École Polytechnique Fédérale de Lausanne

M2 Internship Report

Null controllability for the 1-D heat equation via backstepping approach

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1 Introduction and notations

The null controllability of heat equations is a recurrent subject of research since several decades. Russel and Fattorini were the first to investigate it, using the moment ([11] and [7]). Since then, there have been a few other methods to prove the null controllability of heat equations. A standard proof of the null controllability of the heat equation with a control source term uses the duality between controllability and observation, it can be found in [2]. Recently, in [4], Jean-Michel Coron and Hoài-Minh Nguyên uses a backstepping approach to recover the null controllability for the heat equations with variable coefficients in space in one dimension with Dirichlet boundary control.

In this report, we attempt to extend this method to establish the null controllability for the heat equations with constant coefficients in space in one dimension with a constrained source term control.

1.1 Presentation of the problem

In the following, we denote $\langle \cdot, \cdot \rangle$ the scalar product of $L^2(0,1)$. We consider the controlled heat equation :

$$\begin{cases} \psi_t(t,x) = \psi_{xx}(t,x) + \varphi(x)u(t), & (t,x) \in (0,T) \times (0,1)\,, \\ \psi(t,0) = \psi(t,1) = 0, & t \in (0,1)\,, \\ \psi(0,x) = \psi_0(x), & x \in (0,1)\,. \end{cases} \tag{1}$$

where $T > 0$, $\varphi \in H^3(0,1) \cap H^1_0(0,1)$, $\psi_0 \in L^2(0,1)$ and $u : (0,T) \to \mathbb{R}$ refer to the control.

We recall the Dirichlet laplacian $A : D(A) \subset L^2(0,1) \to L^2(0,1)$ is defined by :

$$D(A) = H^1_0(0,1) \cap H^2(0,1), \quad \forall \psi \in D(A),\ A\psi = \psi_{xx}\,. \tag{2}$$

For $n \in \mathbb{N}^*$, we denote φ_n the eigenvector of A associated to the eigenvalue $-\lambda_n$ where $\lambda_n = (n\pi)^2$. We have :

$$\forall x \in (0,1),\ \varphi_n(x) = \sqrt{2}\sin(n\pi x)\,. \tag{3}$$

We recall that $\{\varphi_n\}_{n\in\mathbb{N}^*}$ is an Hilbert basis of $L^2(0,1)$. It is well-know that A is auto-adjoint and dissipative, hence generates a strongly continuous semigroup on $L^2(0,1)$. Now, let B be the continuous operator defined by :

$$B : u \in \mathbb{R} \longmapsto u \cdot \varphi \in L^2(0,1)\,. \tag{4}$$

The operator formulation of the system (1) is :

$$\begin{cases} \psi_t = A\psi + Bu, & t \in (0,T)\,, \\ \psi(0) = \psi_0\,. \end{cases} \tag{5}$$

The goal of this internship is to show the null-controllability of the system (5) using a backstepping approach.

1.2 Strategy

Step 1 : Rapid stabilization via backstepping.

Let $\lambda > 0$. We want to find two operators T^λ and K^λ such that if ψ denotes the solution of (5) with the feedback $u(t) = K^\lambda \psi(t)$ then $\widetilde{\psi} = T^\lambda \psi$ is a solution of :

$$\begin{cases} \widetilde{\psi}_t = A\widetilde{\psi} - \lambda\widetilde{\psi}, \quad t \in (0,T)\,, \\ \widetilde{\psi}(0) = T^\lambda \psi_0\,. \end{cases} \tag{6}$$

It is easy to find out that $\widetilde{\psi}$ verifies $\left\|\widetilde{\psi}(t)\right\| \leq e^{-\lambda t}\left\|T^\lambda \psi_0\right\|$. Hence, if T^λ is continuous invertible, we have :

$$\|\psi\| \leq \|T^\lambda\| \|(T^\lambda)^{-1}\| \, \|\psi_0\| \, e^{-\lambda t}\,, \tag{7}$$

This means the system (5) is stabilizable.

Step 2 : Null-controllability in finite time.

Now, if $\|T^\lambda\|\|(T^\lambda)^{-1}\|$ is sub-exponential in λ (for exemple if $\|T^\lambda\|\|(T^\lambda)^{-1}\| = O(\lambda^m)$ for some $m \geq 0$) then we will be able to find two increasing non negative sequences $\{\lambda_n\}_{n\in\mathbb{N}^*}$ and $\{t_n\}_{n\in\mathbb{N}^*}$ verifying :

$$\lambda_n \longrightarrow +\infty \quad \text{and} \quad t_n \longrightarrow T\,, \tag{8}$$

and such that if ψ is the solution of (5) with the following piecewise constant feeback control :

$$\forall n \in \mathbb{N}^*, \; \forall t \in]t_n\,, t_{n+1}[, \; u(t) = K^{\lambda_n}\psi(t)\,, \tag{9}$$

then :

$$\psi(T) = 0\,. \tag{10}$$

The system (5) is null-controllable.

1.3 Assumptions

As in [11] and [3], we will assume there exists a constant $c \geq 0$ such that :

$$\forall n \in \mathbb{N}^*, \; |\langle \varphi, \varphi_n \rangle| \geq \frac{c}{n^3}\,. \tag{11}$$

A straightforward computation gives :

$$\forall n \in \mathbb{N}^*, \; \langle \varphi, \varphi_n \rangle = \frac{\sqrt{2}}{(n\pi)^3}((-1)^n\varphi''(1) - \varphi''(0)) - \frac{1}{(n\pi)^3}\int_0^1 \varphi'''(x)\varphi_n(x)\,\mathrm{d}x\,. \tag{12}$$

Since :

$$\int_0^1 \varphi'''(x)\varphi_n(x)\,\mathrm{d}x \longrightarrow 0 \quad \text{when} \quad n \longrightarrow +\infty\,, \tag{13}$$

necessarily :

$$|\varphi''(1)| \neq |\varphi''(0)| \ , \tag{14}$$

otherwise, the assumption (11) would be absurd.

Moreover, in the following we will always assume there exists $c >$ such that :

$$\mathrm{d}\left(\lambda, \left\{\lambda_n - \lambda_k \,\middle|\, (n,k) \in (\mathbb{N}^*)^2\right\}\right) \geq c\,. \tag{15}$$

This assumption implies :

$$\forall (n,k) \in (\mathbb{N}^*)^2,\ |\lambda_n - \lambda_k - \lambda| \geq \min\left(\frac{1}{2}, \frac{c}{2\lambda}\right) |\lambda_n - \lambda_k| \ . \tag{16}$$

1.4 General setup

To see what kind of condition we have on the operator T^λ and K^λ, we start by applying T^λ to (5) :

$$T^\lambda \psi_t = \widetilde{\psi}_t = (T^\lambda A + T^\lambda B K^\lambda)\psi\,. \tag{17}$$

We want T^λ and K^λ to verify :

$$\widetilde{\psi}_t = (AT^\lambda - \lambda T^\lambda)\psi\,. \tag{18}$$

Hence, we will seek T^λ and K^λ such that :

$$T^\lambda A + T^\lambda B K^\lambda = AT^\lambda - \lambda T^\lambda\,. \tag{19}$$

As in [3], in a computation simplification purpose, we introduce the following condition :

$$T^\lambda B = B\,. \tag{20}$$

So, we are interested into the operator system :

$$T^\lambda A + BK^\lambda = AT^\lambda - \lambda T^\lambda\,, \tag{21}$$
$$T^\lambda B = B\,. \tag{22}$$

Now, we introduce some analytic tools we will use through the whole article. For $s \geq 0$, we define :

$$H^s_{(0)}(0,1) \coloneqq D(A^{s/2})\,. \tag{23}$$

To lighten the notation, we will denote $H^s_{(0)}(0,1) = H^s_{(0)}$. The following bilinear product :

$$\langle \phi, \psi \rangle_{H^s_{(0)}} = \sum_{n\geq 1} \lambda_n^s \langle \phi, \varphi_n \rangle \langle \psi, \varphi_n \rangle \tag{24}$$

defines an inner-product of $H^s_{(0)}$ which gives it an Hilbert space structure. In this paper, we will mainly use $H^2_{(0)}$ and $H^3_{(0)}$. We can give an explicit formula for these :

$$H^2_{(0)} = H^2(0,1) \cap H^1_0(0,1) \quad \text{and} \quad H^3_{(0)} = \left\{ \psi \in H^3(0,1) \cap H^1_0(0,1) \;\middle|\; \psi''(0) = \psi''(1) = 0 \right\} . \tag{25}$$

Using the backstepping approach as in [4] and [3], we will search T^λ and K^λ with the following form :

$$\begin{array}{cccccc} T^\lambda & : & H^s_{(0)}(0,1) & \longrightarrow & H^s_{(0)}(0,1) & \\ & & \psi & \longmapsto & \left[x \mapsto \int_0^1 k^\lambda(x,s)\psi(s)\,\mathrm{d}s \right] & , \end{array} \tag{26}$$

$$\begin{array}{cccccc} K^\lambda & : & H^r_{(0)}(0,1) & \longrightarrow & \mathbb{R} & \\ & & \psi & \longmapsto & \int_0^1 \alpha^\lambda(s)\psi(s)\,\mathrm{d}s & , \end{array} \tag{27}$$

where k^λ and α^λ are functions to determined and r and s some integers.

1.5 Heuristic

By assuming that :

$$k^\lambda(x,0) = k^\lambda(x,1) = k^\lambda(0,y) = k^\lambda(1,y) = 0, \quad \forall (x,y) \in (0,1)^2 , \tag{28}$$

we kill the border terms when we inject (26) in (21) :

$$\begin{cases} \Delta_y k^\lambda(x,y) - \Delta_x k^\lambda(x,y) + \lambda k^\lambda(x,y) + \alpha^\lambda(y)\varphi(x) = 0, & (x,y) \in (0,1)^2 , \\ \varphi(x) = \int_0^1 k^\lambda(x,s)\varphi(s)\,\mathrm{d}s, & x \in (0,1) , \\ k^\lambda(x,0) = k^\lambda(x,1) = k^\lambda(0,y) = k^\lambda(1,y) = 0, & (x,y) \in (0,1)^2 . \end{cases} \tag{29}$$

Following a spectral decomposition approach as in [4] and [11], we decompose k^λ and α^λ with the Hilbert basis $\{\varphi_n\}_{n\in\mathbb{N}^*}$. For all $y \in (0,1)$, we have :

$$k^\lambda(x,y) = \sum_{n\in\mathbb{N}^*} f^\lambda_n(x)\varphi_n(y) , \tag{30}$$

$$\alpha^\lambda(y) = \sum_{n\in\mathbb{N}^*} \alpha^\lambda_n \varphi_n(y) . \tag{31}$$

Hence, if we inject this in (29), we obtain for all $n \in \mathbb{N}^*$:

$$\begin{cases} -\lambda_n f^\lambda_n(x) - (f^\lambda_n)''(x) + \lambda f^\lambda_n(x) + \alpha^\lambda_n \varphi(x) = 0, & x \in 0,1) \\ f^\lambda_n(0) = f^\lambda_n(1) = 0 , & \\ \varphi(x) = \sum_{n\in\mathbb{N}^*} f^\lambda_n(x) \langle \varphi, \varphi_n \rangle , & x \in (0,1) . \end{cases} \tag{32}$$

We develop f^λ_n through the Fourier basis :

$$f^\lambda_n(x) = \sum_{k\in\mathbb{N}^*} \left\langle f^\lambda_n, \varphi_k \right\rangle \varphi_k(x), \quad x \in (0,1) . \tag{33}$$

(32) and (33) give :

$$\forall x \in (0,1),\ f_n^\lambda(x) = \alpha_n^\lambda \sum_{k\in\mathbb{N}^*} \frac{\langle \varphi, \varphi_k\rangle}{\lambda_n - \lambda_k - \lambda}\varphi_k(x)\,. \tag{34}$$

α^λ is determined by the relation $T^\lambda B = B$:

$$\varphi(x) = \sum_{n\in\mathbb{N}^*} f_n^\lambda(x)\,\langle \varphi, \varphi_n\rangle\ . \tag{35}$$

2 Properties of the transformations T^λ and K^λ

In all this section, we fix $\lambda > 0$ verifying (15).

2.1 Riesz basis for $H^2_{(0)}$ and $H^3_{(0)}$

The goal of this section is to determine a Riesz basis of $H^s_{(0)}$ for some $s \in \mathbb{N}^*$ which is adapted to the problem (see the appendix A for a quick introduction to Riesz bases).

We suppose $\alpha_n^\lambda \neq 0$ for all $n \in \mathbb{N}^*$. We will show a posteriori this assumption is verified (see section 2.7). Following the ideas in [3], we introduce a sequence of reals $\{\beta_n^\lambda\}_{n\in\mathbb{N}^*}$ and a sequence of functions g_n^λ by :

$$g_n^\lambda = \frac{\beta_n^\lambda}{\alpha_n^\lambda} f_n^\lambda = \beta_n^\lambda \sum_{k\in\mathbb{N}^*} \frac{\langle \varphi, \varphi_k\rangle}{\lambda_n - \lambda_k - \lambda}\varphi_k(x)\,, \tag{36}$$

In particular, g_n^λ verify the following equation :

$$\begin{cases} -(g_n^\lambda)''(x) + (\lambda - \lambda_n) g_n^\lambda(x) + \beta_n^\lambda \varphi(x) = 0, \quad x \in (0,1) \\ g_n^\lambda(0) = g_n^\lambda(1) = 0\,. \end{cases} \tag{37}$$

which is independent of α^λ. We will choose $\{\beta_n^\lambda\}_{n\in\mathbb{N}^*}$ such that $\{g_n^\lambda\}_{n\in\mathbb{N}^*}$ is a Riesz basis of $H^s_{(0)}$.

Remark 1. Because $\{\varphi_n\}_{n\in\mathbb{N}^*}$ is a Hilbert basis for $L^2(0,1)$, it is easy to show that $\left\{\varphi_n \lambda_n^{-s/2}\right\}_{n\in\mathbb{N}^*}$ is a Hilbert basis for $H^s_{(0)}$.

Proposition 1. *We define :*

$$\beta_n^\lambda = \frac{-\lambda}{\langle \varphi, \varphi_n\rangle\,\lambda_n}\,. \tag{38}$$

Then, $\left\{g_n^\lambda\right\}_{n\in\mathbb{N}^*}$ *defined by* (36) *is a Riesz basis for* $H^2_{(0)}$ *and* $\left\{g_n^\lambda \lambda_n^{-1/2}\right\}_{n\in\mathbb{N}^*}$ *is a Riesz basis for* $H^3_{(0)}$.

Remark 2. With the assumption (11) on the decay of the Fourier coefficients of φ, we have the following estimate on β_n^λ :

$$c\lambda n \leq \left|\beta_n^\lambda\right| \leq C\lambda n\,. \tag{39}$$

Proof of the proposition 1. The proof is based on the two characterization theorems 4 and 5 in the appendix A. We adapt the proof of the similar proposition in [3].

We begin by showing that $\{g_n^\lambda\}_n$ is quadratically close to $\left\{\varphi_n \lambda_n^{-s/2}\right\}_{n\in\mathbb{N}^*}$ for $s \in \{2,3\}$.

Lemma 1. *Let $s \in \{2,3\}$. We have :*

$$\sum_{n\in\mathbb{N}^*} \left\| \frac{\varphi_n}{\lambda_n^{s/2}} - \frac{g_n^\lambda}{\lambda_n^{(s-2)/2}} \right\|_{H^s_{(0)}}^2 < +\infty \,. \tag{40}$$

Proof of the lemma 1. Let $s \in \{2,3\}$. Let denote $S = \sum_{n\in\mathbb{N}^*} \left\| \frac{\varphi_n}{\lambda_n^{s/2}} - \frac{g_n^\lambda}{\lambda_n^{(s-2)/2}} \right\|_{H^s_{(0)}}^2$. We have :

$$\begin{aligned} S &= \sum_{n\in\mathbb{N}^*} \sum_{k\in\mathbb{N}^*} \lambda_k^s \left| \left\langle \frac{\varphi_n}{\lambda_n^{s/2}} - \frac{g_n^\lambda}{\lambda_n^{(s-2)/2}}, \varphi_k \right\rangle \right|^2 && (41)\\ &= \sum_{n\in\mathbb{N}^*} \left[\sum_{k\neq n} \lambda_k^s \left(\frac{\beta_n^\lambda}{\lambda_n - \lambda_k - \lambda} \right)^2 \frac{|\langle \varphi, \varphi_k\rangle|^2}{\lambda_n^{s-2}} \right] + \lambda_n^s \left| \frac{1}{\lambda_n^{s/2}} + \frac{\beta_n^\lambda}{\lambda} \frac{\langle \varphi, \varphi_n\rangle}{\lambda_n^{(s-2)/2}} \right|^2 . && (42) \end{aligned}$$

We begin by treating the case $s = 2$. Because of the definition of β_n^λ, we have :

$$\begin{aligned} S &\leq C\lambda^2 \sum_{n\in\mathbb{N}^*} \sum_{k\neq n} \frac{n^2}{k^2} \frac{1}{(n^2-k^2)^2} && (43)\\ &\leq C\lambda^2 \left(\sum_{n\in\mathbb{N}^*} \sum_{k<n} \frac{n^2}{k^2} \frac{1}{(n^2-k^2)^2} + \sum_{n\in\mathbb{N}^*} \sum_{k>n} \frac{n^2}{k^2} \frac{1}{(n^2-k^2)^2} \right) . && (44) \end{aligned}$$

For the first term, we have :

$$\begin{aligned} \sum_{n\in\mathbb{N}^*} \sum_{k<n} \frac{n^2}{k^2} \frac{1}{(n^2-k^2)^2} &= \sum_{n\in\mathbb{N}^*} \sum_{k<n} \frac{n^2}{k^2} \frac{1}{(n-k)^2(n+k)^2} && (45)\\ &\leq \sum_{n\in\mathbb{N}^*} \sum_{k<n} \frac{1}{(n-k)^2 k^2} \leq \left(\sum_{n\in\mathbb{N}^*} \frac{1}{n^2} \right)^2 < +\infty \,. && (46) \end{aligned}$$

For the second term, we have :

$$\begin{aligned} \sum_{n\in\mathbb{N}^*} \sum_{k>n} \frac{n^2}{k^2} \frac{1}{(k^2-n^2)^2} &\leq \sum_{n\in\mathbb{N}^*} \sum_{k>1} \frac{1}{((k+n)^2-n^2)^2} && (47)\\ &\leq \sum_{n\in\mathbb{N}^*} \sum_{k>1} \frac{1}{k^2(k+2n)^2} \leq \frac{1}{4} \left(\sum_{n\in\mathbb{N}^*} \frac{1}{n^2} \right)^2 < +\infty \,. && (48) \end{aligned}$$

For the case $s = 3$ there exists a constant $C \geq 0$ such that :

$$S \leq C\lambda^2 \sum_{n\in\mathbb{N}^*} \sum_{k\neq n} \frac{1}{(n^2-k^2)^2} \,. \tag{49}$$

We deal with this sum the same way than previously. This ends the proof of the lemma. □

Let $s \in \{2,3\}$. We denote :

$$\mathcal{B}_s^\lambda := \left\{ g_n^\lambda \lambda_n^{-(s-2)/2} \right\}_{n\in\mathbb{N}^*} . \tag{50}$$

Thanks to the lemma 1 and the theorem 4 (see appendix A), it remains to prove the ω-independance of $\mathcal{B}_s^\lambda$. Let $\{a_n\}_{n\in\mathbb{N}^*}$ be a sequence of reals such that :

$$\sum_{n\in\mathbb{N}^*} a_n g_n^\lambda = 0 \quad \text{in} \quad H^s_{(0)} . \tag{51}$$

We want to show that $a_n = 0$ for all $n \in \mathbb{N}^*$. Let $\ell \in \mathbb{N}$. Because convergence in $H^s_{(0)}$ implies convergence in $L^2(0,1)$, we can apply $A^{-\ell}$ to (51), we get :

$$\sum_{n\in\mathbb{N}^*} a_n A^{-\ell} g_n^\lambda = 0 \quad \text{in} \quad L^2(0,1) . \tag{52}$$

In fact, this series converges in $H^s_{(0)}$. To prove this statement, let $N \in \mathbb{N}$. We have, thank to the auto-adjointness of $\mathcal{A}^{-\ell}$:

$$\left\| \sum_{n=1}^N a_n A^{-\ell} g_n^\lambda \right\|^2_{H^s_{(0)}} = \sum_{k\in\mathbb{N}^*} \lambda_k^s \left| \left\langle \sum_{n=1}^N a_n A^{-\ell} g_n^\lambda, \varphi_k \right\rangle \right|^2 = \sum_{k\in\mathbb{N}^*} \lambda_k^s \left| \sum_{n=1}^N a_n \left\langle g_n^\lambda, A^{-\ell} \varphi_k \right\rangle \right|^2 \tag{53}$$

$$= \sum_{k\in\mathbb{N}^*} \frac{\lambda_k^s}{\lambda_n^\ell} \left| \left\langle \sum_{n=1}^N a_n g_n^\lambda, \varphi_k \right\rangle \right|^2 \leq \left\| \sum_{n=1}^N a_n g_n^\lambda \right\|^2_{H^s_{(0)}} . \tag{54}$$

So :

$$\sum_{n\in\mathbb{N}^*} a_n A^{-\ell} g_n^\lambda = 0 \quad \text{in} \quad H^s_{(0)} . \tag{55}$$

In view of (37), we find that :

$$A^{-1} g_n^\lambda = \frac{g_n^\lambda + \beta_n^\lambda A^{-1} \varphi}{\lambda_n - \lambda} . \tag{56}$$

And by recurrence, we see that :

$$\forall \ell \in \mathbb{N},\ A^{-\ell} g_n^\lambda = \frac{g_n^\lambda}{(\lambda_n - \lambda)^\ell} + \beta_n^\lambda \sum_{k=1}^{\ell} \frac{A^{-k}\varphi}{(\lambda_n - \lambda)^{\ell-k+1}} . \tag{57}$$

We first inject (56) in (55) :

$$\sum_{n\in\mathbb{N}^*} \frac{\beta_n^\lambda g_n^\lambda}{\lambda_n - \lambda} + \frac{a_n \beta_n^\lambda}{\lambda_n - \lambda} A^{-1} \varphi = 0 . \tag{58}$$

Let denote for $\ell \in \mathbb{N}^*$:

$$c_\ell^\lambda = \sum_{n\in\mathbb{N}^*} \frac{a_n \beta_n^\lambda}{(\lambda_n - \lambda)^\ell} . \tag{59}$$

By the previous equation, c_1^λ is a convergent serie and we have :

$$c_1^\lambda A^{-1}\varphi = -\sum_{n\in\mathbb{N}^*} \frac{\beta_n^\lambda g_n^\lambda}{\lambda_n - \lambda}\,. \tag{60}$$

By recurrence, we successively found that for all $\ell \in \mathbb{N}^*$, the series defining c_ℓ^λ is convergent and that :

$$\sum_{k=1}^{\ell} c_{\ell-k+1}^\lambda A^{-\ell}\varphi = -\sum_{n\in\mathbb{N}^*} \frac{\beta_n^\lambda g_n^\lambda}{(\lambda_n - \lambda)^\ell}\,. \tag{61}$$

Two cases are in front of us.

First, assume that for every $\ell \in \mathbb{N}^*$, we have $c_\ell = 0$. We will show that necessarily for all $n \in \mathbb{N}^*$, we have $a_n = 0$ and hence $\mathcal{B}_s^\lambda$ is a Riesz basis for $H^s_{(0)}$. We define :

$$G\colon z \in \mathbb{C} \longmapsto \sum_{n\in\mathbb{N}^*} \frac{a_n \beta_n^\lambda}{\lambda_n - \lambda} e^{\frac{z}{\lambda_n - \lambda}}\,. \tag{62}$$

Because c_1^λ is convergent, by uniform convergence on every compact set, G defines an entire function on $\mathbb{C}$. By assumption, we have :

$$\forall \ell \in \mathbb{N},\ G^{(\ell)}(0) = \sum_{n\in\mathbb{N}^*} \frac{a_n \beta_n^\lambda}{(\lambda_n - \lambda)^{\ell+1}} = 0\,, \tag{63}$$

hence $G = 0$. Let $\mathcal{C} := \{n \mid a_n \neq 0\}$ and assumme that $\mathcal{C} \neq \emptyset$. Let n_0 be the minimum of $\mathcal{C}$. Hence, we have :

$$\forall z \in \mathbb{C},\ 0 = G(z) e^{-\frac{z}{\lambda_{n_0} - \lambda}} = \frac{a_{n_0}\beta_{n_0}^\lambda}{\lambda_{n_0} - \lambda} + \sum_{n > n_0} \frac{a_n \beta_n^\lambda}{\lambda_n - \lambda} e^{z\left(\frac{1}{\lambda_n - \lambda} - \frac{1}{\lambda_{n_0} - \lambda}\right)}\,. \tag{64}$$

If we let $z \in \mathbb{R}$ goes to $+\infty$ then we found :

$$\frac{a_{n_0}\beta_{n_0}^\lambda}{\lambda_{n_0} - \lambda} = 0\,, \tag{65}$$

and so $a_{n_0} = 0$ which is absurd. Hence, $\mathcal{C} = \emptyset$ and $a_n = 0$ for all n.

Now, assume there exists $\ell \in \mathbb{N}^*$ such that $c_\ell \neq 0$. We will show that $\operatorname{span} \mathcal{B}_s^\lambda$ is dense in $H^s_{(0)}$ which is enough to prove that $\mathcal{B}_s^\lambda$ is a Riesz basis of $H^s_{(0)}$ (see theorem 5 in the appendix A). By recurrence, we can see from (61) that :

$$\forall \ell \in \mathbb{N}^*, A^{-\ell}\varphi \in \overline{\operatorname{span} \mathcal{B}_s^\lambda}\,. \tag{66}$$

Now, let $d \in \left[\operatorname{span} \mathcal{B}_s^\lambda\right]^\perp$. Hence :

$$\forall \ell \in \mathbb{N}^*, \left\langle A^{-\ell}\varphi, d\right\rangle = 0 = \sum_{n\in\mathbb{N}^*} \lambda_n^s \left\langle A^{-\ell}\varphi, \varphi_n\right\rangle \langle d, \varphi_n\rangle = \sum_{n\in\mathbb{N}^*} \lambda_n^{s-\ell} \langle \varphi, \varphi_n\rangle \langle d, \varphi_n\rangle\,. \tag{67}$$

Let's define :

$$G\colon z \in \mathbb{C} \longmapsto \sum_{n\in\mathbb{N}^*} \lambda_n^s \langle \varphi, \varphi_n \rangle \langle d, \varphi_n \rangle e^{\frac{z}{\lambda_n}} . \tag{68}$$

Because d and φ are in $H^s_{(0)}$, the serie above congerges uniformly, G is an entire function and we have :

$$\forall \ell \in \mathbb{N},\ G^{(\ell)}(0) = \sum_{n\in\mathbb{N}^*} \lambda_n^{s-\ell} \langle \varphi, \varphi_n \rangle \langle d, \varphi_n \rangle = 0 , \tag{69}$$

hence $G = 0$. As above, we found out that :

$$\forall n \geq 1,\ \langle \varphi, \varphi_n \rangle \langle d, \varphi_n \rangle = 0 \tag{70}$$

which implies $\langle d, \varphi_n \rangle = 0$. Because $d \in L^2(0,1)$, we have $d = 0$. Hence, $\mathcal{B}_s^\lambda$ is a Riesz basis of $H^s_{(0)}$. □

2.2 Definition of $\{\alpha_n^\lambda\}_{n\in\mathbb{N}^*}$ and of the operators T and K

Because φ is in $H^2_{(0)}$, there exists $\{\phi_n^\lambda\}_{n\in\mathbb{N}^*} \in \ell^2(\mathbb{N}^*)$ such that :

$$\varphi = \sum_{n\geq 1} \phi_n^\lambda g_n^\lambda \quad \text{in} \quad H^2_{(0)} . \tag{71}$$

Heuristically, the condition (20) implies :

$$\sum_{n\geq 1} \phi_n^\lambda g_n^\lambda = \int_0^1 k^\lambda(x,s)\varphi(s)\, \mathrm{d}s = \sum_{n\geq 1} \frac{\alpha_n^\lambda}{\beta_n^\lambda} \langle \varphi, \varphi_n \rangle g_n^\lambda . \tag{72}$$

The formal unicity of the decomposition (71) comply us to choose α_n^λ the following form :

$$\forall n \in \mathbb{N}^*,\ \alpha_n^\lambda = \frac{\phi_n^\lambda \beta_n^\lambda}{\langle \varphi, \varphi_n \rangle} = \frac{-\lambda \phi_n^\lambda}{|\langle \varphi, \varphi_n \rangle|^2 \lambda_n} . \tag{73}$$

We will see that a posteriori that α_n^λ defined as above is not equal to zero (see section 2.7).

The heuristic of the section 1.5 leads to the following definition of the operator T^λ and K^λ :

$$T^\lambda\colon \psi \longmapsto \sum_{n\in\mathbb{N}^*} f_n^\lambda \langle \psi, \varphi_n \rangle , \tag{74}$$

$$K^\lambda\colon \psi \longmapsto \sum_{n\in\mathbb{N}^*} \alpha_n^\lambda \langle \psi, \varphi_n \rangle . \tag{75}$$

Recall there exists constants $0 < c \leq C$ such that :

$$c\lambda n \leq \left|\beta_n^\lambda\right| \leq C\lambda n , \tag{76}$$

which leads to the following estimate for α_n^λ :

$$\left|\alpha_n^\lambda\right| = O\Big(\left|\phi_n^\lambda\right| n^4\Big) . \tag{77}$$

Unfortunately, this estimate is not sharp enough to show the continuity of T^λ and K^λ.

Now, we emphasize an another issue by computing a formal form for $(T^\lambda)^{-1}$. Let $\chi \in H^3_{(0)}$. Because $\left\{g_n^\lambda \lambda_n^{-1/2}\right\}_{n\in\mathbb{N}^*}$ is a Riesz basis of $H^3_{(0)}$, there exists $\{\chi_n^\lambda\}_{n\in\mathbb{N}^*} \in \ell^2(\mathbb{N}^*)$ such that :

$$\chi = \sum_{n\in\mathbb{N}^*} \frac{\chi_n^\lambda}{\lambda_n^{1/2}} g_n^\lambda \quad \text{in} \quad H^3_{(0)} . \tag{78}$$

Let $\psi \in H^3_{(0)}$. Then, because of the Riesz basis properties :

$$T^\lambda \psi = \chi \Longleftrightarrow \sum_{n\in\mathbb{N}^*} \frac{\alpha_n^\lambda}{\beta_n^\lambda} \langle \psi, \varphi_n \rangle g_n^\lambda = \sum_{n\in\mathbb{N}^*} \frac{\chi_n^\lambda}{\lambda_n^{1/2}} g_n^\lambda \Longleftrightarrow \forall n \in \mathbb{N}^*, \ \langle \psi, \varphi_n \rangle = \frac{\beta_n^\lambda \chi_n^\lambda}{\lambda_n^{1/2} \alpha_n^\lambda} . \tag{79}$$

Hence, formally, we have :

$$(T^\lambda)^{-1} \colon \chi = \sum_{n\in\mathbb{N}^*} \frac{\chi_n^\lambda}{\lambda_n^{1/2}} g_n^\lambda \longmapsto \sum_{n\in\mathbb{N}^*} \frac{\beta_n^\lambda \chi_n^\lambda}{\lambda_n^{1/2} \alpha_n^\lambda} \varphi_n . \tag{80}$$

Hence, we see that if we aim to estimate the norm of $(T^\lambda)^{-1}$, we have to find a lower bound for $\{\alpha_n^\lambda\}_{n\in\mathbb{N}^*}$.

2.3 Regularity of $\left\{\alpha_n^\lambda\right\}_{n\in\mathbb{N}^*}$

We will use the same kind of regularization techniques as in [3] to find a shaper estimate for $\{\alpha_n^\lambda\}_{n\in\mathbb{N}^*}$. We decompose φ as the sum of two functions. We denote :

$$h(x) = \frac{1}{6}\left(\varphi''(1) - \varphi''(0)\right) x^3 + \frac{\varphi''(0)}{2} x^2 - \frac{1}{6}\left(2\varphi''(0) + \varphi''(1)\right) x . \tag{81}$$

h verifies :

$$\forall k \in \mathbb{N}^*, \ \langle h, \varphi_k \rangle = \frac{\sqrt{2}}{(k\pi)^3} \Big((-1)^k \varphi''(1) - \varphi''(0) \Big) . \tag{82}$$

Now, we denote :

$$g = \varphi - h . \tag{83}$$

Lemma 2. *We have :*

$$\Big\{ n(\phi_n^\lambda - \lambda_n \langle h, \varphi_n \rangle) \Big\}_{n\in\mathbb{N}^*} \in \ell^2(\mathbb{N}^*) . \tag{84}$$

This result gives the following estimates for ϕ_n^λ and α_n^λ :

Corollary 1. *There exists two constants $C_1(\lambda) > 0$ and $C_2(\lambda) > 0$ such that :*

$$\left|\phi_n^\lambda\right| \leq \frac{C_1(\lambda)}{n} \quad \text{and} \quad \left|\alpha_n^\lambda\right| \leq C_2(\lambda) n^3 . \tag{85}$$

There exists $N(\lambda) \in \mathbb{N}^*$ and an λ independent constant $c > 0$ such that :

$$\forall n \geq N(\lambda), \ \left|\alpha_n^\lambda\right| \geq c n^3 . \tag{86}$$

Proof of the corollary 1. The first two estimates are easy consequences of the lemma 2, of the expression (82) of the Fourier coefficients of h and of the previous estimate (77) for α_n^λ.

Now, we using that :

$$c\lambda n^4 \leq \left| \frac{\beta_n^\lambda}{\langle \varphi, \varphi_n \rangle} \right| \leq C\lambda n^4 \,, \tag{87}$$

and the lemma 2, we found :

$$\left\{ \frac{\alpha_n^\lambda}{n^3} - \frac{\lambda_n \beta_n^\lambda}{n^3} \frac{\langle h, \varphi_n \rangle}{\langle \varphi, \varphi_n \rangle} \right\}_{n \in \mathbb{N}^*} \in \ell^2(\mathbb{N}^*) \,. \tag{88}$$

Hence :

$$\exists N(\lambda) \in \mathbb{N}^*, \ \forall n \geq N(\lambda), \ \left| \frac{\alpha_n^\lambda}{n^3} - \frac{\lambda_n \beta_n^\lambda}{n^3} \frac{\langle h, \varphi_n \rangle}{\langle \varphi, \varphi_n \rangle} \right| \leq \frac{c}{2} \,. \tag{89}$$

Using triangular inequality and the estimate (87), we have :

$$\frac{\left|\alpha_n^\lambda\right|}{n^3} \geq \frac{c}{2} > 0 \,. \tag{90}$$

□

Proof of the lemma 2. Through the computation of g'', we can see that $g \in H^3_{(0)}$. Because $\{g_n^\lambda \lambda_n^{-1/2}\}_{n\in\mathbb{N}^*}$ is a Riesz basis for $H^3_{(0)}$ (see proposition 1), there exists $\{\delta_n^\lambda\}_{n\in\mathbb{N}^*} \in \ell^2(\mathbb{N}^*)$ such that :

$$g = \sum_{n\geq 1} \frac{\delta_n^\lambda}{\lambda_n^{1/2}} g_n^\lambda \quad \text{in} \quad H^3_{(0)} \,. \tag{91}$$

Moreover, $\{g_n^\lambda\}_{n\in\mathbb{N}^*}$ is a Riesz basis for $H^2_{(0)}$ and $h \in H^2_{(0)}$, hence there exists $\{\rho_n^\lambda\}_{n\in\mathbb{N}^*} \in \ell^2(\mathbb{N}^*)$ such that :

$$h = \sum_{n\geq 1} \rho_n^\lambda g_n^\lambda \quad \text{in} \quad H^2_{(0)} \,. \tag{92}$$

Let γ_n^λ such that :

$$\rho_n^\lambda = \lambda_n \langle h, \varphi_n \rangle + \lambda_n \langle \varphi, \varphi_n \rangle \gamma_n^\lambda \,. \tag{93}$$

This equality expresses the fact a Riesz basis is a bounded perturbation of the an Hilbert basis (see [1]). Let $k \in \mathbb{N}^*$. We have :

$$\begin{aligned}
\langle h, \varphi_k \rangle &= \sum_{n\in\mathbb{N}^*} \lambda_n \langle h, \varphi_n \rangle \left\langle g_n^\lambda, \varphi_k \right\rangle + \sum_{n\in\mathbb{N}^*} \lambda_n \gamma_n^\lambda \langle \varphi, \varphi_n \rangle \left\langle g_n^\lambda, \varphi_k \right\rangle && (94)\\
&= \sum_{n\in\mathbb{N}^*} \frac{\lambda_n \beta_n^\lambda \langle h, \varphi_n \rangle \langle \varphi, \varphi_k \rangle}{\lambda_n - \lambda_k - \lambda} + \sum_{n\in\mathbb{N}^*} \frac{\lambda_n \beta_n^\lambda \gamma_n^\lambda \langle \varphi, \varphi_n \rangle \langle \varphi, \varphi_k \rangle}{\lambda_n - \lambda_k - \lambda} && (95)\\
&= \langle h, \varphi_k \rangle + \sum_{n\neq k} \frac{\lambda_n \beta_n^\lambda \langle h, \varphi_n \rangle \langle \varphi, \varphi_k \rangle}{\lambda_n - \lambda_k - \lambda} + \sum_{n\in\mathbb{N}^*} \frac{\lambda_n \beta_n^\lambda \gamma_n^\lambda \langle \varphi, \varphi_n \rangle \langle \varphi, \varphi_k \rangle}{\lambda_n - \lambda_k - \lambda} \,. && (96)
\end{aligned}$$

Hence :

$$-\sum_{n\neq k}\frac{\lambda_n\beta_n^\lambda\left\langle h,\varphi_n\right\rangle\left\langle \varphi,\varphi_k\right\rangle}{\lambda_n-\lambda_k-\lambda}=\sum_{n\in\mathbb{N}^*}\frac{\lambda_n\beta_n^\lambda\gamma_n^\lambda\left\langle \varphi,\varphi_n\right\rangle\left\langle \varphi,\varphi_k\right\rangle}{\lambda_n-\lambda_k-\lambda} \tag{97}$$

We denote :

$$\widetilde{h}^\lambda=-\sum_{k\in\mathbb{N}^*}\left(\sum_{n\neq k}\frac{\lambda_n\beta_n^\lambda\left\langle h,\varphi_n\right\rangle\left\langle \varphi,\varphi_k\right\rangle}{\lambda_n-\lambda_k-\lambda}\right)\varphi_k\,. \tag{98}$$

We assert that $\widetilde{h}^\lambda\in H^3_{(0)}$. Indeed, we have :

$$\sum_{k\in\mathbb{N}^*}\lambda_k^3\left|\left\langle\widetilde{h}^\lambda,\varphi_k\right\rangle\right|^2=\sum_{k\in\mathbb{N}^*}\lambda_k^3\left|\sum_{n\neq k}\frac{\lambda_n\beta_n^\lambda\left\langle h,\varphi_n\right\rangle\left\langle \varphi,\varphi_k\right\rangle}{\lambda_n-\lambda_k-\lambda}\right|^2\leq C\sum_{k\in\mathbb{N}^*}\left|\sum_{n\neq k}\frac{1}{|k^2-n^2|}\right|^2. \tag{99}$$

We decompose $\sum_{n\neq k}\frac{1}{|k^2-n^2|}$ in two terms. First :

$$\sum_{n=1}^{k-1}\frac{1}{|k^2-n^2|}=\frac{1}{2k}\left(\sum_{n=1}^{k-1}\frac{1}{k+n}+\frac{1}{k-n}\right) \tag{100}$$

$$=\frac{1}{2k}\left[\sum_{n=1}^{2k-1}\frac{1}{n}-\frac{1}{k}\right]=O\left(\frac{\ln k}{k}\right) \tag{101}$$

Now let $N>3k$, for the second term we have :

$$\sum_{n=k+1}^{N}\frac{1}{|k^2-n^2|}=\frac{1}{2k}\left(\sum_{n=k+1}^{N}-\frac{1}{k+n}+\frac{1}{n-k}\right) \tag{102}$$

$$=\frac{1}{2k}\left(\sum_{n=2k+1}^{N+k}-\frac{1}{n}+\sum_{n=1}^{N-k}\frac{1}{n}\right) \tag{103}$$

$$=\frac{1}{2k}\left(\sum_{n=1}^{2k}\frac{1}{n}-\sum_{n=N-k+1}^{N+k}\frac{1}{n}\right). \tag{104}$$

We deduce :

$$\sum_{n\geq k+1}\frac{1}{|k^2-n^2|}=O\left(\frac{\ln k}{k}\right). \tag{105}$$

Hence :

$$\widetilde{h}^\lambda\in H^3_{(0)}\,. \tag{106}$$

Using (97) and the Fubini theorem, we have :

$$\widetilde{h}^\lambda=\sum_{k\in\mathbb{N}^*}\left(\sum_{n\in\mathbb{N}^*}\frac{\lambda_n\beta_n^\lambda\gamma_n^\lambda\left\langle \varphi,\varphi_n\right\rangle\left\langle \varphi,\varphi_k\right\rangle}{\lambda_n-\lambda_k-\lambda}\right)\varphi_k=\sum_{n\in\mathbb{N}^*}\left\langle\varphi,\varphi_n\right\rangle\lambda_n\gamma_n^\lambda\left(\sum_{k\in\mathbb{N}^*}\frac{\beta_n^\lambda\left\langle\varphi,\varphi_k\right\rangle}{\lambda_n-\lambda_k-\lambda}\varphi_k\right) \tag{107}$$

$$=\sum_{n\in\mathbb{N}^*}\left\langle\varphi,\varphi_n\right\rangle\lambda_n\gamma_n^\lambda g_n^\lambda\,. \tag{108}$$

From (106) and (107), we found :

$$\langle \varphi, \varphi_n \rangle \, \lambda_n^{3/2} \gamma_n^\lambda \in \ell^2(\mathbb{N}^*) \,. \tag{109}$$

Using the decomposition (71) for φ and the previous computation, we have :

$$\phi_n^\lambda = \lambda_n \langle h, \varphi_n \rangle + \lambda_n \langle \varphi, \varphi_n \rangle \gamma_n^\lambda + \frac{\delta_n^\lambda}{\lambda_n^{1/2}} \,, \tag{110}$$

which prove the lemma. □

The first consequence of these estimates is the continuity of K^λ.

Corollary 2. *The linear operator $K^\lambda \colon H^4_{(0)} \to \mathbb{R}$ is continuous.*

Remark 3. K^λ can not be continuous as an operator $H^3_{(0)} \to \mathbb{R}$. Indeed, from the second part of the lemma 1, we have :

$$\forall n \geq N(\lambda), \ \left| K^\lambda \left(\frac{\varphi_n}{\lambda_n^{3/2}} \right) \right| = \left| \frac{\alpha_n^\lambda}{\lambda_n^{3/2}} \right| \geq c > 0 \,. \tag{111}$$

And for :

$$\psi^N := \frac{1}{\sqrt{N}} \sum_{n=N(\lambda)}^{N(\lambda)+N-1} \frac{\varphi_n}{\lambda_n^{3/2}} \operatorname{sign}(\alpha_n^\lambda) \,, \tag{112}$$

we have :

$$\left\| \psi^N \right\|_{H^3(0)} = 1 \quad \text{and} \quad \left| K(\psi^N) \right| \geq \sqrt{N} c \underset{N \to +\infty}{\longrightarrow} +\infty \tag{113}$$

2.4 Estimate of $\left\{ \alpha_n^\lambda \right\}_{n \in \mathbb{N}^*}$ in λ

If we want to follow the strategy of the section 1.2, we have to estimate $\|T^\lambda\|$ in function of λ. It is the purpose of this section. We will use the Jaffard lemma [8] and some results about localized frames (see appendix A or [1]).

First, we show some estimates on $\{g_n^\lambda \lambda_n^{-1/2}\}_{n \in \mathbb{N}^*}$. In particular, $\{g_n^\lambda \lambda_n^{-1/2}\}_{n \in \mathbb{N}^*}$ is self-localized with decay coefficient 2 (see appendix A for the definition).

Lemma 3. *$\{g_n^\lambda \lambda_n^{-1/2}\}_{n \in \mathbb{N}^*}$ verifies for all $\gamma \in \,]0\,, \frac{1}{2}[$:*

$$\left| \left\langle g_n^\lambda \lambda_n^{-1/2}, g_k^\lambda \lambda_k^{-1/2} \right\rangle_{H^3_{(0)}} \right| \leq \frac{C\lambda^2}{(1 + |n-k|)^2} \left(\frac{1}{|\lambda_k - \lambda|^\gamma} + \frac{1}{|\lambda_n - \lambda|^\gamma} \right) , \tag{114}$$

and if $\lambda > \max(\lambda_n, \lambda_k)$:

$$\left| \left\langle g_n^\lambda \lambda_n^{-1/2}, g_k^\lambda \lambda_k^{-1/2} \right\rangle_{H^3_{(0)}} \right| \geq c\sqrt{\lambda} \,. \tag{115}$$

Proof of the lemma 3. Let $n \neq k \in \mathbb{N}^* \times \mathbb{N}^*$. We have :

$$\left\langle g_n^\lambda \lambda_n^{-1/2}, g_k^\lambda \lambda_k^{-1/2} \right\rangle_{H^3_{(0)}} = \frac{\beta_n \beta_k}{\lambda_n^{1/2} \lambda_k^{1/2}} \sum_{\ell \in \mathbb{N}^*} \frac{\lambda_\ell^3 \langle \varphi, \varphi_\ell \rangle^2}{(\lambda + \lambda_\ell - \lambda_k)(\lambda + \lambda_\ell - \lambda_n)} . \tag{116}$$

Now, using (11), (76), the relation :

$$\frac{1}{(\lambda + \lambda_\ell - \lambda_k)(\lambda + \lambda_\ell - \lambda_n)} = \frac{1}{\lambda_k - \lambda_n} \left(\frac{1}{\lambda + \lambda_\ell - \lambda_k} - \frac{1}{\lambda + \lambda_\ell - \lambda_n} \right) , \tag{117}$$

and for all $a > 0$ such that $a \notin \mathbb{N}^2$:

$$\sum_{\ell \in \mathbb{N}^*} \frac{1}{|\ell^2 + a|} = O\left(\frac{1}{\sqrt{a}} \right) , \tag{118}$$

$$\sum_{\ell \in \mathbb{N}^*} \frac{1}{|\ell^2 - a|} = O\left(\frac{\ln(\sqrt{a})}{\sqrt{a}} \right) , \tag{119}$$

we have for all $\gamma \in \left]0 , \frac{1}{2}\right[$:

$$\left| \left\langle g_n^\lambda \lambda_n^{-1/2}, g_k^\lambda \lambda_k^{-1/2} \right\rangle_{H^3_{(0)}} \right| \leq \frac{C\lambda^2}{|\lambda_n - \lambda_k|} \sum_{\ell \in \mathbb{N}^*} \left(\frac{1}{|\lambda + \lambda_\ell - \lambda_k|} + \frac{1}{|\lambda + \lambda_\ell - \lambda_n|} \right) \tag{120}$$

$$\leq \frac{C\lambda^2}{|\lambda_n - \lambda_k|} \left(\frac{1}{|\lambda_k - \lambda|^\gamma} + \frac{1}{|\lambda_n - \lambda|^\gamma} \right) . \tag{121}$$

Using (105) and the relation :

$$\frac{1}{|n^2 - k^2|} \leq \frac{1}{(n-k)^2} \leq \frac{4}{(1 + |n-k|)^2} , \tag{122}$$

we conclude :

$$\left| \left\langle g_n^\lambda \lambda_n^{-1/2}, g_k^\lambda \lambda_k^{-1/2} \right\rangle_{H^3_{(0)}} \right| \leq \frac{C\lambda^2}{(1 + |n-k|)^2} \left(\frac{1}{|\lambda_k - \lambda|^\gamma} + \frac{1}{|\lambda_n - \lambda|^\gamma} \right) . \tag{123}$$

Now, let $(n, k) \in \mathbb{N}^* \times \mathbb{N}^*$. and assume $\lambda > \max(\lambda_n, \lambda_k)$. Then, using (11), (76), we have :

$$\left| \left\langle g_n^\lambda \lambda_n^{-1/2}, g_k^\lambda \lambda_k^{-1/2} \right\rangle_{H^3_{(0)}} \right| \geq c\lambda^2 \sum_{\ell \in \mathbb{N}^*} \frac{1}{(\lambda + \lambda_\ell - \lambda_k)(\lambda + \lambda_\ell - \lambda_n)} \tag{124}$$

$$\geq c\lambda^2 \sum_{\ell \in \mathbb{N}^*} \frac{1}{(\lambda + \lambda_\ell)^2} \geq c\lambda^2 \sum_{\ell \geq \left\lceil \sqrt{\frac{\lambda}{\pi^2}} \right\rceil} \ell^{-4} \tag{125}$$

$$\geq c\sqrt{\lambda} . \tag{126}$$

Which ends the proof of the lemma. □

Let $\{h_n^\lambda\}_{n\in\mathbb{N}^*}$ be the Riesz dual basis of $\{g_n^\lambda \lambda_n^{-1/2}\}_{n\in\mathbb{N}^*}$ in $H^3_{(0)}$. Thanks to the previous lemma and the theorem 6 (see appendix A), we know that $\{h_n^\lambda\}_{n\in\mathbb{N}^*}$ is also self-localized with decay coefficient 2. However, this theorem is not sufficient to estimate $\|T^\lambda\|$ in function of λ because it does not provide any estimate in λ. So, we will need to specify the proof.

We introduce some notations and define the space of the infinite matrix with polynomial off-diagonal coefficients decay. Let $\mathcal{M}$ be the space of the infinite matrix indexed by $\mathbb{N}^*\times\mathbb{N}^*$ and let $\gamma, C>0$. We define :

$$Q_\gamma^C := \left\{ A=(A_{n,k})\in\mathcal{M} \;\middle|\; \exists C\geq 0,\ \forall (n,k)\in(\mathbb{N}^*)^2,\ |A_{n,k}|\leq \frac{C}{(1+|n-k|)^\gamma} \right\}. \tag{127}$$

If A is an infinite matrix indexed by $\mathbb{N}^*\times\mathbb{N}^*$, we denote by $\|A\|$ the norm of A as an operator $\ell^2(\mathbb{N}^*)\to\ell^2(\mathbb{N}^*)$. Moreover, for $\delta>0$, we define :

$$\|A\|_\delta = \sup_{n,k\in\mathbb{N}^*\times\mathbb{N}^*} |A_{n,k}|\,(1+|n-k|)^\delta . \tag{128}$$

Clearly, if A is in Q_γ^C then $\|A\|_\gamma\leq C$ and by the Schur lemma (see, for example, [1]), if $A\in Q_\gamma^C$ then $\|A\|\leq C$.

Let $G^\lambda = (G^\lambda_{n,k})_{(n,k)\in(\mathbb{N}^*)^2}$ be the Gram matrix associate to $\{g_n^\lambda\lambda_n^{-1/2}\}_{n\in\mathbb{N}^*}$ which is defined by :

$$\forall (n,k)\in(\mathbb{N}^*)^2,\ G^\lambda_{n,k} = \left\langle g_n^\lambda\lambda_n^{-1/2}, g_k^\lambda\lambda_k^{-1/2}\right\rangle_{H^3_{(0)}} . \tag{129}$$

The lemma 3 shows that $G^\lambda\ \in Q_2^{C\lambda^2}$.

Let $\{h_n^\lambda\}_{n\in\mathbb{N}^*}$ be the Riesz dual basis of $\{g_n^\lambda\lambda_n^{-1/2}\}_{n\in\mathbb{N}^*}$ in $H^3_{(0)}$. Because $\{g_n^\lambda\lambda_n^{-1/2}\}_{n\in\mathbb{N}^*}$ is a Riesz basis for $H^3_{(0)}$, G^λ is invertible and the coefficients of $(G^\lambda)^{-1}$ are given by :

$$\forall (n,k)\in\mathbb{N}^*\times\mathbb{N}^*,\ (G^\lambda)^{-1}_{n,k} = \left\langle h_n^\lambda, h_k^\lambda\right\rangle_{H^3_{(0)}} . \tag{130}$$

The Jaffard lemma shows there exists $C(\lambda)>0$ such that :

$$(G^\lambda)^{-1}\in Q_2^{C(\lambda)} . \tag{131}$$

Let B^λ be the operator such that :

$$G^\lambda = \left\|G^\lambda\right\|(1-B^\lambda) . \tag{132}$$

G^λ is invertible, hence we have :

$$\left\|B^\lambda\right\|<1 \quad \text{and} \quad (G^\lambda)^{-1} = \left\|G^\lambda\right\|^{-1}\sum_{n\in\mathbb{N}}(B^\lambda)^n . \tag{133}$$

And we have :

$$\left\|(G^\lambda)^{-1}\right\|_2 \leq \left\|G^\lambda\right\|^{-1}\sum_{n\in\mathbb{N}^*}\left\|(B^\lambda)^n\right\|_2 . \tag{134}$$

In the proof of the Jaffard lemma (see [8]), $\left\|B^\lambda\right\|$ plays a crucial role in the bootstrap argument. By the functional calculus, we have :

$$\left\|B^\lambda\right\| = 1 - \frac{\inf(\operatorname{Sp} G^\lambda)}{\|G^\lambda\|} \,. \tag{135}$$

In the course of my internship, I did not manage to find a lower bound for $\inf(\operatorname{Sp} G^\lambda)$. So, I have to make an hypothesis if a want to carry on the proof.

Hypothesis 1. *There exists $\delta \in]0\,,2]$ and $c > 0$ such that :*

$$\inf(\operatorname{Sp} G^\lambda) \geq c\lambda^\delta \,. \tag{136}$$

In consequence, if λ is big enough, we have :

$$\left\|B^\lambda\right\| \leq 1 - \tilde{c}\lambda^{\delta-2} < 1 \,. \tag{137}$$

Now, using this assumption, if we follow the proof of the Jaffard lemma, we can see that there exists $m > 0$ such that :

$$\left|\langle h_n, h_k\rangle_{H^3_{(0)}}\right| \leq \frac{C\lambda^m}{(1+|n-k|)^2} \,, \tag{138}$$

We recall the decomposition :

$$\phi_n^\lambda = \lambda_n \langle h, \varphi_n\rangle + \lambda_n \langle \varphi, \varphi_n\rangle \gamma_n^\lambda + \frac{\delta_n^\lambda}{\lambda_n^{1/2}} \,. \tag{139}$$

Lemma 4. *If λ is big enough, there exists $\{\tilde{\delta}_n\}_{n\in\mathbb{N}^*} \in \ell^2(\mathbb{N}^*)$ such that :*

$$\left|\delta_n^\lambda\right| \leq \lambda^{m+2}\tilde{\delta}_n \,. \tag{140}$$

Proof of the lemma 4. Thanks to the Riesz basis properties, we have :

$$g = \sum_{n\in\mathbb{N}^*} \frac{\delta_n^\lambda}{\lambda_n^{1/2}} g_n^\lambda = \sum_{n\in\mathbb{N}^*} \left\langle g, g_n^\lambda \lambda_n^{-1/2}\right\rangle_{H^3_{(0)}} h_n^\lambda \,, \tag{141}$$

which leads to :

$$\delta_n^\lambda = \left\langle g, h_n^\lambda\right\rangle_{H^3_{(0)}} = \sum_{k\in\mathbb{N}^*} \left\langle g, g_k^\lambda \lambda_k^{-1/2}\right\rangle_{H^3_{(0)}} \left\langle h_n^\lambda, h_k^\lambda\right\rangle_{H^3_{(0)}} \tag{142}$$

Now, we have :

$$\left\langle g, g_k^\lambda \lambda_k^{-1/2}\right\rangle_{H^3_{(0)}} = \sum_{\ell\in\mathbb{N}^*} \lambda_\ell^3 \langle g, \varphi_\ell\rangle \left\langle g_k^\lambda \lambda_k^{-1/2}, \varphi_\ell\right\rangle \tag{143}$$

$$= \frac{\beta_k^\lambda}{\lambda_k^{1/2}} \sum_{\ell\in\mathbb{N}^*} \lambda_\ell^3 \left(\frac{-1}{(\ell\pi)^3} \langle \varphi''', \varphi_\ell\rangle\right) \frac{\langle \varphi, \varphi_\ell\rangle}{\lambda_k - \lambda_\ell - \lambda} = a_k^1(\lambda) + a_k^2(\lambda) \,, \tag{144}$$

where :

$$a_k^1(\lambda) = \frac{\beta_k^\lambda \langle \varphi, \varphi_k \rangle \lambda_k^{5/2}}{(k\pi)^3 \lambda} \langle \varphi''', \varphi_k \rangle \,, \tag{145}$$

$$a_k^2(\lambda) = \frac{\beta_k^\lambda}{\lambda_k^{1/2}} \sum_{\ell \neq k} \lambda_\ell^3 \left(\frac{-1}{(\ell\pi)^3} \langle \varphi''', \varphi_\ell \rangle \right) \frac{\langle \varphi, \varphi_\ell \rangle}{\lambda_k - \lambda_\ell - \lambda} \,. \tag{146}$$

Using (76) and (11), we have :

$$c \left| \langle \varphi''', \varphi_k \rangle \right| \leq \left| a_k^1(\lambda) \right| \leq C \left| \langle \varphi''', \varphi_k \rangle \right| \,. \tag{147}$$

For the second term, using (76), (11) and the Cauchy-Schwarz inequality, we have :

$$\left| a_k^2(\lambda) \right| \leq C\lambda^2 \sum_{\ell \neq k} \frac{|\langle \varphi''', \varphi_\ell \rangle|}{|k^2 - \ell^2|} \tag{148}$$

$$\leq C\lambda^2 \left\| \varphi''' \right\|_{L^2} \sqrt{\sum_{\ell \neq k} \frac{1}{|k^2 - \ell^2|^2}} := \lambda^2 \tilde{c}_k \,. \tag{149}$$

Clearly, $\tilde{c} := \{\tilde{c}_k\}_{k \in \mathbb{N}^*} \in \ell^2(\mathbb{N}^*)$. Now, coming back to $\left\langle g, g_k^\lambda \lambda_k^{-1/2} \right\rangle_{H^3_{(0)}}$, using (138), we have :

$$\left| \delta_n^\lambda \right| = \left| \left\langle g, g_n^\lambda \lambda_k^{-1/2} \right\rangle_{H^3_{(0)}} \right| \leq \sum_{\ell \in \mathbb{N}^*} \lambda_\ell^3 \left| \langle g, \varphi_\ell \rangle \right| \left| \left\langle g_k^\lambda \lambda_k^{-1/2}, \varphi_\ell \right\rangle \right| \tag{150}$$

$$\leq C\lambda^{m+2} \sum_{k \in \mathbb{N}^*} \frac{|\langle \varphi''', \varphi_k \rangle| + \tilde{c}_k}{(1 + |n-k|)^2} \,. \tag{151}$$

We denote :

$$\tilde{\delta}_n := C \sum_{k \in \mathbb{N}^*} \frac{|\langle \varphi''', \varphi_k \rangle| + \tilde{c}_k}{(1 + |n-k|)^2} \,. \tag{152}$$

Then, the estimate :

$$\left| \tilde{\delta}_n \right| \leq C \left(|\langle \varphi''', \varphi_n \rangle| + \tilde{c}_n + (\|\tilde{c}\|_{\ell^2(\mathbb{N}^*)} + \|\varphi'''\|_{L^2}) \sqrt{\sum_{k \neq n} \frac{1}{|n-k|^4}} \right) , \tag{153}$$

shows that $\{\tilde{\delta}_n\}_{n \in \mathbb{N}^*} \in \ell^2(\mathbb{N}^*)$. Finally :

$$\left| \delta_n^\lambda \right| \leq \lambda^{m+2} \tilde{\delta}_n \,. \tag{154}$$

□

We have have the same kind of estimate for γ_n^λ.

Lemma 5. *If λ is big enough, there exists $\{\tilde{\gamma}_n\}_{n\in\mathbb{N}^*} \in \ell^2(\mathbb{N}^*)$ such that :*

$$\left|\gamma_n^\lambda\right| \leq \lambda^{m+4}\tilde{\gamma}_n\,. \tag{155}$$

Proof of the lemma 5. We follow the same previous proof, replacing g by $\widetilde{h}^\lambda$ and using the following estimation for $\left\langle \widetilde{h}^\lambda, \varphi_\ell \right\rangle$:

$$\left|\left\langle \widetilde{h}^\lambda, \varphi_\ell\right\rangle\right| = \left|\sum_{n\neq\ell} \frac{\lambda_n \beta_n^\lambda \langle h, \varphi_n\rangle \langle \varphi, \varphi_\ell\rangle}{\lambda_n - \lambda_\ell - \lambda}\right| \tag{156}$$

$$\leq \frac{C\lambda^2}{\ell^3} \sum_{n\neq\ell} \frac{1}{|n^2-\ell^2|} \leq C\lambda^2 \frac{\ln(\ell)}{\ell^4}\,. \tag{157}$$

□

Now, from the decomposition (139), we have :

Proposition 2. *If λ is big enough, we have for all $n \in \mathbb{N}^*$*

$$\left|n\phi_n^\lambda\right| \leq C\lambda^{m+4} \quad \text{and} \quad \left|\alpha_n^\lambda\right| \leq C\lambda^{m+4} n^3\,. \tag{158}$$

2.5 Continuity of T^λ

We recall T^λ is defined by :

$$T^\lambda \colon \psi \longmapsto \sum_{n\in\mathbb{N}^*} f_n^\lambda \langle \psi, \varphi_n\rangle\ . \tag{159}$$

Lemma 6. *Let $\psi \in H^3_{(0)}$. Then :*

$$\sum_{n\in\mathbb{N}^*} f_n^\lambda \langle \psi, \varphi_n\rangle \tag{160}$$

converges in $H^2_{(0)}$.

Proof of the lemma 6. Let $\psi \in H^3_{(0)}$ and $M \geq N \geq 1$. We have :

$$\left\|\sum_{n=N}^{M} f_n^\lambda \langle \psi, \varphi_n\rangle\right\|^2_{H^2_{(0)}} = \sum_{k\in\mathbb{N}^*} \lambda_k^2 \left|\left\langle \sum_{n=N}^{M} f_n^\lambda \langle \psi, \varphi_n\rangle, \varphi_k\right\rangle\right|^2 \tag{161}$$

$$= \sum_{k\in\mathbb{N}^*} \lambda_k^2 \left|\sum_{n=N}^{M} \langle \psi, \varphi_n\rangle \left\langle f_n^\lambda, \varphi_k\right\rangle\right|^2 \tag{162}$$

$$\leq \sum_{k\in\mathbb{N}^*} \lambda_k^2 \left(\sum_{n=N}^{M} \lambda_n^3 \left|\langle \psi, \varphi_n\rangle\right|^2\right)\left(\sum_{n=N}^{M} \frac{\left|\left\langle f_n^\lambda, \varphi_k\right\rangle\right|^2}{\lambda_n^3}\right) \tag{163}$$

$$\leq \|\psi\|^2_{H^3_{(0)}} \sum_{n=N}^{M} \frac{(\alpha_n^\lambda)^2}{(\beta_n^\lambda)^2\lambda_n^3} \sum_{k\in\mathbb{N}^*} \lambda_k^2 \left|\left\langle g_n^\lambda, \varphi_k\right\rangle\right|^2 \tag{164}$$

$$\leq \|\psi\|^2_{H^3_{(0)}} \sum_{n=N}^{M} \left\|g_n^\lambda\right\|^2_{H^2_{(0)}} \frac{(\alpha_n^\lambda)^2}{(\beta_n^\lambda)^2\lambda_n^3} \tag{165}$$

Thank to corollary 1 and to the estimate (76) for β_n^λ , we have :

$$\frac{(\alpha_n^\lambda)^2}{(\beta_n^\lambda)^2 \lambda_n^3} = O\left(\frac{1}{n^2}\right) . \tag{166}$$

Because $\left\{g_n^\lambda\right\}_{n\in\mathbb{N}^*}$ is a Riesz basis for $H^2_{(0)}$, there exists a constant $C > 0$ such that :

$$\forall n \in \mathbb{N}^*, \ \left\|g_n^\lambda\right\|^2_{H^2_{(0)}} \leq C . \tag{167}$$

Hence $\left\{\sum_{n=1}^N f_n^\lambda \langle \psi, \varphi_n \rangle\right\}_{N\in\mathbb{N}^*}$ is a Cauchy sequence of $H^2_{(0)}$, then a convergent series in $H^2_{(0)}$. □

Now, we have everything we need to show the continuity of T^λ.

Proposition 3. *If λ is big enough, then :*

$$T^\lambda : H^3_{(0)} \longrightarrow H^3_{(0)} \tag{168}$$

is a continuous linear operator. Moreover :

$$\left\|T^\lambda\right\| = O\left(\lambda^{m+5}\right) . \tag{169}$$

Proof. Let $\psi \in H^3_{(0)}$. By the lemma 6 the series defining T^λ is convergent in $H^2_{(0)}$ hence in $L^2(0,1)$, which justifying the following equality :

$$\forall n \in \mathbb{N}^*, \ \left\langle T^\lambda \psi, \varphi_n \right\rangle = \sum_{k\in\mathbb{N}^*} \langle \psi, \varphi_k \rangle \left\langle f_k^\lambda, \varphi_n \right\rangle \tag{170}$$

Using the corollary 1 and the Cauchy-Schwarz inequality, we have :

$$\left\|T^\lambda \psi\right\|^2_{H^3_{(0)}} = \sum_{n\in\mathbb{N}^*} \lambda_n^3 \left|\left\langle T^\lambda \psi, \varphi_n \right\rangle\right|^2 = \sum_{n\in\mathbb{N}^*} \lambda_n^3 \left|\sum_{k\in\mathbb{N}^*} \langle \psi, \varphi_k \rangle \left\langle f_k^\lambda, \varphi_n \right\rangle\right|^2 \tag{171}$$

$$= \sum_{n\in\mathbb{N}^*} \lambda_n^3 \left|\langle \varphi, \varphi_n \rangle\right|^2 \left|\sum_{k\in\mathbb{N}^*} \frac{\alpha_k^\lambda \langle \psi, \varphi_k \rangle}{\lambda_k - \lambda_n - \lambda}\right|^2 \tag{172}$$

$$\leq 2 \sum_{n\in\mathbb{N}^*} \lambda_n^3 \left|\langle \varphi, \varphi_n \rangle\right|^2 \frac{(\alpha_n^\lambda)^2 \left|\langle \psi, \varphi_n \rangle\right|^2}{\lambda^2} + 2 \sum_{n\in\mathbb{N}^*} \lambda_n^3 \left|\langle \varphi, \varphi_n \rangle\right|^2 \left|\sum_{k\neq n} \frac{\alpha_k^\lambda \langle \psi, \varphi_k \rangle}{\lambda_k - \lambda_n - \lambda}\right|^2 \tag{173}$$

$$\leq C\lambda^{2m+6} \|\psi\|^2_{H^3_{(0)}} + C\lambda^{2m+10} \|\psi\|^2_{H^3_{(0)}} \sum_{n\in\mathbb{N}^*} \sum_{k\neq n} \frac{1}{(n^2-k^2)^2} . \tag{174}$$

The second term is a finite double series (see the proof of the lemma 1). Hence, T^λ is continuous from $H^3_{(0)}$ to $H^3_{(0)}$ and we have the estimate :

$$\|T^\lambda\| = O\left(\lambda^{m+5}\right) . \tag{175}$$

□

2.6 Operator equality

In the rest of this section, we aim to show the operator equality :

$$T^\lambda(A + BK^\lambda) = AT^\lambda - \lambda T^\lambda\,. \tag{176}$$

Following the ideas in the article [3] from Coron, we first specify the space where this equality holds. At least, we need :

$$(A + BK^\lambda)\psi \in H^3_{(0)}\,. \tag{177}$$

This leads to the following space :

$$D(A + BK^\lambda) := \left\{\psi \in H^4_{(0)} \,\middle|\, \Delta\psi + K^\lambda(\psi)\varphi \in H^3_{(0)}\right\}\,. \tag{178}$$

Proposition 4. *If λ is big enough, then : $D(A + BK^\lambda)$ is dense in $H^3_{(0)}$.*

Proof of the proposition 4. Let $\psi \in D(A+BK^\lambda)^\perp$ in $H^3_{(0)}$. By the asymptotic behavior of α_n^λ (see lemma 2), there exists $N(\lambda) \in \mathbb{N}^*$ such that $\alpha_n^\lambda \neq 0$ for all $n \geq N(\lambda)$. Let $m \in \mathbb{N}^*$ and $n \geq N(\lambda)$. We have :

$$\varphi_m - \frac{\alpha_m^\lambda}{\alpha_n^\lambda}\varphi_n \in H^5(0,1) \cap H^4_{(0)}\,. \tag{179}$$

Moreover :

$$K^\lambda(\varphi_m - \frac{\alpha_m^\lambda}{\alpha_n^\lambda}\varphi_n) = \alpha_m^\lambda - \frac{\alpha_m^\lambda}{\alpha_n^\lambda}\alpha_n^\lambda = 0\,. \tag{180}$$

Hence $\varphi_m - \frac{\alpha_m^\lambda}{\alpha_n^\lambda}\varphi_n \in D(A + BK^\lambda)$. We have :

$$0 = \left\langle \psi, \varphi_m - \frac{\alpha_m^\lambda}{\alpha_n^\lambda}\varphi_n \right\rangle_{H^3_{(0)}} = \langle\psi,\varphi_m\rangle_{H^3_{(0)}} - \frac{\alpha_m^\lambda}{\alpha_n^\lambda}\langle\psi,\varphi_n\rangle_{H^3_{(0)}} \tag{181}$$

$$= \langle\psi,\varphi_m\rangle_{H^3_{(0)}} - \frac{\alpha_m^\lambda\lambda_n^{3/2}}{\alpha_n^\lambda}(\lambda_n^{3/2}\langle\psi,\varphi_n\rangle)\,. \tag{182}$$

Using the lemma 2 and $\lambda_n^{3/2}\langle\psi,\varphi_n\rangle \in \ell^2(\mathbb{N}^*)$, we have :

$$0 = \left\langle \psi, \varphi_m - \frac{\alpha_m^\lambda}{\alpha_n^\lambda}\varphi_n \right\rangle_{H^3_{(0)}} \underset{n\to+\infty}{\longrightarrow} \langle\psi,\varphi_m\rangle_{H^3_{(0)}}\,, \tag{183}$$

and so : $\langle\psi,\varphi_m\rangle_{H^3_{(0)}} = 0$. We conclude $\psi = 0$. □

Proposition 5. *If λ is big enough, then for all $\psi \in D(A + BK^\lambda)$, the following holds :*

$$T^\lambda(A + BK^\lambda)\psi = (AT^\lambda - \lambda T^\lambda)\psi\,. \tag{184}$$

Proof of the proposition 5. Let $\psi \in D(A+BK^\lambda)$.

We recall that $D(A) = H^2_{(0)}$ and that A is a closed operator. On one hand, thank to the lemma 6, we have :

$$(AT^\lambda - \lambda T^\lambda)\psi = (A-\lambda)\sum_{n\in\mathbb{N}^*} f_n^\lambda \langle \psi, \varphi_n\rangle = \sum_{n\in\mathbb{N}^*}(A-\lambda) f_n^\lambda \langle \psi, \varphi_n\rangle \ , \tag{185}$$

the convergence being in $L^2(0,1)$. Because f_n^λ is the solution of the equation (32), we have :

$$(AT^\lambda - \lambda T^\lambda)\psi = \sum_{n\in\mathbb{N}^*} \varphi\alpha_n^\lambda \langle \psi, \varphi_n\rangle - \lambda_n f_n^\lambda \langle \psi, \varphi_n\rangle \ . \tag{186}$$

On the other hand :

$$T^\lambda(A+BK^\lambda)\psi = \sum_{n\in\mathbb{N}^*} f_n^\lambda \left\langle (A+BK^\lambda)\psi, \varphi_n\right\rangle = \sum_{n\in\mathbb{N}^*} f_n^\lambda \langle \psi, A\varphi_n\rangle + K^\lambda(\psi) f_n^\lambda \langle \varphi, \varphi_n\rangle \tag{187}$$

$$= K^\lambda(\psi)\sum_{n\in\mathbb{N}^*} f_n^\lambda \langle \varphi, \varphi_n\rangle - \sum_{n\in\mathbb{N}^*} \lambda_n f_n^\lambda \langle \psi, \varphi_n\rangle \ , \tag{188}$$

the convergence being in $H^2_{(0)}$ hence in $L^2(0,1)$. Recall the "$T^\lambda B = B$" condition which says that :

$$\varphi = \sum_{n\in\mathbb{N}^*} f_n^\lambda \langle \varphi, \varphi_n\rangle \quad \text{in} \quad L^2(0,1)\,. \tag{189}$$

Injecting (189) in (188), we found that :

$$T^\lambda(A+BK^\lambda)\psi = (AT^\lambda - \lambda T^\lambda)\psi\,. \tag{190}$$

□

Remark 4. If $\psi \in D(A+BK^\lambda)$, then, using the equality :

$$AT^\lambda\psi = \lambda T^\lambda\psi - T^\lambda(A+BK^\lambda)\psi \in H^3_{(0)}\,, \tag{191}$$

we deduce that $T^\lambda\psi \in H^5_{(0)}$.

2.7 Invertibility of T^λ

We can't prove the invertibility of T^λ by a direct method because we don't have a lower boundary for $\left\{\alpha_n^\lambda\right\}_{n\in\mathbb{N}^*}$. Following [3], we first prove that T^λ is a Fredholm operator then we conclude showing that $\ker[(T^\lambda)^*] = \{0\}$.

Proposition 6. *If λ is big enough, there exists $\widetilde{T}^\lambda : H^3_{(0)} \to H^3_{(0)}$ invertible such that $T^\lambda - \widetilde{T}^\lambda$ is a compact operator of $H^3_{(0)}$.*

Proof of the proposition 6. We define $\widetilde{T}^\lambda$ in the following way :

$$\widetilde{T}^\lambda : \psi \in H^3_{(0)} \longmapsto \sum_{n\in\mathbb{N}^*} \lambda_n^{3/2} \frac{\langle h, \varphi_n\rangle}{\langle \varphi, \varphi_n\rangle} \langle \psi, \varphi_n\rangle \frac{g_n^\lambda}{\lambda_n^{1/2}} \in H^3_{(0)}\,. \tag{192}$$

$\widetilde{T}^\lambda$ is well-defined and continuous. Indeed, if ψ is in $H^3_{(0)}$ then :

$$\left|\lambda_n^{3/2} \frac{\langle h, \varphi_n\rangle}{\langle \varphi, \varphi_n\rangle} \langle \psi, \varphi_n\rangle\right| \leq C \left|\lambda_n^{3/2} \langle \psi, \varphi_n\rangle\right| \in \ell^2(\mathbb{N}^*)\,. \tag{193}$$

Let's show that $\widetilde{T}^\lambda$ is invertible. To see that, let $\chi \in H^3_{(0)}$. There exists $\left\{\chi_n^\lambda\right\}_{n\in\mathbb{N}^*} \in \ell^2(\mathbb{N}^*)$ such that :

$$\chi = \sum_{n\in\mathbb{N}^*} \chi_n^\lambda \frac{g_n^\lambda}{\lambda_n^{1/2}} \in H^3_{(0)}\,. \tag{194}$$

Because of the Riesz bases properties, if $\psi \in H^3_{(0)}$ verifies $T^\lambda \psi = \chi$ if and only if for all $n \in \mathbb{N}^*$:

$$\langle \psi, \varphi_n\rangle = \frac{\chi_n^\lambda \langle \varphi, \varphi_n\rangle}{\langle h, \varphi_n\rangle \lambda_n^{3/2}}\,. \tag{195}$$

Conversely, if $\chi \in L^2(0,1)$ verifies (195), then :

$$|\langle \psi, \varphi_n\rangle| = O\left(\frac{\left|\chi_n^\lambda\right|}{n^3}\right)\,, \tag{196}$$

which implies $\chi \in H^3_{(0)}$.

We show that $T^\lambda - \widetilde{T}^\lambda$ is compact by the Hilbert-Schmidt criterium, which is :

$$\sum_{n\in\mathbb{N}^*} \left\|(T^\lambda - \widetilde{T}^\lambda)\frac{\varphi_n}{\lambda_n^{3/2}}\right\|^2_{H^3_{(0)}} < +\infty\,. \tag{197}$$

However :

$$(T^\lambda - \widetilde{T}^\lambda)\frac{\varphi_n}{\lambda_n^{3/2}} = \frac{(\phi_n^\lambda - \lambda_n \langle h, \varphi_n\rangle)}{\lambda_n \langle \varphi, \varphi_n\rangle} \frac{g_n^\lambda}{\lambda_n^{1/2}}\,. \tag{198}$$

Hence, using the lemma 2 :

$$\sum_{n\in\mathbb{N}^*} \left\|(T^\lambda - \widetilde{T}^\lambda)\frac{\varphi_n}{\lambda_n^{3/2}}\right\|^2_{H^3_{(0)}} \leq C \sum_{n\in\mathbb{N}^*} \left|(\phi_n^\lambda - \lambda_n \langle h, \varphi_n\rangle) n\right|^2 < +\infty\,. \tag{199}$$

This concludes the proof. □

Proposition 7. *If λ is big enough, the operator T^λ is invertible from $H^3_{(0)}$ into itself.*

Proof of the proposition 7. Using the proposition 6, there exists an invertible operator $\widetilde{T}^\lambda : H^3_{(0)} \to H^3_{(0)}$ and a compact operator $\widetilde{K}^\lambda : H^3_{(0)} \to H^3_{(0)}$ invertible such that :

$$T = \widetilde{T}^\lambda \left(1 + (\widetilde{T}^\lambda)^{-1}\widetilde{K}^\lambda\right) . \tag{200}$$

Thank to the Fredholm theory, to show that T^λ is invertible it is sufficient to show that :

$$\ker[(T^\lambda)^*] = \{0\} . \tag{201}$$

We write the main ideas of the proof. The details can be found in [3].

We rewrite (176) :

$$T^\lambda(A + BK^\lambda + \lambda + \rho) = (A + \rho)T^\lambda , \tag{202}$$

where $\rho \in \mathbb{C}$ will be chosen later such that :

$$(A + \rho) \text{ is an invertible operator from } D(A + BK^\lambda) \text{ to } H^3_{(0)} , \tag{203}$$

$$(A + BK^\lambda + \lambda + \rho) \text{ is an invertible operator from } D(A) \text{ to } H^3_{(0)} . \tag{204}$$

All the spaces are complexified. We deduce :

$$(A + \rho)^{-1}T^\lambda = T^\lambda(A + BK^\lambda + \lambda + \rho)^{-1} . \tag{205}$$

Then, by a rapid computation, we show that $\ker[(T^\lambda)^*]$ is stable by $[(A + \rho)^*]^{-1}$. Hence, if $\ker[(T^\lambda)^*] \neq \{0\}$, $[(A+\rho)^*]^{-1}$ has an eigenfunction $\psi \neq 0$ in $\ker[(T^\lambda)^*]$ which is also an eigenfunction of $(A^*)^{-1}$. There exists $\nu \in \mathbb{C}$ such that :

$$(A^*)^{-1}\psi = \nu\psi . \tag{206}$$

Now, we have :

$$\forall j \in \mathbb{N}^*, \ \nu \langle \psi, \varphi_j \rangle = \left\langle (A^*)^{-1}\psi, \varphi_j \right\rangle = -\frac{1}{\lambda_j} \langle \psi, \varphi_j \rangle . \tag{207}$$

Because ψ is not zero, there exists one and only one $k \in \mathbb{N}^*$, such that :

$$\nu = -\frac{1}{\lambda_k} . \tag{208}$$

Hence, there exists $c \in \mathbb{C}$ such that $\psi = c\varphi_k$. Now, using the $T^\lambda B = B$ condition (189), we have :

$$c \langle \varphi, \varphi_k \rangle = \langle \varphi, \psi \rangle = \left\langle T^\lambda \varphi, \psi \right\rangle = \left\langle \varphi, (T^\lambda)^* \psi \right\rangle = 0 . \tag{209}$$

Because $\langle \varphi, \varphi_k \rangle \neq 0$, we have $c = 0$ and $\psi = 0$. This is absurd and we must have :

$$\ker[(T^\lambda)^*] = \{0\} . \tag{210}$$

It remains to show there exists $\rho \in \mathbb{C}$ such that (203) holds. Denoting $\kappa = \rho + \lambda$ and applying A^{-1} to $A + BK^\lambda + \kappa$, we just have to prove the set of $\kappa \in \mathbb{C}$ such that $1 + A^{-1}BK^\lambda + \kappa A^{-1}$ is invertible from $D(A + BK^\lambda)$ to $D(A)$ is not empty.

If $K^\lambda(A^{-1}\varphi) \neq -1$ then it is easy to show that $1 + A^{-1}BK^\lambda$ is invertible from $D(A + BK^\lambda)$ to $D(A)$ and the proof is over.

Now, assume $K^\lambda(A^{-1}\varphi) = -1$. It corresponds to the case $A^{-1}\varphi \in D(A + BK^\lambda)$ (see the definition of $D(A + BK^\lambda)$ (178)). 0 is an eigenvalue of $A + A^{-1}BK^\lambda$ of algebraic multiplicity 1. From the chapter 7 of [9], there exists an open set $\Omega \subset \mathbb{C}$ of $0 \in \mathbb{C}$, there exists an holomorphic function $\kappa \in \Omega \mapsto \lambda(\kappa) \in \mathbb{C}$ and an holomorphic function $\kappa \in \Omega \mapsto \psi(\kappa) \in D(A + BK^\lambda)$ such that for all $\kappa \in \ \Omega$:

$$\psi(0) = A^{-1}\varphi \,, \tag{211}$$

$$(1 + A^{-1}BK^\lambda + \kappa A^{-1})\psi(\kappa) = \lambda(\kappa)\psi(\kappa) \,. \tag{212}$$

If $\lambda(\kappa) \neq 0$ in a small neighborhood near 0 then $(1 + A^{-1}BK^\lambda + \kappa A^{-1})$ is invertible for κ near 0 and the proof is over. Assume $\lambda(\kappa) = 0$ in a small neighborhood near 0. We consider the series expansion of ψ around 0 :

$$\psi(\kappa) = A^{-1}\varphi + \sum_{k \in \mathbb{N}^*} \kappa^k \psi_\kappa \,. \tag{213}$$

At the order 0 in (212), we obtain :

$$A^{-1}\varphi + A^{-1}\varphi K^\lambda(A^{-1}\varphi) = 0 \,. \tag{214}$$

At the higher order, we obtain :

$$\psi_k + A^{-1}\varphi K^\lambda(\psi_k) + A^{-1}(\psi_{k-1}) = 0 \,. \tag{215}$$

Taking K^λ of (215), we obtain :

$$\forall k \in \mathbb{N}, \ K^\lambda\left(A^{-1}\psi_k\right) = 0 \tag{216}$$

By recurrence, taking successively A^{-1} and K^λ of (215), we obtain :

$$\forall n \in \mathbb{N}^*, \forall k \in \ \mathbb{N}^*, \ K^\lambda(A^{-n}\psi_k) = 0 \,. \tag{217}$$

However, we have for all $n \geq 2$:

$$K^\lambda\left(A^{-n}\varphi\right) = \sum_{k \in \mathbb{N}^*} \alpha_k^\lambda \left\langle A^{-n}\varphi, \varphi_k \right\rangle = \sum_{k \in \mathbb{N}^*} \frac{(-1)^n \alpha_k^\lambda}{\lambda_k^n} \left\langle \varphi, \varphi_k \right\rangle = 0 \,. \tag{218}$$

Now, we introduce :

$$H(z) := \sum_{k \in \mathbb{N}^*} \frac{(-1)^n \alpha_k^\lambda}{\lambda_k^2} \left\langle \varphi, \varphi_k \right\rangle e^{-\frac{z}{\lambda_k}} \tag{219}$$

H is an entire function and following the same strategy than the proof of 1, we found :

$$\forall k \in \mathbb{N}^*, \ \alpha_k^\lambda = 0 \,, \tag{220}$$

which is in contradiction with the corollary 1. Hence, $1 + A^{-1}BK^\lambda + \kappa A^{-1}$ is invertible for κ in a small neighborhood of 0. Now, we just have to choose κ small enough to simultaneously have $(A + BK^\lambda + \kappa)$ and $A + \kappa$ invertible. □

Now, we can prove a posteriori that :

$$\forall n \in \mathbb{N}^*, \; \alpha_n^\lambda \neq 0 \,. \tag{221}$$

It is not a logic loop because the proof of the invertibility of T^λ does not use this fact and we also give an explicit expression of $(T^\lambda)^{-1}$.

Let $\chi \in H^3_{(0)}$. Because $\left\{g_n^\lambda \lambda_n^{-1/2}\right\}_{n\in\mathbb{N}^*}$ is a Riesz basis of $H^3_{(0)}$, there exists $\{\chi_n^\lambda\}_{n\in\mathbb{N}^*} \in \ell^2(\mathbb{N}^*)$ such that :

$$\chi = \sum_{n\in\mathbb{N}^*} \frac{\chi_n^\lambda}{\lambda_n^{1/2}} g_n^\lambda \quad \text{in} \quad H^3_{(0)} \,. \tag{222}$$

Let $\psi \in H^3_{(0)}$. Then, because of the Riesz basis properties :

$$T^\lambda \psi = \chi \Longleftrightarrow \sum_{n\in\mathbb{N}^*} \frac{\alpha_n^\lambda}{\beta_n^\lambda} \langle \psi, \varphi_n \rangle \, g_n^\lambda = \sum_{n\in\mathbb{N}^*} \frac{\chi_n^\lambda}{\lambda_n^{1/2}} g_n^\lambda \Longleftrightarrow \forall n \in \mathbb{N}^*, \; \frac{\langle \psi, \varphi_n \rangle \, \lambda_n^{1/2} \alpha_n^\lambda}{\beta_n^\lambda} = \chi_n^\lambda \,. \tag{223}$$

Then, if there exists $n_0 \in \mathbb{N}^*$ such that $\alpha_{n_0}^\lambda = 0$, for $\chi_n^\lambda = \delta_{n,n_0}$, the equation $T\psi = \chi$ does not have any solution. Hence :

$$\forall n \in \mathbb{N}^*, \; \alpha_n^\lambda \neq 0 \,. \tag{224}$$

Now, the last equality uniquely defines ψ. Conversely, if ψ is such that $\forall n \in \mathbb{N}^*, \; \langle \psi, \varphi_n \rangle = \frac{\beta_n^\lambda \chi_n^\lambda}{\lambda_n^{1/2} \alpha_n^\lambda}$, then, using the estimate (76) and the corollary 1 :

$$\sum_{n\in\mathbb{N}^*} \lambda_n^3 \left|\langle \psi, \varphi_n \rangle\right|^2 = \sum_{n\in\mathbb{N}^*} \frac{\lambda_n^3}{(\alpha_n^\lambda)^2} \frac{(\beta_n^\lambda)^2}{\lambda_n} (\chi_n^\lambda)^2 \leq C \sum_{n\in\mathbb{N}^*} (\chi_n^\lambda)^2 < +\infty \,. \tag{225}$$

Hence $\psi \in H^3_{(0)}$ and T^λ is invertible. Moreover, this shows that :

$$(T^\lambda)^{-1} \colon \chi = \sum_{n\in\mathbb{N}^*} \frac{\chi_n^\lambda}{\lambda_n^{1/2}} g_n^\lambda \in H^3_{(0)} \longmapsto \sum_{n\in\mathbb{N}^*} \frac{\beta_n^\lambda \chi_n^\lambda}{\lambda_n^{1/2} \alpha_n^\lambda} \varphi_n \,, \tag{226}$$

is continuous from $H^3_{(0)}$ into itself.

Remark 5. Unfortunately, we can't find an estimate for $\|(T^\lambda)^{-1}\|$ by the usual means without a lower bound for $\left|\alpha_n^\lambda\right|$ for all $n \in \mathbb{N}^*$. I did not manage to find one during my internship. Then, it is an open question.

3 Well-posedness of the problem and rapid stabilization

3.1 Well-posedness of the closed-loop system

Now, we focus our attention on the closed-loop system :

$$\begin{cases} \psi_t = (A + BK^\lambda)\psi, \quad t \in (0,T) \,, \\ \psi(0) = \psi_0 \,. \end{cases} \tag{227}$$

Proposition 8. *Let $\lambda > 0$ big enough verifying (15). The unbounded operator $(A + BK^\lambda)$ defined on $D(A + BK^\lambda)$ generates a strongly continuous semigroup on $H^3_{(0)}$. Thus there exists a unique solution $\mathcal{C}([0,T]; H^3_{(0)})$ of (227) with $u(t) = K^\lambda(\psi(t))$.*

Proof. We have already proved that $D(A + BK^\lambda)$ is densely defined (see lemma 4). Now, we prove that $(A + BK^\lambda)$ is a closed operator.

Let $(\psi_n)_{n\in\mathbb{N}^*} \in D(A + BK^\lambda)$ and $(\psi, \phi) \in (H^3_{(0)})^2$ such that :

$$\psi_n \longrightarrow \psi \quad \text{in} \quad H^3_{(0)}\,, \tag{228}$$

$$\Delta\psi_n + K^\lambda(\psi_n)\varphi \longrightarrow \phi \quad \text{in} \quad H^3_{(0)}\,. \tag{229}$$

Using the first equation (228), we have :

$$\Delta\psi_n \longrightarrow \Delta\psi \quad \text{in} \quad H^1_{(0)}\,. \tag{230}$$

Reinjecting in the second equation (229) :

$$K^\lambda(\psi_n)\varphi \longrightarrow \phi - \Delta\psi \quad \text{in} \quad H^1_{(0)}\,. \tag{231}$$

Because $\mathbb{R}\varphi$ is closed in $H^1(0)$, there exists $\kappa \in \mathbb{R}$ such that :

$$K^\lambda(\psi_n)\varphi \longrightarrow \kappa\varphi \quad \text{in} \quad H^1_{(0)}\,. \tag{232}$$

Which implies :

$$K^\lambda(\psi_n) \longrightarrow \kappa\,. \tag{233}$$

And because $\varphi \in H^2_{(0)}$, we also have convergence in $H^2_{(0)}$:

$$K^\lambda(\psi_n)\varphi \longrightarrow \kappa\varphi = \phi - \Delta\psi \quad \text{in} \quad H^2_{(0)}\,. \tag{234}$$

Hence :

$$\Delta\psi \in H^2_{(0)} \implies \psi \in H^4_{(0)}\,, \tag{235}$$

And $K^\lambda(\psi)$ is well defined. Moreover :

$$\|\Delta\psi - \Delta\psi_n\|_{H^2_{(0)}} \leq \left\|K^\lambda(\psi_n) - (\phi - \Delta\psi)\right\|_{H^2_{(0)}} + \left\|(\Delta\psi_n + K^\lambda(\psi_n)) - \phi\right\|_{H^2_{(0)}}\,, \tag{236}$$

hence :

$$\Delta\psi_n \longrightarrow \Delta\psi \quad \text{in} \quad H^2_{(0)} \implies \psi_n \longrightarrow \psi \quad \text{in} \quad H^4_{(0)}\,. \tag{237}$$

Using the lemma 1, we have :

$$\left|K^\lambda(\psi) - K^\lambda(\psi_n)\right| = \left|\sum_{k\in\mathbb{N}^*} \alpha_k^\lambda \langle \psi - \psi_n, \varphi_k\rangle\right| \leq \underbrace{\sqrt{\sum_{k\in\mathbb{N}^*} \left(\frac{\alpha_k^\lambda}{\lambda_k^2}\right)^2}}_{<+\infty} \|\psi - \psi_n\|_{H^4_{(0)}} \longrightarrow 0\,. \tag{238}$$

We successively obtain :

$$\kappa = K^\lambda(\psi)\,, \tag{239}$$
$$\Delta\psi + K^\lambda(\psi)\varphi = \phi \quad \text{and} \tag{240}$$
$$\psi \in D(A + BK^\lambda)\,. \tag{241}$$

Let us now prove the dissipativity of $(A + BK^\lambda)$. Since T^λ is invertible from $H^3_{(0)}$ into itself, $\|\cdot\|_{T^\lambda} \coloneqq \left\|T^\lambda \cdot\right\|_{H^3_{(0)}}$ defines a equivalent norm to $\|\cdot\|_{H^3_{(0)}}$. Let $\psi \in D(A + BK^\lambda)$, from proposition 5 and remark 4, we have :

$$\left\langle (A + BK^\lambda)\psi, \psi\right\rangle_{T^\lambda} = \left\langle T^\lambda(A + BK^\lambda)\psi, T^\lambda\psi\right\rangle_{H^3_{(0)}} \tag{242}$$
$$= \left\langle (A - \lambda)T^\lambda\psi, T^\lambda\psi\right\rangle_{H^3_{(0)}} \tag{243}$$
$$= -\lambda \left\|T^\lambda\psi\right\|^2_{H^3_{(0)}} + \sum_{k\in\mathbb{N}^*} \lambda_k^3 \left\langle AT^\lambda\psi, \varphi_k\right\rangle \left\langle T^\lambda\psi, \varphi_k\right\rangle \tag{244}$$
$$= -\lambda \left\|T^\lambda\psi\right\|^2_{H^3_{(0)}} - \sum_{k\in\mathbb{N}^*} \lambda_k^4 \left|\left\langle T^\lambda\psi, \varphi_k\right\rangle\right|^2 \tag{245}$$
$$= -\lambda \left\|T^\lambda\psi\right\|^2_{H^3_{(0)}} - \left\|T^\lambda\psi\right\|^2_{H^4_{(0)}} \leq 0\,. \tag{246}$$

Concerning the dissipativity of $(A + BK^\lambda)^*$, we first explicit $D((A + BK^\lambda)^*)$.

Here, we will consider A to be the invertible unbounded operator on $H^3_{(0)}$:

$$A\colon D(A) \coloneqq H^5_{(0)} \longrightarrow H^3_{(0)}\,. \tag{247}$$

A is symetric. Moreover, if $\psi \in D(A^*)$ then it exists $w_\psi \in H^3_{(0)}$ such that :

$$\forall n \in \mathbb{N}^*,\ \langle w_\psi, \varphi_n\rangle_{H^3_{(0)}} = \langle \psi, A\varphi_n\rangle_{H^3_{(0)}} = -\lambda_n \langle \psi, \varphi_n\rangle_{H^3_{(0)}}\,. \tag{248}$$

Hence : $\langle w_\psi, \varphi_n\rangle = -\lambda_n \langle \psi, \varphi_n\rangle$ and we have :

$$w_\psi = \sum_{n\in\mathbb{N}^*} \langle \varphi_n, w_\psi\rangle \varphi_n = \sum_{n\in\mathbb{N}^*} -\lambda_n \langle \varphi_n, \psi\rangle \varphi_n\,, \tag{249}$$

and :

$$w_\psi \in H^3_{(0)} \Longleftrightarrow \sum_{n\in\mathbb{N}^*} \lambda_n^3 \left|\langle w_\psi, \varphi_n\rangle\right|^2 = \sum_{n\in\mathbb{N}^*} \lambda_n^5 \left|\langle \psi, \varphi_n\rangle\right|^2 < +\infty \Longleftrightarrow \psi \in H^5_{(0)}\,. \tag{250}$$

So $D(A^*) = H^5_{(0)}$ and A is self-adjoint. Let $\psi \in D((A + BK^\lambda)^*)$ and $\phi \in D(A + BK^\lambda)$. We recall that $T^\lambda\phi \in H^5_{(0)}$ (see remark 4). And we have :

$$\left\langle \psi, (A + BK^\lambda)\phi\right\rangle_{T^\lambda} = \left\langle T^\lambda\psi, (A - \lambda)T^\lambda\phi\right\rangle_{H^3_{(0)}}\,. \tag{251}$$

Let $\psi \in D((A+BK^\lambda)^*)$. Then :

$$\exists w_\psi \in H^3_{(0)},\ \forall \phi \in D(A+BK^\lambda),\ \left\langle T^\lambda \psi, (A-\lambda)T^\lambda \phi \right\rangle_{H^3_{(0)}} = \left\langle \psi, (A+BK^\lambda)\phi \right\rangle_{T^\lambda} \tag{252}$$

$$= \langle w_\psi, \phi \rangle_{T^\lambda} = \left\langle T^\lambda w_\psi, T^\lambda \phi \right\rangle_{H^3_{(0)}} . \tag{253}$$

Because $D(A+BK^\lambda)$ is dense in $H^3_{(0)}$ and T^λ continuous from $H^3_{(0)}$ into itself, if $\psi \in D((A+BK^\lambda)^*)$ then $T^\lambda \psi \in D(A^*) = H^5_{(0)}$. Conversely, if $\psi \in H^5_{(0)}$, then forall $\phi \in D(A+BK^\lambda)$:

$$\left\langle \psi, (A+BK^\lambda)\phi \right\rangle_{T^\lambda} = \left\langle T^\lambda \psi, (A-\lambda)T^\lambda \phi \right\rangle_{H^3_{(0)}} = \left\langle ((T^\lambda)^{-1}AT^\lambda - \lambda)\psi, \phi \right\rangle_{T^\lambda} , \tag{254}$$

and $\psi \in D((A+BK^\lambda)^*)$. Hence : $D((A+BK^\lambda)^*) = H^5_{(0)}$. Now, let $\psi \in D((A+BK^\lambda)^*)$. We have :

$$\left\langle (A+BK^\lambda)^* \psi, \psi \right\rangle_{T^\lambda} = \left\langle ((T^\lambda)^{-1}AT^\lambda - \lambda)\psi, \psi \right\rangle_{H^3_{(0)}} \tag{255}$$

$$= \left\langle AT^\lambda \psi, T^\lambda \psi \right\rangle_{H^3_{(0)}} - \lambda \left\| T^\lambda \psi \right\|^2_{H^3_{(0)}} \leq 0 . \tag{256}$$

Thank to the Lumer-Phillips theorem (see, for example, [6]), $A+BK^\lambda$ generates a strongly continuous semigroup on $H^3_{(0)}$. □

3.2 Rapid stabilization

We are now interested by the proof of the rapid stabilization of initial system :

$$\begin{cases} \psi_t = A\psi + Bu, & t \in (0,T) , \\ \psi(0) = \psi_0 . \end{cases} \tag{257}$$

Theorem 1. *Let $\lambda > 0$ big enough verifying (15) and $\psi_0 \in H^3_{(0)}$. Then, there exists a linear feedback control u^λ such that any solution of (257) with ψ_0 as initial condition verifies :*

$$\exists C \geq 0,\ \| \psi(t) \|_{H^3_{(0)}} \leq C e^{-\lambda t} \| \psi_0 \|_{H^3_{(0)}} . \tag{258}$$

Proof of the theorem 1. Let $\psi_0 \in D(A+BK^\lambda)$. The solution is given by :

$$\psi(t) = e^{t(A+BK^\lambda)} \psi_0 . \tag{259}$$

Let $\chi(t) = T^\lambda \psi(t)$. Thanks to the operator inequality, we have :

$$\frac{\mathrm{d}}{\mathrm{d}t}\chi(t) = T^\lambda \frac{\mathrm{d}}{\mathrm{d}t}\psi(t) = T^\lambda (A+BK^\lambda)\psi(t) \tag{260}$$

$$= (AT^\lambda - \lambda T^\lambda)\psi(t) = A\chi(t) - \lambda \chi(t) . \tag{261}$$

Taking the scalar product with $\chi(t)$ leads to :

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\chi(t)\|^2_{H^3_{(0)}} = 2\left\langle \frac{\mathrm{d}}{\mathrm{d}t}\chi(t), \chi(t)\right\rangle = 2\left\langle A\chi(t), \chi(t)\right\rangle - 2\lambda\|\chi(t)\|^2_{H^3_{(0)}} \tag{262}$$

$$\leq -2\lambda\|\chi(t)\|^2_{H^3_{(0)}} . \tag{263}$$

Hence, by Gronwall lemma :

$$\|\chi(t)\|^2_{H^3_{(0)}} \leq e^{-2\lambda t}\left\|T^\lambda\psi_0\right\|^2_{H^3_{(0)}} . \tag{264}$$

The continuity and the invertibility of T^λ implies :

$$\|\psi(t)\|_{H^3_{(0)}} \leq \|T^\lambda\|\|(T^\lambda)^{-1}\|e^{-\lambda t}\|\psi_0\|_{H^3_{(0)}} . \tag{265}$$

Now, using the density of $D(A+BK^\lambda)$ in $H^3_{(0)}$ and this previous estimate, we can extend it to all $\psi_0 \in H^3_{(0)}$. □

3.3 Null-controllability

We now investigate the null controllability of the system (257). If we want to use the rapid stabilization result of the system (257) as in [4], we need an estimation on $\|(T^\lambda)^{-1}\|$. Unfortunately, because we did not manage to find a lower bound for all α_n^λ, we have to make an new strong assumption :

Hypothesis 2. *There exists $\widetilde{m} > 0$ and $C > 0$ such that for all $\lambda > 0$ big enough and verifying (15), we have :*

$$\|(T^\lambda)^{-1}\| \leq C\lambda^{\widetilde{m}} . \tag{266}$$

Now, we can state the null controllability of the control system 257.

Theorem 2. *Under the assumptions 1 and 2, there exists exists two increasing sequences $\{\lambda_n\}_{n\in\mathbb{N}^*}$ and $\{t_n\}_{n\in\mathbb{N}^*}$ such that for all $n \in \mathbb{N}^*$, λ_n is big enough and verifies (15) and :*

$$t_1 = 0 \quad \textit{and} \quad t_n \underset{n\to+\infty}{\longrightarrow} T , \tag{267}$$

and such that the solution ψ of (257) with :

$$\forall t \in [0,T],\ u(t) = K^{\lambda_n}\psi(t) \quad \textit{if} \quad t \in [t_n, t_{n+1}[,$$

verifies :

$$\psi(T) = 0 .$$

Proof of the theorem 2. Let $\{\lambda_n\}_{n\in\mathbb{N}^*}$ and $\{t_n\}_{n\in\mathbb{N}^*}$ be two increasing sequence such that for all $n \in \mathbb{N}^*$, λ_n is big enough and verifies (15) and :

$$t_1 = 0 \quad \text{and} \quad t_n \underset{n\to+\infty}{\longrightarrow} T . \tag{268}$$

We denote $s_n = t_n - t_{n-1}$. The assumptions 2 and the theorem 1 give $p > 0$ such that for all $n \geq 2$ the solution ψ_n of the control system :

$$\left\{ \begin{array}{l} \psi_{nt} = A\psi_n + BK^{\lambda_n}\psi_n, \quad t \in (t_{n-1}, t_n)\,, \\ \psi_n(0) = \psi_{n0}\,. \end{array} \right. \tag{269}$$

verifies :

$$\|\psi_n(t_n)\|_{H^3_{(0)}} \leq C\lambda_n^p e^{-\lambda_n s_n} \|\psi_{n0}\|_{H^3_{(0)}}\ . \tag{270}$$

Hence, the solution ψ of (257) with :

$$\forall t \in [0\,, T],\ u(t) = K^{\lambda_n}\psi(t) \quad \text{if} \quad t \in [t_n\,, t_{n+1}[\,,$$

verifies :

$$\begin{aligned} \|\psi(t_n)\|_{H^3_{(0)}} &\leq C^n(\lambda_n \cdots \lambda_1)^p e^{-\sum_{k=1}^n \lambda_k s_k} \|\psi_0\|_{H^3_{(0)}} \\ &\leq \exp\left(n\ln(C) + \sum_{k=1}^n p\ln(\lambda_k) - \lambda_k s_k \right) \|\psi_0\|_{H^3_{(0)}}\ . \end{aligned}$$

We choose, for example, $\{\lambda_n\}_{n\in\mathbb{N}^*}$ and $\{t_n\}_{n\in\mathbb{N}^*}$ such that :

$$\forall n \in \mathbb{N}^*,\ s_n = \frac{6T}{n^2\pi^2} \quad \text{and} \quad \exists C_2 > C_1 > 0,\ \forall n \in \mathbb{N}^*,\ C_1 n^3 \leq \lambda_n \leq C_2 n^3\,. \tag{271}$$

Clearly $t_n \to T$ as n goes to $+\infty$ and we have :

$$\|\psi(t_n)\|_{H^3_{(0)}} \leq \exp\left[n(\ln(C) + 3pC_2\ln(n)) - \frac{6C_1 T}{\pi^2}\frac{n(n+1)}{2} \right] \|\psi_0\|_{H^3_{(0)}} \underset{n\to+\infty}{\longrightarrow} 0\,. \tag{272}$$

Hence :

$$\psi(T) = 0\,. \tag{273}$$

□

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A Notions about Riesz bases

Definition 1. A collection of vectors $\{x_n\}_{n\in\mathbb{N}^*}$ in an Hilbert space $\mathcal{H}$ is a *Riesz basis* for $\mathcal{H}$ if it is the image of an hilbertian basis for $\mathcal{H}$ under an invertible linear transformation.

The following theorem gathers the fundamental properties of the Riesz bases.

Theorem 3. *Let $\{x_n\}_{n\in\mathbb{N}^*}$ a collection of vectors in a Hilbert space $\mathcal{H}$.*

(a) If $\{x_n\}_{n\in\mathbb{N}^*}$ is a Riesz basis for $\mathcal{H}$ then there is a unique collection $\{y_n\}_{n\in\mathbb{N}^*}$ such that $\langle x_n, y_k\rangle = \delta_{n,k}$. In this case $\{y_n\}_{n\in\mathbb{N}^*}$ is also a Riesz basis for $\mathcal{H}$ and is called the dual Riesz basis *of $\{x_n\}_{n\in\mathbb{N}^*}$.*

(b) If $\{x_n\}_{n\in\mathbb{N}^*}$ is a Riesz basis for $\mathcal{H}$ then there are constants $0 \leq A \leq B$ such that for $x \in \mathcal{H}$:

$$A \|x\|^2 \leq \sum_{n\in\mathbb{N}^*} |\langle x, x_n\rangle|^2 \leq B \|x\|^2 \ . \tag{274}$$

This inequality is called the frame inequality*.*

(c) $\{x_n\}_{n\in\mathbb{N}^*}$ is a Riesz basis for $\mathcal{H}$ if and only if there are constants $0 \leq A \leq B$ such that for all finite sequences $\{\alpha_n\}_{n\in\mathbb{N}^*}$, we have :

$$A \sum_{n\in\mathbb{N}^*} |\alpha_n|^2 \leq \left\| \sum_{n\in\mathbb{N}^*} \alpha_n x_n \right\|^2 \leq B \sum_{n\in\mathbb{N}^*} |\alpha_n|^2 \ . \tag{275}$$

(d) If $\{x_n\}_{n\in\mathbb{N}^*}$ is a Riesz basis for $\mathcal{H}$ then for each $x \in \mathcal{H}$ there is a unique collection of scalars $\{\alpha_n\}_{n\in\mathbb{N}^*}$ such that $x = \sum_{n\in\mathbb{N}^*} \alpha_n x_n$ and $\sum_{n\in\mathbb{N}^*} |\alpha_n|^2 < +\infty$.

Now, we give two characterization theorems.

Definition 2. Let $\mathcal{H}$ be an infinite-dimensional Hilbert space. An infinite collection $\{x_n\}_{n\in\mathbb{N}^*}$ of vectors in $\mathcal{H}$ is *ω-linearly independent* if a sequence $\{\alpha_n\}_n$ such that $\sum_{n\in\mathbb{N}^*} \alpha_n x_n$ converges in the norm of $\mathcal{H}$ to 0 must be identically zero.

Definition 3. Let $\mathcal{H}$ be an infinite-dimensional separable Hilbert space and $\{e_n\}_{n\in\mathbb{N}^*}$ an orthonormal basis $\mathcal{H}$. An infinite collection $\{x_n\}_{n\in\mathbb{N}^*}$ of vectors in $\mathcal{H}$ is *quadratically close* to $\{e_n\}_{n\in\mathbb{N}^*}$ if

$$\sum_{n\in\mathbb{N}^*} \|x_n - e_n\|^2_{\mathcal{H}} < +\infty \, . \tag{276}$$

Theorem 4. *Let $\mathcal{H}$ be an infinite-dimensional separable Hilbert space and $\{e_n\}_{n\in\mathbb{N}^*}$ an orthonormal basis $\mathcal{H}$. If $\{x_n\}_{n\in\mathbb{N}^*}$ is ω-linearly independent sequence of $\mathcal{H}$ and is quadracally close to $\{e_n\}_{n\in\mathbb{N}^*}$ then $\{x_n\}_{n\in\mathbb{N}^*}$ is a Riesz basis for $\mathcal{H}$.*

Theorem 5. *Let $\mathcal{H}$ be an infinite-dimensional separable Hilbert space and $\{e_n\}_{n\in\mathbb{N}^*}$ an orthonormal basis $\mathcal{H}$. If $\{x_n\}_{n\in\mathbb{N}^*}$ is a quadracally close to $\{e_n\}_{n\in\mathbb{N}^*}$ sequence of $\mathcal{H}$ and if* $\operatorname{span}\{x_n,\ n \in \mathbb{N}^*\}$ *is dense in $\mathcal{H}$ then $\{x_n\}_{n\in\mathbb{N}^*}$ is a Riesz basis for $\mathcal{H}$.*

Definition 4. Let $\{x_n\}_{n\in\mathbb{N}^*}$ be a Riesz basis in a Hilbert space $\mathcal{H}$. The *analysis operator* T associated to $\{x_n\}_{n\in\mathbb{N}^*}$ is the bounded operator define by :

$$T\colon x \in \mathcal{H} \longmapsto \{\langle x, x_n\rangle\}_{n\in\mathbb{N}^*} \in \ell^2(\mathbb{N}^*)\,. \tag{277}$$

The *synthesis operator* T^* associated to $\{x_n\}_{n\in\mathbb{N}^*}$ is the adjoint operator of T and is define by :

$$T^*\colon \alpha = \{\alpha_n\}_{n\in\mathbb{N}^*} \in \ell^2(\mathbb{N}^*) \longmapsto \sum_{n\in\mathbb{N}^*} \alpha_n x_n \in \mathcal{H}\,. \tag{278}$$

The *Gram matrix* G associated to $\{x_n\}_{n\in\mathbb{N}^*}$ is the infinite matrix associated to the auto-adjoint operator T^*T. We have :

$$\forall (n,k)^2 \in (\mathbb{N}^*)^2,\ G_{nk} = \langle x_n, x_k\rangle\ . \tag{279}$$

Proposition 9. *Let $\{x_n\}_{n\in\mathbb{N}^*}$ be a Riesz basis in a Hilbert space $\mathcal{H}$, G its Gram matrix and $\{y_n\}_{n\in\mathbb{N}^*}$ its dual Riesz basis. Then, G is invertible, G^{-1} is the Gram matrix associated to $\{y_n\}_{n\in\mathbb{N}^*}$ and we have :*

$$\forall (n,k) \in (\mathbb{N}^*)^2,\ G^{-1}_{nk} = \langle y_n, y_k\rangle\ . \tag{280}$$

We introduce a specific class of Riesz bases.

Definition 5. Let $\{x_n\}_{n\in\mathbb{N}^*}$ be a sequence in a Hilbert space $\mathcal{H}$. We say that $\{x_n\}_{n\in\mathbb{N}^*}$ is self-localized with decay rate $\alpha > 1$ if there exists $C > 0$ such that :

$$\forall (n,k) \in (\mathbb{N}^*)^2,\ |\langle x_n, x_k\rangle| \leq \frac{C}{(1+|n-k|)^\alpha}\,. \tag{281}$$

Theorem 6. *Let $\{x_n\}_{n\in\mathbb{N}^*}$ be a Riesz basis in a Hilbert space $\mathcal{H}$. If $\{x_n\}_{n\in\mathbb{N}^*}$ is self-localized with decay rate $\alpha > 1$ then its dual Riezs basis is also self-localized with decay rate $\alpha > 1$.*

B Solutions de (32)

We want to resolve the differential system (282) in $L^2(0,1)$.

$$\begin{cases} -\lambda_n f_n(x) - f_n''(x) + \lambda f_n(x) + \alpha_n \varphi(x) = 0, & x \in 0,1) \\ f_n(0) = f_n(1) = 0\,. \end{cases} \tag{282}$$

Let $f \in L^2(0,1)$ be a solution of (282). We decompose f_n the Hilbert basis $\{\varphi_n\}_{n\in\mathbb{N}^*}$:

$$f_n = \sum_{k\in\mathbb{N}^*} \langle f_n, \varphi_k\rangle\, \varphi_k \quad \text{in} \quad L^2(0,1)\,. \tag{283}$$

Let $\psi \in \mathcal{C}_c^\infty(0,1)$. We have :

$$\langle f_n'', \psi\rangle \coloneqq \langle f_n, \psi''\rangle = \sum_{k\in\mathbb{N}^*} \langle f_n, \varphi_k\rangle\, \langle \psi'', \varphi_k\rangle = \sum_{k\in\mathbb{N}^*} -\lambda_k \langle f_n, \varphi_k\rangle\, \langle \psi, \varphi_k\rangle \tag{284}$$

$$= \left\langle \sum_{k\in\mathbb{N}^*} -\lambda_k \langle f_n, \varphi_k\rangle\, \varphi_k, \psi \right\rangle. \tag{285}$$

Hence :

$$f_n'' = \sum_{k\in\mathbb{N}^*} -\lambda_k \langle f_n, \varphi_k \rangle \varphi_k \quad \text{in} \quad L^2(0,1)\,. \tag{286}$$

If we take the scalar product of with φ_k, we found :

$$\langle f_n, \varphi_k \rangle = \frac{\alpha_n \langle \varphi, \varphi_k \rangle}{\lambda_n - \lambda_k - \lambda}\,. \tag{287}$$

Conversely, if we definite f_n by (283) and (287), f_n lie in $L^2(0,1)$ because :

$$\frac{1}{|\lambda_n - \lambda_k - \lambda|} \leq C \quad \text{and} \quad |\langle \varphi, \varphi_k \rangle| \leq \frac{C}{k^3}\,. \tag{288}$$

And, the previous computations show that f_n is solution of (282).

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