The Art of Distance and Geodesic Computations

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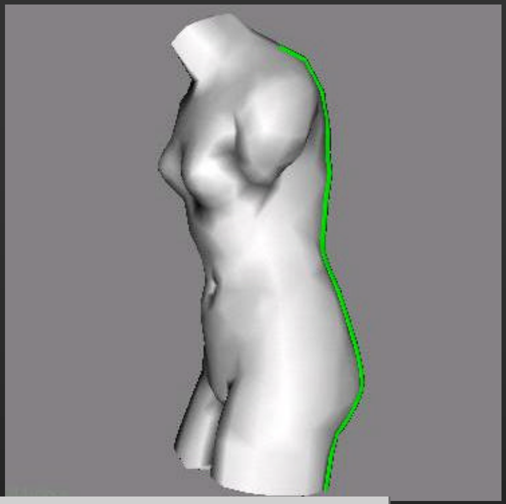
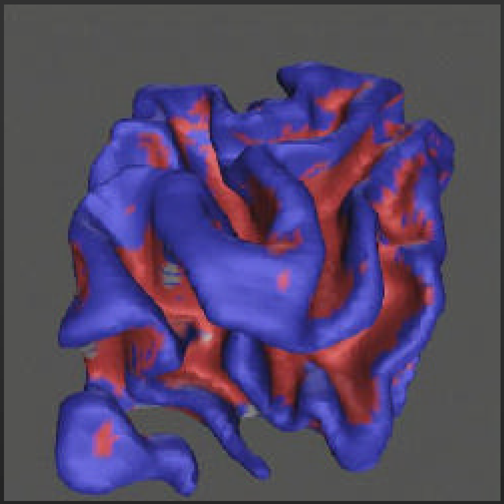
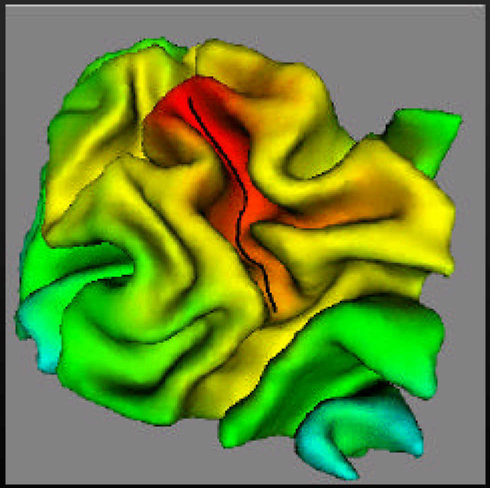
Overview

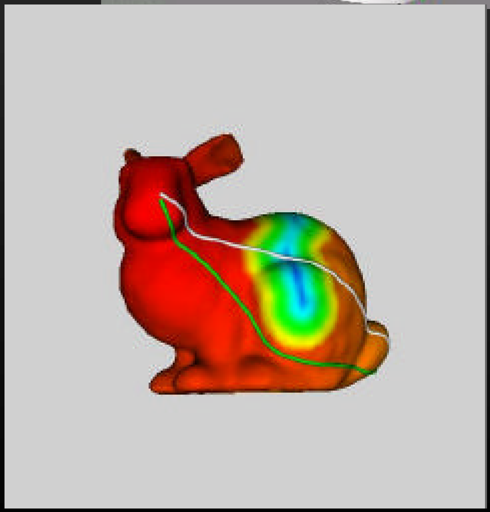
- Motivation
- Background on fast/accurate geodesic computations
- Distance functions and geodesics on implicit hyper-surfaces
- Unorganized points
- Generalized geodesics
- The future and concluding remarks

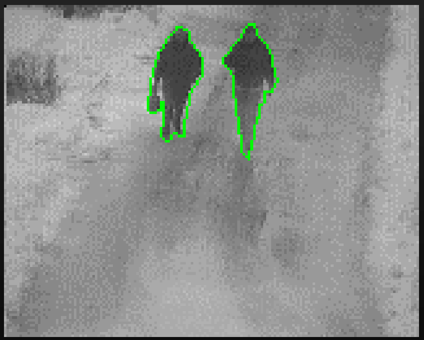
Motivation: A Few Examples





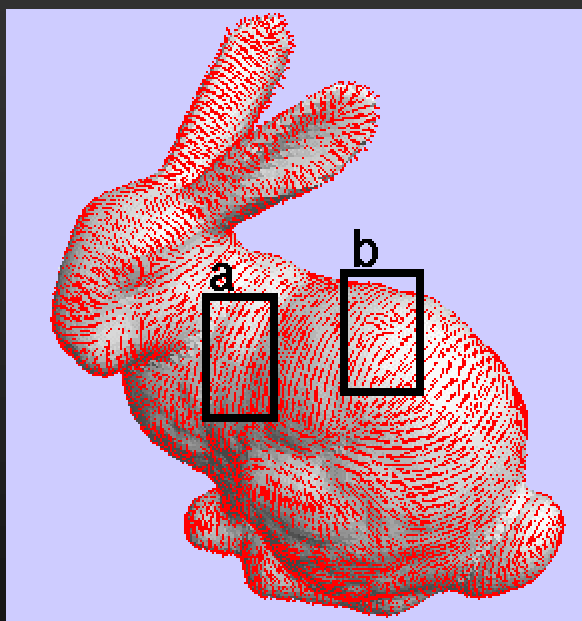




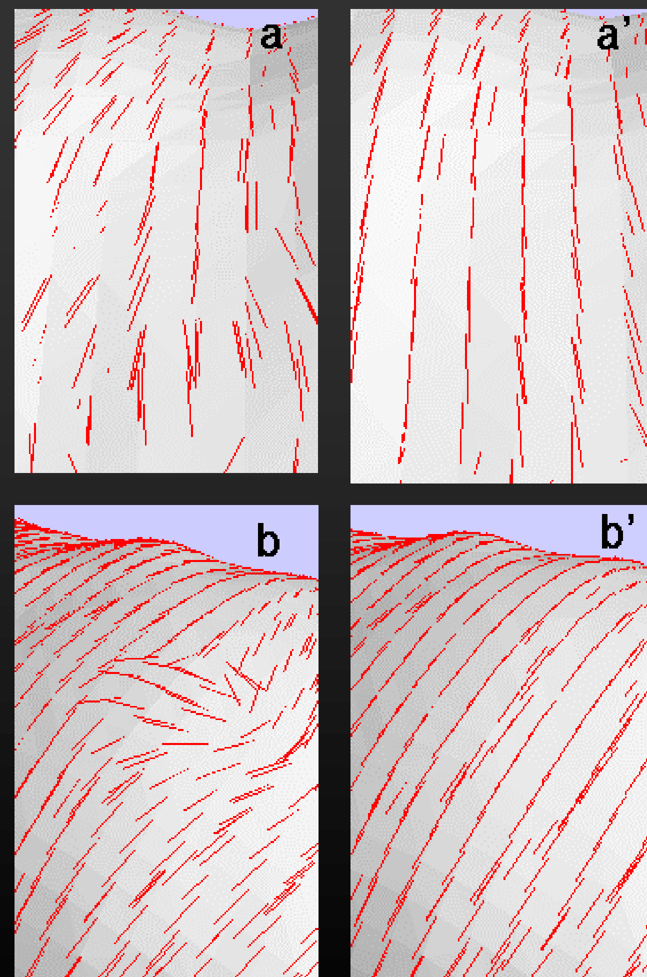


Show me!!!

Motivation: A Few Examples (cont.)



noisy



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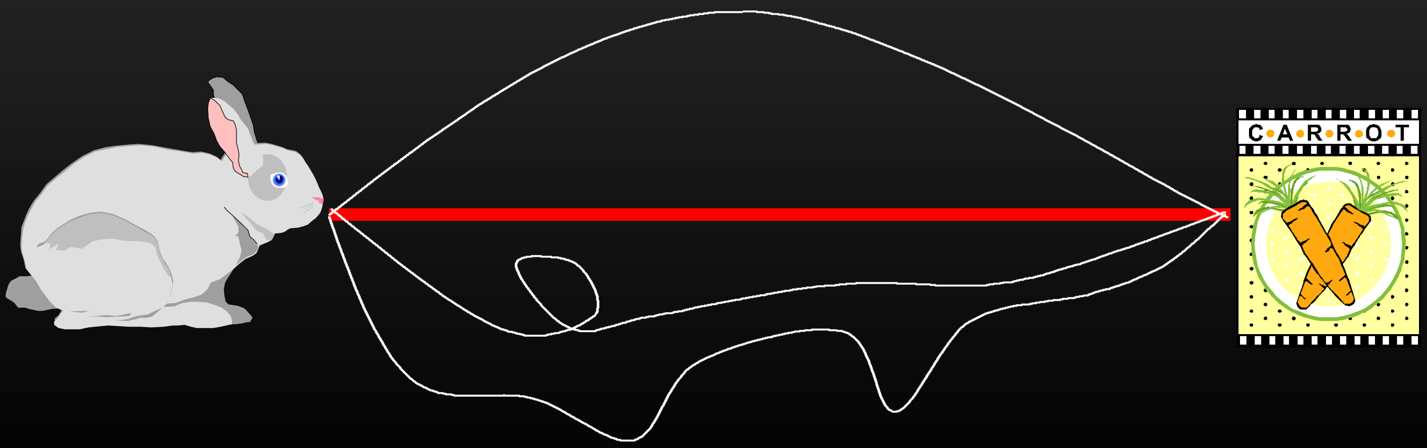
Motivation: What is a Geodesic?

$$d_S^g(p,x) = \inf_C \int_p^x g(C)\,ds$$



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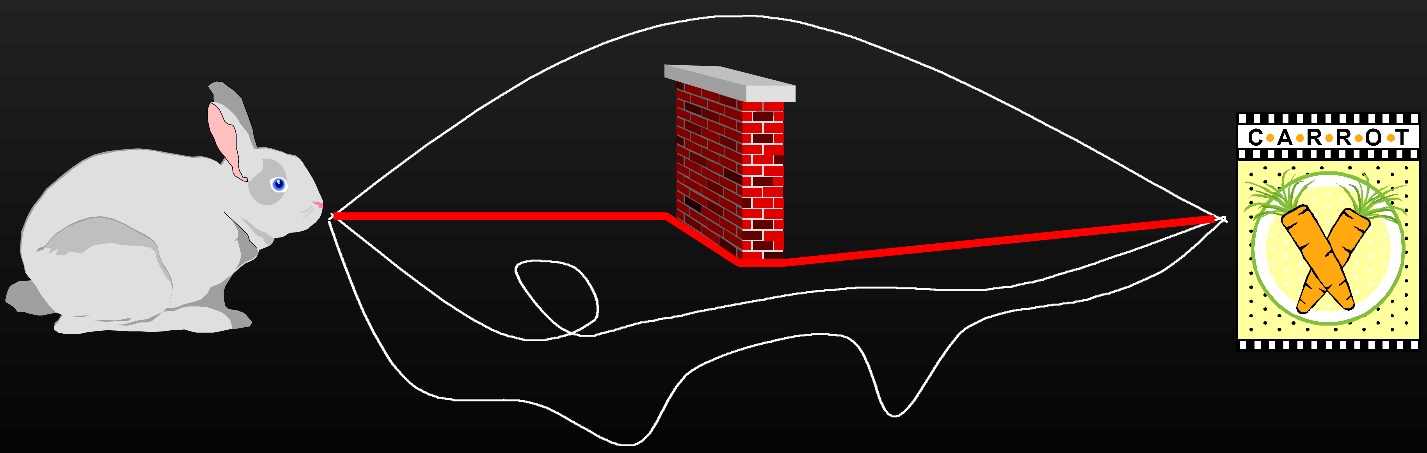
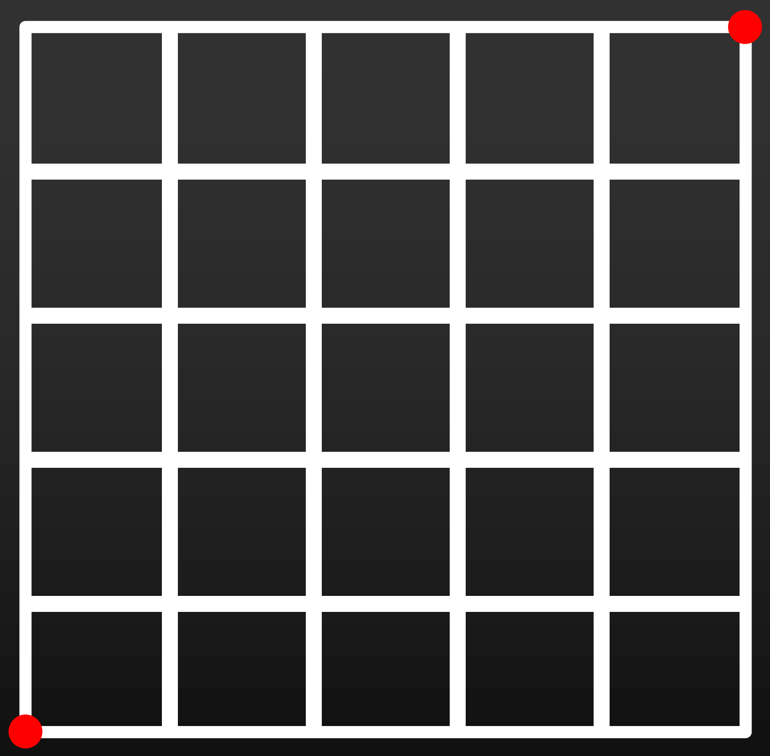


Background: Distance and Geodesic Computation via Dijkstra



- **Complexity:** O(n log n)
- **Advantage:** Works in any dimension and with any geometry (graphs)
- **Problems:**
  - Not consistent
  - Unorganized points?
  - Noise?
  - Implicit surfaces?

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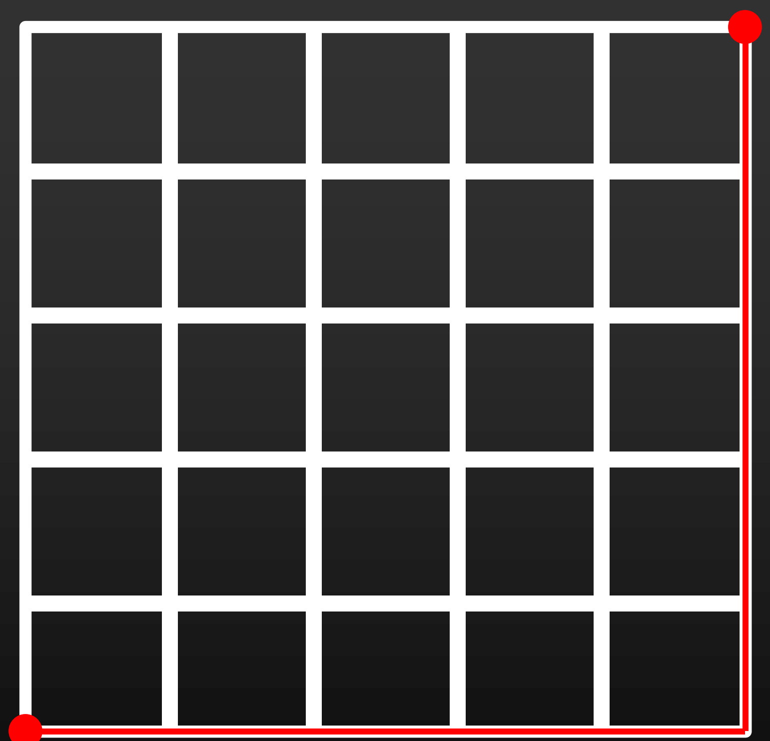

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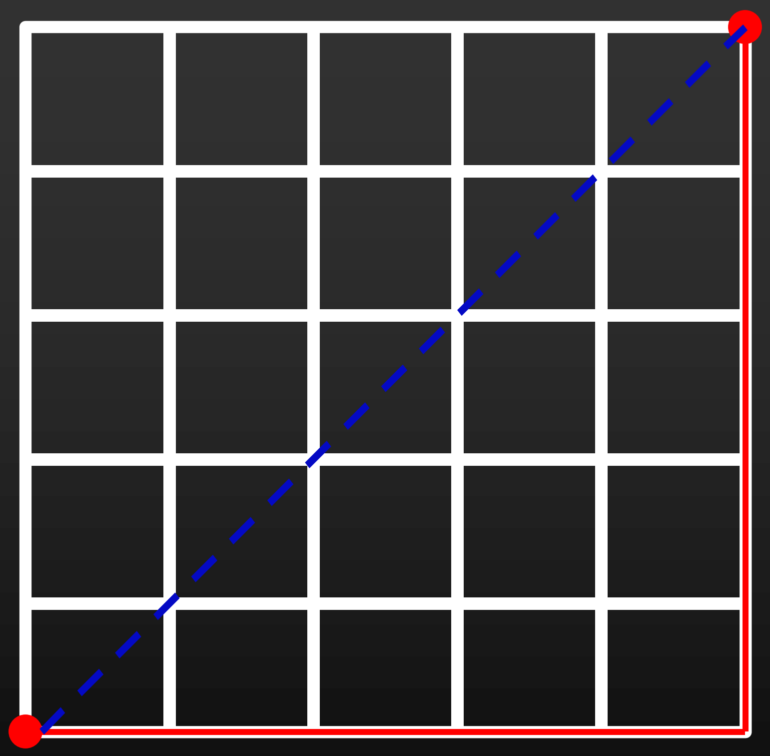

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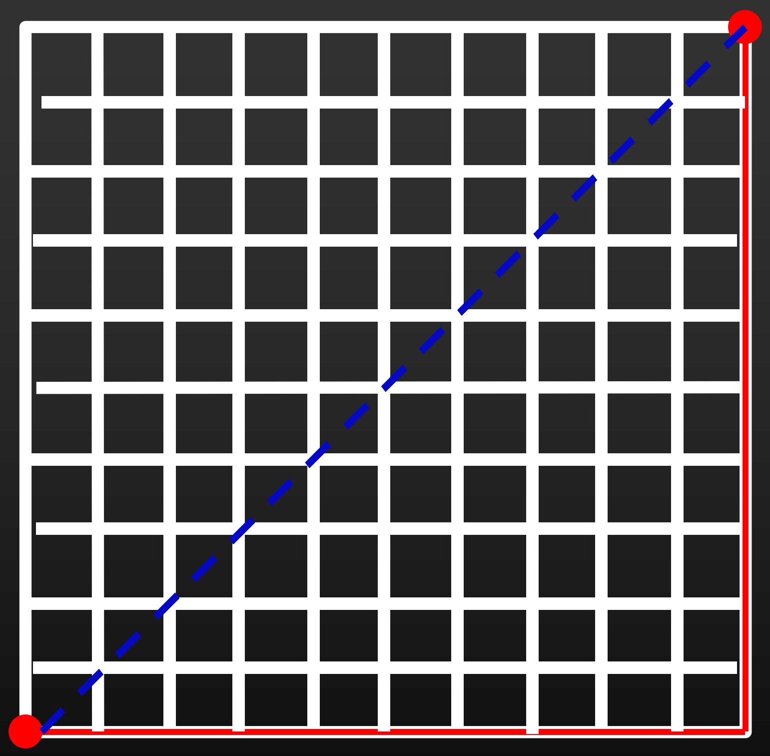

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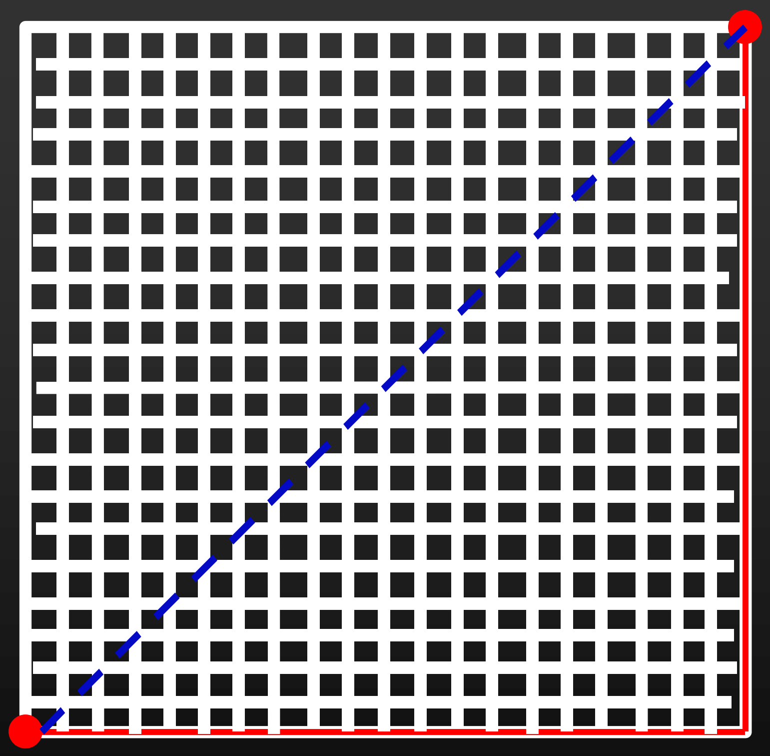

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Background: Distance Functions as Hamilton-Jacobi Equations

- *g* = weight on the hyper-surface
- The *g*-weighted distance function between two points *p* and *x* on the hyper-surface *S* is:

$$\left\| \nabla_S d_S^g(p, x) \right\| = g$$

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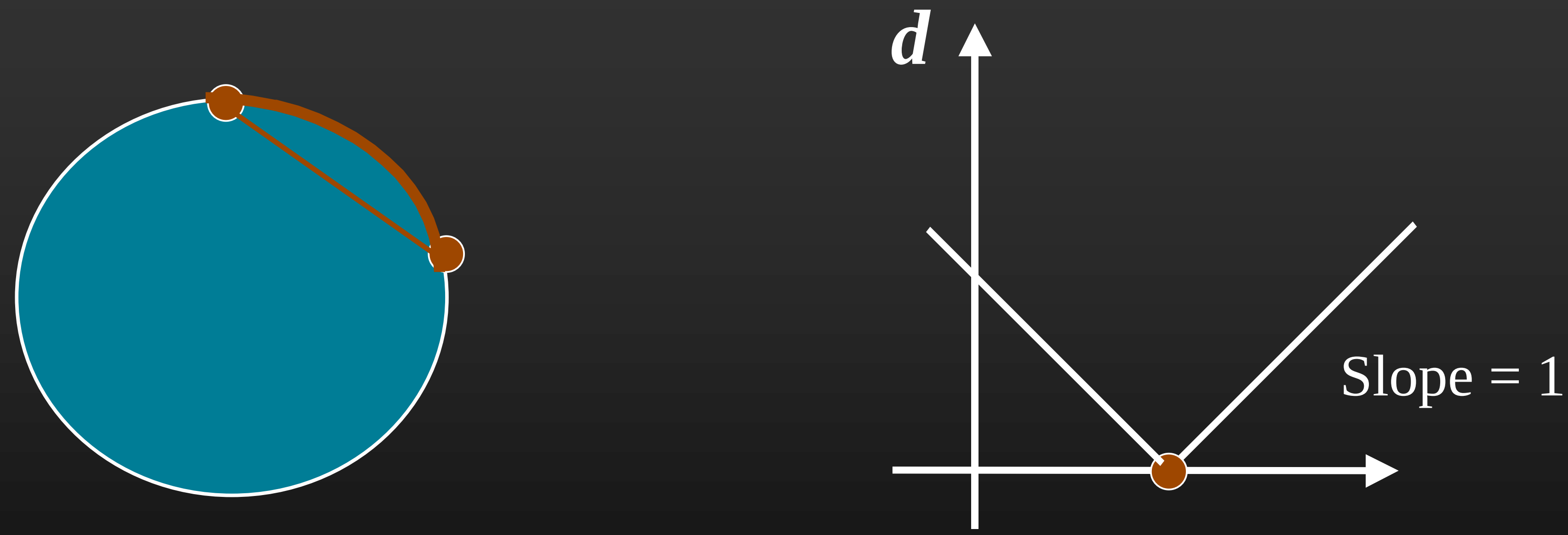
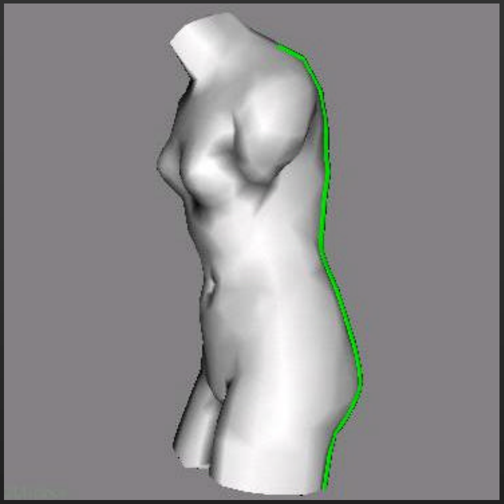
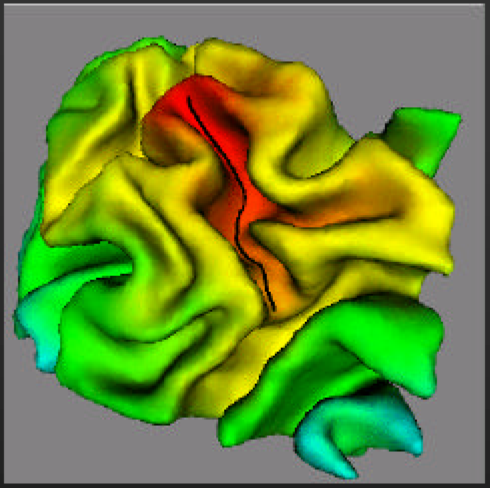


Background: Computing Distance Functions as Hamilton-Jacobi Equations

- Solved in O(n log n) by Tsitsiklis, by Sethian, and by Helmsen, only for Euclidean spaces and Cartesian grids

$$\left\| \nabla d^{g}(p, x) \right\| = g$$

- Solved only for acute 3D triangulations by Kimmel and Sethian

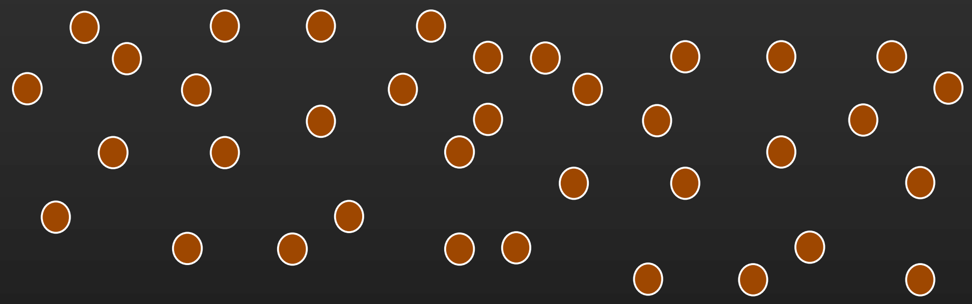
  

[A real time example]

The Problem

- **How to compute intrinsic distances and geodesics for**
  - General dimensions
  - Implicit surfaces
  - Unorganized noisy points (hyper-surfaces just given by examples)

Intermezzo:
Tenenbaum, de Silva, et al...



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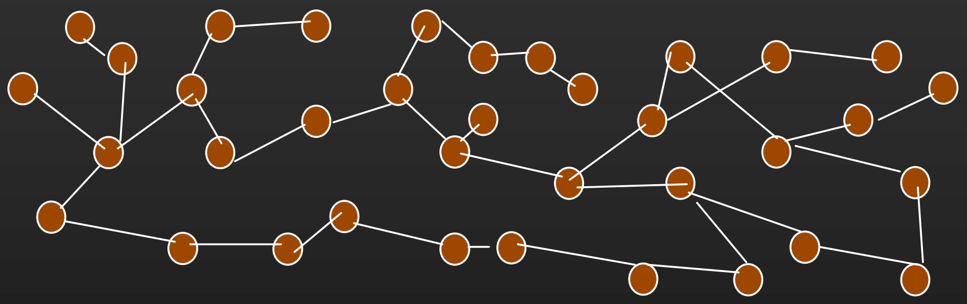

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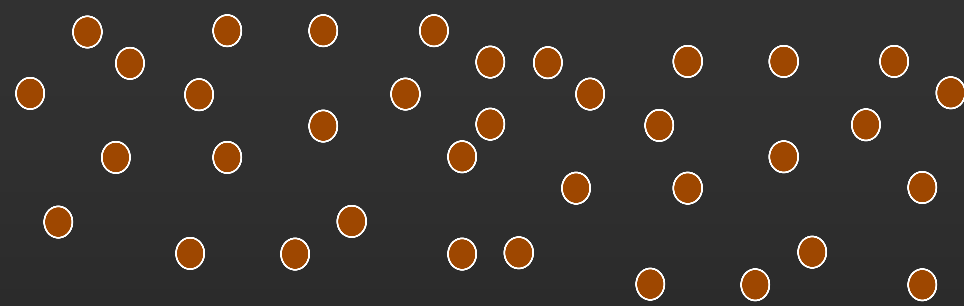


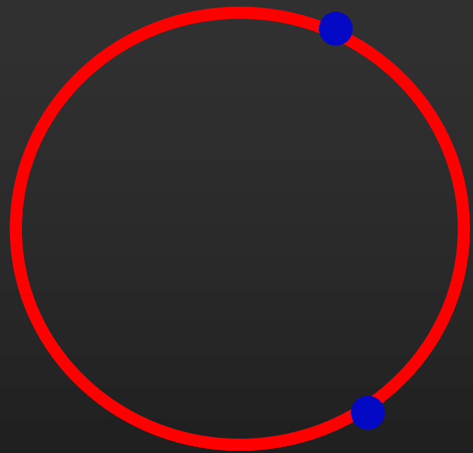
- **Problems:**
  - Doesn’t address noisy examples/measurements
  - Restriction on sampling density and manifolds
  - Uses Dijkestra (back to non consistency)
  - Doesn’t work for implicit surface representations

Our Approach

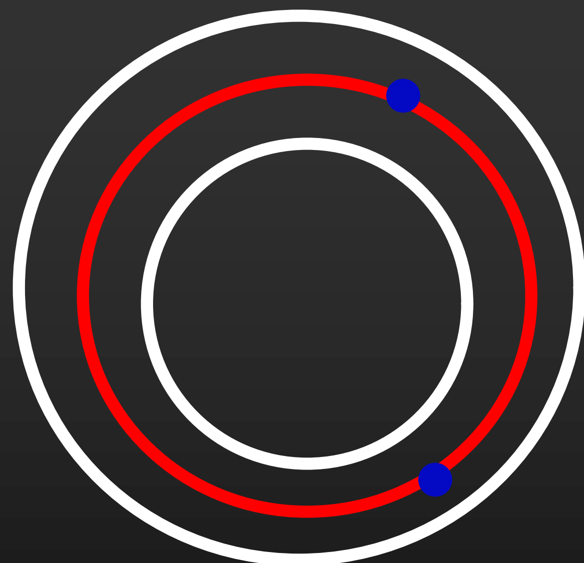
- We have to solve

$$\left\| \nabla_S d_S^g(p, x) \right\| = g$$

Basic Idea



Basic Idea



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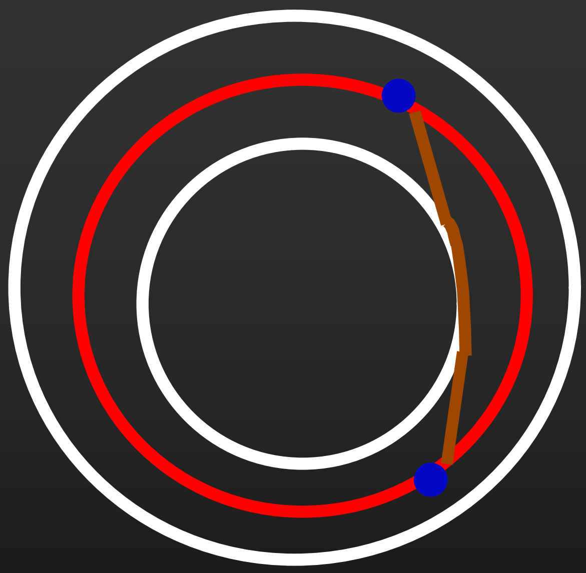
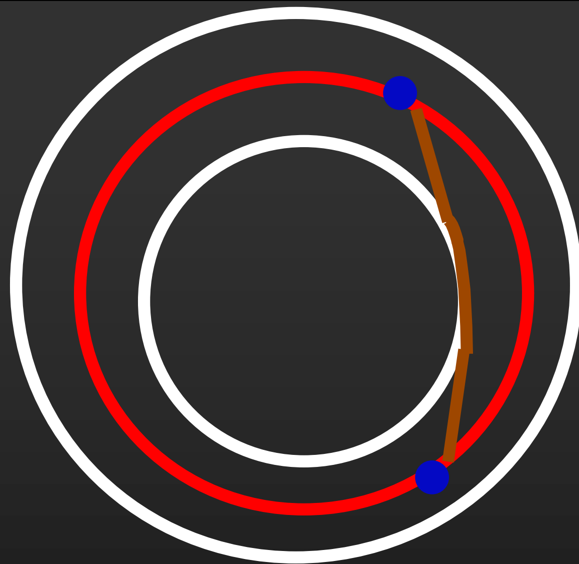


Theorem (Memoli-Sapiro):

(open/closed -- any co-dimension)

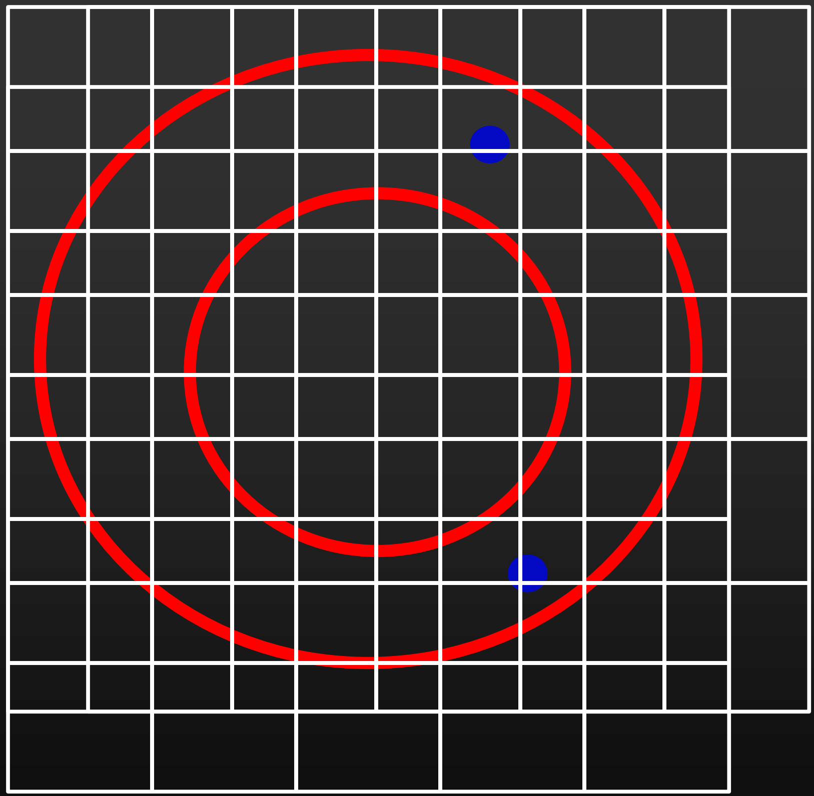
$$\left| d^{g} - d_{S}^{g} \right| \to 0$$

Basic idea



$$|d^g - d_S^g| \rightarrow \begin{cases} h^{1/2} & \textbf{general} \\ h & \textbf{local analytic} \\ h^g, g > 1 & \textbf{"smart" metric} \end{cases}$$

Why is this good?



$$\left\| \nabla_S d_S^g(p,x) \right\| = g$$

$$\Downarrow$$

$$\left\| \nabla d^g(p,x) \right\| = g$$

Implicit Form Representation

$$\text{S} = \text{level} - \text{set of } \Psi : R^n \rightarrow R = \{x : \Psi(x) = 0\}$$

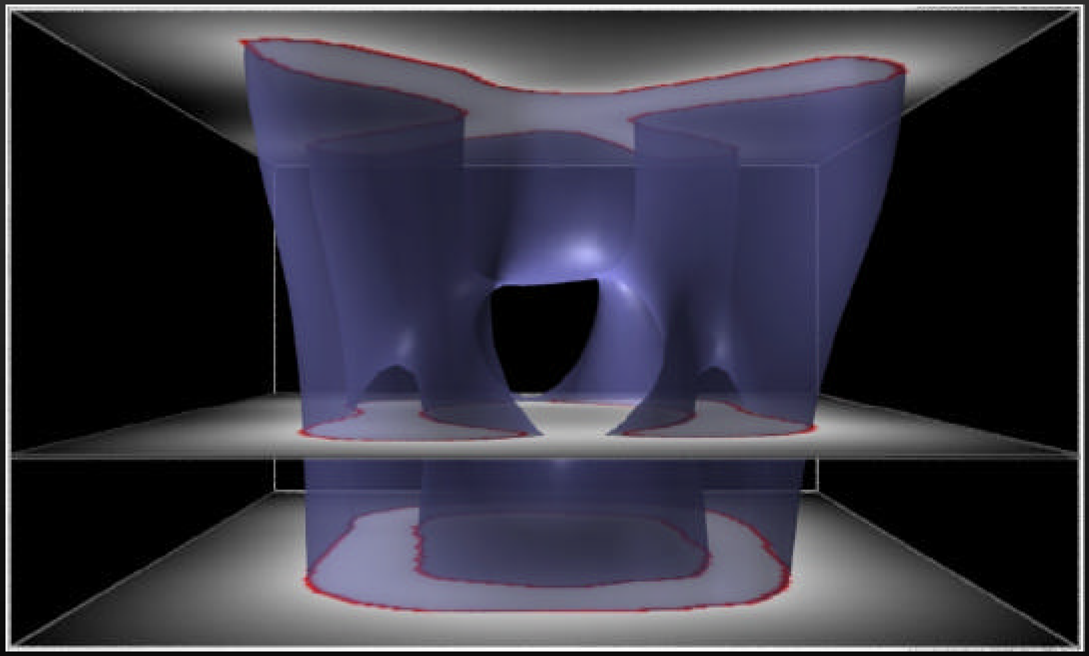


Figure from G. Turk

Data extension

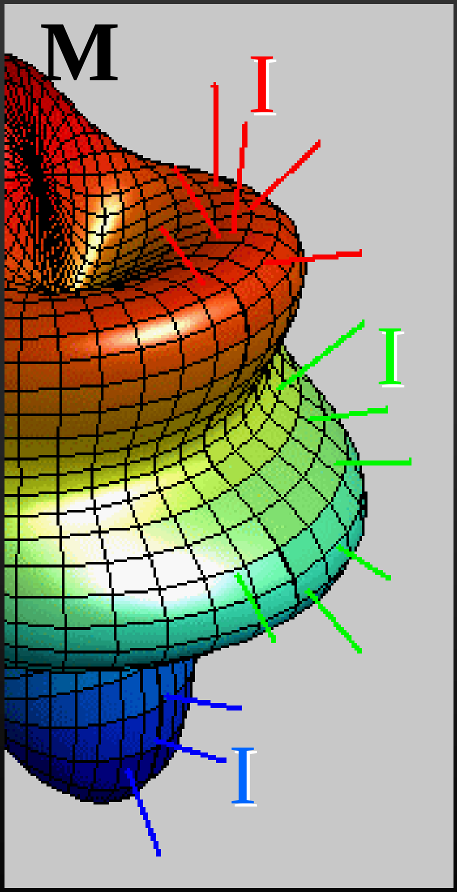
$$\mathbf{I} : \mathbf{M} \rightarrow \mathbf{R}$$

- **Embed M:**

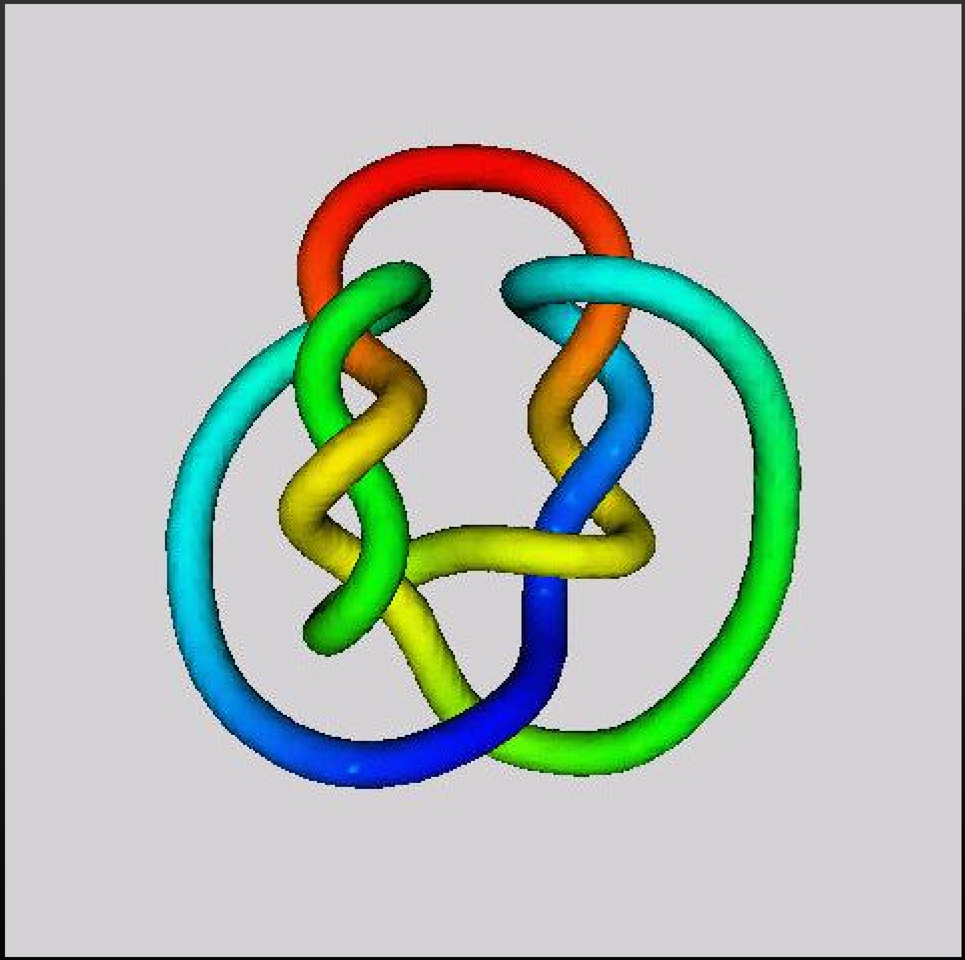
$$\mathrm{M} = \{x : \Psi(x) = 0\}$$

- **Extend I outside M:**

$$\frac{\partial \mathbf{I}}{\partial \mathbf{t}} + \boldsymbol{sign(\o)}\,\boldsymbol{(}\nabla I \cdot \nabla\, \boldsymbol{\o)} = \mathbf{0}$$



Examples



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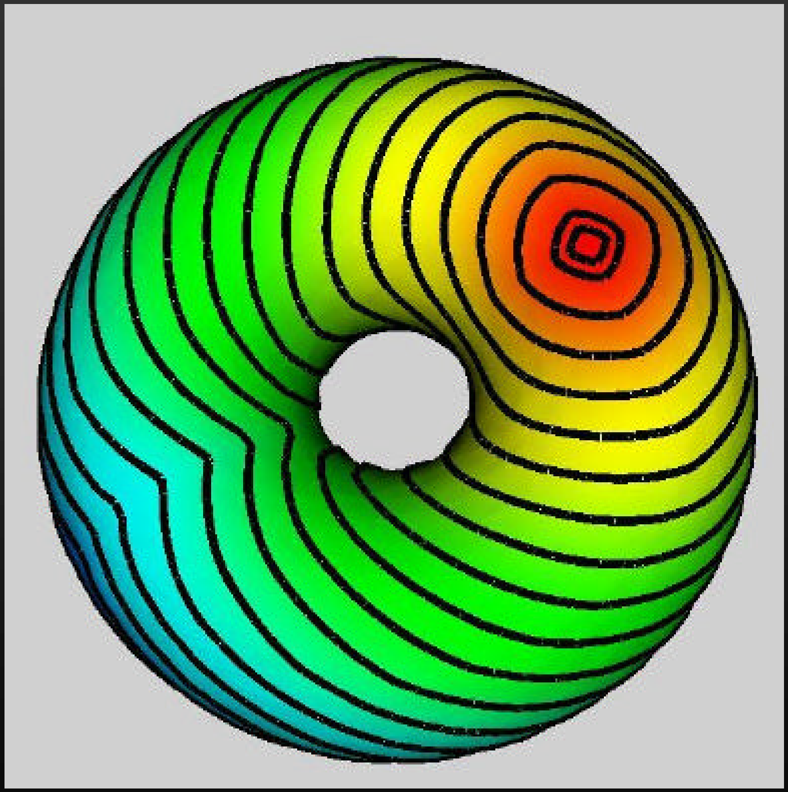

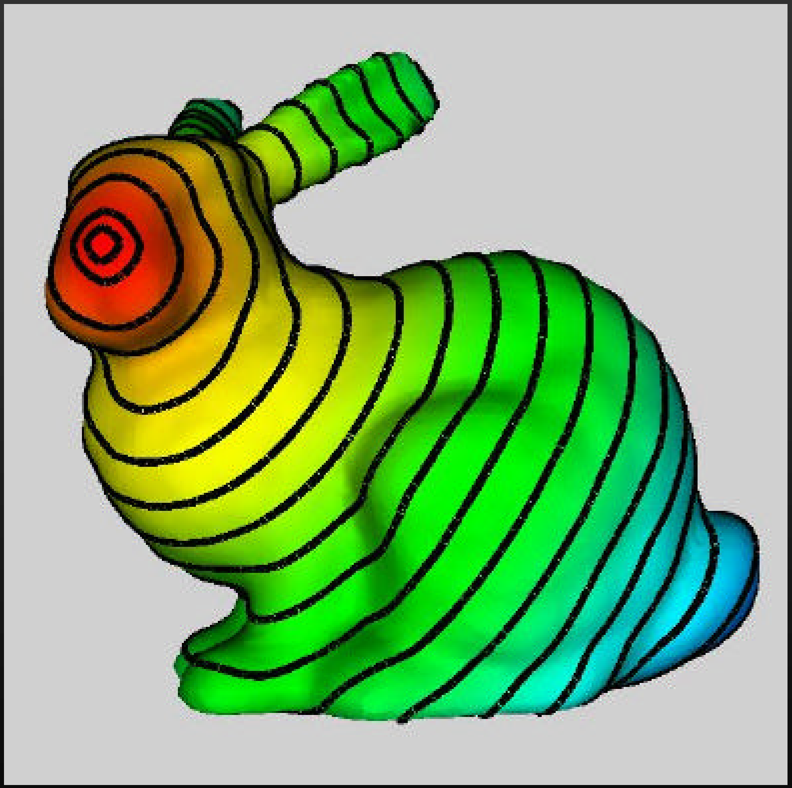

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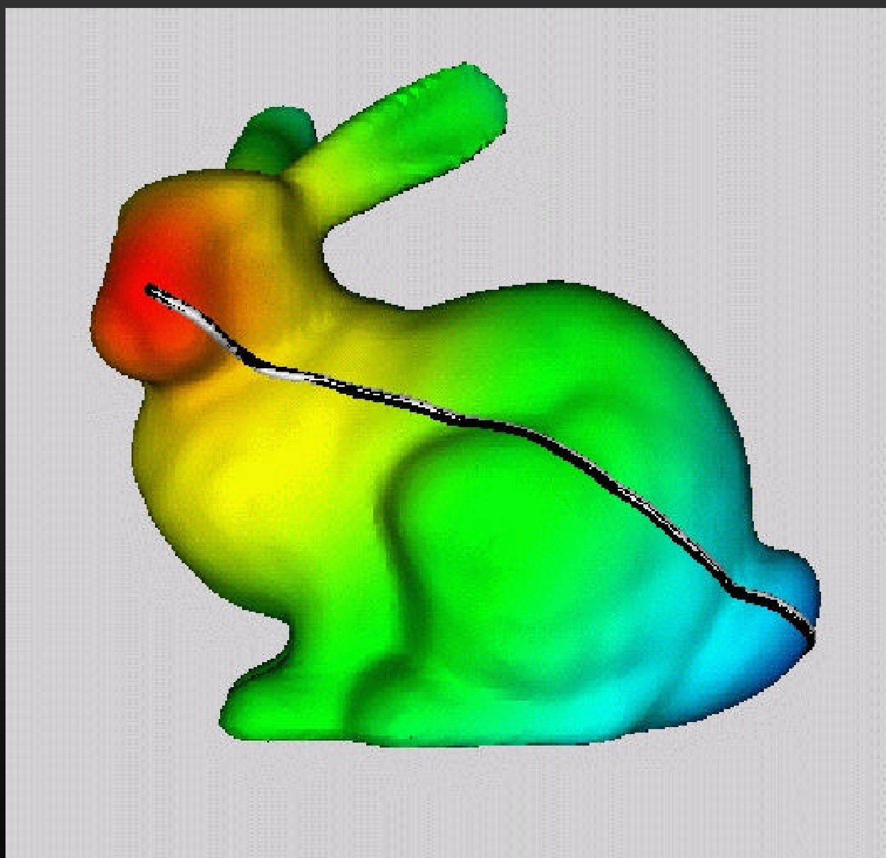

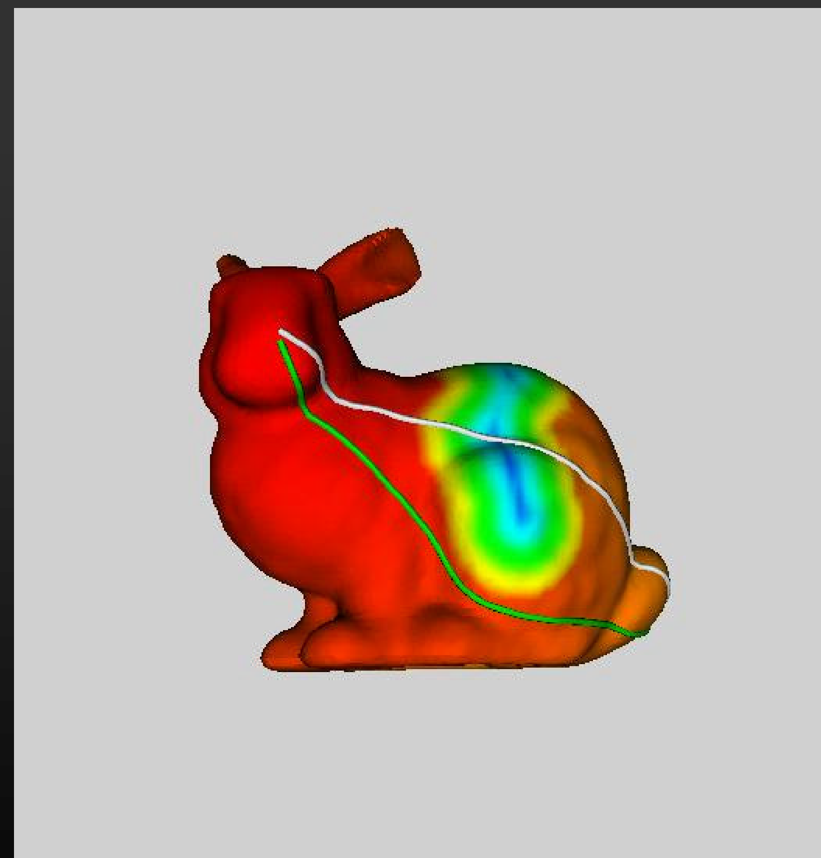

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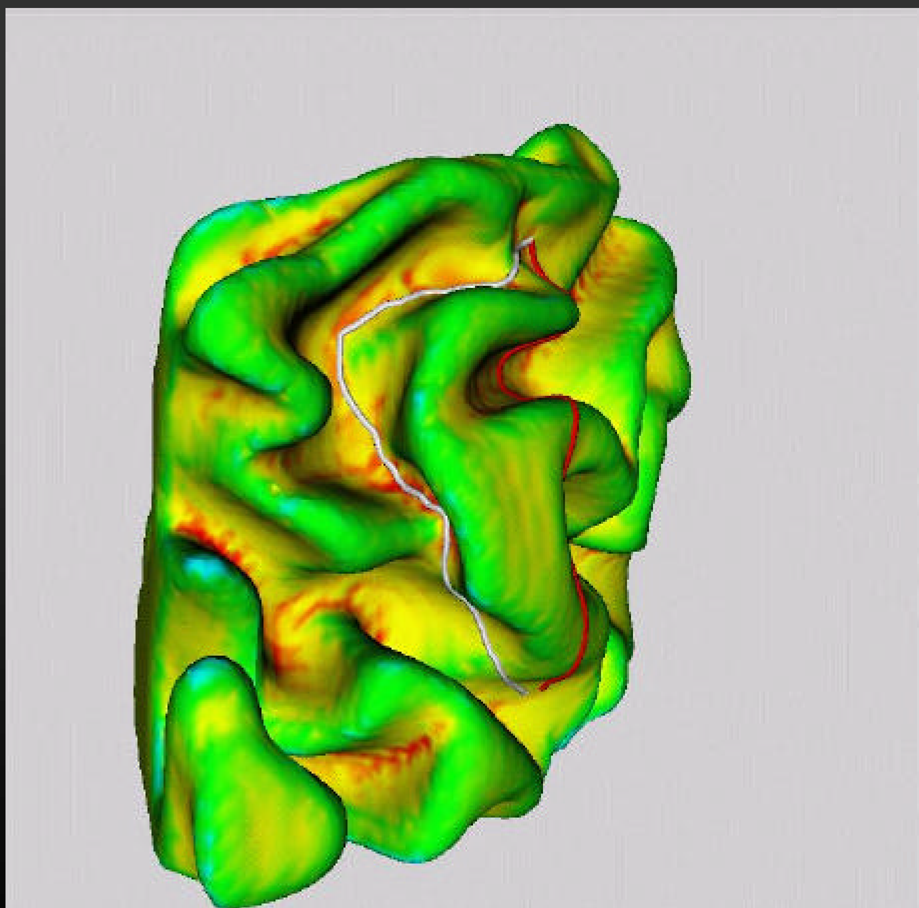

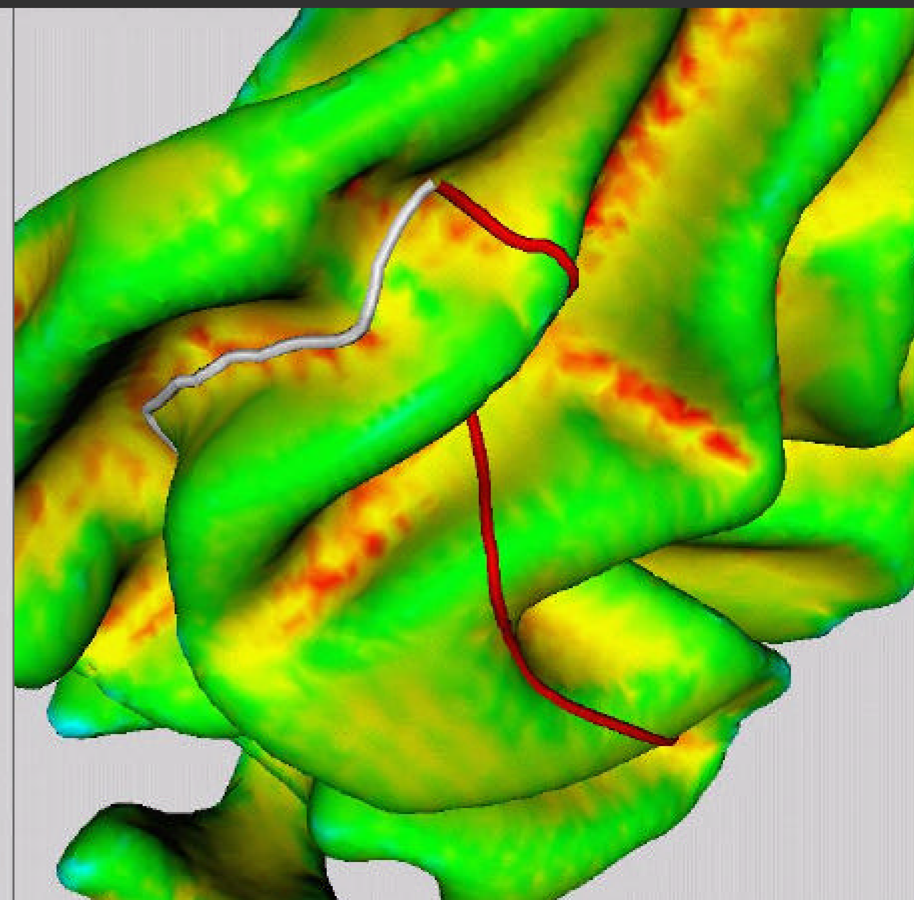

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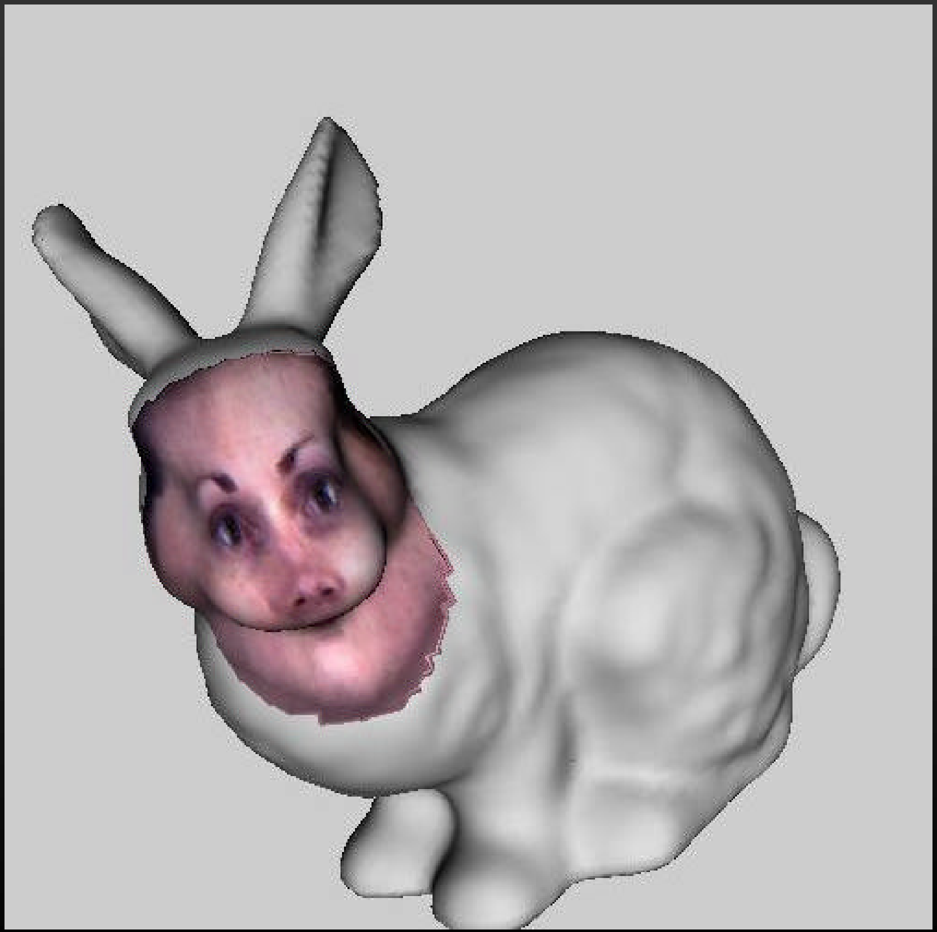
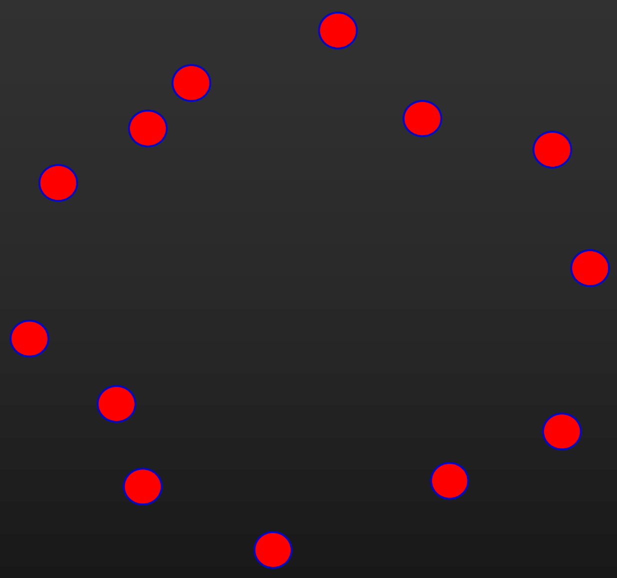
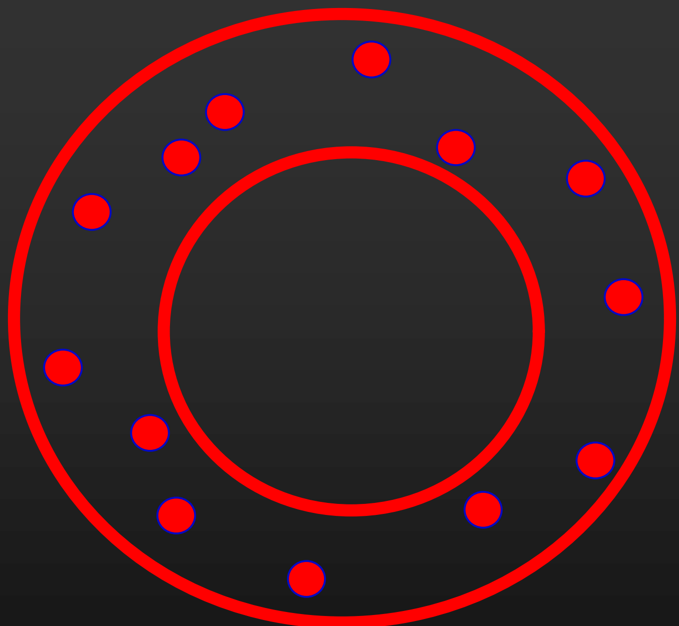


Unorganized points



Unorganized points (cont.)



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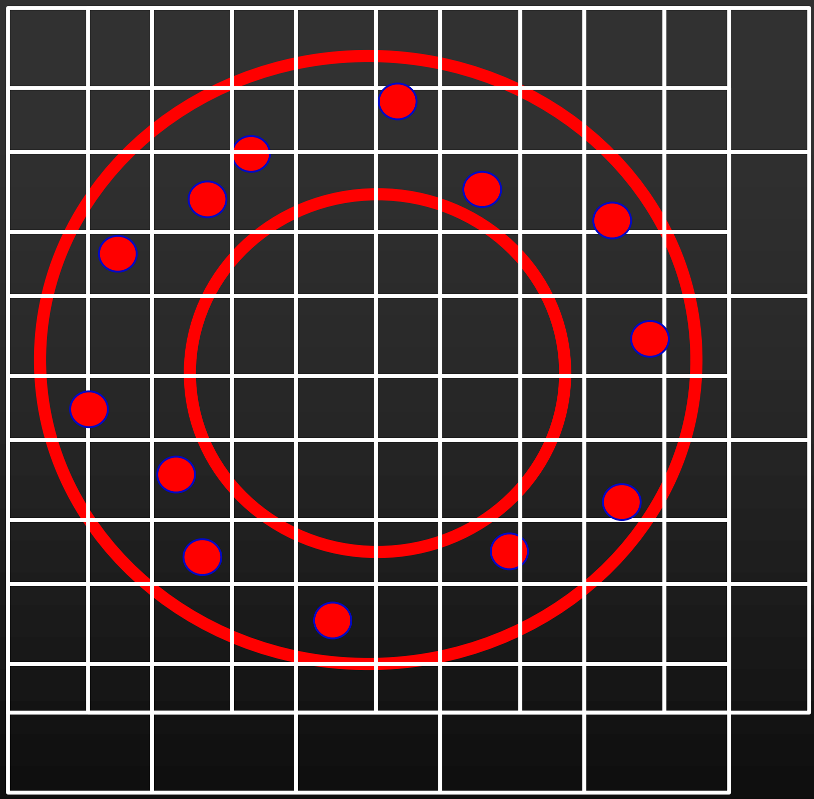
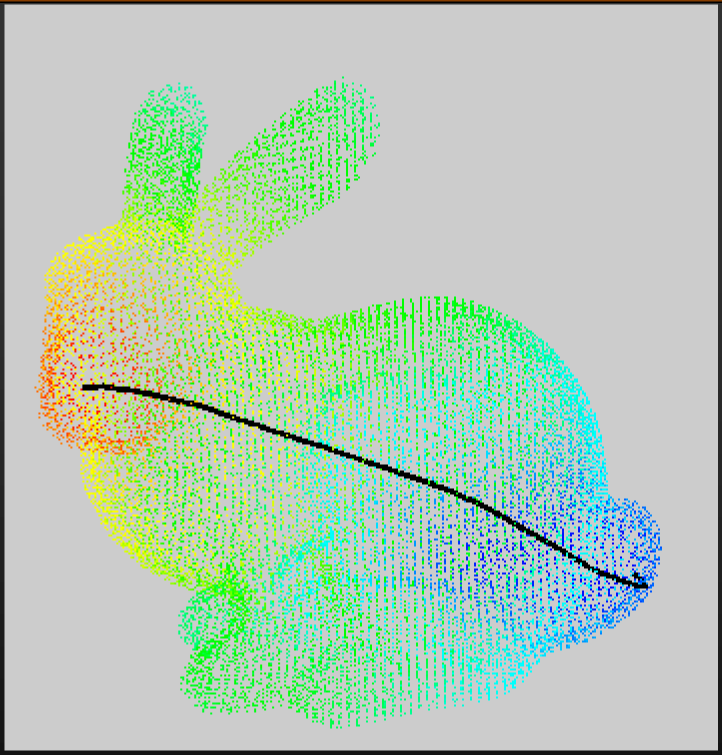
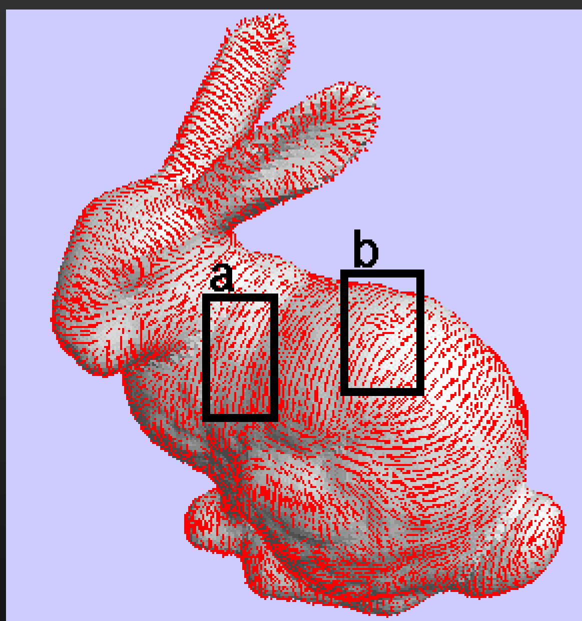


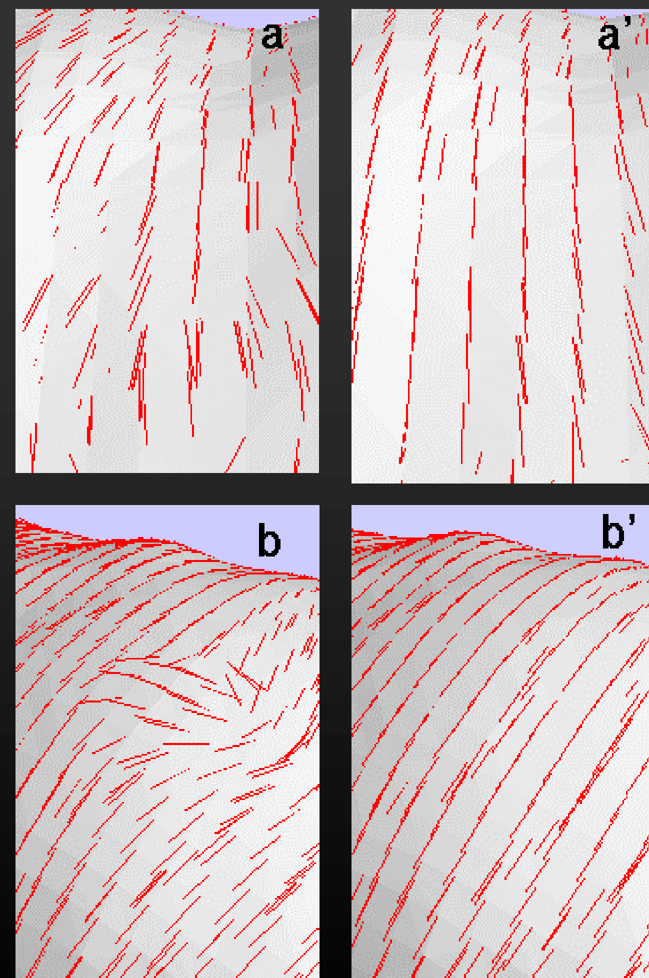


$$\left| d^{g} - d_{S}^{g} \right| \xrightarrow[h,N]{} 0$$

Is this a geodesic?



noisy



cleaned

Generalized geodesics: Harmonic maps

- Find a smooth map from two manifolds *(M,g)* and *(N,h)* such that

$$\min_{I:M\rightarrow N}\int_{\Omega}\left\|\nabla_M I\right\|^p\, dvol_M$$

$$\left(\frac{\P\ I}{\P\ t}=\right)\qquad \Delta_M I + A_N(I) < \nabla_M I, \nabla_M I >= 0$$

Examples

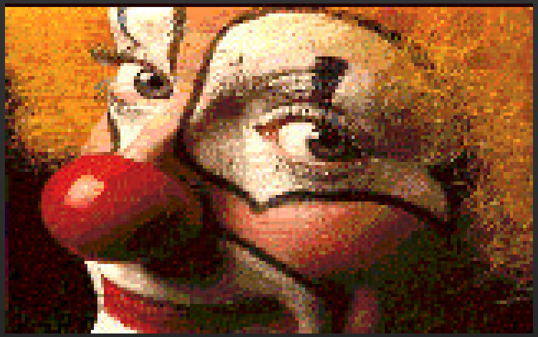
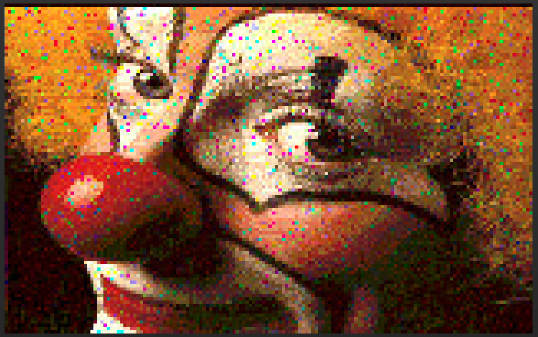
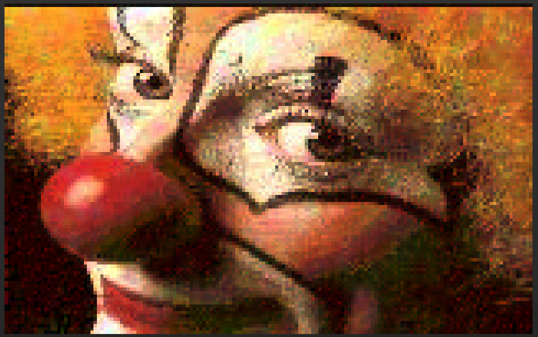
- M is an Euclidean space and N the real line

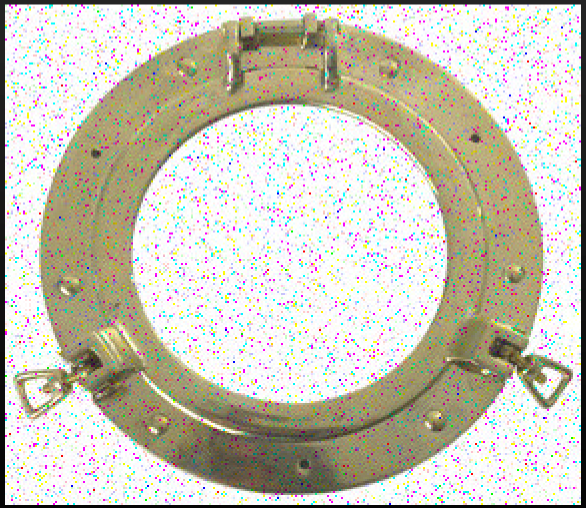
$$\Delta I = 0$$

- M = [0,1], geodesics!

$$\frac{\partial^2 I}{\partial t^2} + A_N(I) < \nabla_M I, \nabla_M I >= 0$$

Color Image Enhancement

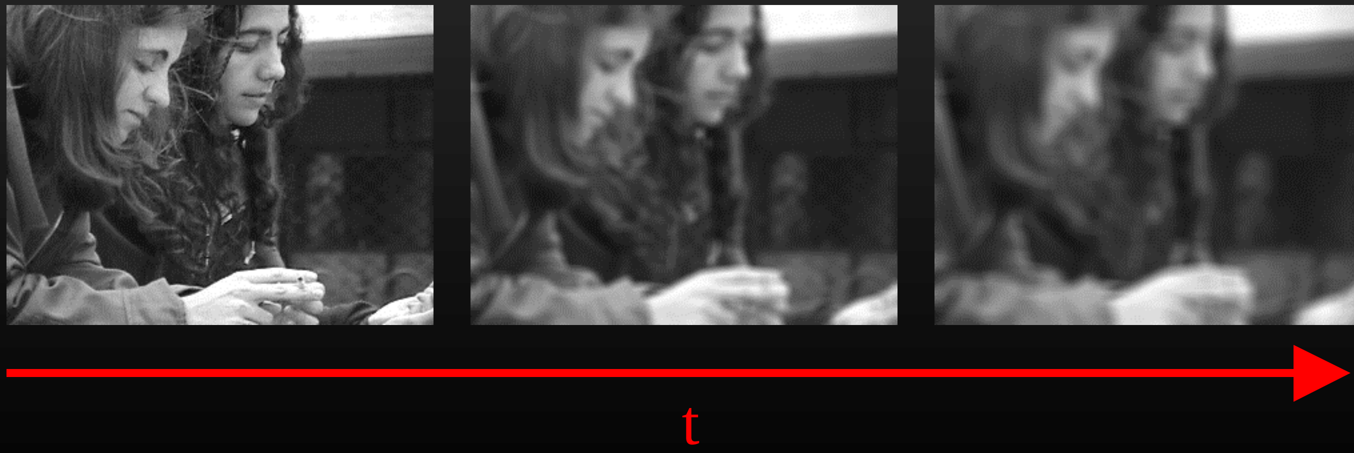



Generalized geodesics: Harmonic maps

- **How we implement this?**
  - Consider M and N defined in implicit form.

Heat flow on the plane

$$E = \int \|\nabla I\|^2 \Rightarrow \frac{\partial I}{\partial t} = \Delta I$$



Embedding the domain surface

- **Example: I:M->R**
  - A map from a generic domain surface onto the real line

$$\min_{I:M\to R} \int_M \left\|\nabla_M I\right\|^2 dvol_M$$

$$\frac{\partial I}{\partial t} = \Delta_M I$$

Embedding the domain surface (cont.)

$$\int_M \left\| \nabla_M I \right\|^2 dvol_M = \int_M \left\| P_{\nabla\Psi} \nabla I \right\|^2 dvol_M$$

$$= \int_{R^3} \left\| P_{\nabla\Psi} \nabla I \right\|^2 d(\Psi) \left\| \nabla\Psi \right\| d\mathbf{x}$$

Embedding the domain surface (cont.)

- The gradient descent flow: Heat flow on intrinsic surfaces

$$\frac{\partial \mathbf{I}}{\partial \mathbf{t}} = \frac{1}{\|\nabla \boldsymbol{\phi}\|} \mathbf{div}\left(\mathbf{P}_{\nabla \phi} \nabla \mathbf{I} \, \|\nabla \boldsymbol{\phi}\|\right)$$

- All the computations are done in the Cartesian grid!

Framework

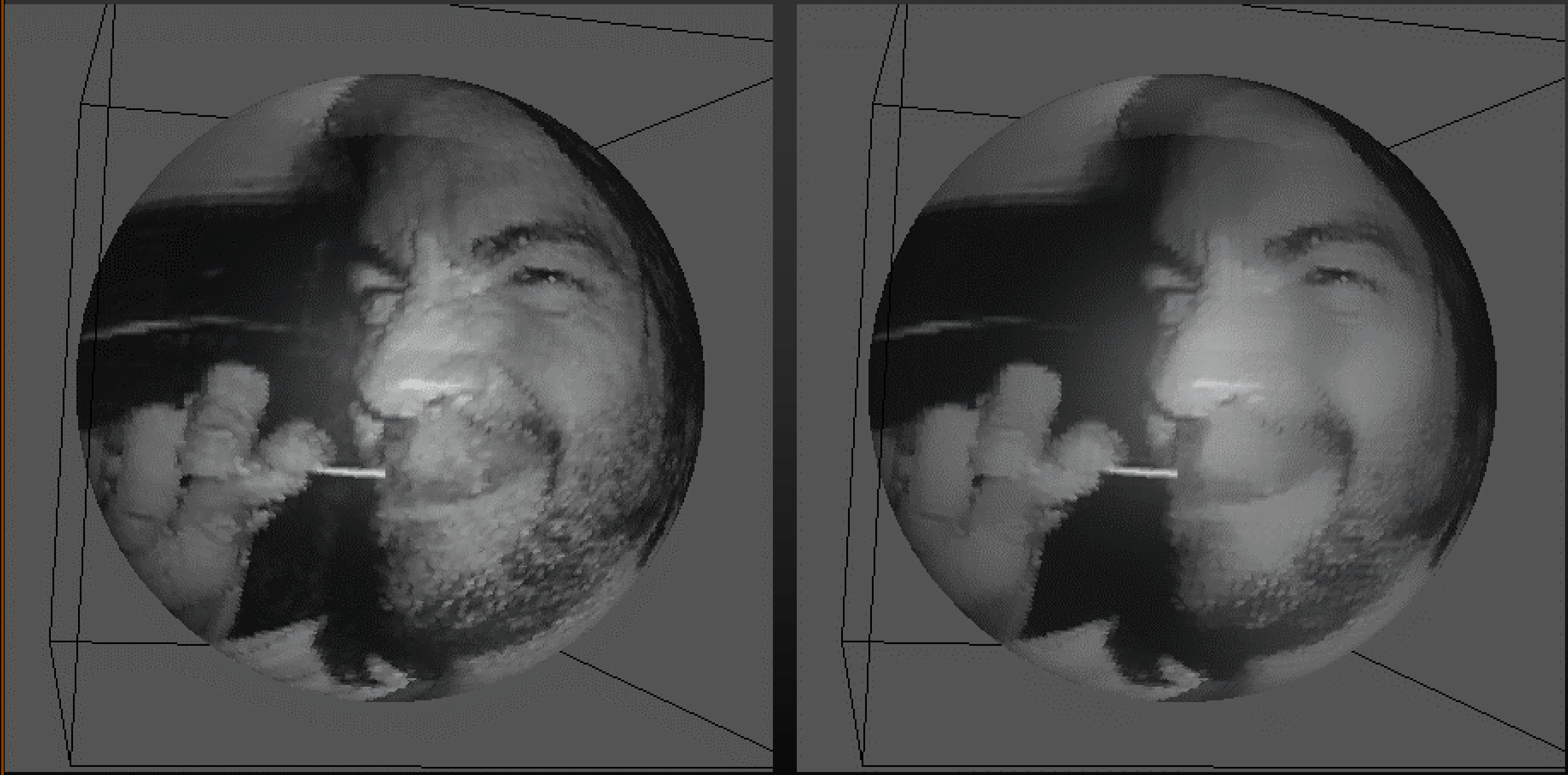
- If embedding with distance function:

$$\frac{\partial I}{\partial t} = \mathrm{div}\left(P_{\nabla \varnothing} \nabla I\right)$$

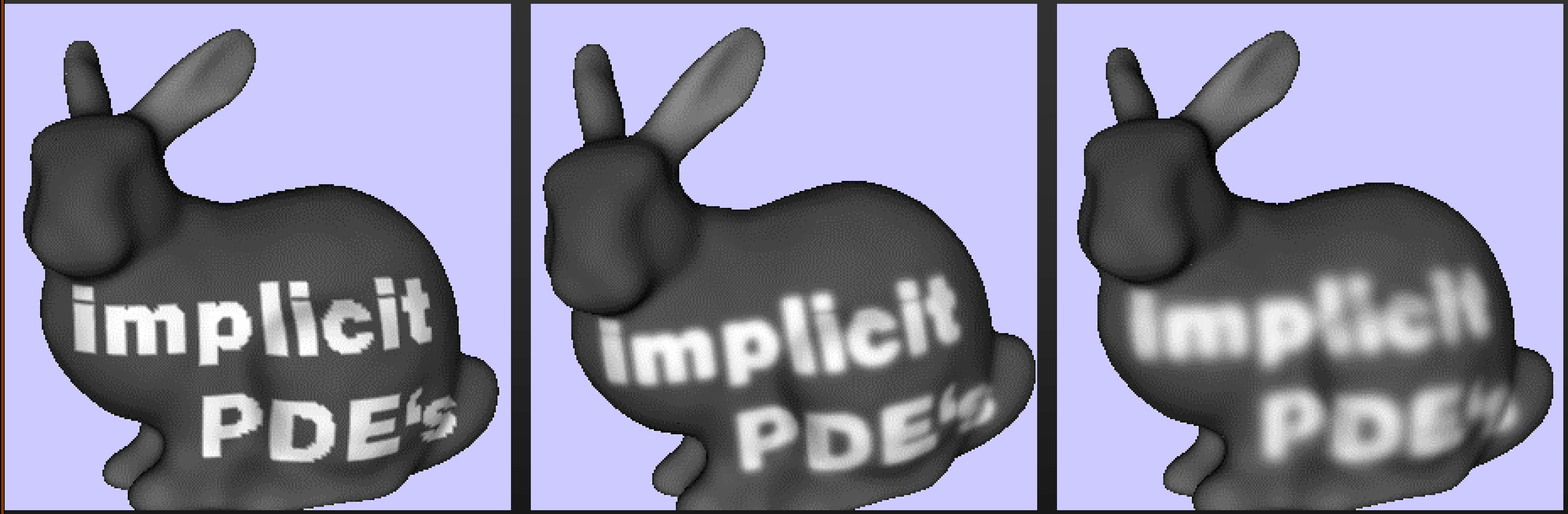
- Compare with planar case:

$$\frac{\partial I}{\partial t} = \mathrm{div}\left(\nabla I\right) = \Delta I$$

Example



Example: intrinsic heat flow



L1 denoising on implicit surfaces

$$\int_M \|\nabla_M I\| \, ds = \int_M \|P_{\nabla\Psi} \nabla I\| \; ds$$

$$= \int_{R^3} \|P_{\nabla\Psi} \nabla I\| d(\bar{\Psi}) \|\nabla\Psi\| \, dx$$

$$\frac{\partial \mathrm{I}}{\partial \mathrm{t}} = \frac{1}{\|\nabla\Psi\|} \mathrm{div}\left(\frac{P_{\nabla\Psi} \nabla I}{\|P_{\nabla\Psi} \nabla I\|} \|\nabla\Psi\| \right)$$

L1 denoising on implicit surfaces

$$\frac{\partial I}{\partial t} = \frac{1}{\|\nabla\Psi\|}\operatorname{div}\left(\frac{P_{\nabla\Psi}\nabla I}{\|P_{\nabla\Psi}\nabla I\|}\|\nabla\Psi\|\right)$$

intrinsic

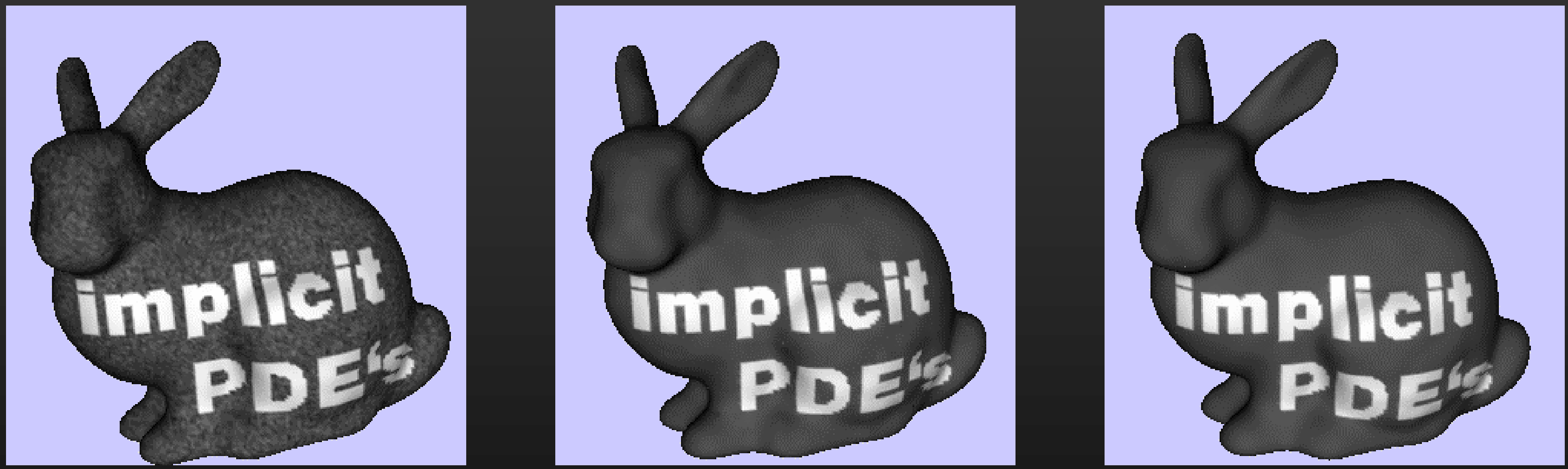
$$\frac{\partial I}{\partial t} = \operatorname{div}\left(\frac{P_{\nabla\Psi}\nabla I}{\|P_{\nabla\Psi}\nabla I\|}\right)$$

intrinsic, embedding w/ signed distance

$$\frac{\partial I}{\partial t} = \operatorname{div}\left(\frac{\nabla I}{\|\nabla I\|}\right)$$

planar

Example:
L1 denoising with constraints



Unit vector/color denoising on implicit surfaces

- **I is a map from the 3D surface to the 3D unit sphere**

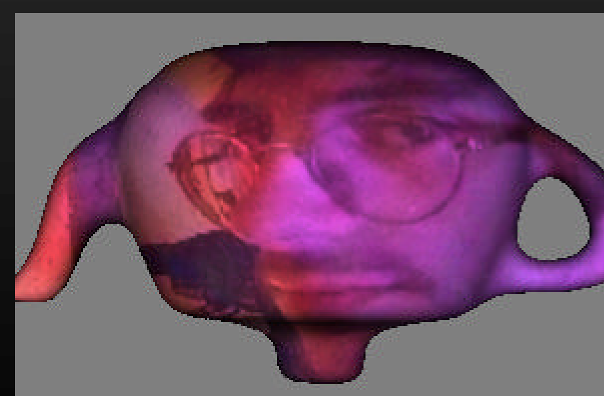
$$\frac{\partial \mathbf{I}}{\partial \mathbf{t}} = \frac{1}{\|\nabla\Psi\|}\mathbf{div}\left(\frac{P_{\nabla\Psi}\nabla I}{\|P_{\nabla\Psi}\nabla I\|}\|\nabla\Psi\|\right) + \mathbf{I}\,\|P_{\nabla\Psi}\nabla I\|$$

Example: Chroma denoising on a surface

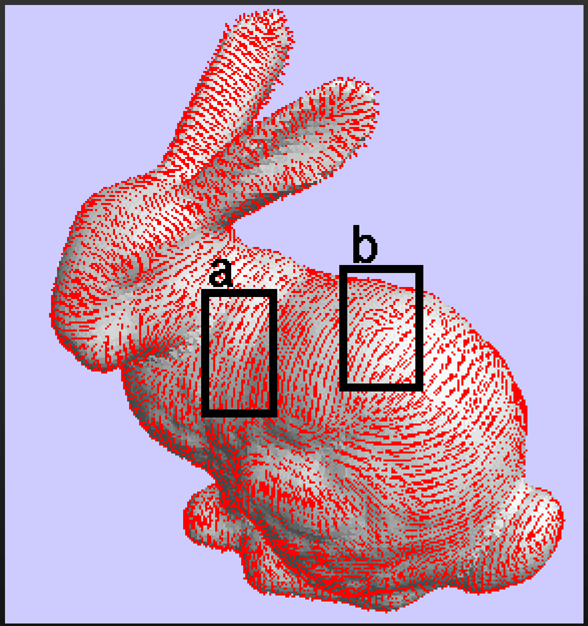
original

noisy

enhanced



Example: Direction denoising



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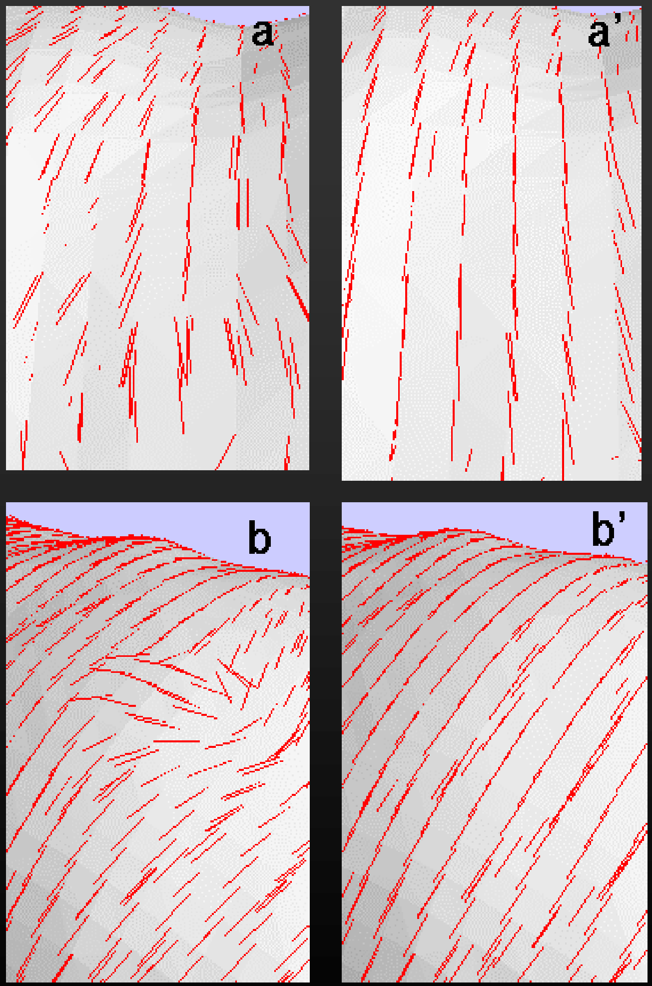
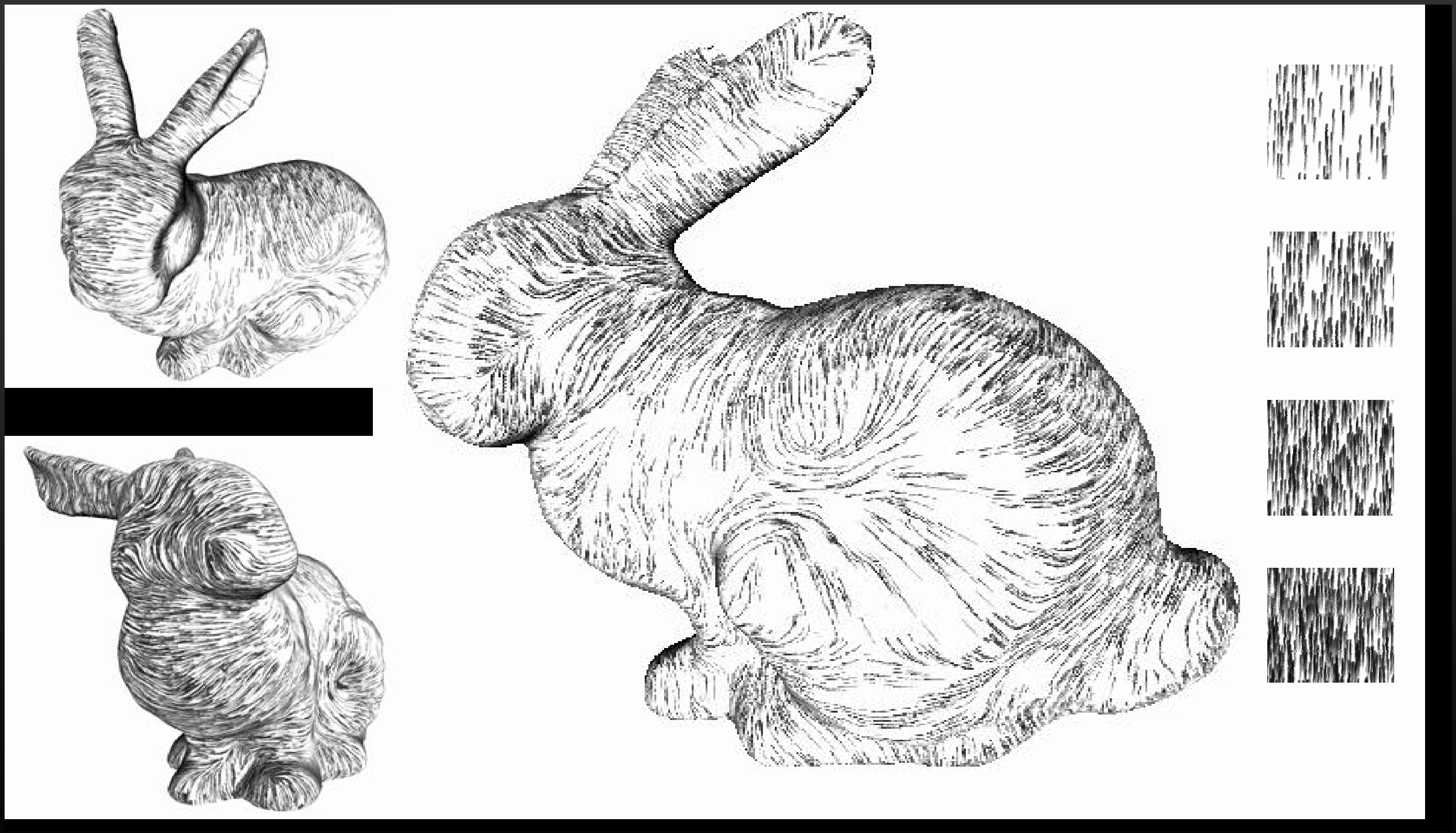


cleaned

Application (with G. Gorla and V. Interrante)



Pattern formation on implicit 3D surfaces

- Follows Turing, Kass-Witkin, Turk

$$\left.\begin{aligned}\frac{\partial \boldsymbol{a}}{\partial \boldsymbol{t}} &= \boldsymbol{f}(\boldsymbol{a},\boldsymbol{b}) + a\ \Delta_M \boldsymbol{a} \\ \frac{\partial \boldsymbol{b}}{\partial \boldsymbol{t}} &= \boldsymbol{g}(\boldsymbol{a},\boldsymbol{b}) + b\ \Delta_M \boldsymbol{b}\end{aligned}\right\} \Rightarrow \left\{\begin{aligned}\frac{\partial \boldsymbol{a}}{\partial \boldsymbol{t}} &= \boldsymbol{f}(\boldsymbol{a},\boldsymbol{b}) + a\ \frac{\mathbf{1}}{\|\nabla\Psi\|}\,\mathbf{div}\left(\mathbf{P}_{\nabla\Psi}\mathbf{I}\ \|\nabla\Psi\|\right) \\ \frac{\partial \boldsymbol{b}}{\partial \boldsymbol{t}} &= \boldsymbol{g}(\boldsymbol{a},\boldsymbol{b}) + b\ \frac{\mathbf{1}}{\|\nabla\Psi\|}\,\mathbf{div}\left(\mathbf{P}_{\nabla\Psi}\mathbf{I}\ \|\nabla\Psi\|\right)\end{aligned}\right.$$

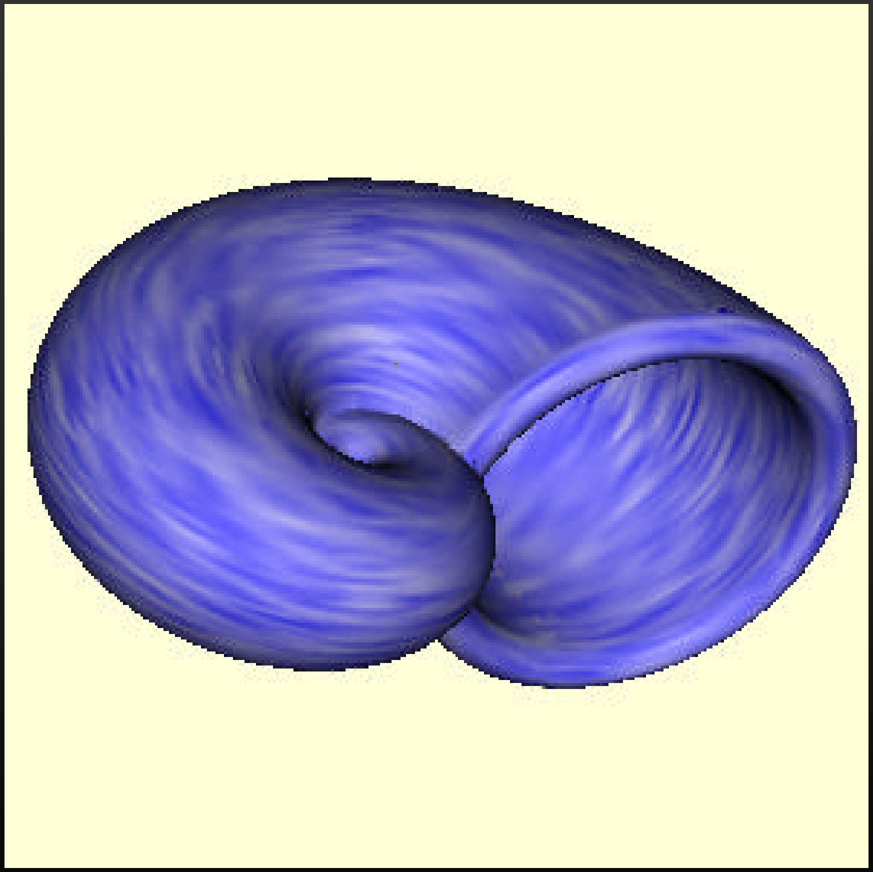
Examples

Vector field visualization

- I is random noise, diffused in direction *v*

$$\left.\begin{aligned}&\frac{\partial I}{\partial t}=\mathrm{div}(\mathrm{A}\nabla I)\\&\mathrm{A}=\vec{v}^{T}\vec{v}\end{aligned}\right\}\Rightarrow\left\{\begin{aligned}&\frac{\partial I}{\partial t}=\frac{1}{\left\|\nabla\Psi\right\|}\mathrm{div}\left(\mathrm{AP}_{\nabla\Psi}\nabla\mathrm{I}\left\|\nabla\Psi\right\|\right)\\&\mathrm{A}=\vec{v}^{T}\vec{v}\end{aligned}\right.$$

Vector field visualization (e.g., principal directions)



Embedding the target manifold

- I:M->N

$$\text{N} = \text{level} - \text{set of } \Phi = \{x : \Phi(x) = 0\}$$

$$\min_{I:M \to \{\Phi=0\}} \int_M \left\| \nabla_M I \right\|^p dvol_M$$

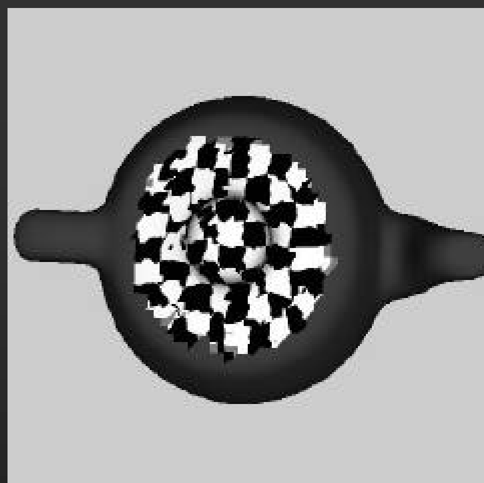
$$\frac{\partial I}{\partial t} = \Delta_M I + A_{\{\Phi=0\}}(I) < \nabla_M I, \nabla_M I >$$

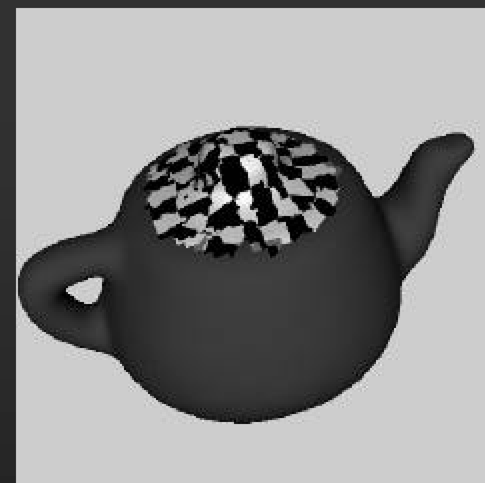
Embedding the target manifold (cont.)

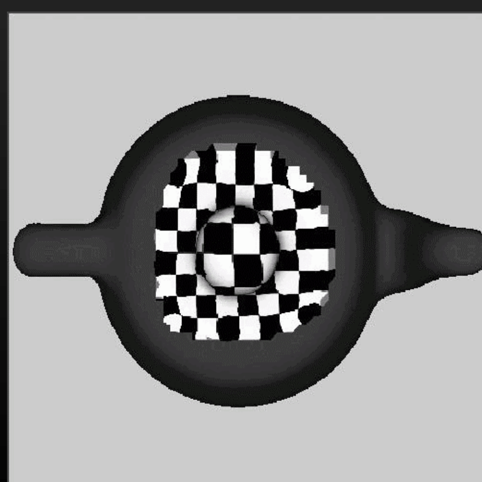
$$\min_{\mathbf{I}:R^k \to \{\Phi=0\}} \int_{R^k} \left\| \nabla_M \mathbf{I} \right\|^2 dx$$

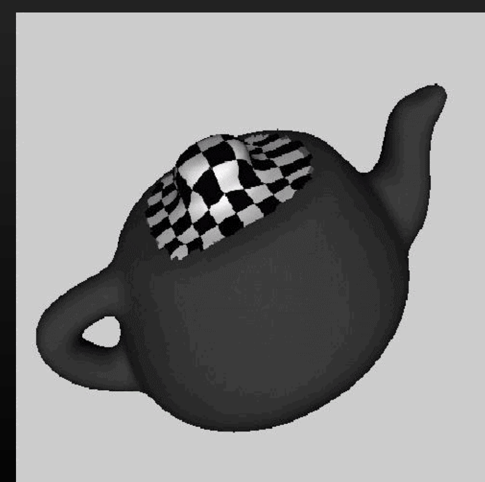
$$\frac{\partial \mathbf{I}}{\partial t} = \ddot{A}\,\mathbf{I} + \left(\sum_k \mathbf{H}_\Phi \left\langle \frac{\partial \mathbf{I}}{\partial \mathbf{x_k}}, \frac{\partial \mathbf{I}}{\partial \mathbf{x_k}} \right\rangle \right) \left\| \nabla \Phi \right\|$$

Texture mapping denoising









Texture mapping denoising

Complete Implicit Surfaces Representation

- Domain and target are implicitly represented: Simple Cartesian numerics

$$\frac{\partial \mathbf{I}}{\partial \mathbf{t}} = \mathbf{div}\left(\mathbf{P}_{\nabla \varnothing} \nabla \mathbf{I}\right) + \left(\sum_{\mathbf{k}} \mathbf{H}_{\Phi} \left\langle \frac{\partial \mathbf{I}}{\partial \mathbf{x}_{\mathbf{k}}}, \frac{\partial \mathbf{I}}{\partial \mathbf{x}_{\mathbf{k}}} \right\rangle \right) \| \nabla \Phi \|$$

- Extended also to sub-manifolds via intersection of implicit surfaces

Concluding remarks

- A general computational framework for distance functions, geodesics, and generalized geodesics
- Implicit hyper-surfaces and points clouds

Thanks

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B. Tang, G. Sapiro, and V. Caselles, "Diffusion of general data on non-flat manifolds via harmonic maps theory: The direction diffusion case," *Int. Journal Computer Vision* 36:2, pp. 149-161, February 2000.

M. Bertalmio, L. T. Cheng, S. Osher, and G. Sapiro, "Variational problems and partial differential equations on implicit surfaces,'' *Journal of Computational Physics* 174:2, pp. 759-780, 2001.

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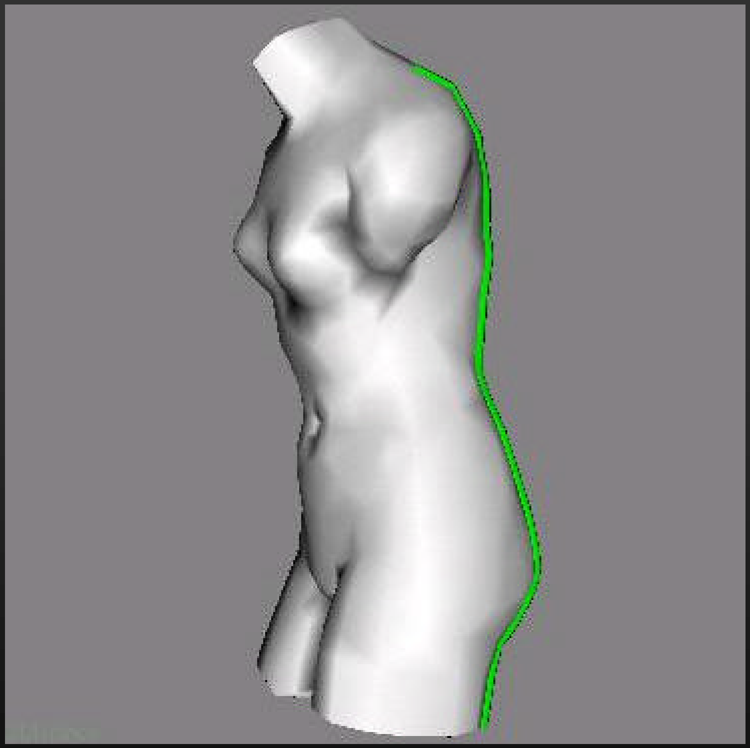
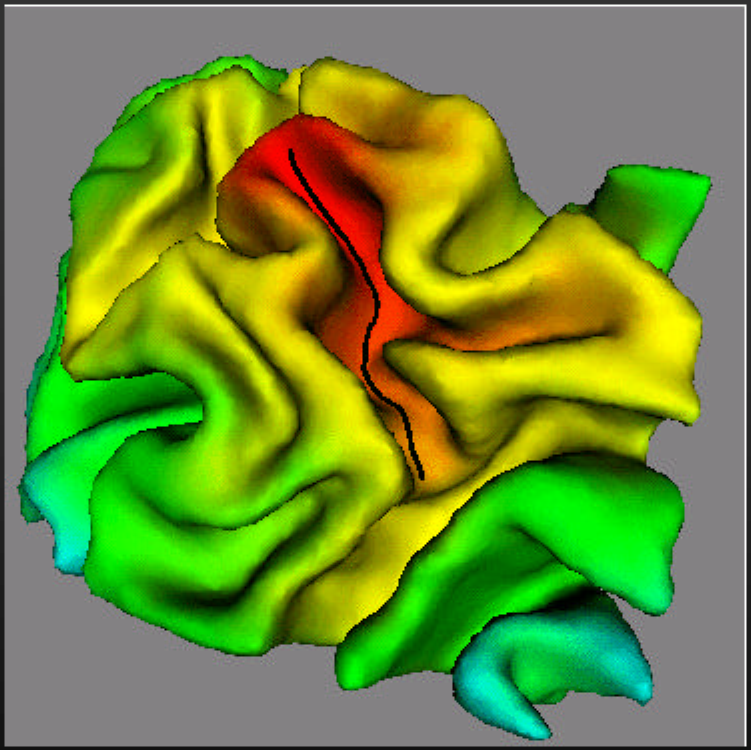
In honor of the victims of terror, worldwide

Sulcii extraction on meshes

(with A. Bartesaghi)





Follows Kimmel-Sethian