Approximation of variational problems with a convexity constraint by PDEs of Abreu type

Guillaume Carlier* Teresa Radice †

April 30, 2018

Abstract

Motivated by some variational problems subject to a convexity constraint, we consider an approximation using the logarithm of the Hessian determinant as a barrier for the constraint. We show that the minimizer of this penalization can be approached by solving a second boundary value problem for Abreu's equation which is a well-posed nonlinear fourth-order elliptic problem. More interestingly, a similar approximation result holds for the initial constrained variational problem.

Keywords: Abreu equation, Monge-Ampère operator, calculus of variations with a convexity constraint.

MS Classification: 35G30, 49K30.

1 Introduction

Given Ω, a bounded, open, convex subset of $\mathbb{R}^d$ with $d \geq 2$, $F : \Omega \times \mathbb{R} \to \mathbb{R}$ strictly convex in its second argument, and φ a uniformly convex and smooth function defined in a neighbourhood of Ω, we are interested in the variational problem with a convexity constraint:

$$\inf_{u \in \overline{S}[\varphi,\Omega]} \mathcal{J}_0(u) := \int_\Omega F(x, u(x))\mathrm{d}x \tag{1.1}$$

*CEREMADE, UMR CNRS 7534, Université Paris IX Dauphine, Pl. de Lattre de Tassigny, 75775 Paris Cedex 16, FRANCE and INRIA-Paris, MOKAPLAN, carlier@ceremade.dauphine.fr

†Department of Mathematics R. Caccioppoli, University of Naples Federico II, Via Cintia–80126, Napoli, ITALY, teresa.radice@unina.it

where $\overline{S}[\varphi, \Omega]$ consists of all convex functions on Ω which admit a convex extension by φ in a neighbourhood of Ω. This is a way to express in some weak sense the boundary conditions

$$u = \varphi \text{ and } \partial_\nu u \leq \partial_\nu \varphi \text{ on } \partial\Omega, \tag{1.2}$$

where ν denotes the outward normal to $\partial\Omega$ and ∂_ν denotes the normal derivative.

Due to the convexity constraint, it is really difficult to write a tractable Euler-Lagrange equation for (1.1) (see [7], [2]). One may therefore wish to construct suitable penalizations for the convexity constraint which force the minimizers to somehow remain in the interior of the constraint and thus to be a critical point of the penalized functional. Since the seminal work of Trudinger and Wang [9, 10] on the prescribed affine mean curvature equation, the regularity of convex solutions of fourth-order nonlinear PDEs which are Euler-Lagrange equations of convex functionals involving the Hessian determinant have received a lot of attention. In particular, the Abreu equation which corresponds to the logarithm of the Hessian determinant has been studied by Zhou [11] in dimension 2 and more recently by Chau and Weinkove [3] and Le [5, 6] in higher dimensions. What the well-posedness and regularity results of these references in particular suggest is that a penalization involving the logarithm of the Hessian determinant should act as a good barrier for the convexity constraint in problems like (1.1). This was indeed confirmed numerically at a discretized level, see [1].

Our goal is precisely to show that one can indeed approximate (1.1) by a suitable boundary value problem for the Abreu equation. To do so, we first introduce a penalized version of (1.1) with a small parameter $\varepsilon > 0$:

$$\inf_{v \in \overline{S}[\varphi, \Omega]} \mathcal{J}_\varepsilon(v) := \mathcal{J}_0(v) - \varepsilon \mathfrak{F}_\Omega(v) \tag{1.3}$$

where, when $v \in \overline{S}[\varphi, \Omega]$ is smooth and strongly convex, (see section 2 for the definition for an arbitrary $v \in \overline{S}[\varphi, \Omega]$), $\mathfrak{F}_\Omega(v)$ is defined by

$$\mathfrak{F}_\Omega(v) := \int_\Omega \log(\det D^2 v).$$

Using the convexity of $\mathcal{J}_\varepsilon$ setting $f(x, u) := \partial_u F(x, u)$, one can easily see that if u is smooth and uniformly convex up to $\partial\Omega$, and solves the first-boundary problem for Abreu equation

$$\varepsilon U^{ij} w_{ij} = f(x, u) \text{ in } \Omega, \; u = \varphi \; \text{ and } \partial_\nu u = \partial_\nu \varphi \text{ on } \partial\Omega \tag{1.4}$$

where $w := \det(D^2u)^{-1}$ and U denotes the cofactor matrix of D^2u then it is indeed the solution of (1.3). It turns out however that the second-boundary value problem (where instead of prescribing both values of u and ∇u one rather prescribes u and $\det(D^2u)$ on $\partial\Omega$) is much more well-behaved, see [3, 5, 6] and it was indeed used as an approximation for the affine Plateau problem in [9]. We shall also consider an extra approximation parameter and a second-boundary value problem on a larger domain and show that it approximates correctly not only (1.3) but also the intial problem (1.1) as the parameter converges to zero.

The paper is organized as follows. Section 2 gives some preliminaries. In section 3, we show a Γ-convergence result for $\mathcal{J}_\varepsilon$. In section 4, we consider an approximation by a second boundary value problem on a ball B containing $\overline{\Omega}$, with a further penalization $\frac{1}{\delta}(u-\varphi)$ on $B\backslash\Omega$, for which we prove existence and uniqueness of a smooth solution. In section 5, we show that when $\delta \to 0$, we recover the minimizer of the problem from section 3. Finally, we also show full convergence of the second boundary value problem to the initial constrained variational problem (1.1) when $\delta = \delta_\varepsilon \to 0$ as $\varepsilon \to 0$, provided F satisfies a suitable uniform convexity assumption.

2 Preliminaries

In the sequel, Ω will be an open, bounded and convex subset of $\mathbb{R}^d$, $d \geq 2$. We are also given an open ball B containing $\overline{\Omega}$ and assume that the boundary datum φ satisfies for some $\lambda > 0$:

$$\varphi \in C^{3,1}(\overline{B}),\ \varphi = 0 \text{ on } \partial B,\ D^2\varphi \geq \lambda \,\mathrm{id} \text{ on } B. \tag{2.1}$$

We then define $\overline{S}[\varphi, \Omega]$ as the set of convex functions on Ω, which, once extended by φ on $B \setminus \Omega$, are convex on B. Note that elements of $\overline{S}[\varphi, \Omega]$ coincide with φ on $\partial\Omega$ and are Lipschitz continuous with Lipschitz constant at most $\|\nabla\varphi\|_{L^\infty(B)}$ so that $\overline{S}[\varphi, \Omega]$ is compact for the topology of uniform convergence.

Finally, we assume that the integrand F: $(x, u) \in \Omega \times \mathbb{R} \mapsto F(x, u)$ in the definition of $\mathcal{J}_0$ in (1.1) is measurable with respect to x, strictly convex and differentiable with respect to u and such that that $F(., 0) \in L^1(\Omega)$ and $f(x, u) := \partial_u F(x, u)$ satisfies $f(., u) \in L^\infty(\Omega)$ for every $u \in \mathbb{R}$. These assumptions in particular guarantee that the convex functional $\mathcal{J}_0$ is everywhere continuous and Gâteaux differentiable on $\overline{S}[\varphi, \Omega]$.

Following [9, 10, 11], let us recall how to define $\mathfrak{F}_\Omega(v)$ for an arbitrary convex function v on Ω, first recall that the subdifferential of v at $x \in \Omega$ is given by

$$\partial v(x) := \{p \in \mathbb{R}^d \ : \ v(y) - v(x) \geq p \cdot (y - x), \ \forall y \in \Omega\}.$$

The Monge-Ampère measure of v, denoted $\mu[v]$ is then defined by

$$\mu[v](A) := |\partial v(A)|$$

for every Borel subset A of Ω. From the seminal results of Alexandrov (see [4]), $\mu[v]$ is indeed a Radon measure and $v \mapsto \mu[v]$ is weakly continuous in the sense that whenever v_n are convex functions which locally uniformly converge to v then

$$\limsup_n \mu[v_n](F) \leq \mu[v](F), \ \forall \ F \subset \Omega, \text{ closed }.$$

Decomposing the Monge-Ampère measure into its absolutely continuous part and its singular part (with respect to the Lebesgue measure $\mathcal{L}^d$) as

$$\mu[v] = \mu_r[v] + \mu_s[v], \ \mu_r[v] \ll \mathcal{L}^d, \ \mu_s[v] \perp \mathcal{L}^d.$$

Thanks to Alexandrov's theorem, v is differentiable twice a.e., at such points of twice differentiability, we denote by $\partial^2 v$ its Hessian matrix, Trudinger and Wang proved in [9] that $\det(\partial^2 v)$ is the density of $\mu_r[v]$ with respect to $\mathcal{L}^d$, and following their approach, one can define the functional $\mathfrak{F}_\Omega$ by

$$\mathfrak{F}_\Omega(v) := \int_\Omega \log(\det \partial^2 v(x)) \mathrm{d}x, \ \forall v \in \overline{S}[\varphi, \Omega]. \tag{2.2}$$

It is well-known that $\mathfrak{F}_\Omega$ is a concave functional and we refer to [8, 9, 11] for a proof of the useful properties of $\mathfrak{F}_\Omega$ recalled below in Lemmas 2.1 and 2.2

Lemma 2.1. *The functional $v \in \overline{S}[\varphi, \Omega] \mapsto \mathfrak{F}_\Omega(v)$ defined in (2.2) is concave, upper semi-continuous for the topology of local uniform convergence and bounded from above on $\overline{S}[\varphi, \Omega]$ with the explicit bound (where c_d denotes the measure of the unit ball of $\mathbb{R}^d$)*

$$\mathfrak{F}_\Omega(v) \leq C_{\Omega,\varphi} := |\Omega| \log\Big(\frac{c_d \|\nabla \varphi\|^d_{L^\infty}}{|\Omega|}\Big), \ \forall v \in \overline{S}[\varphi, \Omega]. \tag{2.3}$$

As we shall also work on the larger domain B, it will be also convenient to consider for every open subset ω of B and every convex function u on B the concave functional

$$\mathfrak{F}_\omega(v) := \int_\omega \log(\det \partial^2 v(x)) \mathrm{d}x. \tag{2.4}$$

Following the same lines as Lemma 6.4 in Trudinger-Wang [8], we also have:

Lemma 2.2. *If ω is an open subset of B with $\omega \subset\subset B$, then for every sequence of convex functions u_n converging locally uniformly on B to u, one has*

$$\limsup_n \mathfrak{F}_\omega(u_n) \leq \mathfrak{F}_\omega(u).$$

3 Logarithmic penalization

Given $\varepsilon > 0$, we consider

$$\inf_{v \in \overline{S}[\varphi,\Omega]} \mathcal{J}_\varepsilon(v) := \mathcal{J}_0(v) - \varepsilon \mathfrak{F}_\Omega(v). \tag{3.1}$$

Since $\mathcal{J}_\varepsilon$ is strictly convex and lsc on the convex compact set $\overline{S}[\varphi,\Omega]$, we immediately have:

Proposition 3.1. *Problem (3.1) admits a unique solution v_ε.*

Arguing exactly as in [9, 11] by using Alexandrov's maximum principle, one can show:

Lemma 3.2. *Let $\varepsilon > 0$ and v_ε be the solution of (3.1) then $\mu_s[v_\varepsilon] = 0$ i.e. $\mu[v_\varepsilon]$ has no singular part.*

Remark 3.3. Let us remark that Lemma 3.2 enables one to express $-\mathfrak{F}_\Omega(v_\varepsilon)$ in an alternative way as the entropy of the push-forward of the Lebesgue measure on Ω by ∇v_ε. Also, thanks to Lemma 3.2, one can prove uniqueness of the solution of (3.1) when $\mathcal{J}_0$ is convex but not necessarily strictly convex.

In dimension 2, we actually even have a uniform local bound on $\det(\partial^2 v_\varepsilon)$:

Proposition 3.4. *Let $d = 2$, $\varepsilon > 0$ and v_ε be the solution of (3.1), then $\mu[v_\varepsilon] = \det(\partial^2 v_\varepsilon) \in L^\infty_{\mathrm{loc}}(\Omega)$.*

Proof. It follows from Theorem 5.1 and Proposition 4.3 that v_ε is the uniform limit as $\delta \to 0$ of a sequence of smooth functions (v_ε^δ) in $\overline{S}[\varphi,\Omega]$ such that, for every open subset ω with $\omega \subset\subset \Omega$, $\| \det(D^2 v_\varepsilon^\delta)\|_{L^\infty(\omega)} \leq C$ where C is a constant that depends on ε and ω but not on δ . By weak convergence of Monge-Ampère measures we deduce that $\det(\partial^2 v_\varepsilon) \in L^\infty_{\mathrm{loc}}(\Omega)$.

□

Let us now state a Γ-convergence result for $\mathcal{J}_\varepsilon$:

Proposition 3.5. *The family of functionals $\mathcal{J}_\varepsilon$ defined on $\overline{S}[\varphi,\Omega]$ equipped with the topology of uniform convergence Γ-converges to $\mathcal{J}_0$ in particular v_ε converges uniformly to the solution of (1.1).*

Proof. Assume u_ε is a family in $\overline{S}[\varphi,\Omega]$ that converges uniformly as $\varepsilon \to 0$ to u, thanks to (2.3) and Fatou's Lemma, we have

$$\liminf_{\varepsilon} \mathcal{J}_\varepsilon(u_\varepsilon) \geq \liminf_{\varepsilon}(\mathcal{J}_0(u_\varepsilon) - \varepsilon C_{\Omega,\varphi}) \geq \mathcal{J}_0(u).$$

Given $u \in \overline{S}[\varphi,\Omega]$, we now look for a recovery sequence $u_\varepsilon \in \overline{S}[\varphi,\Omega]$ converging to u and such that $\limsup_\varepsilon \mathcal{J}_\varepsilon(u_\varepsilon) \leq \mathcal{J}_0(u)$, we simply take

$$u_\varepsilon := (1-\varepsilon)u + \varepsilon\varphi$$

since $\partial^2 u_\varepsilon \geq \varepsilon D^2\varphi$ we have

$$\mathfrak{F}_\Omega(u_\varepsilon) \geq d|\Omega|\log(\varepsilon) + \int_\Omega \log(\det(D^2\varphi))$$

with the convexity of $\mathcal{J}_0$, we then have

$$\limsup_{\varepsilon} \mathcal{J}_\varepsilon(u_\varepsilon) \leq \limsup_{\varepsilon}((1-\varepsilon)\mathcal{J}_0(u) + \varepsilon\mathcal{J}_0(\varphi)) + O(\varepsilon\log(\varepsilon))) \leq \mathcal{J}_0(u).$$

□

4 Second boundary value approximation

Having Proposition 3.5 in mind, we now fix the value of ε. Throughout this section, to simplify notations, we therefore take $\varepsilon = 1$ and we are interested in approximating the solution of

$$\inf_{v\in\overline{S}[\varphi,\Omega]} \mathcal{J}_1(v) := \int_\Omega F(x,v(x))\mathrm{d}x - \mathfrak{F}_\Omega(v), \tag{4.1}$$

by a second-boundary value problem for Abreu equation. More precisely given $\delta > 0$, we consider

$$U^{ij}w_{ij} = f_\delta(x,u), \quad \text{in } B,\ u=\varphi,\ w=\psi \text{ on } \partial B \tag{4.2}$$

where $\psi := \det((D^2\varphi)^{-1})$ and

$$f_\delta(x,u) := \begin{cases} f(x,u) \text{ if } x\in\Omega \\ \frac{1}{\delta}(u-\varphi(x)) \text{ if } x\in B\setminus\Omega \end{cases}$$

and as before $w = \det(D^2u)^{-1}$ and U is the cofactor matrix of D^2u. In view of (4.2) and the definition of f_δ, it is natural to introduce the functional defined over convex functions on B by

$$\mathcal{J}_1^\delta(v) := \int_\Omega F(x, v(x))\mathrm{d}x + \frac{1}{2\delta}\int_{B\setminus\Omega}(v-\varphi)^2 - \mathfrak{F}_B(v)$$

where

$$\mathfrak{F}_B(v) := \int_B \log(\det(\partial^2 v))$$

so that

$$\mathcal{J}_1^\delta(v) = \mathcal{J}_1(v) + \frac{1}{2\delta}\int_{B\setminus\Omega}(v-\varphi)^2 - \int_{B\setminus\Omega}\log(\det(\partial^2 v)). \tag{4.3}$$

4.1 A priori estimates for the second boundary value problem

Following a similar convexity argument as in Lemma 2.2 in Chau and Weinkove [3], we first have

Proposition 4.1. *Let u be a smooth and uniformly convex solution of (4.2), then*

$$\max_B |u| + \int_{\partial B}|\partial_\nu u|^d + \frac{1}{\delta}\int_{B\setminus\Omega}|u-\varphi|^2 \le C \tag{4.4}$$

for some constant C only depending on B, $\|\varphi\|_{C^{3,1}(\overline{B})}$ and the constant λ in (2.1).

Proof. First observe that by convexity and (2.1), $u < 0$ in B and $\partial_\nu u > 0$ on ∂B. Define $\widetilde{u} := \varphi$, $\widetilde{U}$ as the cofactor matrix of $D^2\varphi$, $\widetilde{w} := \det(D^2\varphi)^{-1}$ and $\widetilde{f} := \widetilde{U}^{ij}\widetilde{w}_{ij}$ (whose L^∞ norm only depends on $\|\varphi\|_{C^{3,1}(\overline{B})}$ and the constant λ in (2.1)) we have by the concavity of $\mathfrak{F}_B$, (4.2) and the monotonicity of

$f(x,.)$:

$$
\begin{aligned}
0 &\geq (\mathfrak{F}'_B(u) - \mathfrak{F}'_B(\widetilde{u}))(u-\widetilde{u}) \\
&= \int_B (U^{ij}w_{ij} - \widetilde{U}^{ij}\widetilde{w}_{ij})(u-\varphi) + \int_{\partial B} \psi(U^{ij}-\widetilde{U}^{ij})\partial_i(u-\varphi)\nu_j \\
&= \int_\Omega f(x,u)(u-\varphi) - \int_B \widetilde{f}(u-\varphi) + \frac{1}{\delta}\int_{B\setminus\Omega}(u-\varphi)^2 \\
&+ \int_{\partial B}\psi(U^{\nu\nu}-\widetilde{U}^{\nu\nu})\partial_\nu(u-\varphi) \\
&\geq \int_\Omega f(x,\varphi)(u-\varphi) - \int_B \widetilde{f}(u-\varphi) + \frac{1}{\delta}\int_{B\setminus\Omega}(u-\varphi)^2 \\
&+ \int_{\partial B}\psi(U^{\nu\nu}-\widetilde{U}^{\nu\nu})\partial_\nu(u-\varphi)
\end{aligned}
$$

where, in the last line, we have used the fact that $\nabla u - \nabla\varphi = \partial_\nu(u-\varphi)\nu$ on ∂B and set $U^{\nu\nu} = U\nu\cdot\nu$, $\widetilde{U}^{\nu\nu} = \widetilde{U}\nu\cdot\nu$. Using the fact that $f(x,\varphi)$, $\widetilde{f}$, φ, $\nabla\varphi$ and $\widetilde{U}$ are bounded, we thus get

$$
\frac{1}{\delta}\int_{B\setminus\Omega}(u-\varphi)^2 + \int_{\partial B}\psi U^{\nu\nu}\partial_\nu u \leq C\Big(1 + \int_B |u| + \int_{\partial B}\partial_\nu u + \int_{\partial B} U^{\nu\nu}\Big). \tag{4.5}
$$

Denoting by R the radius of B and by the same argument as in Lemma 2.2 in [3], one has

$$
U^{\nu\nu} = \frac{1}{R^{d-1}}\partial_\nu u^{d-1} + E \text{ with } |E| \leq C(1+\partial_\nu u^{d-2}) \text{ on } \partial B. \tag{4.6}
$$

Moreover since u is convex and $u=\varphi=0$ on ∂B, one has

$$
\max_B |u| = -\min_B u \leq 2R\partial_\nu u(x) \text{ for all } x\in\partial B. \tag{4.7}
$$

Putting together (4.5), (4.6), (4.7) and the fact that $\inf_{\partial B}\psi > 0$, we obtain

$$
\int_{\partial B}(\partial_\nu u)^d \leq C(1 + \int_{\partial B}(\partial_\nu u)^{d-1})
$$

which gives a bound on $\|\partial_\nu u\|_{L^d(\partial B)}$ hence also on $\max_B |u|$ by (4.7) so that finally the bound on $\delta^{-1}\int_{B\setminus\Omega}(u-\varphi)^2$ follows from the latter bounds and (4.5). ☐

4.2 Existence and uniqueness of a smooth uniformly convex solution

Thanks to Theorem 1.1 in [3], a Leray-Schauder degree argument and the a priori estimate (4.4), one easily deduces the following:

Theorem 4.2. *For every $\delta > 0$, the second boundary value problem (4.2) admits a unique uniformly convex solution which is $W^{4,p}(B)$ for every $p \in [1, +\infty)$.*

Proof. Let $D := \{u \in C(\overline{B}),\ \|u\|_{C(\overline{B})} \leq C+1\}$ where C is the constant from (4.4). For $t \in [0,1]$ and $u \in D$, it follows from Theorem 1.1 in [3] that there exists a unique $W^{4,p}$ for every $p \in [1, \infty)$ and uniformly convex solution of

$$V^{ij} w_{ij} = t f_\delta(x, u), w = \det(D^2 v)^{-1} \text{ in } B,\ \ v = \varphi, w = \psi \text{ on } \partial B \tag{4.8}$$

where V denotes the cofactor matrix of $D^2 v$. We denote by $v = T_t(u)$ the solution of (4.8). Moreover, by Theorem 2.1 of [3], for every $\alpha \in (0,1)$ there are a priori bounds on $\|v\|_{C^{3,\alpha}}$ and on $\sup_B(\det(D^2 v) + \det(D^2 v)^{-1})$ that only depend on C, α, δ, $\|\varphi\|_{C^{3,1}}$ and the constant λ in (2.1). Therefore $(t,u) \in [0,1] \times D \mapsto T_t(u)$ is continuous on $[0,1] \times D$ and T_t is compact in $C(\overline{B})$ for every $t \in [0,1]$. Since T_0 is constant and by (4.4) it has a unique fixed point in D, again by (4.4), T_t has no fixed point on ∂D, it thus follows from the Leray-Schauder Theorem that T_1 has a fixed point in D, this proves the existence claim for (4.2).

Finally, uniqueness follows from the same argument as in Lemma 7.1 from [10] where it is proven that two smooth solutions actually have the same gradient on ∂B and then are the minimizers of the same strictly convex minimization problem hence coincide.

□

In dimension $d = 2$, following the argument of Remark 4.2 of Trudinger and Wang [9] and taking advantage of the fact that the right-hand side of the Abreu equation (4.2) does not depend on δ on Ω, we have the following local bound (which we have used in the proof of Proposition 3.4):

Proposition 4.3. *Let $d = 2$ and u be the solution of (4.2) then for every open set $\omega \subset\subset \Omega$, $\|\det(D^2 u)\|_{L^\infty_{\text{loc}}(\omega)}$ is bounded independently of δ.*

Proof. Let $B_r := B_r(0) \subset\subset \Omega$, and observe that thanks to (4.4) both $\|f_\delta(., u(.))\|_{L^\infty(\Omega)} = \|f(., u(.))\|_{L^\infty(\Omega)}$ and $\|\nabla u\|_{L^\infty(\Omega)}$ are bounded independently of δ. Define then $\eta(x) := \frac{1}{2}(r^2 - |x|^2)$ and consider $z := \log(w) -$

$2\log(\eta) - \frac{1}{2}|\nabla u|^2$, by construction z achieves its minimum at an interior point x_0 of B_r at such a point, we have

$$\frac{\nabla w}{w} = 2\frac{\nabla \eta}{\eta} + D^2 u \nabla u. \tag{4.9}$$

We also have

$$z_{ij} = \frac{w_{ij}}{w} - \frac{w_i w_j}{w^2} + 2\frac{\delta_{ij}}{\eta} + 2\frac{\eta_i \eta_j}{\eta^2} - u_{ijk}u_k - u_{ik}u_{jk}, \tag{4.10}$$

multiplying by $wU = [D^2u]^{-1}$, using $wU^{ij}z_{ij} \geq 0$ at x_0, $U^{ij}w_{ij} = f(x,u) \leq C$ and the identities

$$wU^{ij}u_{ik}u_{jk} = u_{ii} = \Delta u, \ wU^{ij}u_{ijk}u_k = -\frac{w_k}{w}u_k = -\frac{\nabla w}{w} \cdot \nabla u, \tag{4.11}$$

(the second identity is classically obtained by first differentiating the relation $-\log(w) = \log(\det D^2 u)$ and then taking the scalar product with ∇u) as well as the fact that $\mathrm{Tr}(U) = \Delta u$ in dimension $d = 2$, we get

$$0 \leq C - wU\frac{\nabla w}{w} \cdot \frac{\nabla w}{w} + 2\frac{w}{\eta}\Delta u + 2wU\frac{\nabla \eta}{\eta} \cdot \frac{\nabla \eta}{\eta} + \frac{\nabla w}{w} \cdot \nabla u - \Delta u. \tag{4.12}$$

Using (4.9) and using again $wU = [D^2u]^{-1}$, we then obtain

$$wU\frac{\nabla w}{w} \cdot \frac{\nabla w}{w} = 4wU\frac{\nabla \eta}{\eta} \cdot \frac{\nabla \eta}{\eta} + D^2 u \nabla u \cdot \nabla u + 4\frac{\nabla \eta}{\eta} \cdot \nabla u \tag{4.13}$$

and

$$\frac{\nabla w}{w} \cdot \nabla u = 2\frac{\nabla \eta}{\eta} \cdot \nabla u + D^2 u \nabla u \cdot \nabla u. \tag{4.14}$$

Replacing (4.13), (4.14) in (4.12), multiplying by η and rearraging gives

$$\Delta u(\eta - 2w) \leq C\eta - 2\frac{w}{\eta}U\nabla\eta \cdot \nabla\eta - 2\nabla\eta \cdot \nabla u \leq C\eta + \|\nabla \eta\|_{L^\infty} \|\nabla u\|_{L^\infty(\Omega)} \leq C' \tag{4.15}$$

If $\eta(x_0) \geq 4w(x_0)$, (4.15) gives $\eta(x_0)\Delta u(x_0) \leq 2C'$ and since $\Delta u(x_0) w(x_0)^{1/2} \geq 2$ we get the desired lower bound on the minimum of $\eta^{-2}w$. In the remaining case $w(x_0) \geq \frac{1}{4}\eta(x_0) \geq \frac{\eta^2(x_0)}{2r^2}$ and we reach the same conclusion. This gives a local lower bound on w i.e. the desired local upper bound on $\det(D^2 u)$.

□

5 Convergence

5.1 Letting $\delta \to 0$ for fixed ε

In this paragraph, we fix ε (and thus normalize it to $\varepsilon = 1$ as we did in the whole of section 4).

Theorem 5.1. *Let u_δ be the unique smooth strictly convex solution of (4.2), then u_δ converges uniformly on Ω to the unique minimizer of (4.1) as $\delta \to 0^+$.*

Proof. We already know from (4.4) that (possibly up to an extraction) u_δ converges locally uniformly on B to some convex u and it also follows from (4.4) that $u \in \overline{S}[\varphi, \Omega]$. Let $v \in \overline{S}[\varphi, \Omega]$ (extended by φ on $B \setminus \Omega$), thanks to (4.2) and the convexity of $\mathcal{J}_1^\delta$ we first have

$$\mathcal{J}_1^\delta(v) - \mathcal{J}_1^\delta(u_\delta) \geq \int_{\partial B} U_\delta^{\nu\nu} \psi \partial_\nu (u_\delta - \varphi)$$

i.e.

$$\begin{aligned}\mathcal{J}_1(v) - \mathcal{J}_1(u_\delta) \geq \frac{1}{2\delta} \int_{B\setminus\Omega} (u_\delta - \varphi)^2 + \int_{B\setminus\Omega} (\log(\det(D^2\varphi)) - \log(\det(D^2 u_\delta)))\\ + \int_{\partial B} U_\delta^{\nu\nu} \psi \partial_\nu (u_\delta - \varphi)\\ \geq \int_{B\setminus\Omega} (\log(\det(D^2\varphi)) - \log(\det(D^2 u_\delta))) + \int_{\partial B} U_\delta^{\nu\nu} \psi \partial_\nu (u_\delta - \varphi).\end{aligned}$$

It follows from Lemma 5.2 below that

$$\liminf_{\delta \to 0} \int_{B\setminus\Omega} (\log(\det(D^2\varphi)) - \log(\det(D^2 u_\delta))) \geq 0.$$

We now have to pay attention to the boundary term, we know from (4.6) that $\theta_\delta := \psi U_\delta^{\nu\nu}$ satisfies $0 \leq \theta_\delta \leq C(1 + (\partial_\nu u_\delta)^{d-1})$ so that thanks to (4.4), θ_δ is bounded in $L^{\frac{d}{d-1}}(\partial B)$, up to an extraction we may therefore assume that it weakly converges in $L^{\frac{d}{d-1}}(\partial B)$ to some nonnegative function θ. By convexity we also have that for $\tau > 0$

$$\partial_\nu u_\delta(x) \geq D_{\tau,\nu} u_\delta(x) := \frac{1}{\tau}\Big(u_\delta(x - \tau\nu(x)) - u_\delta(x - 2\tau\nu(x))\Big), \ \forall x \in \partial B$$

For small fixed $\tau > 0$ note that $D_{\tau,\nu} u_\delta$ is bounded independently of δ thanks to (4.4) and that it converges as $\delta \to 0$ pointwise to $D_{\tau,\nu}\varphi$, we thus have

$$\begin{aligned}\liminf_{\delta\to 0} \int_{\partial B} \theta_\delta \partial_\nu (u_\delta - \varphi) \geq \liminf_{\delta \to 0} \int_{\partial B} \theta_\delta (D_{\tau,\nu} u_\delta - \partial_\nu \varphi)\\ = \int_{\partial B} \theta (D_{\tau,\nu}\varphi - \partial_\nu\varphi)\end{aligned}$$

where in the last line we have passed to the limit using the fact that we have the product of a weakly convergent sequence with a strongly convergent sequence. Now letting $\tau \to 0$ and using the smoothness of φ, we deduce that

$$\liminf_{\delta\to 0} \int_{\partial B} U_\delta^{\nu\nu} \psi \partial_\nu (u_\delta - \varphi) \geq 0.$$

Since $\mathcal{J}_1$ is lower semi-continuous thanks to Lemma 2.2, we can conclude that

$$\mathcal{J}_1(v) \geq \liminf_{\delta\to 0} \mathcal{J}_1(u_\delta) \geq \mathcal{J}_1(u)$$

so that u solves (4.1) and by uniqueness of the minimizer there is in fact convergence of the whole sequence.

□

In the previous proof we have used:

Lemma 5.2. *Let u_δ be the unique smooth strictly convex solution of (4.2) as before, then*

$$\limsup_{\delta\to 0} \int_{B\setminus\Omega} \log(\det(D^2 u_\delta)) \leq \int_{B\setminus\Omega} \log(\det(D^2\varphi)).$$

Proof. The key point here is the estimate $\int_B \Delta u_\delta = \int_{\partial B} \partial_\nu u_\delta \leq C$ which follows from (4.4). Let ω be an arbitrary Borel subset of B, we have (for some constant C varying from a line to another):

$$\begin{aligned}
\int_\omega \log(\det(D^2 u_\delta)) &\leq C(|\omega| + \int_\omega \det(D^2 u_\delta)^{1/2d}) \\
&\leq C\Big(|\omega| + \int_\omega \sqrt{\Delta u_\delta}\Big) \leq C\Big(|\omega| + |\omega|^{1/2}\Big(\int_B \Delta u_\delta\Big)^{1/2}\Big) \\
&= C\Big(|\omega| + |\omega|^{1/2}\Big(\int_{\partial B} \partial_\nu u_\delta\Big)^{1/2}\Big)
\end{aligned}$$

so that

$$\int_\omega \log(\det(D^2 u_\delta)) \leq C(|\omega| + |\omega|^{1/2}). \tag{5.1}$$

Take $0 < R' < R$ with Ω contained in $B_{R'}$ (recall R is the radius of B), we then have, thanks to Lemma 2.2, the fact that $\log(\det(D^2\varphi))$ is bounded and (5.1):

$$\begin{aligned}\limsup_{\delta\to 0}\mathfrak{F}_{B\setminus\Omega}(u_\delta) &= \limsup_{\delta\to 0}\mathfrak{F}_{B\setminus\overline{\Omega}}(u_\delta)\\ &\le \limsup_{\delta\to 0}\mathfrak{F}_{B_{R'}\setminus\overline{\Omega}}(u_\delta)+\limsup_{\delta\to 0}\mathfrak{F}_{B\setminus B_{R'}}(u_\delta)\\ &\le \mathfrak{F}_{B_{R'}\setminus\overline{\Omega}}(\varphi)+C(|B\setminus B_{R'}|+|B\setminus B_{R'}|^{1/2})\\ &\le \mathfrak{F}_{B\setminus\Omega}(\varphi)+C'(|B\setminus B_{R'}|+|B\setminus B_{R'}|^{1/2}).\end{aligned}$$

The desired result follows by letting R' tend to R. □

5.2 Full convergence

We now take $\delta = \delta_\varepsilon > 0$ with

$$\lim_{\varepsilon\to 0^+}\delta_\varepsilon = 0, \tag{5.2}$$

i.e. we only have a single small parameter ε and we consider the second-boundary value problem

$$\varepsilon U_\varepsilon^{ij} w_{ij}^\varepsilon = g_\varepsilon(x,u_\varepsilon), \quad \text{in } B,\ u_\varepsilon=\varphi,\ w^\varepsilon=\psi \text{ on } \partial B \tag{5.3}$$

where $\psi := \det((D^2\varphi)^{-1})$,

$$g_\varepsilon(x,u) := \begin{cases} f(x,u) \text{ if } x\in\Omega\\ \frac{1}{\delta_\varepsilon}(u-\varphi(x)) \text{ if } x\in B\setminus\Omega\end{cases},$$

$w^\varepsilon = \det(D^2u_\varepsilon)^{-1}$ and U_ε is the cofactor matrix of D^2u_ε. We further assume that there is an $\alpha>0$ such that

$$(f(x,u)-f(x,v))(u-v)\ge\alpha(u-v)^2,\ \forall(u,v)\in\mathbb{R}^d,\ \text{and a.e. } x\in\Omega \tag{5.4}$$

which amounts to say that the integrand F is uniformly convex in its second argument. Under these assumptions, we have a *full* convergence result:

Theorem 5.3. *Let u_ε be the unique smooth strictly convex solution of (5.3), then u_ε converges uniformly on Ω to the unique minimizer of (1.1) as $\varepsilon\to 0^+$.*

Proof. **Step 1: a priori estimates.** The first step of the proof is similar to the proof of Proposition 4.1. Again define $\widetilde{u} := \varphi$, $\widetilde{U}$ as the cofactor matrix of $D^2\varphi$, $\widetilde{w} := \det(D^2\varphi)^{-1}$ and $\widetilde{f}_\varepsilon := \varepsilon \widetilde{U}^{ij}\widetilde{w}_{ij}$. We then have together with (5.4):

$$\begin{aligned}
0 &\geq \varepsilon(\mathfrak{F}'_B(u_\varepsilon) - \mathfrak{F}'_B(\widetilde{u}))(u_\varepsilon - \widetilde{u}) \\
&\geq \int_\Omega (f(x,\varphi) - \widetilde{f}_\varepsilon)(u_\varepsilon - \varphi) + \alpha \int_\Omega (u_\varepsilon - \varphi)^2 + \frac{1}{\delta_\varepsilon}\int_{B\setminus\Omega} (u_\varepsilon - \varphi)^2 \\
&+ \varepsilon \int_{\partial B} \psi(U_\varepsilon^{\nu\nu} - \widetilde{U}^{\nu\nu})\partial_\nu(u_\varepsilon - \varphi)
\end{aligned}$$

thanks to the fact that $f(x,\varphi) - \widetilde{f}_\varepsilon$ is bounded uniformly with respect to ε, using Young's inequality and invoking (4.6), we get

$$\int_\Omega (u_\varepsilon - \varphi)^2 + \frac{1}{\delta_\varepsilon}\int_{B\setminus\Omega}(u_\varepsilon - \varphi)^2 + \varepsilon\int_{\partial B}(\partial_\nu u_\varepsilon)^d \leq C. \tag{5.5}$$

Step 2: convergence. Thanks to (5.5), up to taking a subsequence of vanishing ε_n, we may assume that u_ε converges locally uniformly in B to some u such that $u = \varphi$ in $B \setminus \Omega$ so that the restriction of u to Ω belongs to $\overline{S}[\varphi, \Omega]$. For every v convex on B such that $v = \varphi$ on ∂B, define

$$\widetilde{\mathcal{J}}_\varepsilon(v) := \int_\Omega F(x, v(x))\mathrm{d}x + \frac{1}{2\delta_\varepsilon}\int_{B\setminus\Omega}(v - \varphi)^2 - \varepsilon\int_B \log(\det(\partial^2 v)).$$

Let then $v \in \overline{S}[\varphi, \Omega]$ (extended by φ on $B \setminus \Omega$), we then have

$$\widetilde{\mathcal{J}}_\varepsilon(v) - \widetilde{\mathcal{J}}_\varepsilon(u_\varepsilon) \geq \varepsilon\int_{\partial B}\psi U_\varepsilon^{\nu\nu}\partial_\nu(u_\varepsilon - \varphi)$$

hence

$$\mathcal{J}_0(v) \geq \liminf_\varepsilon \mathcal{J}_0(u_\varepsilon) + \liminf_\varepsilon \varepsilon(\mathfrak{F}_B(v) - \mathfrak{F}_B(u_\varepsilon)) - \limsup_\varepsilon \varepsilon\int_{\partial B}\psi U_\varepsilon^{\nu\nu}\partial_\nu\varphi.$$

Arguing as in the proof of Proposition 3.5, we may actually assume that $\mathfrak{F}_\Omega(v) > -\infty$ so that $\liminf_\varepsilon \varepsilon\mathfrak{F}_B(v) \geq 0$. As for an upper bound for $\varepsilon\mathfrak{F}_B(u_\varepsilon)$ we use the fact that thanks to (5.5), we have $\int_{\partial B}\partial_\nu u_\varepsilon \leq C\varepsilon^{-1/d}$ and argue in a similar way as in the proof of Lemma 5.2, to obtain

$$\varepsilon\mathfrak{F}_B(u_\varepsilon) \leq C\varepsilon(1 + \int_B \det(D^2 u_\varepsilon)^{1/d}) \leq C\varepsilon(1 + \int_{\partial B}\partial_\nu u_\varepsilon) \leq C(\varepsilon + \varepsilon^{1-1/d}),$$

which yields

$$\liminf_\varepsilon \varepsilon(\mathfrak{F}_B(v) - \mathfrak{F}_B(u_\varepsilon)) \geq 0.$$

Thanks to (4.6), we have

$$\int_{\partial B} \psi U_\varepsilon^{\nu\nu} \partial_\nu \varphi \leq C \int_{\partial B} (1 + (\partial_\nu u_\varepsilon)^{d-1})$$

but, thanks to (5.5) and Hölder's inequality, we deduce

$$\varepsilon \int_{\partial B} (\partial_\nu u_\varepsilon)^{d-1} \leq C \varepsilon^{\frac{1}{d}}$$

so that

$$\mathcal{J}_0(v) \geq \liminf_{\varepsilon} \mathcal{J}_0(u_\varepsilon) = \mathcal{J}_0(u)$$

hence u solves (1.1) (and the whole family u_ε converges uniformly on Ω to u by uniqueness of the minimizer of $\mathcal{J}_0$ on $\overline{S}[\varphi, \Omega]$).

□

Acknowledgments: G.C. is grateful to the Agence Nationale de la Recherche for its support through the project MAGA (ANR-16-CE40-0014).

References

[1] Jean-David Benamou, Guillaume Carlier, Quentin Mérigot, and Édouard Oudet. Discretization of functionals involving the Monge-Ampère operator. *Numer. Math.*, 134(3):611–636, 2016.

[2] G. Carlier and T. Lachand-Robert. Representation of the polar cone of convex functions and applications. *J. Convex Anal.*, 15(3):535–546, 2008.

[3] Albert Chau and Ben Weinkove. Monge-Ampère functionals and the second boundary value problem. *Math. Res. Lett.*, 22(4):1005–1022, 2015.

[4] Cristian E. Gutiérrez. *The Monge-Ampère equation*, volume 89 of *Progress in Nonlinear Differential Equations and their Applications.* Birkhäuser/Springer, [Cham], 2016. Second edition [of MR1829162].

[5] Nam Q. Le. Global second derivative estimates for the second boundary value problem of the prescribed affine mean curvature and Abreu's equations. *Int. Math. Res. Not. IMRN*, (11):2421–2438, 2013.

[6] Nam Q. Le. $W^{4,p}$ solution to the second boundary value problem of the prescribed affine mean curvature and Abreu's equations. *J. Differential Equations*, 260(5):4285–4300, 2016.

[7] Pierre-Louis Lions. Identification du cône dual des fonctions convexes et applications. *C. R. Acad. Sci. Paris Sér. I Math.*, 326(12):1385–1390, 1998.

[8] Neil S. Trudinger and Xu-Jia Wang. The Bernstein problem for affine maximal hypersurfaces. *Invent. Math.*, 140(2):399–422, 2000.

[9] Neil S. Trudinger and Xu-Jia Wang. The affine Plateau problem. *J. Amer. Math. Soc.*, 18(2):253–289, 2005.

[10] Neil S. Trudinger and Xu-Jia Wang. Boundary regularity for the Monge-Ampère and affine maximal surface equations. *Ann. of Math. (2)*, 167(3):993–1028, 2008.

[11] Bin Zhou. The first boundary value problem for Abreu's equation. *Int. Math. Res. Not. IMRN*, (7):1439–1484, 2012.