We prove here the following: let $f : X \to (-\infty, +\infty]$ be a proper, lower semicontinuous convex function and D its domain, and we assume that D has non empty interior, $f \in C^1(\mathrm{int}(D))$, $\lim_{n\to\infty} \|df(x_n)\|_* \to \infty$ for any sequence $x_n \to \partial D$. Then $\partial f(x) = \emptyset$ for any $x \in \partial D$. (In finite dimension, one can replace "interior" by "relative interior" and the same is true.)

Proof: We first observe that f is locally bounded from above in the interior of the domain, because we have assumed it is C^1. (If we want to relax this assumption, it also relies on Baires's theorem, using that for y in the interior we have $D = \cup_{n\geq 1} y + n[\{x : f(x) \leq f(y)+1\} - y]$ so that the set $\{f(x) \leq f(y)+1\}$ contains a ball, then one can show that f is bounded in some ball around y.)

We consider y, δ with $B(y,\delta) \subset \mathrm{int}(D)$ and $\sup_{B(y,\delta)} f < +\infty$. We let for $x \in D$, $t \in]0,1[$, $|z| < \delta$,

$$f_t(z) = \frac{1}{t}(f((1-t)x + t(y+z)) - f(x)).$$

By convexity, $f_t(z) \leq f(y+z) - f(x) \leq \sup_{B(y,\delta)} f - f(x) < +\infty$.

Assume $x \in \partial D$ with $\partial f(x) \neq \emptyset$. Then if $p \in \partial f(x)$,

$$f((1-t)x + t(y+z)) - f(x) \geq (p, (1-t)x + t(y+z) - x) = t(p, y+z-x)$$

so that for $|z| < \delta$, $f_t(z) \geq (p, y-x) - \delta\|p\|_*$. Letting $C = \max\{\sup_{B(y,\delta)} f - f(x), \delta\|p\|_* - (p, y-x)\}$, we find that $|f_t(z)| \leq C$ in $B(0,\delta)$. For $L = 4C/\delta$, we deduce that f_t is L-Lipschitz in $B(0,\delta/2)$. In particular,

$$\|df(x + t(y-x))\|_* = \|df_t(0)\|_* \leq L$$

for any $t > 0$, which contradicts the assumption. We could even remove the assumption that f is C^1 in the interior of the domain and assume that for (x_n, p_n) with $p_n \in \partial f(x_n)$ and $x_n \to \partial D$, $\|p_n\|_* \to \infty$, and would get a contradiction as well.