previous paragraph

$$h_{\varepsilon,\lambda} := \frac{U'(\overline{x}_\lambda)}{\lambda}.$$

The optimality conditions may be expressed in terms of the $h_{\varepsilon,\lambda}$. First define:

$$\frac{\Lambda(t)}{\lambda} = \varepsilon h_{\varepsilon,\lambda}(0) + (1-\varepsilon)\int_0^t (f'(1-s)h_{\varepsilon,\lambda} - q).$$

Then the optimality conditions read as:

$$\varepsilon h_{\varepsilon,\lambda}(0) + (1-\varepsilon)\int_0^t (f'(1-s)h_{\varepsilon,\lambda} - q) \geq 0,\ \forall t \in [0,1]\ , \tag{26}$$

$h_{\varepsilon,\lambda}$ positive nonincreasing and

$$h'_{\varepsilon,\lambda}(t)\left(\varepsilon h_{\varepsilon,\lambda}(0) + (1-\varepsilon)\int_0^t (f'(1-s)h_{\varepsilon,\lambda} - q) = 0\right) \tag{27}$$

$$\varepsilon h_{\varepsilon,\lambda}(0) + (1-\varepsilon)\int_0^1 (f'(1-s)h_{\varepsilon,\lambda} - q) = 0. \tag{28}$$

Since these conditions do not depend on λ and define $h_{\varepsilon,\lambda}$ in a unique way as previously, $h_{\varepsilon,\lambda}$ is independent of λ and will simply be denoted h_ε. As in the continuous case, the optimal solution $\overline{x}$ and the associated multiplier $\lambda^* > 0$ are determined by the budget constraint.

Since $\Lambda(t) > 0$ for small $t > 0$ (and this holds for any f' and F_ψ^{-1}), h_ε is constant and so is $\overline{x}$ for small values of t. In other words, for any distortion, utility index and distribution of prices when $\varepsilon > 0$, the demand always exhibits a flat zone for high prices:

Proposition 8 *The demand of an RDU agent with a discontinuous distortion is constant for high values of the pricing density. If furthermore $f'(0) = 0$, then the demand is also constant for low values of the pricing density.*

As in the continuous case, the optimal solution $\overline{x}$ and the associated multiplier $\lambda^* > 0$ are determined by the budget constraint. We therefore have the following expression of the demand:

$$\overline{X}(\psi) = I(\lambda^* H_\varepsilon(\psi)) \text{ with: } H_\varepsilon(\psi) = h_\varepsilon(1 - F_\psi(\psi)). \tag{29}$$

and $\lambda = \lambda^*$ is the root of : $\int_0^1 I(\lambda h_\varepsilon) q = w$.

When F_ψ^{-1}/f' is nonincreasing as in proposition 6 the demand is totally flat. In the polar case where F_ψ^{-1}/f' is increasing, it is no longer true that the demand is decreasing in the price:

Proposition 9 *If F_ψ^{-1}/f' is nonincreasing, the demand is constant. If F_ψ^{-1}/f' is increasing, then there exists $t_1 \in (0,1)$ such that:*

$$h_\varepsilon(t) = \min\left\{(1-\varepsilon)\frac{F_\psi^{-1}(1-t)}{f'(1-t)}, (1-\varepsilon)\frac{F_\psi^{-1}(1-t_1)}{f'(1-t_1)}\right\} \quad .$$

When F_ψ^{-1}/f' increasing, by formula (29), let us remark that the demand $\overline{X}$ is decreasing for prices smaller than $F_\psi^{-1}(1-t_1)$ and then constant for prices larger than $F_\psi^{-1}(1-t_1)$.

5 Examples

5.1 RDU with a continuous distortion

Example 1

To illustrate the method discussed in the previous section, let us consider an RDU example with $u(x) = \ln(x)$, $f(t) = t^2$ and $q(t) = F_\psi^{-1}(1-t) = 1 - 2t^2 + t^3$. Given $\lambda > 0$, we then have to solve first

$$\sup_{x \in \mathcal{A}} \int_0^1 2(1-t)\ln(x(t))dt - \lambda \int_0^1 (1 - 2t^2 + t^3)x(t)dt. \tag{30}$$

The function $\tilde{x}_\lambda$ that maximizes pointwise the integrand in (30) is given by:

$$\tilde{x}_\lambda(t) = \frac{2}{\lambda(1 + t - t^2)}$$

is symmetric with respect to 1/2 and attains a minimum at this point. Hence the solution $\overline{x}_\lambda$ of (30) cannot equal $\tilde{x}_\lambda$ on the whole of $[0, 1]$. We then know that $\overline{x}_\lambda$ has at least one flat zone. In this example, due to the shape of $\tilde{x}_\lambda$, we look for a solution with a single flat zone of the form $[0, t_0]$ with $t_0 \in [1/2, 1]$. In other words, we look for $\overline{x}_\lambda$ of the form $\overline{x}_\lambda = 1/(\lambda h)$ with:

$$h(t) = \begin{cases} (1 + t_0 - t_0^2)/2 & \text{if } t \in [0, t_0], \\ (1 + t - t^2)/2 & \text{if } t \in [t_0, 1]. \end{cases} \tag{31}$$

with $t_0 \in [1/2, 1]$ determined by the optimality condition:

$$\frac{\Lambda(t_0)}{\lambda} := h(t_0) \int_0^{t_0} f'(1-s)ds - \int_0^{t_0} q(s)ds = 0.$$

After elementary computations, this equation simplifies to $t_0^2(3t_0^2 - 10t_0 + 6) = 0$ which admits as unique root in $[1/2, 1]$ $t_0 = (5 - \sqrt{7})/3$. All the sufficient optimality conditions are satisfied by the function $\overline{x}_\lambda$ determined above, hence $\overline{x}_\lambda$ solves (30). To determine the demand, we finally determine λ such that $\overline{x}_\lambda$ satisfies the budget constraint.

It is possible to construct examples where the demand has an arbitrary number of constant parts. The demand in the next example has exactly two flat pieces : one for high values of the pricing density and one for low values of the pricing density. For other choices of the distortion f, we could generate examples of demands with an arbitrary large number of flat pieces.

Example 2

We now consider a case where the demand has exactly two flat pieces. In this example, we assume that $u(x) = \ln(x)$, $F_\psi^{-1}(t) = q(1-t) = e^t$ and that the distortion f is given by:

$$f(t) = \frac{9}{2} + e^t(-\frac{9}{2} + \frac{19}{2}t - \frac{9}{2}t^2 + t^3)$$

Given $\lambda > 0$, we first consider the problem

$$\sup_{x \in \mathcal{A}} \int_0^1 f'(1-t)\ln(x(t))dt - \lambda \int_0^1 e^{1-t}x(t)dt. \tag{32}$$

The function $\tilde{x}_\lambda$ that maximizes pointwise the integrand in (32) is given by:

$$\lambda \tilde{x}_\lambda(t) := \frac{f'(1-t)}{e^{1-t}} = 5 - t(t-1)(t-\frac{1}{2}) =: z_0(t).$$

Since z_0 is not nondecreasing, $\tilde{x}_\lambda$ cannot solve problem (32). However the shape of z_0 together with the optimality conditions of proposition 5 suggests to look for a solution $\overline{x}_\lambda$ of (32), with $\overline{x}_\lambda = 1/(\lambda h)$ where:

$$\frac{1}{h(t)} = \begin{cases} z_0(t_1) & \text{if } t \in [0, t_1], \\ z_0(t) & \text{if } t \in [t_1, t_2], \\ z_0(t_2) & \text{if } t \in [t_2, 1]. \end{cases} \tag{33}$$

Define for all $t \in [0, 1]$:

$$\frac{\Lambda(t)}{\lambda} = \int_0^t \left(f'(1-s)h(s) - e^{1-s} \right) ds.$$

From (21), $\Lambda(1) = 0$. If $\overline{x}_\lambda = 1/(\lambda h)$ is the solution, we must have $\Lambda(t_1) = \Lambda(t_2) = 0$ and $\Lambda = 0$ on $[t_1, t_2]$ for a pair $t_1 \in (0, 1/2)$ and $t_2 \in (1/2, 1)$. Hence

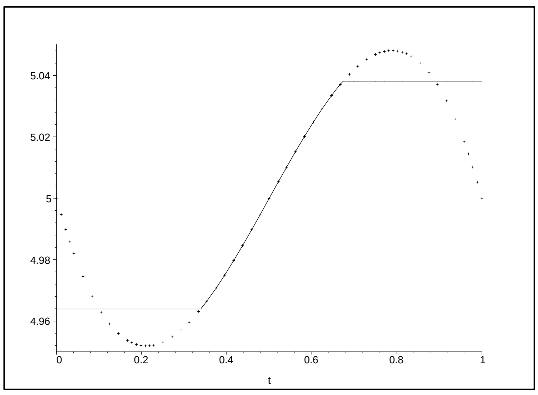
$$\frac{1}{z_0(t_1)}[f(1) - f(1-t_1)] = e - e^{1-t_1}.$$

$$\frac{f(1-t_2)}{z_0(t_2)} = e^{1-t_2} - 1$$

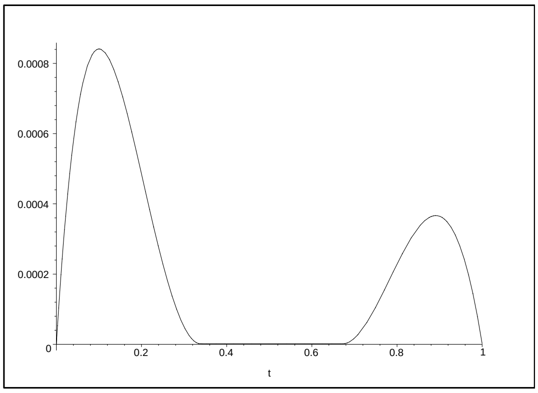
We obtain $t_1 \approx 0.339$, $t_2 \approx 0.672$. By construction, h defined by (33) is nonincreasing and it can be checked (see graph) that $\Lambda \geq 0$ on $[0, 1]$. The optimality conditions for (32) are thus satisfied. It finally remains to determine the multiplier λ by using the budget constraint. One then gets $\lambda = C/w$ for $C \approx 8.577$. Hence

$$\overline{x} = \frac{w}{Ch} \quad \text{equivalently} \quad \overline{X} = \frac{w}{Ch(1 - \ln \psi)}$$

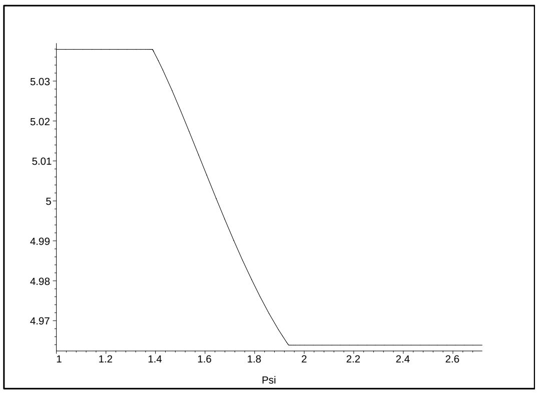
The next figure represents the graph of the unconstrained solution z_0 and that of the constrained solution $\lambda \overline{x}$.



Here follows the graph of Λ/λ.



Finally, the demand function is represented in the next figure



5.2 RDU with discontinuous distortions

As an example, we study the demand of an ε-contamined RDU with logarithmic utility index and a power distortion function in the case of uniformly distributed prices on $[1,2]$ (i.e. $F_\psi(t) = t-1$, $q(t) = 2-t$). We then have to study first for given $\lambda > 0$:

$$\sup_{x\in\mathcal{A}} v_\lambda(x) := \int_0^1 ((1-\varepsilon)\beta(1-t)^{\beta-1}\ln(x(t)) - \lambda(2-t)x(t))dt + \varepsilon\ln(x(0)). \quad (34)$$

Denoting by $\overline{x}_\lambda$ the solution of (34), we recall that $\overline{x}_\lambda = 1/(\lambda h_\varepsilon)$ for some nonincreasing function h_ε independent of λ (when $\varepsilon = 0$, we will simply write $h_0 = h$). Let us also recall that the demand is given by $\overline{X}(\psi) = \overline{x}(2-\psi) = Cw/h_\varepsilon(2-\psi)$ (see Example A of section 4.1) for some constant $C > 0$.

Our aim is to discuss the dependence of the demand with respect to the parameters $\varepsilon \in [0,1)$ and $\beta > 1$. The interpretation of those two parameters is the following: β is a measure of distortion and ε a measure of aversion to the worst case (or extreme cautiousness).

Let us define for every $t \in [0,1]$ and $\lambda > 0$

$$\frac{\Lambda(t)}{\lambda} = \varepsilon h_\varepsilon(0) + \int_0^t (1-\varepsilon)\beta(1-s)^{\beta-1} h_\varepsilon(s) ds - \left(2t - \frac{t^2}{2}\right) \tag{35}$$

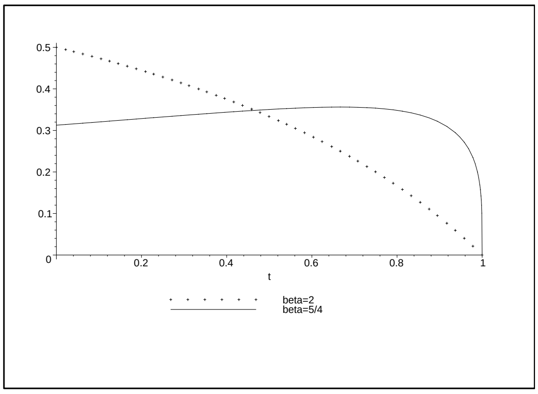
and

$$z_\varepsilon(t) := \lambda \tilde{x}_\lambda(t) = \frac{(1-\varepsilon)\beta(1-t)^{\beta-1}}{(2-t)}.$$

An easy computation shows that $\tilde{x}_\lambda$ is decreasing for every $\beta \geq 3/2$. Hence h_ε and $\overline{x}_\lambda$ are constant in that case. Since $\Lambda(1) = 0$, we obtain from (35) that $1/h_\varepsilon = \lambda \overline{x}_\lambda = 2/3$. Using the budget constraint, we then obtain

$$\overline{x} \equiv \frac{2w}{3},\ \lambda = \frac{1}{w}. \tag{36}$$

When $\beta \in (1, 3/2)$, we denote by t_{max} the point where $\tilde{x}_\lambda$ attains its maximum. The shape of $\tilde{x}$ is represented in the next figure



The next statement, proved in the appendix, characterizes the form of the demand according to the values of the parameters ε and β:

Proposition 10 *Let $\beta \in (1, 3/2)$, the demand is given by $\overline{X}(\psi) = \overline{x}(2-\psi)$ where:*

1. *if $\varepsilon = 0$, then either $\overline{x} \equiv \frac{2w}{3}$ or*

$$\overline{x}(t) := \begin{cases} Cwz_0(t) & \text{if } t \in [0, t_1], \\ Cwz_0(t_1) & \text{if } t \in [t_1, 1] \end{cases}$$

for some $t_1 \in [0, t_{max}]$ and some $C > 0$,

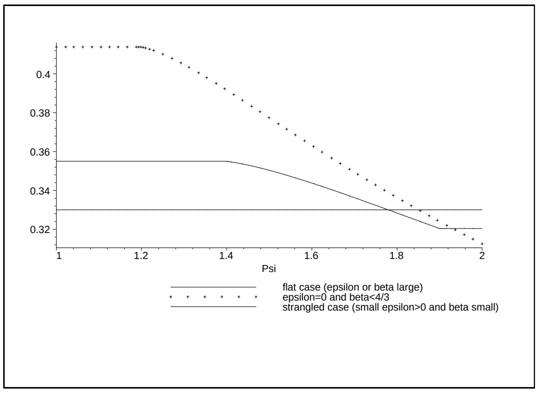
2. *if $\varepsilon = 0$, then $\overline{x} \equiv \frac{2w}{3}$ if and only if $\beta \geq 4/3$,*
3. *if $\varepsilon \in (0,1)$, then either $\overline{x} \equiv \frac{2w}{3}$ or*

$$\overline{x}(t) := \begin{cases} Cwz_\varepsilon(t_0) & \text{if } t \in [0, t_0], \\ Cwz_\varepsilon(t) & \text{if } t \in [t_0, t_1] \\ Cwz_\varepsilon(t_1) & \text{if } t \in [t_1, 1] \end{cases}$$

for some pair $0 \leq t_0 \leq t_1 \leq t_{max}$ and some $C > 0$, *4. if* $\varepsilon \in (0,1)$*, then* $\overline{x} \equiv \frac{2w}{3}$ *if and only if:*

$$\Phi(\varepsilon, \beta) := \max_{t \in [0,1]} \left\{ (1-\varepsilon)(1-t)^{\beta} + \frac{4}{3}t - \frac{t^2}{3} \right\} \leq 1. \tag{37}$$

The three possible shapes of the demand are represented in the next figure.



When $\varepsilon = 0$ and the distortion is high ($\beta \geq 4/3$ in our example), the demand is totally flat. For small distortion ($\beta < 4/3$), the demand is flat only for low values of the pricing density.

When $\varepsilon > 0$, there is an additional effect due to aversion to the worst case. For fixed $\varepsilon > 0$, since Φ is nonincreasing in both arguments, there exists $\beta(\varepsilon)$ such that the demand is constant if and only if $\beta \geq \beta(\varepsilon)$. Note that $\beta(0) = 4/3$ and $\beta(\varepsilon)$ is nonincreasing in ε. For the demand to be constant, it is enough that either ε or β is large. When both aversion to the worst state and ambiguity aversion are small (in the sense $\beta < \beta(\varepsilon)$), then the demand is flat only for low and for high values of the pricing density, in other words, the demand is *strangled*. In that case, it should also be noted that ε and β have quite different effects: ε forces the demand to be constant for high prices whereas β induces constant demand for low prices.

5.3 A class of RLU examples

In this example, we consider an RLU example which is not in the class of RDU's: the case where $L(t,x) = \ln(t+x)$ and as previously, prices are uniformly distributed on $[1,2]$, i.e. $q(t) = 2 - t$. Given an income $w > 0$ we then have to solve:

$$\sup_{x \in \mathcal{A}} v(x) := \int_0^1 \ln(t + x(t))dt \ : \ \int_0^1 qx \leq w. \tag{38}$$

As previously this problem admits a unique solution, whose determination amounts to find a multiplier $\lambda > 0$ such that x solves:

$$\sup_{x \in \mathcal{A}} v_{\lambda}(x) := \int_0^1 \ln(t + x(t))dt - \lambda \int_0^1 (2-t)x(t)dt \tag{39}$$

and such that the budget constraint is satisfied by x. For a given λ the function that maximizes v_λ subject to $x \geq 0$ is given by:

$$\tilde{x}_\lambda(t) := \left(\frac{1}{\lambda(2-t)} - t \right)_+ . \tag{40}$$

Contrary to the RDU case, where it is easy to discuss the monotonicity of $\tilde{x}$ independently of the multiplier λ, the situation is more complicated here because the variations of $\tilde{x}_\lambda$ depend on λ. The next proposition, proved in the appendix, characterizes the form of the solution $\overline{x}$ to (38) depending on the value of the income w:

Proposition 11 *The demand is given by $\overline{X}(\psi) = \overline{x}(2-\psi)$ where the solution $\overline{x}$ to (38) is:*

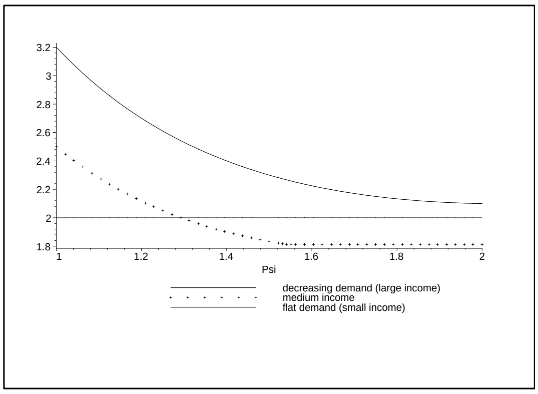
- *constant equal to $2w/3$ when $w \leq 3/(2e^{3/2}-2)$,*
- *increasing, equal to $\tilde{x}_\lambda$ with $\lambda = (w+2/3)^{-1}$ when $w \geq 10/3$,*
- *of the form:*

$$\overline{x}(t) = \begin{cases} \tilde{x}_\lambda(t_0) & \text{if } t \in [0, t_0], \\ \tilde{x}_\lambda(t) & \text{if } t \in [t_0, 1], \end{cases}$$

  for some $\lambda = \lambda(w) \in (1/4, 1)$ and some $t_0 = t(w)$ when $w \in (3/(2e^{3/2} - 2), 10/3)$.

We want to emphasize here an important qualitative difference between the RLU and RDU models. We have seen in the previous proposition that when the income is low, the demand is constant. As the income increases, the demand becomes constant only for high prices. Finally, if w is large, the demand is decreasing. In the RDU model, whether the demand is flat or not does not depend on the income. The RLU model therefore seems to allow richer income effects than the RDU one.

The three possible shapes of the demand are represented in the next figure.



Appendix

Proof of proposition 10

Let us recall that $\overline{x} = \overline{x}_{\lambda^*}$ for some value of the multiplier of the form $\lambda^* = 1/(Cw)$ and define $\tilde{x} = \tilde{x}_{\lambda^*} = Cwz_\varepsilon$.

Let us assume $\varepsilon = 0$. If $\overline{x}$ is given by (36), there is nothing to prove. Let us then assume that $\overline{x}$ is not constant. If $\overline{x}(0) < \tilde{x}(0)$, then $\overline{x}$ is constant on $[0,1]$, a contradiction. Because $\Lambda(0) = 0$ and $\Lambda' < 0$ whenever $\overline{x} > \tilde{x}$, necessarily, $\overline{x}$ and $\tilde{x}$ have to coincide on some (maximal) interval $[0, t_1]$ with $t_1 \leq t_{max}$. On $[t_1, 1]$, we have a.e. either $\tilde{x} = \overline{x}$ or $\overline{x}'(t) = 0$. The first case being impossible, $\overline{x}$ is constant on $[t_1, 1]$ which proves the first claim.

Let us prove now that the solution is constant if and only if $\beta \geq 4/3$. To prove this, let us first remark that $\beta \geq 4/3$ is equivalent to $\tilde{x}_{1/w}(0) \geq 2w/3$. Thus, if $\beta < 4/3$, the constant (36) is above the graph of $\tilde{x}_{1/w}$ for small values of t which implies $\Lambda(t) < 0$ for small $t > 0$. Hence (36) is not optimal in this case. If $\beta \geq 4/3$, then by construction the constant function given by (36) satisfies the budget constraint and all the optimality conditions of proposition 5 except possibly the nonnegativity of Λ that has to be justified. Since the constant (36) is less than $\tilde{x}_{1/w}(0)$, the equation $\tilde{x}_{1/w}(t) = 2w/3$ has a single root t_*. By construction one has, $\Lambda(0) = \Lambda(1) = 0$, Λ non decreasing on $[0, t_*]$ and nonincreasing on $[t_*, 0]$, hence Λ is everywhere nonnegative which proves the optimality of (36).

In the case $\varepsilon > 0$, we have $\Lambda(0) = \varepsilon/\overline{x}(0) > 0$. If $\overline{x}$ is constant, then it is necessarily given by (36). Assume that $\overline{x}$ is not constant, then for small $t > 0$, $\Lambda(t) > 0$, hence $\overline{x}(t) = \overline{x}(0)$ for t in some maximal interval $[0, t_0]$ with $t_0 < 1$. One necessarily has $t_0 \leq t_{max}$, since otherwise one would have $\overline{x} \neq \tilde{x}$ a.e. which would imply that $\overline{x}$ is constant. From (16), $\overline{x}$ has to coincide with $\tilde{x}$ on some maximal interval $[t_0, t_1]$ with $t_1 \leq t_{max}$. On $[t_1, t_{max}]$, $\overline{x} \neq \tilde{x}$, hence $\overline{x}$ is constant. Let t^* be the upperbound of the interval on which $\overline{x} = \overline{x}(t_1)$. If $t^* < 1$, then one should have $\overline{x}(t) = \tilde{x}(t)$ for $t \geq t^*$ close to t^* which is impossible since $\tilde{x}$ is decreasing on $[t_{max}, 1]$. This proves that $t^* = 1$, hence that $\overline{x}$ takes the form:

$$\overline{x}(t) := \begin{cases} \tilde{x}(t_0) & \text{if } t \in [0, t_0], \\ \tilde{x}(t) & \text{if } t \in [t_0, t_1] \\ \tilde{x}(t_1) & \text{if } t \in [t_1, 1]. \end{cases}$$

Since $\tilde{x} = Cwz_\varepsilon$, this proves assertion 3.

Finally, plugging the expression of the constant candidate solution (36) in the expression of Λ given by (35), we get

$$\Lambda(t) = \frac{3}{2w}\left(1 - (1-\varepsilon)(1-t)^\beta - \frac{4}{3}t + \frac{t^2}{3}\right)$$

Hence condition (37) exactly means that $\Lambda \geq 0$, which is equivalent to the fact that the constant given by (36) is optimal.

Proof of proposition 11

For fixed λ, let $\overline{x}_\lambda$ denote the solution to (39). Elementary computations show that:

- **first case:** if $\lambda \leq 1/4$, then $\tilde{x}_\lambda$ is increasing, hence $\tilde{x}_\lambda = \overline{x}_\lambda$,
- **second case:** if $\lambda \geq 1$, then $\tilde{x}_\lambda$ is nonincreasing, hence $\overline{x}_\lambda \equiv c_\lambda$, c_λ a constant. By the optimality conditions, $c_\lambda = (e^{3\lambda/2} - 1)^{-1}$,
- **third case:** if $\lambda \in (1/4, 1)$, then $\tilde{x}_\lambda$ is decreasing on $[0, t_\lambda]$ and increasing on $[t_\lambda, 1]$ with $t_\lambda = 2 - (\lambda)^{-1/2}$. In that case:

$$\overline{x}_\lambda(t) = \begin{cases} \tilde{x}_\lambda(t^*_\lambda) & \text{if } t \in [0, t^*_\lambda], \\ \tilde{x}_\lambda(t) & \text{if } t \in [t^*_\lambda, 1], \end{cases}$$

  for some $t^*_\lambda \in (t_\lambda, 1)$.

We know that there is $\lambda > 0$ such that $\overline{x} = \overline{x}_\lambda$ and $\int_0^1 (2-t)\overline{x}(t)dt = w$.

If $\lambda \leq 1/4$, then the budget constraint $\int_0^1 (2-t)\tilde{x}_\lambda(t)dt = w$ yields $w = 1/\lambda - 2/3$. Hence, we obtain $w \geq 10/3$. Conversely, if $w \geq 10/3$, defining $\lambda = (w + 2/3)^{-1}$, then $\tilde{x}_\lambda$ solves (39) and satisfies the budget constraint so that $\overline{x} = \tilde{x}_\lambda$.

If $\lambda \geq 1$, then $c_\lambda = 2w/3$ so that $w = 3/(2e^{3\lambda/2} - 2) \leq 3/(2e^{3/2} - 2)$. Conversely if $w \leq 3/(2e^{3/2} - 2)$, then the constant $2w/3$ solves (39) for $\lambda = 2\ln(1 + 2w/3)/3$, hence $\overline{x} \equiv 2w/3$.

The only remaining case is $w \in (3/(2e^{3/2} - 2), 10/3)$. In that case, $\lambda = \lambda(w)$ necessarily belongs to $(1/4, 1)$, hence $\overline{x}$ is as in the claim. The values of $\lambda(w)$ and $t(w)$ are (in theory) determined by the budget constraint and the optimality conditions.

References

[1] P. Bank, F. Riedel, Non-Time Additive Utility Optimization - the Case of Certainty, Journal of Mathematical Economics **33** (2000) 271-290.

[2] G. Carlier, R.-A. Dana, Core of a convex distortion of a probability, Journal of Economic Theory, **113** (2003) 199-222.

[3] G. Carlier, R.-A. Dana, Rearrangement inequalities in non convex insurance models, Journal of Mathematical Economics, **41** (2005) 483-503.

[4] G. Carlier, R.-A. Dana, Law invariant concave utility functions and optimization problems with monotonicity and comonotonicity constraints, to appear in Statistics and Decisions (2006).

[5] G. Carlier, R.-A. Dana, ?????????? (2006).

[6] S.H. Chew, L. Epstein, A Unifying Approach to Axiomatic Non-Expected Utility Theories, Journal of Economic Theory **49** (1989) 207-240.

[7] S.H. Chew, E. Karni, E. Safra, Risk aversion in the theory of expected utility with Rank dependent probabilities, Journal of Economic Theory **42** (1987) 370-381.

[8] S.H. Chew, P. Wakker, The Comonotonic Sure-Thing Principle, Journal of Risk and Uncertainty **12** (1996) 5-27.

[9] R.-A. Dana, A representation result for concave Schur concave functions, Mathematical Finance, **15** (2005) 613-634.

[10] D. Denneberg, Non-additive Measures and Integral, Kluwer Academic Publishers, Holland, 1994.

[11] H. Föllmer, A. Schied, Stochastic finance. An introduction in discrete time, De Gruyter editor, Berlin, (2004).

[12] J. Green, B. Jullien, Ordinal Independence in Non-Linear Utility Theory, Journal of Risk and Uncertainty **1** (1988) 355-387 .

[13] R. Guesnerie, J.-J. Laffont, A complete solution to a class of principal-agent problems with an application to the control of a self-managed firm, Journal of Public Economics **25** (1984) 329-369.

[14] E. Jouini, W. Schachermayer, N. Touzi, Optimal risk sharing for law invariant monetary utility functions, working paper, available at `http://www.crest.fr/pageperso/lfa/touzi/touzi.htm`.

[15] M. Landsberger, I. Meilijson, Comonotone allocations, Bickel-Lehmann dispersion and the Arrow-Pratt measure of risk aversion, Annals of Operations Research, **52** (1994) 97-106.

[16] J. Mirrlees, Optimal tax theory : a synthesis, Journal of Public Economics **6** (1976) 327-358.

[17] M. Mussa, S. Rosen, Monopoly and Product Quality, Journal of Economic Theory, **18** (1978) 301-317.

[18] J.-C. Rochet. Sur quelques problèmes de Calcul des Variations de l' Economie Mathématique, Annals of the CEREMADE, Birkhauser Ed. (1989).

[19] A. Schied, On the Neyman-Pearson problem for law- invariant risk measures and robust utility functionals, Annals of Applied Probability, **14** (2004) 1398-1423.

[20] M. Spence, Competitive and optimal responses to signals, Journal of Economic Theory **7** (1974) 296-33.

[21] T. Wang, L_p-Fréchet Differentiable Preference and local utility Analysis, Journal of Economic Theory **61** (1993) 139-159.