## Around Borel Cantelli lemma

**Lemma 1.** Let \((A_n)\) be a sequence of events, and \(B = \bigcap_{N \geq 1} \bigcup_{n > N} A_n = \limsup A_n\) the event “the events \(A_n\) occur for an infinite number of \(n\) (\(A_n\) occurs infinitely often)”. Then:

1. If \(\sum P(A_n) < \infty\), then \(P(B) = 0\).
2. If \(\sum P(A_n)\) diverge and \(A_n\) are independent, then \(P(B) = 1\).

This lemma is quite useful to characterize a.s. convergence, or create counter examples.

### Counter example and application of the lemma

First a counter example to the second point:

\(\Omega = ]0, 1[\), \(\mathcal{F}\) = Borel sets, \(P =\) Lebesgue measure, we define \(A_n = ]0, a_n[\), with \(a_n \to 0\). Then \(\limsup A_n = \emptyset\) but with \(a_n \geq 1/n\), \(\sum a_n = \infty\).

We can even use the Borel Cantelli lemma to show a (weaker) weak law of large numbers:

**Theorem 2.** Let \(X_1, X_2, \ldots\) be a sequence of iid random variables with \(EX_i = \mu\) and \(EX_i^4 < \infty\), then \(S_n/n = (X_1 + \cdots + X_n)/n \overset{P}{\to} \mu\).

**Proof.** By independence of the \(X_i\) and supposing that \(\mu = 0\) without loss of generality, we have

\[
ES_n^4 = E\left(\sum X_i\right)^4 = nEX_1^4 + 3(n^2 - n)(EX_1^4)^2 \leq Cn^2.
\]

Tchebycheff inequality gives us:

\[
P(|S_n| > n\varepsilon) \leq ES_n^4/(n\varepsilon)^4 \leq C/n^2,
\]

by summing on \(n\), the Borel-Cantelli lemma gives us \(P(|S_n| > \varepsilon\text{ infinitely often}) = 0\) which concludes the proof.

\(\square\)

### \((X_n)\) converges in probability to \(X\) and not almost surely

We define \(X_n\) independent such that, \(P(X_n = n) = 1/n\) and \(P(X_n = 0) = 1 - 1/n\). Then, \(X_n \overset{P}{\to} 0\), as \(P(|X| < \varepsilon) = P(X = n) = 1/n \to 0\) as \(n \to \infty\).

We use Borel Cantelli lemma to show that this sequence does not converge a.s. to zero. Here \(A_n = \{X_n(\omega) = n\}\), as \(\sum P(A_n) = \infty\), so by the lemma, \(P(B) = 1\), with \(B = \{\omega, X_n(\omega) = n\text{ for an infinite number of }n\}\), thus \(B^c = \{\omega, X_n(\omega) \to 0\}\) has probability 0. That is the sequence does not converge to 0.

However, if \(X_n \overset{P}{\to} X\), then we can extract a subsequence \(X_{\phi(n)}\) such that \(X_{\phi(n)}\) converge a.s. to \(X\), we show that result with Borel Cantelli lemma:
We define $\phi(n)$ by recurrence on $n$, we initialize the recurrence with $\phi(0)$ arbitrary chosen, suppose $\phi(0), \phi(1), \ldots, \phi(n-1)$ defined, then we can choose an integer $k > \phi(n-1)$ such that:

$$P(|X_k - X| > 1/n) < 1/n^2,$$

by definition of the convergence in probability. We define $\phi(n) = k$. To prove that $X_{\phi(n)}$ converges a.s. we use Borel Cantelli lemma. $A_n = \{|X_{\phi(n)} - X| > 1/n\}$, and $\sum 1/n^2 < \infty$ so by Borel Cantelli lemma, $P(A_n$ occurs infinitely often) = 0, then for almost all $\omega$ for all $N$ greater than some index $N$ (depending on $\omega$), $|X_{\phi(n)}(\omega) - X(\omega)| \leq 1/n$, and thus $X_{\phi(n)} \to X$, a.s.

$(X_n)$ converges a.s. to $X$ and not in $L^p$

The same way, we construct $X_n$ independent such that $P(X_n = n^2) = 1/n^2$ and $P(X_n = 0) = 1 - 1/n^2$. With the same events, and with Borel Cantelli, as $\sum 1/n^2 < \infty$, we conclude that the event “$X_n = n^2$ infinitely often” has probability 0, and that it converges a.s. to zero. However $EX_n = 1$ it cannot converge to 0 in $L^1$. 

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