Pareto optima and equilibria when preferences are incompletely known

G. Carlier [*], R.-A. Dana [†]

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Abstract

An exchange economy in which agents have convex incomplete preferences defined by families of concave utility functions is considered. Sufficient conditions for the set of efficient allocations and equilibria to coincide with the set of efficient allocations and equilibria that result when each agent has a utility in her family are provided. Welfare theorems in an incomplete preferences framework therefore hold under these conditions and efficient allocations and equilibria are characterized by first order conditions.

Keywords: incomplete preferences, efficient allocations and equilibria.

1 Introduction

Since the work of von Neumann and Morgenstern [25], a number of authors have raised objections about the use of the completeness axiom in utility theory (for early work, see for example Luce and Raiffa [15], Aumann[2], Shapley [24]). These objections have on one hand, lead to the development of axiomatic theories and multi-utility representations of incomplete preferences (see for example Aumann [2], Maccheroni et al [16], Ok [19]). Incomplete preferences have, on the other hand, been used in a wide variety of fields, risk

[*] CEREMADE, UMR CNRS 7534, Université Paris IX Dauphine, Pl. de Lattre de Tassigny, 75775 Paris Cedex 16, FRANCE carlier@ceremade.dauphine.fr

[†] CEREMADE, UMR CNRS 7534, Université Paris IX Dauphine, Pl. de Lattre de Tassigny, 75775 Paris Cedex 16, FRANCE dana@ceremade.dauphine.fr , G. Carlier and R.-A. Dana acknowledge the support of the Fondation du Risque, chaire Dauphine-ENSAE-Groupama, "Les particuliers face au risque" and of the ANR project Risque.

theory (Rothschild and Stiglitz [21]), finance (Dybvig [10], Jouini and Kallal [14]), social choice theory, cooperative game theory, asymmetric information and equilibrium theory (Mas-Colell [18], Gale and Mas-Colell [13]).

Recently there has been a renewed interest for incomplete preferences with the work of Bewley [4] and Rigotti and Shannon [22] as a way to model choices in the presence of ambiguity and to explain non participation in markets for which there is ambiguity and more particularly in financial markets (see Dana and Riedel [9] and Easley and O'hara [11]).

This paper considers an exchange economy populated by a finite number I of agents whose preferences are incompletely known. Their utilities may for example depend on unknown parameters or more generally belong to given families $\mathcal{U}_i, i \in I$ of utility functions. For each $\mathcal{U}_i$, the corresponding strict preference is defined by the unanimity rule: $X_i \in E$ is strictly preferred to $Y_i \in E$ (which will be denoted $X_i \succ_{\mathcal{U}_i} Y_i$) if $u_i(X_i) > u_i(Y_i)$ for every $u_i \in \mathcal{U}$. In contrast with the literature on representing incomplete preferences, the agents' multi-utilities $\mathcal{U}_i,\ i \in I$ are taken as primitive.

The purpose of the paper is to identify conditions under which the following two properties are satisfied:

- efficient allocations for the incomplete preferences coincide with the set of efficient allocations that result for some choice of utilities in the sets of each agent,
- equilibria for the incomplete preferences coincide with the set of equilibria that result for some choice of utilities in the sets of each agent.

Specific examples have already been considered in the literature (see for example Carlier et al [5], Dana [7], Dana and Riedel [9], Rigotti and Shannon [22]).

A positive answer to the previous questions has numerous implications. It, in particular, implies that the welfare theorems hold true in an incomplete preferences framework and that efficient allocations may be characterized by first order conditions (namely the intersection of normalized supergradients is non empty for some choice of utilities in the sets of each agent).

The paper considers **concave** multi-utility representations. This covers the case of families of linear preferences often used in asymetric information models, of second order stochastic dominance (Rothschild and Stiglitz [21]), of Bewley's [4] and Rigotti and Shannon [22] ambiguity models with incomplete preferences. It is based on a definition of efficiency corresponding to the strict preferences defined above and provides sufficient conditions for obtaining a positive answer to both of the above questions. In particular compactness conditions on the set of utility functions and on the set of their super differentials at any point are required. We also show that the above questions have a positive answer in the case of second order stochastic dominance for univariate risks although our technical assumptions are not satisfied in this case.

Given a family of concave utility functions on $E, \mathcal{U}_i$, one can also associate to it, the preference relation $X_i \in E$ is preferred to $Y_i \in E$ by agent i if $u_i(X_i) \geq u_i(Y)$ for every $u_i \in \mathcal{U}_i$. The corresponding strict preference is: $X_i \in E$ is strictly preferred to $Y_i \in E$ by agent i if $u_i(X_i) \geq u_i(Y)$ for every $u_i \in \mathcal{U}_i$ with a strict inequality for some u_i. Other concepts of efficiency may be considered for that preference relation. When utilities in $\mathcal{U}_i$ are strictly concave, under the hypotheses of the paper, it is shown that all efficiency concepts coincide. The same holds true for equilibrium concepts

The next example shows that one must consider *convex* families of utilities $\mathcal{U}_i$ whatever strict order or definition of efficiency and equilibria is chosen. Indeed, consider the case of two agents sharing risk X, the first agent has the CARA utility $X_1 \mapsto -\mathbb{E}(e^{-X_1})$ and the second one has incomplete preferences given by the family $\mathcal{U}_2$ of CARA utilities $X_2 \mapsto -\mathbb{E}(e^{-\theta X_2})$ where $\theta \in [1/2, 2]$. Note that $\mathcal{U}_2$ is not a convex family. Efficient risk sharing pairs between the first agent and any agent with utility in $\mathcal{U}_2$ are pairs of affine functions of X, with slopes $(\frac{\theta}{1+\theta}, \frac{1}{1+\theta})$, $\theta \in [1/2, 2]$. Consider the following efficient sharing problem between agent one and an agent with utility $-\frac{1}{5}\mathbb{E}(e^{-2X_2}) - \frac{4}{5}\mathbb{E}(e^{-\frac{X_2}{2}})$, obtained by mixing two utilities in $\mathcal{U}_2$

$$\max_{X_1+X_2=X} -\mathbb{E}(e^{-X_1}) - \frac{1}{5}\mathbb{E}(e^{-2X_2}) - \frac{4}{5}\mathbb{E}(e^{-\frac{1}{2}X_2}). \tag{1.1}$$

The solution of (1.1) is undominated in the incomplete setting (see definition 3.4 below): there does not exist (X_1', X_2') such that $-\mathbb{E}(e^{-X_1'}) > -\mathbb{E}(e^{-X_1})$ and $-\mathbb{E}(e^{-\theta X_2'}) > -\mathbb{E}(e^{-\theta X_2})$ for any $\theta \in [1/2, 2]$. However X_1 which is implicitely defined by the equation $e^{-X_1} + \frac{2}{5}e^{-2(X-X_1)} + \frac{2}{5}e^{-\frac{1}{2}(X-X_1)} = 0$ is not an affine function of X. Therefore it cannot be obtained as an efficient allocation that result when each agent has a utility in her family. This proves that some convexity of the sets $\mathcal{U}_i$ should be imposed if one wants to answer positively the first question raised above.

The method used is to first establish a no trade principle for families of linear utilities parametrized by a family of convex compact subsets of a given hyperplane by using an extension of Samet's theorem [23]. From the no-trade principle, any feasible allocation is efficient if and only if the compact sets have non-empty intersection. When the compact sets have empty intersection, there are no efficient allocations. A characterization of efficient allocations for families of concave utilities is, then obtained by local linearization of utilities. Thus the paper connects results that have been proven in the no trade literature (Samet [23], Man-Chung Ng [17]) mainly for linear utilities and infinite dimensional generalizations of results proven in the general equilibrium literature for concave utilities (Rigotti and Shannon's [22] ambiguity model with incomplete preferences).

The paper is organized as follows. The framework and general assumptions and examples are presented in section 2. Section 3 provides an abstract no-trade principle which nests most of the ex-ante no-trade theorems existing in the literature. Efficiency is then characterized and efficient allocations for the incomplete preferences are shown to coincide with the set of efficient allocations that result when each agent has a utility in her family of concave utility functions. A characterization by first order condition is provided. Section 4 is devoted to equilibria and welfare theorems. Section 5 discusses some concepts used in the paper and links the paper to the literature. Section 6 considers the case of second order stochastic dominance where our main assumptions are not satisfied and which nonetheless has the property studied in the paper.

2 Framework, assumptions and examples

2.1 Incomplete preferences framework

Let E and F be two real vector spaces, and $(P, X) \in F \times E \mapsto P \cdot X \in \mathbb{R}$ be some separating duality mapping which means that this map is bilinear and

- if $X \in E$ is such that $P \cdot X = 0$ for all $P \in F$ then $X = 0$,
- if $P \in F$ is such that $P \cdot X = 0$ for all $X \in E$ then $P = 0$.

We endow F (respectively E) with the locally convex Hausdorff topology $\sigma(F, E)$ (resp. $\sigma(E, F)$) which is the coarsest topology on F (resp. on E) for which $P \in F \mapsto P \cdot X$ is continuous for every $X \in E$ (resp. $X \in E \mapsto P \cdot X$ is continuous for every $P \in F$). With this choice of topologies, the topological dual of F may be identified to E and vice versa (see for instance Aliprantis and Border [1], Theorem 5.83). We shall therefore in the sequel interpret E as the space of goods and F as the space of prices.

Two polar special cases will mainly be considered. In the first, E is a Banach space, $F = E'$ and $\sigma(F, E)$ is the weak star topology on $F = E'$

(examples are $(E, F) = (L^1, L^\infty)$ or E is a space of continuous functions and F a space of Radon measures). In the polar case, F is a Banach space, $E = F'$ and $\sigma(F, E)$ is the weak topology on F (a typical example being $E = L^\infty$ and $F = L^1$). An example covered by our framework but not by the previous two cases is $E = B(\Omega, \mathcal{F})$ the space of real-valued bounded measurable functions on $(\Omega, \mathcal{F})$, a measurable space, $E' =$ba$(\Omega, \mathcal{F})$ the space of finitely additive measures on $(\Omega, \mathcal{F})$ and $F =$ca$(\Omega, \mathcal{F})$ the subspace of countably additive measures on $(\Omega, \mathcal{F})$.

An exchange economy with consumption space E populated by a finite set I of agents is considered. Agent $i \in I$ has an incomplete strict preference over E, defined by a family $\mathcal{U}_i : E \to \mathbb{R}$ of utility functions as follows: $X_i \in E$ is strictly preferred to $Y_i \in E$ by agent i, which will be denoted $X_i \succ_{\mathcal{U}_i} Y_i$, if $u_i(X_i) > u_i(Y_i)$ for every $u_i \in \mathcal{U}_i$.

For $X \in E$ (aggregate endowment), the set of allocations of X are defined as:

$$\mathcal{A}(X) := \{(X_i)_{i \in I} \in E^I : \sum_{i \in I} X_i = X\}.$$

An allocation $(X_i)_{i \in I} \in \mathcal{A}(X)$ is efficient if there is no other allocation $(Y_i)_{i \in I} \in \mathcal{A}(X)$ fulfilling $Y_i \succ_{\mathcal{U}_i} X_i$ for every $i \in I$. [1]

For further use, we recall that for $u : E \to \mathbb{R}$ concave, the superdifferential of u at $X \in E$, denoted $\partial u(X)$, is defined by

$$\partial u(X) := \{P \in E' = F : u(Y) - u(X) \leq P \cdot (Y - X), \text{ for all } Y \in E\}$$

and that u is superdifferentiable at X if $\partial u(X)$ is nonempty. We shall also denote

$$\partial \mathcal{U}_i(X) := \bigcup_{u_i \in \mathcal{U}_i} \partial u_i(X).$$

Note that as

$$\lambda \partial u_i(X) + (1 - \lambda) \partial v_i(X) \subset \partial(\lambda u_i + (1 - \lambda) v_i)(X),\ 0 \leq \lambda \leq 1,\ X \in E$$

$\partial \mathcal{U}_i(X)$ is convex whenever $\mathcal{U}_i$ is convex.

In the sequel, unless otherwise stated, the following assumptions are assumed to hold:

- **(H1)** For every $i \in I$, utilities in $\mathcal{U}_i$ are everywhere finite, concave superdifferentiable functions defined on E. Furthermore, for every i, every $u_i \in \mathcal{U}_i$ and every $X \in E$, $\partial u_i(X)$ is $\sigma(F, E)$ compact.

[1] Other concepts of efficiency will be discussed in section 5.

- **(H2)** For every $i \in I$, the set of utilities $\mathcal{U}_i$ is convex and there is a topology on $\mathcal{U}_i$ which makes it compact and such that the evaluation map $u_i \in \mathcal{U}_i \mapsto u_i(X)$ is continuous for every $X \in E$.

- **(H3)** There exists $\Phi \in E$ such that, for every $i \in I$, every $X_i \in E$, every $u_i \in \mathcal{U}_i$ and every $P \in \partial u_i(X_i)$, one has $\Phi \cdot P > 0$ and the set of normalized marginal utilities

$$V_i(X_i) := \left\{ \frac{P}{\Phi \cdot P} \ : P \in \partial \mathcal{U}_i(X_i) \right\}$$

is $\sigma(F, E)$ compact for every i and every $X_i \in E$.

Classes of examples where the previous assumptions are satisfied are provided in the next subsection. Let us make some general comments. Regarding assumption (**H1**), it always holds true if E is a normed space, $F = E'$ and the elements of $\mathcal{U}_i$ are Gâteaux-differentiable. More generally, (**H1**) is satisfied if E is a Banach space with dual F and the functions in $\mathcal{U}_i$ are concave, u.s.c (for the strong topology) and finite everywhere. Indeed, they are then everywhere continuous and superdifferentiable (see Chapter 1 in [12]). Moreover, their superdifferential being weakly star closed and bounded in E' is $\sigma(F, E)$ compact from Banach-Alaoglu's Theorem. In (**H2**), the requirement that $\mathcal{U}_i$ is convex has already been discussed in the introduction. Loosely put, the second part of (**H2**) says that the set of utilities admits a compact parametrization. To illustrate (**H3**), let $(\Omega, \mathcal{F}, \mathbb{P})$ be given and assume that there is one good in each state of the world. Let $E = L^\infty$ be the set of contingent claims, $F = L^1$ and $\Phi = 1$. Then

$$V_i(X_i) = \left\{ \frac{P}{\mathbb{E}(P)}, \ P \in \partial \mathcal{U}_i(X_i) \right\}$$

is the familiar set of normalized supergradients (or marginal utilities) of the u_i's at X_i.

2.2 Examples

In this subsection, we show that the assumptions above are satisfied in several quite large classes of examples. For the sake of notational simplicity, we will drop the subscript i everywhere in these examples.

The case of finite dimensions

Let $E = F = \mathbb{R}^d$ and $\mathcal{U}$ be a convex set of concave functions: $\mathbb{R}^d \to \mathbb{R}$. We claim that **(H1)** is trivially satisfied. Indeed the elements of $\mathcal{U}$ are everywhere continuous and thus superdifferentiable everywhere with a superdifferential which is closed and bounded, hence compact at every point.

Let us further assume that for every $R > 0$

$$M_R := \sup\{|u(x)|,\ |x| \le R,\ u \in \mathcal{U}\} < +\infty$$

and that $\mathcal{U}$ is closed for the topology of $C(\mathbb{R}^d)$ (that is that of uniform convergence on compact subsets). This implies that **(H2)** is fulfilled and that $\partial\mathcal{U}(x)$ is compact. Indeed for every $R > 0$, every $u \in \mathcal{U}$, every $(x, h) \in \mathbb{R}^d \times \mathbb{R}^d$ such that $|x| \le R$, $|h| \le R$ and every $p \in \partial u(x)$, we have

$$p \cdot h \le u(x) - u(x - h) \le 2M_{2R}$$

hence $|p| \le K_R = 2M_{2R}/R$ which proves that $\partial\mathcal{U}(x)$ is included in the ball of radius K_R and also that u is K_R Lipschitz on the ball of radius R. From Ascoli's theorem, as $\mathcal{U}$ is closed, $\mathcal{U}$ is compact in $C(\mathbb{R}^d)$ and **(H2)** is fulfilled. Let us next show that $\partial\mathcal{U}(x)$ is closed. Let $p_n \in \partial u_n(x)$ converges to some p. Passing to a subsequence if necessary, we may assume that u_n converges uniformly on every compact to some $u \in \mathcal{U}$. Since for every y, we have $u_n(y) - u_n(x) \le p_n(x - y)$, letting n tend to $+\infty$ gives that $p \in \partial u(x) \subset \partial\mathcal{U}(x)$.

Finally, let us assume that there is a common increasing direction, i.e. some $e \in \mathbb{R}^d$ such that $e \cdot p > 0$ for every $(x, u) \in \mathbb{R}^d \times \mathcal{U}$ and every $p \in \partial u(x)$ and let $V(x)$ be defined as above by

$$V(x) := \{p/(e \cdot p),\ p \in \partial\mathcal{U}(x)\}$$

As $V(x)$ is the image of the compact set $\partial\mathcal{U}(x)$ by $p \mapsto p/(e \cdot p)$ which is continuous on $\partial\mathcal{U}(x)$, it is compact and **(H3)** holds true.

This example shows that the assumptions made in the paper are rather harmless in finite dimensions.

Linear utilities and expectations with respect to a family of priors

Let $\Phi \in E$ and K be a convex $\sigma(F, E)$ compact subset of F such that $K \subset \{P \in F\ :\ P \cdot \Phi = 1\}$. Consider the family of linear utilities:

$$u_a(X) = a \cdot X,\ a \in K,\ X \in E \text{ and } \mathcal{U} := \{u_a,\ a \in K\}$$

Identifying $\mathcal{U}$ to K (endowed with $\sigma(F, E)$), as $\partial\mathcal{U}(X) = K \subset \{P \in F\ :\ P \cdot \Phi = 1\}$; $V(X) = K$ and it is immediate to check that **(H1)**, **(H2)** and **(H3)** are fulfilled.

Let us now consider an example. Let A be a compact subset of $\mathbb{R}$, $E = C(A, \mathbb{R})$ and $F = E' = \mathcal{M}(A)$. One may interpret A as the set of states of the world and E as the set of contingent claims. Let K be a convex and weakly-star closed subset of the set of probability measures on A. Let

$$u_\mu(X) := \int_A X(x) d\mu(x),\ X \in E,\ \mu \in K$$

be the expectation of X with respect to the family K of probabilities and let

$$\mathcal{U} := \{u_\mu,\ \mu \in K\}$$

(H1) is trivially fulfilled. Identifying $\mathcal{U}$ to K endowed with the weak-star topology, **(H2)** is fulfilled. Finally, choosing $\Phi = 1$, as $\partial u_\mu(X) = \{\mu\}$, $V(X) = K$ for all X and **(H3)** is in turn fulfilled.

A nonlinear variant of this model satisfying our assumptions as well, is obtained by considering families of utilities of the form

$$v_\mu(X) := \int_A V(X(x)) d\mu(x),\ X \in E,\ \mu \in K$$

where V is a given concave differentiable utility index and μ ranges over a convex and weakly-star closed set of probability measures K. As v_μ is differentiable ($\partial v_\mu(X) = \{V'(X)\mu\}$), **(H1)** is trivially fulfilled. Identifying $\mathcal{U}$ to K endowed with the weak-star topology, **(H2)** is fulfilled. Let $\Phi = 1$, then $V(X) = \{\frac{V'(X)\mu}{\mathbb{E}_\mu(V'(X))},\ \mu \in K\}$ is $\sigma(F, E)$ compact and **(H3)** is fulfilled. This incomplete preference may be viewed as an infinite dimensional extension of Bewley's [4] and Rigotti and Shannon's [22] ambiguity incomplete preference.

Other families of linear or non linear utilities may be interpreted as expectations with respect to a family of priors. For example, one may consider $E = B(\Omega, \mathcal{F})$ and $F =$ba$(\Omega, \mathcal{F})$ and K a weak-star compact subset of probability measures. Fixing a probability P, one may also consider pairs such as $(E, F) = (L^p, L^{p'})$ (with $p \in (1, \infty)$), $(E, F) = (L^1, L^\infty)$, $(E, F) = (L^\infty, L^1)$ and K a $\sigma(F, E)$ compact subset of probability densities. Nonlinear variants of these models also provide infinite dimensional extensions of Bewley's [4] and Rigotti and Shannon [22]'s ambiguity models (see Dana and Riedel [9] for a dynamic example).

Incomplete preferences on random vectors

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. We now consider strict orders on *random vectors* defined by families of expected utilities. More precisely, let $p \in$

$(1,\infty)$ with conjugate exponent $p' = p/(p-1)$ and $E = L^p((\Omega, \mathcal{F}, \mathbb{P}), \mathbb{R}^d)$ and $E' = F = L^{p'}((\Omega, \mathcal{F}, \mathbb{P}), \mathbb{R}^d)$ with the usual duality map between these spaces. As the class of concave functions does not satisfy our assumptions, let us consider a class $\mathcal{V}$ of concave and C^1 functions $\mathbb{R}^d \to \mathbb{R}$ such that:

1. $\mathcal{V}$ is convex and closed for the topology of $C(\mathbb{R}^d)$ (i.e. uniform convergence on compact sets),

2. The utilities and their gradients fulfill a uniform growth conditions: there is a constant C such that

$$|v(x)| \leq C(|x|^p + 1),\ |\nabla v(x)| \leq C(|x|^{p-1} + 1),\ \forall (x, v) \in \mathbb{R}^d \times \mathcal{V},$$

3. The utilities and their gradients are equicontinuous on compact sets: for every compact subset of $\mathbb{R}^d$, K, one has

$$\lim_{\varepsilon \to 0^+} \sup_{(v,x,y) \in \mathcal{V} \times K^2,\ |x-y| \leq \varepsilon} \Big\{ |v(x) - v(y)| + |\nabla v(x) - \nabla v(y)| \Big\} = 0.$$

4. there exists a unit vector $e \in \mathbb{R}^d$ such that

$$e \cdot \nabla v(x) > 0,\ \forall (v, x) \in \mathcal{V} \times \mathbb{R}^d$$

Note that from Ascoli's theorem, assumptions 1 and 3 guarantee that $\mathcal{V}$ is compact for the topology of $C^1(\mathbb{R}^d)$.

Let $\mathcal{U}$ be the set of expected utilities generated by $\mathcal{V}$ i.e.

$$\mathcal{U} := \{u_v := \mathbb{E}(v(.)),\ v \in \mathcal{V}\}$$

Assumption 2 insures that u_v and ∇u_v are well defined. Let $\mathcal{U}$ be endowed with the topology induced by the topology of $C(\mathbb{R}^d)$ on $\mathcal{V}$. From assumption 2 and Lebesgue dominated convergence theorem, the evaluation maps $v \mapsto \mathbb{E}(v(X))$ are continuous and the elements of $\mathcal{U}$ are continuous for the strong topology of E. Hence **(H1)** and **(H2)** are fulfilled. Let us check that **(H3)** is satisfied when one takes $P \cdot \Phi := \mathbb{E}(e \cdot P)$. For $X \in E$, let

$$V(X) := \Big\{ \frac{\nabla v(X)}{\mathbb{E}(e \cdot \nabla v(X))},\ v \in \mathcal{V} \Big\}$$

Note that $V(X)$ is convex. We claim that $V(X)$ is strongly compact in F. Indeed, let

$$Y_n = \lambda_n \nabla v_n(X),\ \lambda_n := \frac{1}{\mathbb{E}(e \cdot \nabla v_n(X))}$$

be some sequence in $V(X)$. From assumption 3 and Ascoli's theorem, a subsequence $(v_n, \nabla v_n)$ converges in $C(\mathbb{R}^d)$ to some $(v, \nabla v)$ with $v \in \mathcal{V}$. Using again assumption 3 and Lebesgue's dominated theorem, we deduce that $\nabla v_n(X) - \nabla v(X)$ converges to 0 strongly in F. This implies that λ_n also converges and that Y_n converges strongly in F to

$$\frac{\nabla v(X)}{\mathbb{E}(e \cdot \nabla v(X))}.$$

Since $V(X)$ is convex, we thus have **(H3)**.

Families of state dependent, additively separable utilities on random vectors

$$u_v(X) := \int_\Omega v(\omega, X(\omega)) d\mathbb{P}(\omega).$$

may as well be considered **(H1)**, **(H2)** and **(H3)** are fulfilled under similar assumptions as above, uniform in ω.

In the previous examples, except in the finite dimensional case, utilities were assumed differentiable. Let us now consider an example of families of superdifferentiable utilities.

Families of Rank-linear Utilities

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a non-atomic probability space, $E = L^\infty((\Omega, \mathcal{F}, \mathbb{P}), \mathbb{R})$ and $F = L^1((\Omega, \mathcal{F}, \mathbb{P}), \mathbb{R})$. Rank-linear-utilities both generalize expected utilities and Choquet's integrals with respect to a distortion and are of the form

$$X \in E \mapsto u_l(X) := \int_0^1 l(t, F_X^{-1}(t)) dt$$

where $F_X^{-1} = \inf\{z \in \mathbb{R} : F_X(z) > t\}$ denotes the quantile of X and l is a smooth function, concave nondecreasing in its second argument and submodular (i.e. $\partial^2_{tx} l \geq 0$). The expected utility is obtained for $l(t, x) = v(x)$ and $v : \mathbb{R} \to \mathbb{R}$ concave nondecreasing, the Choquet expectation with respect to a convex C^1 distortion f for $l(t, x) = f'(1-t)x$ and the risk averse RDU for $l(t, x) = f'(1-t)v(x)$ with f convex and v concave nondecreasing.

Let us consider the family of utilities

$$\mathcal{U} := \{u_l, \ l \in \mathcal{L}\}$$

where $\mathcal{L}$ is a convex and compact (for the $C([0,1] \times \mathbb{R})$ topology) set of C^1 concave and nondecreasing in their second argument functions fulfilling

$\partial^2_{tx} l > 0$. We identify $\mathcal{U}$ with $\mathcal{L}$. From a supermodular version of Hardy-Littlewood's theorem, one has for every $l \in \mathcal{L}$

$$u_l(X) = \min\{E(l(U,X)),\ U \text{ uniform}\}$$

The superdifferentiability properties of such utilities have been studied in [6] where it is proved that

$$\partial u_l(X) = \overline{\text{co}}\{\partial_x l(U,X),\ U \in \mathcal{V}_X\}$$

where $\overline{\text{co}}$ denotes closed convex hull operation for the $L^1(\Omega, \mathcal{F}, \mathbb{P})$ topology and $\mathcal{V}_X$ denotes the set of uniformly distributed random variables such that $u_l(X) = \mathbb{E}(l(U,X))$. Note that since $\partial u_l(X)$ is a bounded subset of L^∞, it is $\sigma(L^1, L^\infty)$- compact. The class $\mathcal{U}$ therefore satisfies **(H1)** and **(H2)**. If we further assume that for every compact K of $\mathbb{R}$, one has

$$\inf_{(t,x)\in[0,1]\times K,\ l\in\mathcal{L}} \partial_x l(t,x) > 0$$

then taking again $\Phi = 1$, we claim that **(H3)** is also satisfied. Indeed, let $X \in E$ and consider the set

$$V(X) := \Big\{\frac{Z}{\mathbb{E}(Z)},\ Z \in \bigcup_{l\in\mathcal{L}} \partial u_l(X)\Big\}$$

This set is bounded in L^∞ and thus $\sigma(L^1, L^\infty)$ relatively compact in L^1. To show that $V(X)$ is $\sigma(L^1, L^\infty)$ closed, since it is convex, it suffices to prove that it is strongly closed. Let us then suppose that $Y_n = Z_n/\mathbb{E}(Z_n)$ with $Z_n \in \partial u_{l_n}(X)$ for some $l_n \in \mathcal{L}$ converges in L^1 to some Y. Since $\mathbb{E}(Z_n)$ is bounded and bounded away from 0, up to a subsequence, we may assume that Z_n converges in L^1 to some Z and that l_n converges to l. Passing to the limit in the inequality

$$u_{l_n}(Y) - u_{l_n}(X) \le Z_n \cdot (Y - X),\ \text{for all} Y \in E$$

and using the continuity of the evaluation maps, we obtain that $Z \in \partial u_l(X)$. Hence $Y = Z/\mathbb{E}(Z) \in V(X)$ which proves **(H3)**.

3 No-trade principle and efficient allocations

3.1 No-trade principle for sets of linear utilities

We first consider families of linear utilities on E. Agent i's set of utilities is parametrized by K_i, a convex and $\sigma(F,E)$ compact subset of F. It is further assumed that the family $(K_i)_{i\in I}$ lies in a common hyperplane of F: there exists $\Phi \in E$ such that $a_i \cdot \Phi = 1$ for all $a_i \in K_i$ and all i. We show that no subset of agents can make a profitable trade if and only if $\bigcap_{i\in I} K_i \neq \emptyset$. The main part of the proof of the theorem that follows (that is the equivalence between 1 and 2), is an infinite dimensional version of Samet's theorem [23].

Theorem 3.1. *The following assertions are equivalent:*

1. *There exists no* $(X_i)_{i\in I} \in E^I$ *with* $\sum_{i\in I} X_i = 0$ *such that* $a_i \cdot X_i > 0$ *for all* $a_i \in K_i$ *and all* $i \in I$,
2. $\bigcap_{i\in I} K_i \neq \emptyset$,
3. *There exists* $P \in F$ *such that for all* $i \in I$, $a_i \cdot X_i > 0$ *for all* $a_i \in K_i$ *implies* $P \cdot X_i > 0$.

Proof.

- 1. implies 2. Assume 1. and that $\bigcap_{i\in I} K_i = \emptyset$. Let $K = K_1 \times K_2 \ldots \times K_I$ and $L = \{(P, P, \ldots, P),\ P \in F\}$ be the diagonal of F^I. We then have $K \cap L = \emptyset$. Since K is $\sigma(F,E)$ compact and convex and L is $\sigma(F,E)$-closed convex, from Hahn-Banach's theorem, there exists $(X_i)_{i\in I} \in E^I$ and $c \in \mathbb{R}$ such that

$$\sum_i a_i \cdot X_i > c \geq P \cdot (\sum_i X_i), \text{ for all } P \in F,\ a_i \in K_i,\ i \in I$$

  From the right-hand side, we obtain that $\sum_{i\in I} X_i = 0$ and $c \geq 0$. Let $\overline{a}_i \in \operatorname{argmin}_{a_i \in K_i} a_i \cdot X_i$ and $\overline{c}_i = \overline{a}_i \cdot X_i$. We claim that there exists $c_i < \overline{c}_i$ for all $i \in I$ with $\sum_i c_i = 0$. If not $\sum_i c_i$ would have a constant sign (strictly negative) on $\{(c_i) \in \mathbb{R}^I \mid \overline{c}_i - c_i > 0,\ \forall\, i \in I\}$ contradicting $\sum_{i\in I} \overline{c}_i > 0$. Let $\tilde{X}_i = X_i - c_i \Phi$. We have $\sum_i \tilde{X}_i = 0$ and $a_i \cdot \tilde{X}_i = a_i \cdot X_i - c_i a_i \cdot \Phi = a_i \cdot X_i - c_i > 0$ for every $a_i \in K_i,\ i \in J$ contradicting 1. Therefore $\bigcap_{i=I} K_i \neq \emptyset$.

- 2 implies 3. Indeed, any $P \in \bigcap_{i\in I} K_i$ fulfills 3.

- 3. implies 1. If $\sum_{i\in I} X_i = 0$ and $a_i \cdot X_i > 0$ for all $a_i \in K_i$ and all $i \in I$, then $P \cdot X_i > 0$ for all $i \in I$ contradicting $\sum_{i\in I} X_i = 0$.

□

Remark 3.2. It follows from assertion 3 that the assertions of theorem 3.1 are also equivalent to: there exists no $J \subset I$ and no $(X_i)_{i\in J} \in E^J$ with $\sum_{i\in J} X_i = 0$ such that $a_i \cdot X_i > 0$ for all $a_i \in K_i$ and all $i \in J$.

The hypothesis that there exists $\Phi \in E$ such that $a_i \cdot \Phi = 1$ for all $a_i \in K_i$ and all i is in particular verified for $\Phi = 1$ when $(K_i)_{i \in I}$ are subsets of probabilities or densities interpreted as sets of possible priors of the agents. The equivalence between assertions 1 and 2 of theorem 3.1 provides an ex-ante no-trade principle. By applying theorem 3.1 to various pairs (E, F), one recovers most of the no-trade results that have been proved in the literature. Samet [23] considers the case of $E = \mathbb{R}^m$, $F = E'$ and of a collection of non-empty convex closed subsets of e probabilities $(K_i)_{i \in I}$. Man-Chung Ng [17] considers $E = C(A)$ the set of continuous functions on a compact set A endowed with the sup-norm and $F = E'$ the set of finite Borel measures on A and a collection $(K_i)_{i \in I} \subset F$ of non-empty convex weak star compact subsets of probability measures. For a given probability space $(\Omega, \mathcal{F}, P)$, Dana and Riedel [9] consider $E = L^\infty(\Omega, \mathbb{R})$, $F = L^1(\Omega, \mathbb{R})$ and $(K_i)_{i \in I}$ a collection of non-empty convex $\sigma(L^\infty, L^1)$ compact subsets of densities. Finally Billot et al [3] consider $E = B(\Omega, \mathcal{F})$ the space of real-valued bounded measurable functions on Ω, $F = E' =\text{ba}(\Omega, \mathcal{F})$ the space of finitely additive measures on $(\Omega, \mathcal{F})$ and $(K_i)_{i \in I}$ a collection of non-empty weak star ($\sigma(F, E)$) closed, convex subsets of countably additive probability measures on $(\Omega, \mathcal{F})$.

3.2 No-trade principle for sets of concave utilities

We next consider the case where agents' preferences are given by families of concave utilities. The next lemma characterizes the directions in which preferences are increasing in a neighborhood of X_i and shows that locally these directions coincide with the directions in which the linear utilities associated to the family of compact sets $(V_i(X_i))_{i \in I}$ are increasing.

Lemma 3.3. *Let $X_i \in E$ and $Y_i \in E$ be given. The following are equivalent:*

1. *There exists $t_0 > 0$ such that for all $t \in (0, t_0]$, $X_i + tY_i \succ_{\mathcal{U}_i} X_i$,*
2. *there exists $t_0 > 0$ such that $X_i + t_0 Y_i \succ_{\mathcal{U}_i} X_i$,*
3. *for any $P \in V_i(X_i)$, $P \cdot Y_i > 0$ (or equivalently $P \cdot Y_i > 0$ for all $P \in \partial u_i(X_i)$ and $u_i \in \mathcal{U}_i$).*

Proof. Let us first remark that since u_i is concave, the map $t \to t^{-1}(u_i(X_i + tY_i) - u_i(X_i))$ is nonincreasing on $(0, \infty)$. Hence $u_i(X_i + t_0 Y_i) - u_i(X_i) > 0$ implies that $u_i(X_i + tY_i) - u_i(X_i) > 0$ for all $t \leq t_0$. Therefore, assertions 1 and 2 are equivalent. Let us prove that 2 and 3 are equivalent. Let Y_i fulfill 3 and $u_i \in \mathcal{U}_i$ be fixed. From **(H1)** $\partial u_i(X_i)$ is $\sigma(F, E)$ compact. As $P \cdot Y_i > 0$

for all $P \in \partial u_i(X_i)$, there exists $m > 0$ such that $\min_{\{P \in \partial u_i(X_i)\}} P \cdot Y_i \geq m$. Since u_i is concave, for every $(X_i, Y_i) \in E^2$,

$$\frac{u_i(X_i + tY_i) - u_i(X_i)}{t} \to \min_{\{P \in \partial u_i(X_i)\}} P \cdot Y_i \quad \text{as } t \to 0^+,$$

hence for every $u_i \in \mathcal{U}_i$, there exists $t_{u_i} > 0$ such that if $t \leq t_{u_i}$, $u_i(X + tY) - u_i(X) > 0$. Let $t > 0$ be given and let

$$W_i(t) = \{u_i \in \mathcal{U}_i \mid u_i(X_i + tY_i) - u_i(X_i) > 0\}$$

From what preceeds, $\mathcal{U}_i = \cup_{t>0} W_i(t)$. From **(H2)**, the evaluation maps being continuous, $V_i(t)$ is open. From **(H2)**, $\mathcal{U}_i$ is compact, hence there exists a finite subcovering of $\mathcal{U}_i$, $V_i(t_1), ..., V_i(t_k)$. Let $t_0 = \min_j t_j$. We then have that $X_i + t_0 Y_i \succ_{\mathcal{U}_i} X_i$.

Conversely, let Y_i fulfill 2. Then that there exists $t_0 > 0$ such that $X_i + t_0 Y_i \succ_{\mathcal{U}_i} X_i$. For any $u_i \in \mathcal{U}_i$, we have

$$0 > u_i(X_i) - u_i(X_i + t_0 Y_i) \geq -t_0 P \cdot Y_i, \text{ for any } \; Y_i \in \partial u_i(X_i)$$

Hence $P \cdot Y_i > 0$ for any $u_i \in \mathcal{U}_i$ and $P \in \partial u_i(X_i)$, equivalently $P \cdot Y_i > 0$ for any $P \in V_i(X_i)$ proving that 2 implies 3. ☐

A non linear no-trade principle now follows:

Theorem 3.4. *The following assertions are equivalent:*

1. *There exists no $(Y_i)_{i \in I} \in E^I$ with $\sum_{i \in I} Y_i = 0$ such that $X_i + Y_i \succ_{\mathcal{U}_i} X_i$ for all $i \in I$,*
2. *$\bigcap_{i \in I} V_i(X_i) \neq \emptyset$,*
3. *there exists $P \in F$ such that for all $i \in I$, $X_i + t_i Y_i \succ_{\mathcal{U}_i} X_i$ for some $t_i > 0$ implies $P \cdot Y_i > 0$.*

Proof. Let us show that 1 implies 2. From lemma 3.3, if assertion 1 holds true, there exists no $(Y_i)_{i \in I} \in E^I$ with $\sum_{i \in I} Y_i = 0$ such that $P \cdot Y_i > 0$ for all $P \in V_i(X_i)$ and all $i \in I$. From theorem 3.1 and **(H3)**, we have $\bigcap_{i \in I} V_i(X_i) \neq \emptyset$.
2 implies 3. Assume 2, then from theorem 3.1, there exists $P \in F$ such that for all i, $Q \cdot Y_i > 0$ for all $Q \in V_i(X_i)$ implies $P \cdot Y_i > 0$. From lemma 3.3, there exists $P \in F$ such that for all i $X_i + t_i Y_i \succ_{\mathcal{U}_i} X_i$ for some $t_i > 0$ implies $P \cdot Y_i > 0$. Finally, the fact that 3 implies 1 is obvious. ☐

Remark 3.5. It follows from remark 3.2 that the assertions above are also equivalent to: "there exists no $J \subset I$ and $(Y_i)_{i \in J} \in E^I$ with $\sum_{i \in J} Y_i = 0$ such that $X_i + Y_i \succ_{\mathcal{U}_i} X_i$ for all $i \in J$".

3.3 Efficient allocations and no-trade

Setting $\mathcal{U} := \Pi_{i\in I}\mathcal{U}_i$, we may now define $\mathcal{U}$-efficient allocations.

Definition 3.6. *Let $(X_i)_{i\in I} \in \mathcal{A}(X)$, then $(X_i)_{i\in I}$ is $\mathcal{U}$-efficient if there is no $(Y_i)_{i\in I} \in \mathcal{A}(X)$ such that $Y_i \succ_{\mathcal{U}_i} X_i$ for every $i \in I$.*

Let us first remark that if for each i, there exists $u_i \in \mathcal{U}_i$ such that $(X_i)_{i\in I}$ is efficient for the economy with complete preferences represented by the $(u_i)_{i\in I}$, then $(X_i)_{i\in I}$ is $\mathcal{U}$- efficient. If not there would exist $(Y_i)_{i\in I} \in \mathcal{A}(X)$ such that, for all i $Y_i \succ_{\mathcal{U}_i} X_i$ and in particular $u_i(Y_i) > u_i(X_i)$ contradicting the efficiency of $(X_i)_{i\in I}$ for the $(u_i)_{i\in I}$.

We next characterize $\mathcal{U}$-efficiency and state our first main result that provides a positive answer to the first question adressed by the paper.

Theorem 3.7. *The following assertions are equivalent:*

1. *The allocation $(X_i)_{i\in I} \in \mathcal{A}(X)$ is $\mathcal{U}$-efficient,*
2. *there exists no trade $(Y_i)_{i\in I} \in E^I$ with $\sum_{i\in I} Y_i = 0$ such that $X_i + Y_i \succ_{\mathcal{U}_i} X_i$ for all $i \in I$,*
3. $\bigcap_{i\in I} V_i(X_i) \neq \emptyset$,
4. *there exists $(u_i)_{i\in I}$, $u_i \in \mathcal{U}_i$ for each i such that $(X_i)_{i\in I}$ is efficient for the economy with complete preferences represented by the $(u_i)_{i\in I}$.*

Proof. 1. implies 2 follows directly from the definition of efficiency. 2. implies 3. follows from theorem 3.4. 3. implies 4. since $\bigcap_{i\in I} V_i(X_i) \neq \emptyset$ implies that for each i, there exists $u_i \in \mathcal{U}_i$ and $\lambda_i > 0$ such that $\bigcap_{i\in I} \lambda_i \partial u_i(X_i) \neq \emptyset$ which implies 4. 4 implies 1 was already discussed. □

Corollary 3.8. *Let utilities be linear and fulfill the hypotheses of subsection 3.1. Then the following assertions are equivalent:*

1. *Any allocation $(X_i)_{i\in I} \in \mathcal{A}(X)$ is $\mathcal{U}$-efficient,*
2. *there exists no $(X_i)_{i\in I} \in E^I$ with $\sum_{i\in I} X_i = 0$ such that $a_i \cdot X_i > 0$ for all $a_i \in K_i$ and all $i \in I$,*
3. $\bigcap_{i\in I} K_i \neq \emptyset$.

Let agents have endowments $W_i \in E,\ i \in I$ with $X = \sum_i W_i$. When is no-trade efficient, in other words when is $(W_1, \ldots, W_I)$ efficient? The following corollary follows directly from theorem 3.7.

Corollary 3.9. *The following assertions are equivalent:*

1. *No-trade is* $\mathcal{U}$*-efficient,*
2. $\bigcap_{i \in I} V_i(W_i) \neq \emptyset$
3. *there exists* $(u_i)_{i \in I}$ *such that no-trade is efficient for the economy with complete preferences represented by the* $(u_i)_{i \in I}$ *.*

Remark 3.10. A further assumption that can be made is that E is ordered by a closed convex cone E_+ in E with non nonempty interior, polar to a convex cone F_+ in F (the cone of nonnegative prices). Adding nonnegativity constraints in the model, agents choose consumptions in E_+ and their utility functions defined on E_+, are assumed to be concave, super differentiable on the interior of E_+ and monotone with respect to the order associated to E_+. Let $\mathcal{A}_+(X) := \{(X_i)_i \in I \in \mathcal{A}(X)\ X_i \in E_+,\ \forall i \in I\}$. Assuming **(H2)** and $\partial u_i(X) \in F$ and $\sigma(F, E)$ compact and $V_i(X)\,\sigma(F, E)$ compact at any interior point X of E_+ and any i, theorem 3.7 can be stated as: an *interior* allocation $(X_i)_{i \in I}$ is $\mathcal{U}$-efficient if and only if there exists $(u_i)_{i \in I},\ u_i \in \mathcal{U}_i$ for each i such that$(X_i)_{i \in I}$ is efficient for the economy with complete preferences represented by the $(u_i)_{i \in I}$.

4 Equilibria

4.1 Equilibria and welfare theorems

A price $P \in F$ supports the preferred set to X_i if $Y_i \succ_{\mathcal{U}_i} X_i$ implies $P \cdot Y_i > P \cdot X_i$. Supporting prices to a preferred set have appeared in the previous section in theorem 3.1 and lemma 3.3. The next lemma characterizes them.

Lemma 4.1. *For* $P \in F$*, the following are equivalent*

1. $Y_i \succ_{\mathcal{U}_i} X_i$ *implies* $P \cdot Y_i > P \cdot X_i$*,*
2. $\lambda P \in V_i(X_i)$ *for some* $\lambda > 0$*,*
3. *there exists* $u_i \in \mathcal{U}_i$ *such that* X_i *maximizes* $u_i(Y)$ *s.t.* $P \cdot Y \leq P \cdot X_i$*.*

Proof. To show that 1 implies 2, let $P \in F$ be such that $Y_i \succ_{\mathcal{U}_i} X_i$ implies $P \cdot Y_i > P \cdot X_i$. Assume that $\lambda P \notin V_i(X_i)$ for all $\lambda > 0$. Since from **(H3)**, $V_i(X_i)$ is $\sigma(F, E)$ compact, from Hahn-Banach's theorem there exists $Z_i \in E$ such that

$$\lambda P \cdot Z_i \leq 0 < \min_{H \in V_i(X_i)} H \cdot Z_i$$

Since $H \cdot Z_i > 0$, for all $H \in V_i(X_i)$, from lemma 3.3, for $t > 0$ sufficiently small $X_i + tZ_i \succ_{\mathcal{U}_i} X_i$ while $P \cdot Z_i < 0$ contradicting assertion 1.
2 implies 3, since if $\lambda P \in V_i(X_i)$ for some $\lambda > 0$, there exists $u_i \in \mathcal{U}_i$ such that $\lambda P \in \partial u_i(X_i)$. Hence X_i maximizes $u_i(Y)$ subject to $P \cdot Y \leq P \cdot X_i$.
Finally to show that 3 implies 1, if 3 holds true, then there exists $u_i \in \mathcal{U}_i$ and $\lambda > 0$ such that $\lambda P \in \partial u_i(X_i)$. Let $Y \succ_{\mathcal{U}_i} X_i$, we then have

$$0 > u_i(X_i) - u_i(Y) \geq P \cdot (X_i - Y)$$

and therefore $P \cdot Y > P \cdot X_i$ proving 1. □

The second main result of the paper, the characterization of equilibria for the incomplete preferences follows directly from the previous lemma as well as the characterisation of the demand correspondence.

The demand correspondence for the incomplete preference $\succ_{\mathcal{U}_i}$ at price P and wealth $P \cdot W_i$ is defined by:

$$\xi_i(P, P \cdot W_i) = \{X_i \in E \ : \ P \cdot X_i = P \cdot W_i \text{ and } X' \succ_{\mathcal{U}_i} X \text{ implies } P \cdot X' > P \cdot W_i\}$$

Thanks to lemma 4.1, the demand correspondence is characterized as follows.

Corollary 4.2. *For $P \in F$, the following are equivalent*

1. $X_i \in \xi_i(P, P \cdot W_i)$,
2. $\lambda P \in V_i(X_i)$ *for some* $\lambda > 0$,
3. *there exists* $u_i \in \mathcal{U}_i$ *such that* X_i *maximizes* $u_i(Y)$ *s.t.* $P \cdot Y \leq P \cdot W_i$.

We now turn to the definition of concepts of equilibria and their characterization.

Definition 4.3. *An allocation $\mathcal{X}^* = (X_i^*)_{i \in I} \in \mathcal{A}(X)$ with a price $P^* \in F$, is a $\mathcal{U}$-equilibrium with transfer payments if for every i, $X_i \succ_{\mathcal{U}_i} X_i^*$ implies $P^* \cdot X_i > P^* \cdot X_i^*$. An allocation $(X_i^*)_{i \in I} \in \mathcal{A}(X)$ with a price $P^* \in F$, is a $\mathcal{U}$-equilibrium if for every i, $P^* \cdot X_i^* = P^* \cdot W_i$ and for every i, $X_i \succ_{\mathcal{U}_i} X_i^*$ implies $P^* \cdot X_i > P^* \cdot W_i$.*

Theorem 4.4. *The following are equivalent*

1. $(\mathcal{X}^*, P^*)$ *is a* $\mathcal{U}$*-equilibrium with transfer payments,*
2. $\lambda P^* \in \bigcap_{i\in I}^d V_i(X_i^*)$ *for some* $\lambda > 0$*,*
3. *there exists* $(u_i) \in \mathcal{U}$ *such that* $(\mathcal{X}^*, P^*)$ *is an equilibrium with transfer payment of the economy with utilities* (u_i)*.*

The following are equivalent:

1. $(\mathcal{X}^*, P^*)$ *is a* $\mathcal{U}$*-equilibrium,*
2. $\lambda P^* \in \bigcap_{i\in I}^d V_i(X_i^*)$ *for some* $\lambda > 0$ *and for every* i*,* $P^* \cdot X_i^* = P^* \cdot W_i$*,*
3. *there exists* $(u_i) \in \mathcal{U}$ *such that* $(\mathcal{X}^*, P^*)$ *is an equilibrium of the economy with utilities* (u_i)*.*

The proof of Proposition 4.4 follows directly from lemma 4.1.

Remark 4.5. As in remark 3.10, the case of consumptions in E_+ may be considered. Under the same assumptions as those of remark 3.10), one can state the second part of theorem 4.4 as *interior* $\mathcal{U}$-equilibria coincide with interior equilibria of the economies with utilities (u_i) for some $(u_i) \in \mathcal{U}$.

We may now prove the welfare theorems for incomplete preferences.

Theorem 4.6. *The following assertions hold:*

1. *Any* $\mathcal{U}$*-equilibrium is* $\mathcal{U}$*-efficient.*
2. *Any* $\mathcal{U}$*-efficient allocation is a* $\mathcal{U}$*-equilibrium with transfer payments for some* $P \in F$*.*

Proof. Proof of assertion 1. From theorem 4.4, any $\mathcal{U}$-equilibrium $(\mathcal{X}^*, P^*)$ is an equilibrium for the some economy with complete preferences u_i, $i \in I$, hence is efficient in that economy. From theorem 3.7, $(X_i^*)_{i\in I}$ is $\mathcal{U}$-efficient. Let us now prove assertion 2: if $(X_i^*)_{i\in I}$ is $\mathcal{U}$-efficient, from theorem 3.7, it is efficient for the some economy with complete preferences u_i, $i \in I$, hence there exists P^*, such that $((X_i^*)_{i\in I}, P^*)$ is an equilibrium with transfer payments u_i, $i \in I$. From 4.4, $((X_i^*)_{i\in I}, P^*)$ is a $\mathcal{U}$-equilibrium with transfer payments. □

4.2 Equilibria with inertia

Bewley [4] introduced the principle of inertia: agents never trade to a position which is not unanimously preferred to their status quo. As noted by Rigotti and Shannon [22], the restriction of inertia plays the role of a natural equilibrium refinement. The definition of equilibrium is modified as follows. An equilibrium of the economy $(\mathcal{X}^*, P^*)$ satisfies the inertia condition if for all agents i with $X_i^* \neq W_i$, we have $X_i^* \succ_{\mathcal{U}_i} W_i$. Following Dana and Riedel [9], sufficient conditions for existence of equilibrium with inertia are provided.

From **(H2)**, the following utility is well defined:

$$\psi_i(X) = \min_{\mathcal{U}_i}(u_i(X) - u_i(W_i))$$

Let us introduce a new assumption:

(H4) The economy with utilities (ψ_i), $i \in I$ has an equilibrium which furthermore fulfills $\psi_i(X_i^*) > \psi_i(W_i)$ if $X_i^* \neq W_i$.

Proposition 4.7. *Assume* **(H2)** *and* **(H4)**. *Then any equilibrium of the economy with complete preferences $(\psi_i)_{i\in I}$ is an equilibrium with inertia.*

Proof. Let (X^*, P^*) be an equilibrium of the economy with complete preferences $(\psi_i)_{i\in I}$. If $X_i \succ_{\mathcal{U}_i} X_i^*$, then $u_i(X_i) > u_i(X_i^*)$ for any $u_i \in \mathcal{U}_i$, hence from **(H2)**, $\psi_i(X_i) > \psi_i(X_i^*)$ and $P^* \cdot X_i > P^* \cdot X_i^*$. Therefore (X^*, P^*) is a $\mathcal{U}$-equilibrium. By assumption if $X_i^* \neq W_i$, $\psi_i(X_i^*) > \psi_i(W_i) = 0$, hence $u_i(X_i^*) > u_i(W_i)$ for any $u_i \in \mathcal{U}_i$, thus $X_i^* \succ_{\mathcal{U}_i} W_i$. Thus (X^*, P^*) is a $\mathcal{U}$-equilibrium with inertia. □

Lemma 4.8. *If the u_i are strictly concave, then any equilibrium of the economy with complete preferences $(\psi_i)_{i\in I}$ verifies $\psi_i(X_i^*) > \psi_i(W_i)$ if $X_i^* \neq W_i$.*

Proof. Since, under **(H2)**, ψ_i is strictly concave, any equilibrium allocation X^* verifies $\psi_i(X_i^*) > \psi_i(W_i) = 0$ if $X_i^* \neq W_i$. If not, we would have $\psi_i(X_i^*) = \psi_i(W_i) = 0$ and $X_i^* \neq W_i$. Hence $\psi_i((X_i^* + W_i)/2) > \psi_i(X_i^*)$, implying $P^* \cdot W_i > P^* \cdot X_i^*$ contradicting the definition of an equilibrium. □

It follows from lemma 4.8 that if utilities are strictly concave, the existence of an equilibrium with inertia may be brought down to the existence of an equilibrium for the economy with utilities (ψ_i), $i \in I$.

Remark 4.9. In the proof of proposition 4.7, we have used the following properties on $(\psi_i)_{i\in I}$:

1. **(H4)** is fulfilled,
2. $X \succ_{\mathcal{U}_i} Y$ implies that $\psi_i(X) > \psi_i(Y)$,
3. $\psi_i(X) > \psi_i(W_i)$ is equivalent to $X \succ_{\mathcal{U}_i} W_i$.

From assertion 1, one deduces existence of an equilibrium for the economy with complete preferences $(\psi_i)_{i\in I}$, from assertion 2 that the equilibrium is an equilibrium for the economy with incomplete preferences and from assertion 3 that the equilibrium fulfills the inertia property. Assuming **(H4)** fulfilled, we could have used other utilities such as $\tilde{\psi}_i(X) = \min_{\mathcal{U}_i}(\frac{u_i(X)}{u_i(W_i)})$ or $\tilde{\psi}_i(X) = \min_{\mathcal{U}_i} \phi_i(u_i(X) - u_i(W_i))$ with $\phi_i : \mathbb{R} \to \mathbb{R}$ increasing strictly concave fulfilling $\phi_i(0) = 0$ suggesting the indeterminacy of equilibria with inertia.

5 Discussions of concepts

5.1 Preferences

Given the family of utilities $\mathcal{U}_i$, we have considered the strict incomplete preference defined by: $Y_i \succ_{\mathcal{U}_i} X_i$ if and only if $u_i(Y_i) > u_i(X_i)$ for all $u_i \in \mathcal{U}_i$.

Given $\mathcal{U}_i$, another preference may be considered : $Y_i \succeq_{\widetilde{\mathcal{U}}_i} X_i$ if and only if $u_i(Y_i) \geq u_i(X_i)$ for all $u_i \in \mathcal{U}_i$. The strict associated preference is $Y_i \succ_{\widetilde{\mathcal{U}}_i} X_i$ if and only if $u_i(Y_i) \geq u_i(X_i)$ for all $u_i \in \mathcal{U}_i$ with a strict inequality for some $u_i \in \mathcal{U}_i$. Clearly $Y_i \succ_{\mathcal{U}_i} X_i$ implies that $Y_i \succ_{\widetilde{\mathcal{U}}_i} X_i$.

5.2 Concepts of efficiency and equilibria

Two concepts of efficiency may be associated to the order $Y_i \succeq_{\widetilde{\mathcal{U}}_i} X_i$:

1. $(X_i)_{i\in I}$ is strongly-$\widetilde{\mathcal{U}}$-efficient if there does not exist $(Y_i)_{i\in I} \in \mathcal{A}(X)$ such that $Y_i \succeq_{\widetilde{\mathcal{U}}_i} X_i$ for all i, strictly for some i,
2. $(X_i)_{i\in I}$ is weakly-$\widetilde{\mathcal{U}}$ efficient if there does not exist $(Y_i)_{i\in I} \in \mathcal{A}(X)$ such that $Y_i \succ_{\widetilde{\mathcal{U}}_i} X_i$ for all $i \in I$.

Let us now discuss the relations between the different concepts. By definition, strong-$\widetilde{\mathcal{U}}$-efficiency implies weak-$\widetilde{\mathcal{U}}$-efficiency which implies $\mathcal{U}$ efficiency.

Proposition 5.1. *Assume* **(H1-H3)** *If the u_i are strictly concave, $\mathcal{U}$ efficiency , strong-$\widetilde{\mathcal{U}}$-efficiency and weak-$\widetilde{\mathcal{U}}$-efficiency are equivalent.*

Proof. It remains to show that $\mathcal{U}$-efficiency implies strong-$\widetilde{\mathcal{U}}$-efficiency. Let $(X_i)_{i\in I}$ be $\mathcal{U}$-efficient and assume that there exists $(Y_i)_{i\in I} \in \mathcal{A}(X)$ such that $u_i(Y_i) \geq u_i(X_i)$ for all i with a strict inequality for some i and some u_i. Let $(Z_i = (Y_i + X_i)/2) \in \mathcal{A}(X),\ u_i(Z_i) \geq u_i(X_i)$ for all i, and for all i such that $Z_i \neq X_i,\ u_i(Z_i) > u_i(X_i)$ for all $u_i \in \mathcal{U}_i$. From remark 3.5, this contradicts $\mathcal{U}_i$-efficiency.

□

A counterexample Let us now give a counterexample where the set of strong-$\widetilde{\mathcal{U}}$-efficient allocations is strictly smaller than the set of $\mathcal{U}$ efficient allocations. Assume that there are k states of the world with probabilities $\pi_i,\ i = 1, \ldots, k$ and one good in each state of the world. A contingent claim $X = (x_1, \ldots, x_k)$ where x_j is the amount to be received in state j is identified to an element of $\mathbb{R}^k$. Hence $E = F = \mathbb{R}^k$. Let $(\mathcal{P}_i)_{i\in I}$ be a family of compact convex subsets of the probability simplex and denote by ri $\mathcal{P}_i$ the relative interior of $\mathcal{P}_i$. Let

$$\mathcal{U}_i = \{u_i : \mathbb{R}^k \to \mathbb{R} \text{ s.t. } u_i(X) = E_\pi(X) = \sum \pi_i x_i,\ \pi \in \mathcal{P}_i,\ X \in \mathbb{R}^k\}$$

Let $Y_i \succeq_{\widetilde{\mathcal{U}}_i} X_i$ if and only if $E_\pi(Y_i) \geq E_\pi(X_i)$ for all $\pi \in \mathcal{P}_i$ and $Y_i \succ_{\widetilde{\mathcal{U}}_i} X_i$ if and only if $E_\pi(Y_i) \geq E_\pi(X_i)$ for all $\pi \in \mathcal{P}_i$ with a strict inequality for some $\pi \in \mathcal{P}_i$. Let $X \in \mathbb{R}^k$ be the aggregate endowment. When utilities are linear, either all feasible allocations are efficient or there is no efficient allocation. From Dana and Le Van [8], any feasible allocation is $\widetilde{\mathcal{U}}$-efficient if and only if $\cap_i \text{ri } \mathcal{P}_i \neq \emptyset$, while from section 3, any feasible allocation is $\mathcal{U}$-efficient if and only if $\cap_i \mathcal{P}_i \neq \emptyset$. It is easy to construct examples where $\cap_i \text{ri} \mathcal{P}_i = \emptyset$ and $\cap_i \mathcal{P}_i \neq \emptyset$.

An allocation $(X_i^*)_{i\in I} \in \mathcal{A}(X)$ with a price $P^* \in F$, is a $\tilde{\mathcal{U}}$-equilibrium if for every $i,\ P^* \cdot X_i^* = P^* \cdot W_i$ and $X_i \succ_{\tilde{\mathcal{U}}_i} X_i^*$ implies $P^* \cdot X_i > P^* \cdot W_i$. Note that any $\tilde{\mathcal{U}}$-equilibrium is a $\mathcal{U}$-equilibrium since $X_i \succ_{\mathcal{U}_i} X_i^*$ implies $X_i \succ_{\tilde{\mathcal{U}}_i} X_i^*$.

Proposition 5.2. *If the u_i are strictly concave, an allocation $(X_i^*)_{i\in I} \in \mathcal{A}(X)$ with a price $P^* \in F$, is a $\tilde{\mathcal{U}}$-equilibrium if an only if it is a $\mathcal{U}$-equilibrium*

Proof. It remains to show that if $(\mathcal{X}^*, P^*)$ is a $\mathcal{U}$-equilibrium, then it is a $\tilde{\mathcal{U}}$-equilibrium. Let $(\mathcal{X}^*, P^*)$ be a $\mathcal{U}$-equilibrium and assume that there exists for some $i,\ X_i$ such that $u_i(X_i) \geq u_i(X_i^*)$ for all i with a strict inequality for some

u_i and $P^*{\cdot}X_i \leq P^*{\cdot}X_i^* = P^*{\cdot}W_i$. Let $Z_i = (X_i{+}X_i^*)/2$. Then $P^*{\cdot}Z_i \leq P^*{\cdot}W_i$ and $Z_i \succ_{\mathcal{U}_i} X_i^*$. Indeed, since $X_i \neq X_i^*$, $u_i(Z_i) = u_i((X_i + X_i^*)/2) > u_i(X_i^*)$ for all u_i, contradicting the assumption that $(\mathcal{X}^*, P^*)$ is a $\mathcal{U}$-equilibrium of the economy with utilities (u_i). □

6 Second order stochastic dominance

The aim of this final section is to extend theorems 3.7 and 4.4 to the case of the concave order in one dimension, a case in which neither **(H2)** nor **(H3)** are fulfilled. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a non-atomic probability space and consider the set of bounded *real-valued* random variables, $L^\infty = L^\infty((\Omega, \mathcal{F}, \mathbb{P}), \mathbb{R})$. Let $\mathcal{U}$ be the set of strictly concave increasing utilities on $\mathbb{R}$. For X and Y in L^∞, let us denote $X \succ Y$ if and only if $\mathbb{E}(v(X)) > \mathbb{E}(v(Y))$, for all $v \in \mathcal{U}$. As is well-know, the corresponding large preference, $X \succeq Y$, is defined by $X \succeq Y$ if and only if $E(v(X)) \geq \mathbb{E}(v(Y))$, for all v concave nondecreasing. Let $\Phi = 1$, and

$$V(X) = \left\{ \frac{Y}{\mathbb{E}(Y)} \ : \ Y \in \partial u(X), \ u \in \mathcal{U} \right\}.$$

be the set of normalized supergradients at $X \in L^\infty$.

Proposition 6.1. *Let $(X_i)_{i \in I} \in \mathcal{A}(X)$. Then the following statements are equivalent:*

1. *$(X_i)_{i \in I}$ is strongly-$\widetilde{\mathcal{U}}$ efficient,*
2. *$(X_i)_{i \in I}$ is $\mathcal{U}$-efficient,*
3. *$(X_i)_{i \in I}$ is a comonotone allocation of X,*
4. *there exist continuous and strictly concave increasing functions $(u_1, ..., u_I)$ such that $(X_i)_{i \in I}$ is efficient for the corresponding expected utility economy,*
5. *$\bigcap_{i \in I} V(X_i) \neq \emptyset$.*

Proof. The equivalence between assertions 1 and 2 follow from Proposition 5.1, that between 1, 3 and 4 from Carlier Dana and Galichon [5]. The equivalence between 4 and 5 follows from standard first order conditions for strictly concave utilities. □

Given individual endowments $(W_i)_{i\in I}$ and defining $\mathcal{U}$-equilibria as previously, under some technical assumptions on the aggregate risk $X = \sum_{i\in I} W_i$, we have the following characterization

Theorem 6.2. *Assume that F_X, the cumulative distribution function of X is increasing and continuous, then the following assertions are equivalent:*

1. *$(\mathcal{X}^*, P^*)$ is a $\mathcal{U}$-equilibrium,*
2. *there exists $\lambda > 0$ such that $\lambda P^* \in \bigcap_{i\in I}^d V_i(X_i^*)$ and $\mathbb{E}(P^* X_i^*) = \mathbb{E}(P^* W_i)$ for every $i \in I$,*
3. *there exists $(u_i)_{i\in I} \in \mathcal{U}^I$ such that $(\mathcal{X}^*, P^*)$ is an equilibrium of the economy with utilities $(u_i)_{i\in I}$.*

Proof. Clearly, assertions 2 and 3 are equivalent and assertion 3 implies assertion 1. Let us now assume that $(\mathcal{X}^*, P^*) = ((X_i^*)_i, P^*)$ is a $\mathcal{U}$-equilibrium, which means that $(X_i^*)_i \in \mathcal{A}(X)$ and for every i, one has

$$Y \succ X_i^* \Rightarrow \mathbb{E}(P^* Y) > \mathbb{E}(P^* X_i^*) = \mathbb{E}(P^* W_i). \tag{6.1}$$

First, we claim that the X_i^*'s are P^*-measurable, since if not taking $Y_i := \mathbb{E}(X_i^* | P^*) \succ X_i^*$ as $\mathbb{E}(P^* Y_i) = \mathbb{E}(P^* X_i^*)$, we would derive a contradiction to (6.1). Hence $X_i^* = f_i(P^*)$ and $X = f(P^*)$ with $f = \sum_{i\in I} f_i$. This implies that P^* is non-atomic since otherwise X would have an atom, contradicting the assumption that F_X is continuous. Since P^* is non-atomic, we may find nonincreasing functions g_i such that $Y_i := g_i(P^*)$ has the same law as $X_i^* = f_i(P^*)$. If $X_i^* \neq Y_i$ ae, then $(X_i^* + Y_i)/2 \succ X_i^*$ implying from (6.1) that $\mathbb{E}(P^* Y_i) > \mathbb{E}(P^* X_i^*)$ which contradicts Hardy-Littlewood's inequality $\mathbb{E}(P^* Y_i) \leq \mathbb{E}(P^* X_i^*)$. Hence it must be the case that $X_i^* = g_i(P^*)$ a.s.: the X_i^*'s are nonincreasing functions of P^*. We may therefore find concave functions u_i such that $P^* \in \partial u_i(X_i^*)$ for every i; furthermore, since $P^* \geq 0$, u_i can also be taken nondecreasing. It remains to prove that the u_i's are strictly concave on the closed convex hull of the range of X_i^*. To prove this fact, given $Y \in L^\infty$, let F_Y^{-1} be the generalized inverse (or quantile function) of the cumulative distribution function F_Y. Since $X = \sum_{i\in I} g_i(P^*) =: g(P^*)$ with all the g_i's nonincreasing, for Lebesgue almost-every $t \in [0,1]$, we have

$$F_X^{-1}(t) = g(F_{P^*}^{-1}(1-t)) = \sum_{i\in I} g_i(F_{P^*}^{-1}(1-t)) = \sum_{i\in I} F_{X_i^*}^{-1}(t)$$

Since we have assumed that F_X is increasing, F_X^{-1} is continuous. It thus follows from that previous identity that $F_{X_i^*}^{-1}$ is continuous or, equivalently$F_{X_i}^*$

is increasing for every i. If u_i was affine one some nondegenerate interval $[a, b] \subset [\text{essinf} X_i^*, \text{esssup} X_i^*]$, then P^* would be constant on the set $(X_i^*)^{-1}((a, b))$. Since $F_{X_i^*}$ is increasing, this set would have positive probability and P^* would have an atom and we already know that this cannot happen. All the u_i's can therefore be chosen strictly concave (and thus increasing) implying assertion 2.

□

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