ANALYSIS OF CONTACT OCCURRENCE IN FLUID–STRUCTURE INTERACTION SYSTEM UNDER THE THIN FILM APPROXIMATION

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Abstract. In this paper, we consider the interactions between a 2D film of an incompressible viscous fluid deposited on a solid substrate and an elastic 2D structure delimiting the upper boundary of the film. The system is strongly coupled since we assume continuity of velocities and normal stresses at the fluid/structure interface. A general model including a fractional power of the laplacian is chosen to model the structure dynamics. We derive two asymptotic models in the thin film approximation depending on the relative size of the structure mechanical parameters. We then discuss the influence of the power exponent on possible collapse of the film in the various reduced models.

1. Introduction

This note is concerned with the topological-singularity formation in moving boundary problems. In many fluid/structure interaction problems, one considers the motion of an incompressible viscous fluid within a domain whose shape is deformed under the action of the fluid. For instance, one can cite the motion of rigid bodies in a vessel containing an incompressible viscous fluid. In this case, the fluid domain consists of the vessel minus the bodies which evolve according to fluid force and torque. In this note, we will consider specifically the case of an incompressible viscous fluid filling a container when one part of the container boundary is made of an elastic structure. In both examples, topological singularities may appear in the fluid domain because of a collision between moving boundaries (or between a moving boundary and a fixed one). When viscosity is considered, drag terms prevent from the motion of the boundaries and are thus likely to forbid collisions. A classical issue is then to determine in a given fluid/structure model whether collision may occur in finite time or not. In the case of the motion of rigid bodies in a container, many particular configurations have been studied showing that no collisions have to be expected when considering standard models for interactions [16, 18, 23]. This result might seem contradictory to the physical intuition. Hence, modifications have been proposed to recover collision (considering bodies with low-regular boundaries [8, 9], assuming some slip at the fluid/solid interface [9, 10, 19]). Similar issues with inviscid fluids have also been carefully studied [7, 17, 20, 21].

The case of a container with a partly elastic boundary motivated fewer studies. To our knowledge, only the case of a 2D film whose upper boundary is made of an elastic structure has been considered by the second author [13] (see below for more details on the model). In this case, a no-collision result is obtained when dissipation is added to the elastic-structure equations. Without this dissipation, local-in-time existence results have been obtained but with no information on a possible collision. We underline that the film+elastic structure models have been proposed in the literature as toy-models for blood arteries [22] : the 2D+1D problem can be seen has a projection of a cylindrically invariant problem in which the elastic structure is the artery boundary. A collision then corresponds to a "collapse" of the artery. Hence, in this note, we provide a first study of collapse occurrence in film+elastic structure problems in absence of dissipation. Our study is based on the analysis of reduced models in the thin-film approximation.

We sketch now the original system whose thin film approximations we study below. More details concerning the modelling can be found in [13]. We consider a 2D film of an incompressible viscous fluid deposited on a solid substrate that is at rest. We assume that the upper boundary of the film is made of an elastic structure whose shape is given by the graph of a function h. For simplicity we restrict to configurations that are L-periodic in the horizontal variable. In short, the unknowns of our system are then h with (u,p) the fluid velocity field and pressure. They solve the following coupled system of pdes:

$$\rho_s \partial_{tt} h + \alpha(-\partial_{xx})^{(1+\theta)} h = \phi(u,p,h) + f, \tag{1.1}$$

with

$$\begin{cases} \rho_f(\partial_t u + u\cdot\nabla u) = \mu\Delta u - \nabla p \\ \qquad \text{div} u = 0 \end{cases} \quad \text{in } \{(x,y) | y \in (0, h(t,x))\}. \tag{1.2}$$

In this system, ρ_s, ρ_f are the respective (constant) densities of the structure and fluid. The parameter $\alpha > 0$ represents the mechanical properties of the structure while μ stands for the fluid viscosity. The symbol f stands for an external forcing term while ϕ stands for the trace of the fluid normal stresses on the structure written in the structure referential. Splitting $u = (u_1, u_2)$ according to the x, y-coordinates, it reads:

$$\phi(u,p,h) = p - \mu(2\partial_y u_2 - (\partial_x u_2 + \partial_y u_1)\partial_x h).$$

Finally, we introduced the operator $(-\partial_{xx})^{1+\theta}$ where θ is a given exponent between 0 and 1. In this L-periodic framework, it reads, for arbitrary smooth L-periodic functions ζ :

$$\left[(-\partial_{xx})^{1+\theta}\zeta\right](x) = \sum_{k\in\mathbb{Z}^*} \left(\frac{2\pi k}{L}\right)^{1+\theta} \hat{\zeta}_k \exp\left(\frac{2ik\pi}{L}x\right),$$

where the $(\hat{\zeta}_k)_{k\in\mathbb{Z}}$ are the Fourier coefficients of ζ :

$$\hat{\zeta}_k = \frac{1}{L}\int_0^L \zeta(s) \exp\left(\frac{2ik\pi}{L}s\right) \mathrm{d}s \quad \forall\, k \in \mathbb{Z}.$$

Classical values for θ are 0 and 1. In case $\theta = 0$ our fractional power is actually a simple laplacian and equation (1.1) can be interpreted as a rod equation so that (1.1)-(1.2) is the so-called "rod-fluid equation". In case $\theta = 1$ our power is now a bi-laplacian and our coupled system is known as the beam-fluid equations. In order to study the influence of the interface regularity on the contact occurrence we allow all intermediate values though the physical significance of other possible exponents is less obvious. Existence of solutions to the Cauchy problem associated with this two cases $\theta = 0$ and $\theta = 1$ has been the center of many studies in the last years. In the case $\theta = 0$, the second author in collaboration with C. Grandmont and J. Lequeurre provide a local-in-time existence and uniqueness result [14] in a class of solutions with no loss of regularity between the solution and the data (in contrast with previous studies [1, 6]). Existence of weak solutions up to collapse has also been obtained previously [12]. We refer the reader to the introduction of [14] for a more exhaustive review of the studies on the Cauchy theories for fluid+structure problems.

In film equations an important parameter is the aspect ratio H/L where H is the characteristic height and L the characteristic width. For a thin film, we assume that $H/L << 1$. In this regime and assuming viscosity dominates the flow, two possible systems appear depending on the mechanical properties of the structure. In the case of a light structure acceleration terms in (1.1) vanish and one gets the thin film equation:

$$\begin{cases} \partial_t h - \partial_x(h^3\partial_x p) = \partial_x(h^3\partial_x f) \\ \qquad\qquad p = (-\partial_{xx})^{1+\theta} h. \end{cases} \tag{1.3}$$

We emphasize that there should remain some parameters comparing the relative amplitude of pressure and forcing terms. We have set all these parameters to 1 for simplicity. If the structure is dense, acceleration terms cannot be neglected and one obtains a kind of coupled beam+reynolds system:

$$\begin{cases} \partial_t h - \partial_x(h^3\partial_x p) = 0 \\ \partial_{tt} h + (-\partial_{xx})^{1+\theta} h = p + f. \end{cases} \tag{1.4}$$

Again physical parameters have been set to 1 for simplicity. More details on the computation of these non-dimensional systems are provided in Appendix A. In the following sections, we focus on the handling of contacts by these two reduced models respectively.

The second system (1.4) seems to be original. To our knowledge, only formal estimates are provided in [13] when dissipation is added in the structure equation. System (1.3) is more classical. In case $\theta = 0$ it corresponds to the classical thin film equation. Rigorous derivations are provided in closely related framework when no structure is added on the top boundary [11, 15]. In presence of a structure we refer to [2] for a prescribed motion and to [5] and references therein for the interaction problem. Thin film equations have been thoroughly studied for general mobility functions m (corresponding to the prefactor $m(h) = h^3$ of $\partial_x p$ in our case) and more general pressure laws than $p = -\partial_{xx} h$ (including for instance Van der Waals repulsion forces). We refer to the recent review [24] for an overview. Many references analyse the existence of vanishing self-similar solutions. Such solutions are supposed to highlight the motion of a contact line (i.e. a transition between a wetted and an unwetted region, or a collapsed region and an uncollapsed one with our terminology). However, it seems that such construction could be performed only for $m(h) = h^n$ with $n < 3$ (see [4]) or by adding a Van der Waals term in the pressure (see [24, Section 3]). Considering the thin film equation (1.3) and especially its variants with $\theta \in (0, 1]$ seems open.

To analyse the contact issue for the thin film approximations (1.3) and (1.4) we adopt an approach slightly different to the studies mentioned above. First, we construct solutions locally-in-time and identify a framework for source term f that enable to reduce the possible blow-up to collapse. Then either we show that available estimates enable to prevent from collapse or we construct explicit solutions with collapse associated to explicit source terms (whose regularity is consistent with the Cauchy theory developed previously). This second strategy is reminiscent of previous constructions by V. Starovoitov in the case of fluid/solid interaction problems [25]. Briefly, we achieve the following results. For the thin film equation (1.3), we identify a family of source terms f whose regularity enable to prevent from contact for arbitrary initial data when $\theta > 1/2$. While, in case $\theta < 1/2$ we are able to construct scenarios of solutions with finite-time collapse. As for the beam+Reynolds problem (1.4), we construct a framework for the local-in-time existence of strong solutions. But we show that whatever the exponent $0 \le \theta \le 1$, we can construct explicit solutions with finite-time collapse.

We end up this section by introducing notations we will use throughout the paper below. We consider possibly time-dependent 1-periodic solutions. We denote $\mathbb{T} := \mathbb{R}/\mathbb{Z}$. We will sometime identify $\mathbb{T} \sim (-1/2, 1/2)$ especially when we consider even functions. We denote with sharp indices periodic versions of classical spaces. This will be particularly used with continuous ($C_\sharp$) or smooth ($C^\infty_\sharp$) functions as well as with Lebesgue spaces ($L^p_\sharp$) or Sobolev spaces ($H^m_\sharp$).

2. Analysis of the thin-film equation (1.3)

In this section, we focus on the thin film equation that we recall below:

$$\begin{cases} \partial_t h - \partial_x (h^3 \partial_x p) = \partial_x (h^3 \partial_x f) \\ \qquad\qquad\qquad p = (-\partial_{xx})^{1+\theta} h. \end{cases} \tag{2.1}$$

The system is considered on $\mathbb{T}$ and complemented with initial conditions:

$$h(0, \cdot) = h^0. \tag{2.2}$$

We first construct a theory of strictly positive strong solutions locally-in-time. We envisage blow-up and prove that if θ is sufficiently large then these strong solutions are global under a mild criterion on the forcing term f (whatever the initial data). Finally, for θ sufficiently small, we prove that even with the criterion derived previously, we can construct solutions that blow up in finite time because of a collapse of the film (meaning that h must vanish at some points).

2.1. **Strictly positive strong solutions.** We fix $\theta \in [0,1]$ in this subsection. The construction herein relies on the following estimates. Assume that h is a sufficiently smooth solution to (2.1) on $[0,T]$ that does not vanish and multiply the first equation in (2.1) with $p = (-\partial_{xx})^{1+\theta} h$. Standard integration by parts show that for arbitrary $0 \leq s < t \leq T$

$$\|h(t,\cdot)\|^2_{H^{1+\theta}_\sharp} + 2\int_s^t \int_{\mathbb{T}} h^3 |\partial_x p|^2 = \|h(s,\cdot)\|^2_{H^{1+\theta}_\sharp} - 2\int_s^t \int_{\mathbb{T}} h^3 \partial_x f \partial_x p. \tag{2.3}$$

With a standard Young inequality we infer that:

$$\|h(t,\cdot)\|^2_{H^{1+\theta}_\sharp} + \int_0^t \int_{\mathbb{T}} h^3 |\partial_x p|^2 \leq \|h^0\|^2_{H^{1+\theta}_\sharp} + \int_0^t \int_{\mathbb{T}} h^3 |\partial_x f|^2, \quad \forall\, t \leq T. \tag{2.4}$$

At this point, we realize that, if $h \geq c > 0$ on $(0,T)$ the dissipation term enables to control:

$$\int_0^T \|\partial_x p\|^2_{L^2_\sharp} = \int_0^T \|\partial_x h\|^2_{H^{2(1+\theta)}_\sharp}.$$

Consequently we give the following definition for strong solutions:

Definition 1. Let $h^0 \in H^{1+\theta}_\sharp$ and $T > 0$. Assume that $f \in L^2(0,T;H^1_\sharp)$. We call $h : [0,T] \times \mathbb{T} \to \mathbb{R}$ strong solution on $[0,T]$ to (2.1) with initial data (2.2) if

- $h \in C([0,T];H^{1+\theta}_\sharp) \cap L^2(0,T;H^{3+2\theta}_\sharp)$ is such that $\inf_{[0,T]\times\mathbb{T}} h > 0$,
- h satisfies (2.1) in $L^2(0,T;H^{-1}_\sharp)$ and (2.2) almost everywhere.

If $f \in L^2_{loc}([0,T);H^1_\sharp))$ we call $h : [0,T) \times \mathbb{T}$ strong solution on $[0,T)$ to (2.1) with initial data (2.2) if, for arbitrary $T' < T$ we have that $h_{|_{[0,T']\times\mathbb{T}}}$ is a strong solution on $[0,T']$.

It is classical to associate to these definitions of local-in-time solutions definitions of global solutions (with respect to the time-extent of the data f) and non-extendable solutions. Our first result then states:

Proposition 2. *Given $h^0 \in H^{1+\theta}_\sharp$ such that $\min h^0 > 0$, $T > 0$ and $f \in L^2_{loc}([0,T);H^1_\sharp)$ there exists $T_0 < T$ such that there exists a unique strong solution to (2.1)-(2.2) on $[0,T_0]$. Furthermore, this solution satisfies the energy equality (2.3) for arbitrary $0 \leq s < t \leq T_0$.*

Adapting then classical arguments of dynamical systems to **Proposition 2**, we infer that there also exists a unique non-extendable solution starting from any $h^0 \in H^{1+\theta}_\sharp$. This reads:

Corollary 3. *Let $T > 0$ and $f \in L^2_{loc}([0,T);H^1_\sharp)$. Given $h^0 \in H^{1+\theta}_\sharp$ such that $\min h^0 > 0$, there exists $T_* \leq T$ such that there is a unique non-extendable strong solution to (2.1)-(2.2) on $[0,T_*)$. Furthermore, we have the following blow-up alternative:*

- *either $T_* = T$*
- *or $T_* < T$ and there holds:*

$$\limsup_{t\to T_*} \|h(t,\cdot)\|_{H^{1+\theta}_\sharp} + \|1/h(t,\cdot)\|_{L^\infty_\sharp} = +\infty.$$

By abusive convention, we also refer to (T_*, h) as the "unique non-extendable solution to (2.1) satisfying (2.2)". The proofs of **Proposition 2** and **Corollary 3** must be part of the folklore. Yet, we did not find a reference matching exactly the framework we consider herein, even though previous references on the construction of solutions to thin film equations can be adapted (see [3] for instance). For completeness, we depict here shortly a method leading to our result.

Proof of **Proposition 2**. Let $T > 0$ and fix initial data h^0 and forcing term f matching the assumptions of our proposition.

Existence. We propose to construct solutions via a standard Galerkin method. For this, we introduce P_N the projection operator on the space E_N corresponding to Fourier series containing only modes k with $|k| \leq N$. We approximate then our problem by looking for some $h_N \in C([0,T];E_N)$ solution to

$$\begin{cases} \partial_t h_N + \partial_x P_N\left[(h_N)^3 \partial_x p_N\right] = \partial_x P_N\left[(h_N)^3 \partial_x f\right] \\ p_N = (-\partial_{xx})^{1+\theta} h_N \\ h_N(0,\cdot) = P_N[h^0]. \end{cases}$$

This system reduces to a set of odes that we can solve at least locally-in-time on some short interval $[0,T_N]$ (which depends *a priori* on N). At this point, we realize that the arguments leading to (2.3) can be reproduced for h_N (since $p_N \in C([0,T_N];E_N)$). As long as $\min h^0/2 \leq h_N \leq 2\max h^0$, this estimate yields a bound on h_N in $L^\infty_t H^{1+\theta}_\sharp$ and on p_N in $L^2_t H^1_\sharp$. Applying the equation, we infer then a bound for h_N in $H^1_t H^{-1}_\sharp$ using the equation satisfied by h_N. Moreover, for large N we have that $P_N[h^0] \geq 2\min h^0/3$ and that

$$L^\infty(0,T;H^1_\sharp) \cap H^1(0,T;H^{-1}_\sharp) \subset C^{0,1/8}([0,T];C_\sharp), \tag{2.5}$$

with an embedding constant independent of T (actually, we could choose any exponent smaller than $1/4$ instead of $1/8$). So, we can run a standard continuation argument relying on (2.3) to construct a time-interval $[0,T_0]$ on which, for N sufficiently large, there holds:

$$\|h_N(t,\cdot) - P_N[h^0]\|_{C_\sharp} \leq \frac{\min h_0}{6},$$

and consequently:

$$\|h_N\|_{L^\infty(0,T_0;H^{1+\theta}_\sharp)} \leq \left(\|h^0\|^2_{H^{1+\theta}_\sharp} + 8(\max h^0)^3 \|f\|^2_{L^2(0,T_0;H^1_\sharp)}\right)^{\frac{1}{2}}$$
$$\|p_N\|_{L^2(0,T_0;H^1_\sharp)} \leq K(h^0)\|h^0\|_{H^{1+\theta}_\sharp}.$$

We conclude that, up to the extraction of a subsequence, h_N converges to some

$$h \in C([0,T_0];H^{1+\theta}_\sharp - w) \cap C([0,T_0];H^1_\sharp) \cap L^2(0,T_0;H^{2+\theta}_\sharp),$$

such that $\partial_t h \in L^2(0,T_0;H^{-1}_\sharp)$. This solution has then sufficient regularity to multiply (2.1) by $p = (-\partial_{xx})^{1+\theta} h \in L^2(0,T_0;H^1_\sharp)$. This entails that (2.3) holds true also for h and that $\|h\|_{H^{1+\theta}_\sharp} \in C([0,T_0])$. Consequently, we have also that $h \in C([0,T_0];H^{1+\theta}_\sharp)$.

Uniqueness. Assume that h_1 and h_2 are two strong solutions on $[0,T_0]$. Reproducing the arguments above we note that $p_i = (-\partial_{xx})^{1+\theta} h_i$ is an admissible test-function for (2.1). Thus we have at hand (2.3) and, up to restrict the size of T_0 (depending only on h^0) we have

$$\|h_i\|_{L^\infty(0,T_0;H^{1+\theta}_\sharp)} \leq \left(\|h^0\|^2_{H^{1+\theta}_\sharp} + 8(\max h^0)^3 \|f\|^2_{L^2(0,T_0;H^1_\sharp)}\right)^{\frac{1}{2}}$$
$$\|p_i\|_{L^2(0,T_0;H^1_\sharp)} \leq K(h_0)\|h^0\|_{H^{1+\theta}_\sharp}.$$

We can then write an equation for the difference $h = h_2 - h_1$ between the two solutions. Introducing $p = p_2 - p_1$ and $h_3 = (h_2)^3 - (h_1)^3$ we derive:

$$\partial_t h - \partial_x((h_2)^3 \partial_x p) = \partial_x(h_3(\partial_x f + \partial_x p_1)).$$

Again, we have sufficient regularity on h to multiply this equation by p yielding:

$$\frac{1}{2}\|h(t,\cdot)\|^2_{H^{1+\theta}} + \int_0^t \int_{\mathbb{T}} (h_2)^3 |\partial_x p|^2 \leq -\int_0^t \int_{\mathbb{T}} h_3(\partial_x f + \partial_x p_1)\partial_x p \quad \forall\, t \in (0,T_0).$$

By a standard Cauchy-Schwarz inequality and the embedding $H^{1+\theta}_\sharp \subset C_\sharp$ we infer that:

$$\frac{1}{2}\|h(t,\cdot)\|^2_{H^{1+\theta}_\sharp} \leq K(h^0,f)\int_0^t \|h(s,\cdot)\|^2_{H^{1+\theta}_\sharp}\left(\|f(s,\cdot)\|^2_{H^1_\sharp} + \|p_1(s,\cdot)\|^2_{H^1_\sharp}\right), \quad \forall\, t\in[0,T_0].$$

We conclude that $h\equiv 0$ on $[0,T_0]$ by applying a standard Gronwall inequality (and recalling that f as well as p belong to $L^2(0,T_0;H^1_\sharp)$).

□

In order to derive **Corollary 3** from **Proposition 2**, it is mandatory to extract precisely the dependencies of the time T_0 appearing in **Proposition 2**. So, we argue below that we can choose T_0 in order to control the growth of the $H^{1+\theta}_\sharp$-norm of the solution h to (2.1) and the decrease of $\min h$. Similar arguments would apply to h_N showing that T_0 in the previous proposition can be chosen only with the given dependencies. Assume that, on $[0,T_0]$ we have

$$\frac{\min h^0}{4} \leq h(t,x) \leq 4\max h^0.$$

Since h satisfies energy estimate (2.3), we infer from (2.4) and the previous assumed bounds that:

$$\|h(t,\cdot)\|^2_{H^{1+\theta}_\sharp} + \frac{(\min h^0)^3}{64}\int_0^t\int_{\mathbb{T}}|\partial_x p|^2 \leq \|h^0\|^2_{H^{1+\theta}_\sharp} + 64(\max h^0)^3\int_0^t\int_{\mathbb{T}}|\partial_x f|^2.$$

Consequently, there holds:

$$\|h(t,\cdot)\|^2_{H^{1+\theta}_\sharp} \leq \|h^0\|^2_{H^{1+\theta}_\sharp} + 64(\max h^0)^3\int_0^t\int_{\mathbb{T}}|\partial_x f|^2 \quad \forall\, t\in(0,T_0),$$

and, applying (2.1)

$$\|\partial_t h\|_{L^2(0,T_0;H^{-1}_\sharp)} \leq$$
$$64(\max h^0)^3\left(\sqrt{\frac{64}{(\min h^0)^3}\left(\|h^0\|^2_{H^{1+\theta}_\sharp} + 64(\max h^0)^3\int_0^{T_0}\int_{\mathbb{T}}|\partial_x f|^2\right)} + \|f\|_{L^2(0,T_0;H^1_\sharp)}\right).$$

So, with K_X the constant associated with the embedding (2.5) we conclude that:

$$\|h\|_{C^{0,1/8}([0,T_0];C_\sharp)} \leq$$
$$K_X\left(1+64(\max h^0)^3)\right)\left(\sqrt{\frac{64}{|\min h^0|^3}\left(\|h^0\|^2_{H^{1+\theta}_\sharp} + 64(\max h^0)^3\int_0^{T_0}\int_{\mathbb{T}}|\partial_x f|^2\right)} + \|f\|_{L^2(0,T_0;H^1_\sharp)}\right).$$

Consequently, restricting the size of T_0 if necessary, depending on $\|f\|_{L^2(0,T;H^1_\sharp)}$, $\|h^0\|_{H^{1+\theta}}$ and $\min h^0$, we ensure that:

$$\|h(t,\cdot) - h^0\|_{C_\sharp} \leq \frac{\min h^0}{2},$$

so that, on $(0,T_0)\times\mathbb{T}$ there holds:

$$\frac{\min h^0}{2} \leq h(t,x) \leq \frac{3}{2}\max h^0.$$

This shows that, by a continuation argument, we can ensure this latter bound on the time interval $[0,T_0]$ constructed above and depending only on $\|f\|_{L^2(0,T;H^1_\sharp)}$, $\|h^0\|_{H^{1+\theta}_\sharp}$ and $\min h^0$. This gives at first that $\min h(t,\cdot) \geq \min h^0/2$ on $[0,T_0]$ but, we may again introduce the control from above for h into (2.4) to yield that:

$$\|h(t,\cdot)\|^2_{H^{1+\theta}_\sharp} \leq \|h^0\|^2_{H^{1+\theta}_\sharp} + \frac{27}{8}(\max h^0)^3\|f\|^2_{L^2(0,T_0;H^1_\sharp)} \quad \forall\, t\in[0,T_0].$$

2.2. **Further results on blow-up and the case of large θ.** In this section, we focus on the possible absence of blow-up. In particular, we derive a criterion preventing from collapse of the fluid film. Let first provide a criterion (on the source term f) which reduces blow-up to a possible collapse.

Proposition 4. *Let $T > 0$ and $f \in L^2_{loc}([0,T); H^1_\sharp) \cap H^1(0,T; H^{-(1+\theta)}_\sharp)$. Given $h^0 \in H^{1+\theta}_\sharp$ such that $\min h^0 > 0$ and (T_*, h) the unique non-extendable solution to (2.1) satisfying (2.2), we have the alternative:*

- *either $T_* = T$*
- *or $T_* < T$ and $\liminf_{t \to T^*} \min h(t,\cdot) = 0$.*

We emphasize that, with respect to **Corollary 3**, the new assumption is that $f \in H^1(0,T; H^{-(1+\theta)}_\sharp)$ and the new result is that the $H^{1+\theta}_\sharp$-norm may not blow-up under this assumption.

Proof. Under the assumptions of **Proposition 4**, let consider (T_*, h) the unique non-extendable solution to (2.1) satisfying (2.2). We recall that we have(2.3) valid and more precisely:

$$\frac{1}{2}\|h(t,\cdot)\|^2_{H^{1+\theta}_\sharp} + \int_0^t \int_{\mathbb{T}} h^3 |\partial_x p|^2 \leq \frac{1}{2}\|h^0\|^2_{H^{1+\theta}_\sharp} - \int_0^t \int_{\mathbb{T}} h^3 \partial_x f \partial_x p, \quad \forall\, t < T_*.$$

On the other hand, on any compact subset $[0, T_0]$ of $[0, T_*)$, we have that $f \in L^2(0, T_0; H^1_\sharp)$ so that f is an admissible multiplier of (2.1). We obtain after integration:

$$\int_{\mathbb{T}} f(t,\cdot) h(t,\cdot) + \int_0^t \int_{\mathbb{T}} h^3 |\partial_x f|^2 = \int_{\mathbb{T}} f(0,\cdot) h^0 + \int_0^t \int_{\mathbb{T}} h \partial_t f + \int_0^t \int_{\mathbb{T}} h^3 \partial_x f \partial_x p.$$

By combination, this yields that:

$$Y(t) := \|h(t,\cdot)\|^2_{H^{1+\theta}_\sharp} + 2\int_{\mathbb{T}} f(t,\cdot) h(t,\cdot),$$

satisfies, for any $t < T_*$

$$\frac{1}{2} Y(t) \leq \frac{1}{2} Y(0) + \int_0^t \int_{\mathbb{T}} h \partial_t f. \tag{2.6}$$

We note here that, for any $t < T_*$ there holds:

$$Y(t) \geq \frac{1}{2}\|h(t,\cdot)\|^2_{H^{1+\theta}_\sharp} - 8\|f(t,\cdot)\|^2_{H^{-(1+\theta)}_\sharp}, \tag{2.7}$$

and

$$\begin{aligned}\int_{\mathbb{T}} h \partial_t f &\leq \frac{1}{2}\|h\|^2_{H^{1+\theta}_\sharp} + \frac{1}{2}\|\partial_t f\|^2_{H^{-(1+\theta)}_\sharp} \\ &\leq Y(t) + 8\|f(t,\cdot)\|^2_{H^{-(1+\theta)}_\sharp} + \frac{1}{2}\|\partial_t f\|^2_{H^{-(1+\theta)}_\sharp}.\end{aligned}$$

Introducing this identity into (2.6) and applying a Gronwall inequality, we conclude that:

$$Y(t) \leq \exp(2t) Y(0) + 16 \exp(2T) \|f\|^2_{H^1(0,T;H^{-(1+\theta)}_\sharp)} \quad \forall\, t < T_*.$$

Since $H^1(0,T; H^{-(1+\theta)}_\sharp) \subset C([0,T]; H^{-(1+\theta)}_\sharp)$ we infer from (2.7) that $\|h\|_{H^{1+\theta}_\sharp}$ is bounded on $[0, T_*)$ thus reducing possible blow-up of the non-extendable solution to vanishing of h. □

We remark that, not only the previous proof guarantees that blow-up arises because of collapse, but also that the $H^{1+\theta}_\sharp$-norm of the solution is bounded up to T_* (since we assume integrability of f up to T). Namely, there exists $M(h^0, f)$ such that:

$$\|h(t,\cdot)\|_{H^{1+\theta}_\sharp} \leq M(h^0, f) \quad \forall\, t < T_*. \tag{2.8}$$

We arrive now to the first main result of this section. We prove that no collapse nor blow-up arise, for source term f which has sufficient time-regularity, when θ is sufficiently large. This is the content of the following theorem:

Theorem 5. *Let $T > 0$ and $f \in L^2_{loc}([0,T); H^1_\sharp)$ such that:*

$$f \in L^2(0,T;H^{-\theta}_\sharp), \quad \partial_t f \in L^2(0,T;H^{-(1+\theta)}_\sharp).$$

If $\theta > 1/2$, *for any $h^0 \in H^{1+\theta}_\sharp$ such that $\min h^0 > 0$, the unique non-extendable solution to (2.1) satisfying (2.2) is global.*

Proof. With the assumptions of our theorem, we have in particular that $f \in H^1(0,T;H^{-(1+\theta)}_\sharp)$. In this case, the previous proposition applies and (2.8) holds true. Our proof thus reduces to obtaining a bound from below for the non-extendable solution as long as it exists. This bound shall imply that h cannot vanish prior to T (thus implying that $T_* = T$).

On any compact subset $[0,T_0]$ of $[0,T_*)$ we have that $h \in C([0,T_0];H^1_\sharp)$ with $\min h \geq c_0 > 0$. Consequently $w := -1/h^2 \in L^2(0,T_0;H^1_\sharp)$ is an admissible multiplier for (2.1). Arguing that:

$$\int_0^t \int_{\mathbb{T}} \partial_t h w = \int_{\mathbb{T}} \frac{1}{h(t,\cdot)} - \int_{\mathbb{T}} \frac{1}{h^0}$$

$$\int_0^t \int_{\mathbb{T}} \partial_x (h^3 \partial_x f) w = -2 \int_0^t \int_{\mathbb{T}} f \partial_{xx} h$$

$$\int_0^t \int_{\mathbb{T}} \partial_x (h^3 \partial_x p) w = -2 \int_0^t \|h\|^2_{H^{2+\theta}_\sharp},$$

we conclude that,

$$\int_{\mathbb{T}} \frac{1}{h(t,\cdot)} + 2\int_0^t \|h\|^2_{H^{2+\theta}_\sharp} \leq \int_{\mathbb{T}} \frac{1}{h^0} + \int_0^t \int_{\mathbb{T}} f \partial_{xx} h \quad \forall t < T_*.$$

Applying the duality $(H^{-\theta}_\sharp, H^{\theta}_\sharp)$ to bound the last term, we end up with:

$$\int_{\mathbb{T}} \frac{1}{h(t,\cdot)} + \int_0^t \|h\|^2_{H^{2+\theta}} \leq \int_{\mathbb{T}} \frac{1}{h^0} + \int_0^t \|f\|^2_{H^{-\theta}_\sharp} \quad \forall t < T_*. \tag{2.9}$$

Finally, combining this inequality with (2.8) we obtain that there exists a constant $M(h^0,f)$ such that:

$$\|h\|_{H^{1+\theta}_\sharp} + \int_{\mathbb{T}} \frac{1}{h(t,\cdot)} \leq M(h^0,f) \quad \forall t < T_*.$$

as long as $T_* \leq T$. We apply then Lemma 14 in Appendix B which yields that there exists a constant $c(h^0,f)$ depending only on $M(h^0,f)$ such that:

$$\min h \geq c(h^0,f) \quad \forall t < T_*.$$

This ends up the proof. □

We emphasize that the latter proof not only ensures that the solution is global, but it also proves that the fluid film does not collapse at time T. This is to be compared with the content of the next subsection.

2.3. **Construction of collapsing solutions.** We complete the study of the thin film equation by constructing scenarios of collapsing solutions for θ sufficiently small. We summarize the construction in the following theorem.

Theorem 6. *Let $\theta < 1/2$ and $T > 0$. There exists initial data $h^0 \in C^\infty_\sharp$ and a solution to (2.1)-(2.2) on $[0,T)$ such that the associated source term f satisfies*

$$f \in L^2_{loc}([0,T);H^1_\sharp) \cap L^2(0,T;H^{-\theta}_\sharp),$$

and for which collapse occurs at time T.

We underline that one shortcoming of the construction is that we do not have $\partial_t f \in L^2(0,T;H_\sharp^{-(1+\theta)})$. This condition has been introduced to guarantee that the solution h remains bounded in $H_\sharp^{1+\theta}$. However, this property is *a priori* independent of the collapse issue.

The remainder of this section is devoted to the proof of **Theorem 6**. We fix $\theta < 1/2$ and we construct explicit solutions with explicit source terms. Namely, we introduce an even function $\chi \in C^\infty(\mathbb{R})$ such that:

$$\mathbf{1}_{[-1/8,1/8]} \leq \chi \leq \mathbf{1}_{[-1/4,1/4]}.$$

We then set:

$$h(t,x) = \chi(x)(\eta(t)+x^2)^{\frac{\alpha}{2}} + (1-\chi(x))\lambda(t) \quad \forall\, x \in [-1/2,1/2],$$

that we extend by 1-periodicity on $\mathbb{R}$. Here we introduced η and λ two scalar smooth functions defined on $[0,T)$. We will choose $\eta(t) = (T-t)^\beta$ and λ so that the constant-mean condition is fulfilled by h. The symbols α and β stand for fixed strictly positive exponents. With these conventions, we obtain a smooth h as long as $t < T$. In particular $h(0,\cdot) := h^0 \in C_\sharp^\infty \subset H_\sharp^{1+\theta}$. Our method of proof consists then in plugging the ansatz for h into (2.1), computing the source term f and looking for conditions on α and β so that f satisfies the conditions of our theorem with our given θ.

We start with explicit expressions for λ. The parameter λ is chosen so that the mean of h matches its initial value for all time. Let denote by $\overline{h^0}$ this value. We must then choose:

$$\lambda(t) = \frac{\overline{h^0} - \displaystyle\int_{-1/2}^{1/2} \chi(x)(\eta(t)+x^2)^{\alpha/2}\mathrm{d}x}{\displaystyle\int_{-1/2}^{1/2}(1-\chi(x))\mathrm{d}x}. \tag{2.10}$$

We remark that, choosing $\bar{h}_0$ sufficiently large (depending on T but independent of other parameters as long as they remain bounded) we guarantee that there exists $\lambda_0 > 0$ and $K > 0$ for which:

$$\lambda_0 \leq \lambda(t) \leq K \qquad |\dot{\lambda}| \leq K|\dot{\eta}|\eta^{\frac{\alpha-1}{2}} \qquad \forall\, t \in [0,T). \tag{2.11}$$

As for f, we choose $f = H - p$ where $p = (-\partial_{xx})^{1+\theta}h$ and $H(t,\cdot)$ is a solution to

$$\partial_x(h^3\partial_x H) = \partial_t h \qquad \int_{\mathbb{T}} H = 0. \tag{2.12}$$

Since h contains no singularity prior to T we have obviously that $f \in L^2_{loc}([0,T);H^1_\sharp)$. The main difficulty is to show that

$$f \in L^2(0,T;H_\sharp^{-\theta}).$$

To this end, we simply prove independently that p and H satisfy both estimates.

Computation of p. Since $p = (-\partial_{xx})^{1+\theta}h$ we have:

$$\|p\|_{L^2(0,T;H_\sharp^{-\theta})} \leq \|h\|_{L^2(0,T;H_\sharp^{2+\theta})}.$$

To control the h norms appearing on the right-hand side of these inequalities, we split $h = h_{sing} + h_{reg}$ where:

$$h_{sing}(t,x) = \chi(x)(\eta(t)+x^2)^{\alpha/2} \quad \forall\, x \in (-1/2,1/2)$$
$$h_{reg}(t,x) = (1-\chi(x))\lambda(t).$$

We bound separately both terms by applying the interpolation inequalities:

$$\|u\|_{H_\sharp^{2+\theta}} \leq \|u\|_{H_\sharp^2}^{1-\theta}\|u\|_{H_\sharp^3}^{\theta} \quad \forall\, u \in C_\sharp^\infty.$$

With explicit computations of the Sobolev norms of h_{sing} with integer exponent under the condition that $\alpha < 1$, we have:

$$\|h(t,\cdot)\|_{H_\sharp^{2+\theta}} \leq K\left(\eta^{\frac{1}{2}(\alpha-\frac{3}{2})-\frac{\theta}{2}} + |\lambda|\right), \tag{2.13}$$

for some absolute constant K. At this point, we use the explicit computation (2.10) of λ implying that we have a bound from above on $(0,T)$. In order to obtain that p satisfies the required integrability: $p \in L^2(0,T;H^{-\theta}_\sharp)$ we replace $\eta(t) = (T-t)^\beta$ in (2.13) and verify that we obtain something in $L^2(0,T)$. This yields the condition:

$$\alpha \in (0,1), \quad \beta < \frac{1}{3/2+\theta-\alpha}. \tag{2.14}$$

Computation of H. To derive similar bounds for H, we provide at first an explicit computation of H. First, we note that there exists a constant $C_0 \in \mathbb{R}$ for which:

$$\partial_x H(t,x) = \frac{1}{(h(t,x))^3}\int_0^x \partial_t h(t,y)\mathrm{d}y + \frac{C_0}{(h(t,x))^3} \quad \forall\, x \in (-1/2,1/2).$$

The constant is fixed by requiring that the mean of the right-hand side on $(-1/2,1/2)$ vanishes. Since h is even the first part of the right-hand side is odd and thus $C_0 = 0$. We have then:

$$H(t,x) = c_0 + \int_0^x \frac{1}{(h(t,y))^3}\int_0^y \partial_t h(t,z)\mathrm{d}z\mathrm{d}y \quad \forall\, x \in (-1/2,1/2),$$

with $c_0 \in \mathbb{R}$. The constant c_0 handles the mean-free condition. However, in the family of functions H defined above with c_0 varying, the one with 0 mean minimizes the L^2-norm. Hence, we will derive an estimate on $\|H\|_{L^2_\sharp}$ for a chosen constant c_0. This estimate will then be satisfied *a fortiori* by the mean-free H. We note also that H is even so that we will compute only norms on $(0,1/2)$ in what follows. In the sequel, we fix c_0 so that:

$$H(t,x) = -\int_x^{1/8} \frac{1}{(h(t,y))^3}\int_0^y \partial_t h(t,z)\mathrm{d}z\mathrm{d}y \quad \forall\, x \in (0,1/2).$$

We underline that the upper bound $1/8$ is chosen here in consistency with the definition of χ. With this choice, we have, for $x \in (0,1/8)$:

$$H(t,x) = -\frac{\alpha}{2}\int_x^{1/8} \frac{1}{(\eta+y^2)^{3\alpha/2}}\int_0^y \dot{\eta}(\eta+z^2)^{\alpha/2-1}\mathrm{d}z\mathrm{d}y, \tag{2.15}$$

and for $x \in (1/8,1/2)$:

$$H(t,x) = \int_{1/8}^x \frac{1}{((\eta+y^2)^{\alpha/2}\chi(y)+\lambda(1-\chi(y)))^3}\int_0^y \left[\dot{\eta}(\eta+z^2)^{\alpha/2-1}\chi(z)+\dot{\lambda}(1-\chi(z))\right]\mathrm{d}z\mathrm{d}y. \tag{2.16}$$

Consequently, we derive the following integrability bounds . For $x < 1/8$ we apply a change of variables to get:

$$|H(t,x)| \leq \frac{\alpha}{2}\frac{|\dot{\eta}|}{\eta^\alpha}J_{\sqrt{\eta}}(x/\sqrt{\eta}),$$

where

$$J_e(\xi) = \int_\xi^{1/(8e)} \frac{1}{(1+s^2)^{\frac{3\alpha}{2}}}\int_0^s (1+\tau^2)^{\frac{\alpha}{2}-1}\mathrm{d}\tau\mathrm{d}s, \qquad \forall\, \xi \in (0,1/(8e)), \qquad \forall\, e > 0.$$

When $\alpha \in (1/2,1)$ we obtain that $|J_e(\xi)| \leq (1+\xi)^{1-3\alpha} \in L^2(0,\infty)$. Hence:

$$\int_0^{1/8} |H(t,x)|^2\mathrm{d}x \leq K|\dot{\eta}|^2\eta^{\frac{1}{2}-2\alpha}.$$

For $x > 1/8$ we note that, for $y < x$, there holds $h(t,y) \geq (\chi(y)/8^{\alpha/2} + (1-\chi(y))/C)^3$ and

$$\left|\int_0^y \dot{\eta}(\eta+z^2)^{\alpha/2-1}\chi(z)+\dot{\lambda}(1-\chi(z))\mathrm{d}z\right| \leq K\left(|\dot{\eta}|\eta^{\frac{\alpha-1}{2}}+|\dot{\lambda}|\right).$$

This entails (with the bound (2.11)) that:

$$|H(t,x)| \leq K|\dot{\eta}|\eta^{\frac{(\alpha-1)}{2}} \qquad \int_{1/8}^{1/2} |H(t,x)|^2\mathrm{d}x \leq K|\dot{\eta}|^2\eta^{(\alpha-1)}.$$

Finally, we have:

$$\int_0^T \int_{\mathbb{T}} |H|^2 \leq C \int_0^T |\dot\eta|^2 \left(\eta^{\alpha-1} + \eta^{\frac{1}{2}-2\alpha}\right).$$

Replacing $\eta(t) = (T-t)^\beta$ we infer that $H \in L^2(0,T; H^{-\theta}_\sharp)$ in particular if ($H \in L^2(0,T;L^2_\sharp)$ and)

$$\beta > \frac{1}{\frac{5}{2} - 2\alpha}. \tag{2.17}$$

We recall that, since $\alpha > 1/2$ there holds:

$$\frac{1}{\frac{5}{2}-2\alpha} > \frac{1}{1+\alpha}.$$

Finally, our construction yields a solution with a source term enjoying the expected regularity if we can find $\alpha \in (1/2, 1)$ and $\beta > 0$ such that:

$$\frac{1}{\frac{5}{2}-2\alpha} < \beta < \frac{1}{3/2+\theta-\alpha}.$$

This is possible since $\theta < 1/2$. This concludes the proof.

3. Analysis of the Beam+Reynolds system (1.4)

In this section, we focus on the Beam-Reynolds system (1.4):

$$\begin{cases} \partial_t h - \partial_x(h^3 \partial_x p) = 0 \\ \partial_{tt} h + (-\partial_{xx})^{1+\theta} h = p + f. \end{cases} \tag{3.1}$$

We complement again the system with periodic boundary conditions and initial conditions:

$$h(0,\cdot) = h^0, \qquad \partial_t h(0,\cdot) = \dot h^0. \tag{3.2}$$

Our study follows the scheme of the previous section. First, we construct a theory of strictly positive strong solutions locally-in-time and we analyse then the influence of collapse on the possibility of blow-up.

3.1. **Strictly positive strong solutions.** As in the case of the thin film equation, we define strong solutions relying on suitable *a priori* remarks/estimates that are valid for sufficiently smooth solutions. First, we note that if (h,p) is a sufficiently smooth solution then the first equation of (3.1) implies that $\partial_t h$ has mean zero on $\mathbb{T}$. If we have a little time-regularity, this mean-free condition should then also be satisfied by initial data. At this point we realize, by a straightforward integration of the second equation, that the mean of p and f coincide. For simplicity, we restrict to source term f such that

$$\int_{\mathbb{T}} f = 0, \quad \forall\, t > 0.$$

To conclude, we can reinterpret the system as a coupling of the equation

$$\partial_{tt} h + (-\partial_{xx})^{1+\theta} h = p + f \quad \text{ on } (0,T)\times\mathbb{T}, \tag{3.3}$$

with a standard elliptic equation enabling to construct the pressure p:

$$\partial_x(h^3\partial_x p) = \partial_t h \text{ on } \mathbb{T} \text{ with } \int_{\mathbb{T}} p = \int_{\mathbb{T}} f = 0 \quad \text{for all } t < T. \tag{3.4}$$

To derive an estimate satisfied by (h,p), we multiply now the first equation of (3.1) with p and the second one with $\partial_t h$. Standard combination and integration by parts then imply that for all $0 \leq s < t \leq T$ there holds:

$$\frac{1}{2}\left[\|\partial_t h(t,\cdot)\|^2_{L^2_\sharp} + \|h(t,\cdot)\|^2_{H^{1+\theta}_\sharp}\right] + \int_s^t h^3|\partial_x p|^2 \leq \frac{1}{2}\left[\|\partial_t h(s,\cdot)\|^2_{L^2_\sharp} + \|h(s,\cdot)\|^2_{H^{1+\theta}_\sharp}\right] + \int_s^t \int_{\mathbb{T}} f\partial_t h. \tag{3.5}$$

Since, we can bound
$$\int_s^t \int_{\mathbb{T}} f\partial_t h \leq \int_0^t \int_{\mathbb{T}} \|f\|_{L^2_\sharp}\|\partial_t h\|_{L^2_\sharp},$$
we infer, by a standard Gronwall Lemma, that a source term $f \in L^1(0,T;L^2_\sharp)$ should yield a solution such that
$$\frac{1}{2}\left[\|\partial_t h(t,\cdot)\|^2_{L^2_\sharp} + \|h(t,\cdot)\|^2_{H^{1+\theta}_\sharp}\right].$$
is continuous with time. Adding a little further time-regularity, we propose the following definition of strong solution:

Definition 7. Let $(h^0,\dot{h}^0) \in H^{1+\theta}_\sharp \cap L^2_{\sharp,0}$ and $T>0$. Assume that $f \in L^1(0,T;L^2_{\sharp,0})$. We call $(h,p) : [0,T]\times\mathbb{T} \to \mathbb{R}^2$ a strong solution on $[0,T]$ to (3.1) with initial data (3.2) if

- $h \in C([0,T];H^{1+\theta}_\sharp) \cap C^1([0,T];L^2_\sharp)$ is such that $\inf_{[0,T]\times\mathbb{T}} h > 0$,
- $p \in L^2(0,T;H^1_\sharp \cap L^2_{\sharp,0})$,
- (h,p) satisfies the first equation of (3.1) in $L^2(0,T;H^{-1}_\sharp)$ and the second one in $L^2(0,T;H^{-(1+\theta)}_\sharp)$,
- $(h,\partial_t h)$ matches initial data (3.2) almost everywhere.

If $f \in L^1_{loc}([0,T);L^2_{\sharp,0})$ we call $(h,p) : [0,T)\times\mathbb{T}\to\mathbb{R}^2$ a strong solution on $[0,T)$ to (3.1) with initial data (3.2) if, for arbitrary $T' < T$ we have that $(h,p)_{|_{[0,T']}}$ is a strong solution on $[0,T']$.

The main result of this first part reads:

Proposition 8. *Given $(h^0,\dot{h}^0) \in H^{1+\theta}_\sharp \cap L^2_{\sharp,0}$, $T>0$ and $f \in L^1_{loc}([0,T);L^2_{\sharp,0})$, there exists $T_0 < T$ such that there is a unique strong solution on $[0,T_0]$ to (3.1) satisfying (3.2). Furthermore, this solution satisfies the energy estimate (3.5).*

Again, since strong solutions satisfy estimate (3.5) which implies that $(h,\dot{h})$ remains bounded in $H^{1+\theta}_\sharp \times L^2_{\sharp,0}$ we deduce by a straightforward dynamical system argument the following corollary:

Corollary 9. *Given $(h^0,\dot{h}^0) \in H^{1+\theta}_\sharp \cap L^2_{\sharp,0}$ satisfying $\min h^0 > 0$, $T>0$ and $f \in L^1_{loc}([0,T);L^2_{\sharp,0})$, there exists $T_* < T$ such that there is a unique non-extendable strong solution on $[0,T_*)$ to (3.1) satisfying (3.2). Furthermore, we have the following alternative:*

- *either $T_* = T$,*
- *or $T_* < T$ and there holds:*

$$\limsup_{t\to T^*} \|1/h(t,\cdot)\|_{L^\infty(\mathbb{T})} = \infty.$$

Proof of Proposition 8. We propose a proof of Proposition 8 *via* a classical perturbative approach. Since most of the arguments are classical, we only point out the main difficulties. We recall that we see a solution to (3.1) as a solution to the coupled system (3.3) with (3.4). So, we analyse the properties of both equations independently and then look to the existence of a fixed-point in the coupling.

Let first analyse the equation:
$$\partial_{tt}h + (-\partial_{xx})^{1+\theta}h = F \quad h(0)=\eta^0 \quad \partial_t h(0) = \dot{\eta}^0. \tag{3.6}$$
This is the content of the following proposition.

Proposition 10. *Given $T>0$, $(\eta^0,\dot{\eta}^0) \in H^{1+\theta}_\sharp \times L^2_{\sharp,0}$ and $F \in L^1(0,T;L^2_{\sharp,0})$ there exists a unique solution to (3.6)*
$$h \in C([0,T];H^{1+\theta}_\sharp) \cap C^1([0,T];L^2_\sharp).$$
Furthermore, the application $\mathcal{L}_T : (\eta^0,\dot{\eta}^0,F) \mapsto h$ is linear continuous
$$H^{1+\theta}_\sharp \times L^2_{\sharp,0} \times L^1(0,T;L^2_{\sharp,0}) \to C([0,T];H^{1+\theta}_\sharp) \cap C^1([0,T];L^2_\sharp).$$
with a continuity constant $C_{\mathcal{L}}$ independent of $T<1$.

The proof of this proposition is standard and is left to the reader. We note here that for

$$\|F\|_{L^1(0,T;L^2_\sharp)} \leq 2\|f\|_{L^1(0,T;L^2_\sharp)},$$

we have then that the solution h to (3.6) with initial data $(h^0, \dot{h}^0)$ and source term F satisfies

$$\|h\|_{L^\infty(0,T;H^{1+\theta}_\sharp)} + \|\partial_t h\|_{L^\infty(0,T;L^2_\sharp)} \leq C_{\mathcal{L}} \left(\|h^0\|_{H^{1+\theta}_\sharp} + \|\dot{h}^0\|_{L^2_\sharp} + 2\|f\|_{L^1(0,T;L^2_\sharp)} \right).$$

We denote by M_0 the right-hand side of this inequality. Correspondingly we fix

$$\begin{aligned} X_T := \{ h \in C([0,T]; H^{1+\theta}_\sharp) \cap C^1([0,T]; L^2_\sharp) \quad \text{with} \quad \partial_t h \in C([0,T]; L^2_{\sharp,0}) \\ \text{s.t. } \|h\|_{L^\infty(0,T;H^{1+\theta}_\sharp))} + \|\partial_t h\|_{L^\infty(0,T;L^2_\sharp)} \leq M_0 \quad h(0) = h^0, \quad \partial_t h(0) = \dot{h}^0 \}, \end{aligned}$$

and

$$Y_T := \{ f \in L^1(0,T; L^2_{\sharp,0}) \text{ s.t. } \|F\|_{L^1(0,T;L^2_\sharp)} \leq 2\|f\|_{L^1(0,T;L^2_\sharp)} \}.$$

We note that, by a standard interpolation inequality, any $h \in X_T$ satisfies:

$$h \in C^{0,\frac{1}{1+\theta}}([0,T]; C_\sharp) \text{ with } \|h\|_{C^{0,\frac{1}{1+\theta}}([0,T];C_\sharp)} \leq M_0. \tag{3.7}$$

Consequently, up to restrict the size of T depending only on M_0 and h^0 we can enforce that $\min h \geq \min h^0/2$ for any $h \in X_T$. We assume T_0 satisfies this restriction from now on.

We proceed with the analysis of (3.4). We first provide the following lemma.

Proposition 11. *Let $h \in H^{1+\theta}_\sharp$ such that $\min h > 0$ and $w \in L^2_{\sharp,0}$ there exists a unique $p \in H^1_\sharp$ solution to*

$$\partial_x (h^3 \partial_x p) = w \text{ with } \int_{\mathbb{T}} p = 0. \tag{3.8}$$

By classical arguments, we provide also the following estimates. Given (h, w) satisfying the assumptions of the theorem, the unique solution p to (3.8) satisfies:

$$\|p\|_{H^1_\sharp} \leq \frac{C}{(\min h)^3} \|w\|_{L^2_\sharp},$$

for some absolute constant $C > 0$. Given (h_1, w_1) and (h_2, w_2) two pairs satisfying the assumptions of our theorem and denoting

$$\bar{h}_0 := \min(\min h_1, \min h_2) \quad \tilde{M}_0 := \max(\|h_1\|_{H^{1+\theta}_\sharp}, \|h_2\|_{H^{1+\theta}_\sharp})$$

we also obtain by a difference-multiplier argument that the two associated solutions (p_1, p_2) satisfy:

$$\|p_1 - p_2\|_{H^1_\sharp} \leq C \frac{\tilde{M}_0^2}{\bar{h}_0^3} \left(\|h_1 - h_2\|_{C_\sharp} + \|w_2 - w_1\|_{L^2_\sharp} \right).$$

Combining the previous proposition with these remarks we infer the following proposition.

Proposition 12. *Given $h \in X_T$ there is a unique solution $\mathcal{NL}(h) \in L^2(0,T; H^1_\sharp)$ to (3.8). Furthermore, we have:*

$$\|\mathcal{N}L(h)\|_{L^2(0,T;H^1_\sharp)} \leq \frac{C\sqrt{T}}{(\min h_0/2)^3} \|h\|_{X_T},$$

and, if $(h_1, h_2) \in X_T$ there holds:

$$\|\mathcal{N}L(h_2) - \mathcal{N}L(h_1)\|_{L^2(0,T;H^1_\sharp)} \leq C\sqrt{T} \frac{M_0^2}{\bar{h}_0^2} \left(\|h_1 - h_2\|_{X_T} \right).$$

To conclude, we note that, with the notations of the previous propositions, a solution (h,p) to our problem can be recovered from a fixed point of the mapping:

$$h \mapsto \mathcal{L}(h^0, \dot{h}^0, \mathcal{N}\mathcal{L}(h) + f).$$

With the results of the previous proposition, we remark that we can choose T_0 sufficiently small so that this mapping fixes the set X_T (closed subset of $C([0,T];H^{1+\theta}_\sharp) \cap C^1([0,T];L^2_\sharp)$) and is a contraction on this set. To obtain the energy estimate (3.5) we finally remark that a strong solution (h,p) has sufficient regularity to be a multiplier of (3.1). □

3.2. **Construction of collapsing solutions.** To complete the study, we consider now the construction of collapsing solutions with well-chosen initial data $(h_0, \dot{h}^0)$ and source term f. Our main result is summarized in the following theorem:

Theorem 13. *Let $\theta \in [0,3/4)$ and $T > 0$. There exists initial data $(h^0, \dot{h}^0) \in [C^\infty_\sharp]^2$ and a solution to (3.1)-(3.2) on $[0,T)$ such that the associated source term satisfies $f \in L^1(0,T;L^2_\sharp)$ and for which collapse occurs at time T.*

The remainder of the section is devoted to the proof of this theorem. We fix $\theta \in [0,1]$ and we look again for a solution of the form:

$$h(t,x) = \chi(x)(\eta(t) + x^2)^{\frac{\alpha}{2}} + (1-\chi(x))\lambda(t) \quad \forall\, x \in [-1/2, 1/2].$$

where $\eta(t) = (T-t)^\beta$. We refer to **Section 2.3** for more explanations on the symbols α, β and the truncation function χ. As in this section, we shall choose

$$\lambda(t) = \frac{\overline{h^0} - \displaystyle\int_{-1/2}^{1/2} \chi(x)(\eta(t)+x^2)^{\alpha/2}\mathrm{d}x}{\displaystyle\int_{-1/2}^{1/2}(1-\chi(x))\mathrm{d}x},$$

with a sufficiently large $\overline{h^0}$. This guarantees that the candidate h has constant mean $\overline{h^0}$ and that $\lambda(t) \geq \lambda_0 > 0$ on $(0,T)$.

Our strategy for the proof of **Theorem 13** is the same as for **Theorem 6**. We plug the ansatz for h into (3.1), we compute p by solving the second equation and then compute the source-term $f = \partial_{tt}h + (-\partial_{xx})^{1+\theta}h + p$. We provide sufficient conditions on α, β for the different terms of this expression to be in $L^1(0,T;L^2_\sharp)$ and we prove finally that for $\theta \in [0,1]$ we can construct a pair (α,β) so that all these conditions are matched.

Conditions related to the integrability of h. In the first equation of (3.1), a combination of $\partial_{tt}h$ and $(-\partial_{xx})^{1+\theta}h$ appear. For simplicity we enforce that this combination is in $L^1(0,T;L^2_\sharp)$ by requiring that both terms lie in this very space. So, we split the computations into two blocks.

Computation of h_{tt}. By standard computations we have that:

$$h_{tt}(t,x) = \left(\frac{\alpha}{2}\ddot{\eta}(\eta+x^2)^{\frac{\alpha}{2}-1} + \frac{\alpha}{2}(\frac{\alpha}{2}-1)|\dot{\eta}|^2(\eta+x^2)^{\frac{\alpha}{2}-2}\right)\chi + (1-\chi)\ddot{\lambda} \quad \forall\, x \in (-1/2,1/2),$$

where there exists a constant $K > 0$ for which:

$$|\dot{\lambda}| \leq \begin{cases} K|\dot{\eta}|\eta^{\frac{\alpha-1}{2}} \text{ if } \alpha < 1, \\ K|\dot{\eta}| \text{ if } \alpha > 1, \end{cases} \qquad |\ddot{\lambda}| \leq \begin{cases} K\left(|\ddot{\eta}|\eta^{\frac{\alpha-1}{2}} + |\dot{\eta}|^2\eta^{\frac{\alpha-3}{2}}\right) \text{ if } \alpha < 1, \\ K\left(|\ddot{\eta}| + |\dot{\eta}|^2\eta^{\frac{\alpha-3}{2}}\right) \text{ if } 1 < \alpha < 3, \\ K\left(|\ddot{\eta}| + |\dot{\eta}|^2\right) \text{ if } \alpha > 3. \end{cases} \tag{3.9}$$

By a standard scaling analysis of integral terms, we infer that:

$$\|h_{tt}(t,\cdot)\|_{L^2(\mathbb{T})} \leq \begin{cases} K|\ddot{\eta}|\left(\eta^{\frac{\alpha}{2}-\frac{3}{4}}+\eta^{\frac{\alpha-1}{2}}\right)+K|\dot{\eta}|^2\left(\eta^{\frac{\alpha}{2}-\frac{7}{4}}+\eta^{\frac{\alpha-3}{2}}\right), \text{ if } \alpha<1, \\ K|\ddot{\eta}|(1+\eta^{\frac{\alpha}{2}-\frac{3}{4}})+K|\dot{\eta}|^2\left(\eta^{\frac{\alpha}{2}-\frac{7}{4}}+\eta^{\frac{\alpha-3}{2}}\right), \text{ if } 1<\alpha<3/2, \end{cases}$$

for some absolute constant $K>0$. Replacing $\eta=\eta(t)=(T-t)^\beta$ and restricting to the case $1<\alpha<3/2$, we obtain that $h_{tt}\in L^1(0,T;L^2_\sharp)$ provided that

$$\beta > \frac{4}{2\alpha+1}. \tag{3.10}$$

Computation of $(-\partial_{xx})^{1+\theta}h$.

This term is estimated through interpolation as in **Section 2.3**. More precisely we have

$$\left\|(-\partial_{xx})^{1+\theta}h\right\|_{L^2(\mathbb{T})} \leq C\,\|h\|_{H^{2(1+\theta)}(\mathbb{T})} \leq C\,\|h\|_{H^2(\mathbb{T})}^{(1-\theta)}\,\|h\|_{H^4(\mathbb{T})}^{\theta}.$$

We note here that h holds singularities only in 0 so that, up to bounded terms, the Sobolev norms of h can be computed in $(-1/8,1/8)$ only, where

$$h(t,x)=(\eta+x^2)^{\alpha/2}.$$

Furthermore, by a scaling argument, one notes that a differentiation in x of h inside $(-1/8,1/8)$ corresponds to dividing h by half a power of $(\eta+x^2)$. We have then that:

$$|\partial_{xx}h(t,x)|\leq(\eta+x^2)^{\alpha/2-1} \qquad |\partial_{xxxx}h(t,x)|\leq(\eta+x^2)^{\alpha/2-2} \quad \forall\, x\in(-1/8,1/8),$$

and, by a standard scaling argument, there exists an absolute $K>0$ for which:

$$\|h\|_{H^2(\mathbb{T})}\leq K(1+\eta^{\frac{\alpha}{2}-\frac{3}{4}}), \quad \|h\|_{H^4(\mathbb{T})}\leq K\left(1+\eta^{\frac{\alpha}{2}-\frac{7}{4}}\right).$$

Using interpolation, we obtain

$$\|h\|_{H^{2(1+\theta)}(\mathbb{T})}\leq K\left(1+\eta^{(\frac{\alpha}{2}-\frac{3}{4})-\theta}\right).$$

Plugging again $\eta=\eta(t)=(T-t)^\beta$, we obtain finally that $(-\partial_{xx})^{1+\theta}h\in L^1(0,T;L^2_\sharp)$ under the condition:

$$\beta\left(\left(\frac{\alpha}{2}-\frac{3}{4}\right)-\theta\right)>-1,$$

which yields (recall that $\alpha<3/2$) :

$$\beta<\frac{4}{3+2(2\theta-\alpha)}. \tag{3.11}$$

Conditions related to integrability of the pressure p. . We note that p is computed via solving the same system as H in **Section 2.3** (up to a normalizing factor). Namely, since we assume by convention that p is mean-free, we have:

$$\partial_x(h^3\partial_x p)=\partial_t h \qquad \int_{\mathbb{T}} p = 0.$$

Consequently, we can reproduce here the computations of H to get that:

$$p(t,x)=c_0+\int_0^x\frac{1}{h(t,x)^3}\int_0^y\partial_t h(t,z)\mathrm{d}z\mathrm{d}y \quad \forall\, x\in(-1/2,1/2),$$

and, under the condition that $\alpha>3/2$, that $p\in L^1(0,T;L^2_\sharp)$ if:

$$\int_0^T|\dot{\eta}|(1+\eta^{\frac{1}{4}-\alpha})<\infty.$$

Replacing η with its chosen value, we obtain that this condition is satisfied under the condition that

$$\alpha<\frac{5}{4}.$$

In conclusion, our construction yields a suitable f if we can construct a pair (α,β) satisfying:

$$\alpha \in (1,5/4) \qquad \frac{4}{2\alpha+1} < \beta < \frac{4}{3+2(2\theta-\alpha)}.$$

Such a pair (α,β) exists for $\theta \in [0,3/4)$. Indeed, in that case, we can choose $\alpha > 1/2+\theta$ matching the condition $\alpha \in (1,5/4)$. Furthermore, for such a α, there holds:

$$\frac{4}{2\alpha+1} < \frac{4}{3+2(2\theta-\alpha)}.$$

4. Conclusion

In the computations above, we showed that the two versions of thin-film equations for the film+elastic structure problem handle collapse in a different way. Firstly, the thin film equation exhibits a collapse for $\theta < 1/2$ with a structure-graph with very low regularity (typically $C^{0,\alpha}$ with $\alpha < 1$). On the other hand, the beam+reynolds system yields a collapse for $\theta < 3/4$ and a $C^{1,\alpha}$ structure-graph with $\alpha < 1/4$. One may argue that the choice of source term regularity are different in both case which may explain the differences between the two systems. However, it is worth pointing out that we cannot exchange the regularities between both systems. If we require further time regularity of source terms in the beam+reynolds system, the candidate we chose cannot produce a collapse. If we require further space regularity of source terms in the thin film equation, the candidate we constructed are not eligible neither. In conclusion, the two asymptotic models seem to indicate that collapse is to be expected in the case $\beta < 1/2$. But, one may wonder which of these approximate models captures the exact behaviour of solutions to the initial system. To this end, one possible lead is to construct a lifting operator transforming systematically solutions to the approximating system (thin film equations or reynolds+structure system) into a solution to the original film+structure system.

Appendix A. Non-dimensionalization

In this section, we give some details on the non-dimensionalization of equations (1.1)-(1.2). We recall this non-dimensionalization is computed by assuming that the motion of the system is prescribed by the initial pinching of the structure. Source terms will be computed in order to fit the scales corresponding to this assumption. So, we fix characteristics scales:

time: T	width: L	height: H
speed: U	pressure: P	external force: F

with the compatibility conditions $U = H/T$. Correspondingly, we perform the change of unknowns:

$$h(t,x) = Hh^*\left(\frac{t}{T},\frac{x}{L}\right),$$

$$u_1(t,x,y) = \frac{UL}{H}u_1^*\left(\frac{t}{T},\frac{x}{L},\frac{y}{H}\right), \qquad u_2(t,x,y) = Uu_2^*\left(\frac{t}{T},\frac{x}{L},\frac{y}{H}\right),$$

$$p(t,x,y) = P\,p^*\left(\frac{t}{T},\frac{x}{L},\frac{y}{H}\right), \qquad f(t,x,y) = Ff^*\left(\frac{t}{T},\frac{x}{L},\frac{y}{H}\right).$$

We then plug this anzatz into (1.1)-(1.2). We focus on thin films in the sense that the aspect ratio H/L is small and we normalize the pressure, resp. the forcing term, so that

$$\frac{PH^2}{\mu LU} = \kappa_p \quad \text{resp.} \quad \frac{FH^2}{\mu LU} = \kappa_f,$$

are both of order 1. Assuming that the motion is slow so that $U << 1$ and especially that the viscosity dominates the flow (so that the Reynolds number $\rho_f UH/\mu << 1$) we obtain that at first order, the fluid equations reduce to the classical Reynolds system:

$$\partial_1 u_1^* + \partial_2 u_2^* = 0 \qquad \partial_{22} u_1^* = \kappa_p \partial_1 p^* \quad \partial_2 p^* = 0.$$

Introducing the no-slip boundary conditions on the bottom boundary of the film, we can compute explicitly the fluid velocity-field w.r.t. p and reinterpret the no-slip boundary conditions on the upper boundary as a compatibility condition between pressure and structure velocity which reads:

$$\partial_t h^* - \frac{\kappa_p}{12} \partial_1 \left((h^*)^3 \partial_1 p^* \right) = 0. \tag{A.1}$$

As for the structure equation, we obtain in non-dimensional form, keeping only dominating terms on the right-hands side:

$$\frac{\rho_s HU}{\mu L} \partial_{tt} h^* + \frac{\alpha HT}{\mu L^{(2+2\theta)}} (-\partial_{11})^{1+\theta} h^* = \kappa_p p^* + \kappa_f f^*.$$

At this point, we assume that the mechanical properties of the structure impose that

$$\frac{\alpha HT}{\mu L^{(2+2\theta)}} = \alpha^*,$$

is of order 1. We distinguish then two cases. Either we have a light structure so that $\rho_s HU/\mu L << 1$ and we delete the inertia term in the previous equation. We can then interpret this equation as computing the pressure with respect to h^* and source term f^*. This leads to the following system:

$$\begin{cases} \partial_t h^* - \partial_1 \Big((h^*)^3 \partial_1 q^* \Big) = \partial_1 \left((h^*)^3 \partial_1 g^* \right) \\ \qquad\qquad\qquad\qquad q^* = (-\partial_{11})^{1+\theta} h^*. \end{cases} \tag{A.2}$$

Here we renormalized the source term f^* into g^* and pressure p^* into q^* to avoid unnecessary coefficients.

On the contrary, if we consider a thick structure such that

$$\frac{\rho_s HU}{\mu L} = Re^*,$$

is of order 1, we cannot turn the structure equation into an algebraic one and need to consider the coupled system:

$$\begin{cases} \partial_t h^* - \frac{\kappa_p}{12} \partial_1 \left((h^*)^3 \partial_1 q^* \right) = 0 \\ Re^* \partial_{tt} h^* + \alpha^* (-\partial_{11})^{1+\theta} h^* = q^* + g^*. \end{cases} \tag{A.3}$$

Appendix B. An abstract lemma on positive functions

To control by below a periodic function, we apply above the following lemma:

Lemma 14. *Let $\theta > 1/2$ and $M > 0$. There exists $c > 0$ depending only on M such that, for arbitrary $h \in H^{1+\theta}_\sharp$ satisfying;*

$$h > 0, \qquad \|h\|_{H^{1+\theta}_\sharp} + \int_{\mathbb{T}} \frac{1}{h} \le M,$$

there holds $\min h \ge c$.

Proof. We apply here that, since $\theta > 1/2$ we have that $H^{1+\theta}_\sharp \subset C^{1,\theta-1/2}_\sharp$. In particular, since $h > 0$ there exists $x_0 \in \mathbb{T}$ realizing $\underline{h}_0 := \min h$. For any $x \in [x_0 - 1/4, x_0 + 1/4]$ we have then:

$$h(x) \ge M|x - x_0| + \underline{h}_0,$$

and

$$\int_{-1/4}^{1/4} \frac{\mathrm{d}x}{M|x| + \underline{h}_0} \le \int_{\mathbb{T}} \frac{1}{h} \le M.$$

By a standard argument enabling to bound by below the left-hand side of these inequalities, we conclude that:

$$\frac{2}{M}\ln\left(1+\frac{M}{4\underline{h}_0}\right)\leq M \quad \text{and} \quad \underline{h}_0 \geq \frac{M}{4\left(\exp\left(M^2/2\right)-1\right)}.$$

This concludes the proof. □

References

[1] M. Badra and T. Takahashi. Gevrey regularity for a system coupling the Navier-Stokes system with a beam equation. *SIAM J. Math. Anal.*, 51(6):4776–4814, 2019.

[2] G. Bayada, M. Chambat, and I. Ciuperca. Comportement asymptotique d'un fluide dans un domaine mince variable en temps. *C. R. Acad. Sci. Paris Sér. I Math.*, 326(2):265–268, 1998.

[3] F. Bernis and A. Friedman. Higher order nonlinear degenerate parabolic equations. *J. Differential Equations*, 83(1):179–206, 1990.

[4] A. L. Bertozzi and M. Pugh. The lubrication approximation for thin viscous films: regularity and long-time behavior of weak solutions. *Comm. Pure Appl. Math.*, 49(2):85–123, 1996.

[5] B. Muha M. Bukal. A review on rigorous derivation of reduced models for fluid-structure interaction systems. arXiv:2003.00066, Feb. 2020.

[6] C. H. Arthur Cheng, D. Coutand, and S. Shkoller. Navier-Stokes equations interacting with a nonlinear elastic biofluid shell. *SIAM J. Math. Anal.*, 39(3):742–800 (electronic), 2007.

[7] Daniel Coutand. Finite-time singularity formation for incompressible Euler moving interfaces in the plane. *Arch. Ration. Mech. Anal.*, 232(1):337–387, 2019.

[8] D. Gérard-Varet and M. Hillairet. Regularity issues in the problem of fluid structure interaction. *Arch. Ration. Mech. Anal.*, 195(2):375–407, 2010.

[9] D. Gérard-Varet and M. Hillairet. Computation of the drag force on a sphere close to a wall: the roughness issue. *ESAIM Math. Model. Numer. Anal.*, 46(5):1201–1224, 2012.

[10] D. Gérard-Varet, M. Hillairet, and C. Wang. The influence of boundary conditions on the contact problem in a 3D Navier-Stokes flow. *J. Math. Pures Appl. (9)*, 103(1):1–38, 2015.

[11] L. Giacomelli and F. Otto. Rigorous lubrication approximation. *Interfaces Free Bound.*, 5(4):483–529, 2003.

[12] C. Grandmont. Existence of weak solutions for the unsteady interaction of a viscous fluid with an elastic plate. *SIAM J. Math. Anal.*, 40(2):716–737, 2008.

[13] C. Grandmont and M. Hillairet. Existence of global strong solutions to a beam-fluid interaction system. *Arch. Ration. Mech. Anal.*, 220(3):1283–1333, 2016.

[14] C. Grandmont, M. Hillairet, and J. Lequeurre. Existence of local strong solutions to fluid-beam and fluid-rod interaction systems. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 36(4):1105–1149, 2019.

[15] M. Günther and G. Prokert. A justification for the thin film approximation of Stokes flow with surface tension. *J. Differential Equations*, 245(10):2802–2845, 2008.

[16] M. Hillairet. Lack of collision between solid bodies in a 2D incompressible viscous flow. *Comm. Partial Differential Equations*, 32(7-9):1345–1371, 2007.

[17] M. Hillairet, D. Seck, and L. Sokhna. Note on the fall of an axisymmetric body in a perfect fluid over a horizontal ramp. *C. R. Math. Acad. Sci. Paris*, 356(11-12):1156–1166, 2018.

[18] M. Hillairet and T. Takahashi. Collisions in three-dimensional fluid structure interaction problems. *SIAM J. Math. Anal.*, 40(6):2451–2477, 2009.

[19] M. Hillairet and T. Takahashi. Existence of contacts for the motion of a rigid body into a viscous incompressible fluid with the tresca boundary conditions. arXiv:1912.01882, December 2019.

[20] J.-G. Houot and A. Munnier. On the motion and collisions of rigid bodies in an ideal fluid. *Asymptot. Anal.*, 56(3-4):125–158, 2008.

[21] A. Munnier and K. Ramdani. Asymptotic analysis of a Neumann problem in a domain with cusp. Application to the collision problem of rigid bodies in a perfect fluid. *SIAM J. Math. Anal.*, 47(6):4360–4403, 2015.

[22] A. Quarteroni, M. Tuveri, and A. Veneziani. Computational vascular fluid dynamics: problems, models and methods. *Comput. Visual Sci.*, 2:163–197, 2000.

[23] L. Sabbagh. On the motion of several disks in an unbounded viscous incompressible fluid. *Nonlinearity*, 32(6):2157–2181, 2019.

[24] T. Witelski. Nonlinear dynamics of dewetting thin films. *AIMS Mathematics*, 5(5):4229–4259, 2020.

[25] V.N. Starovoĭtov. Nonuniqueness of a solution to the problem on motion of a rigid body in a viscous incompressible fluid. *J. Math. Sci.*, 130(4):4893–4898, 2005.

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