Using a dictionary to define the constraint for image restoration

François Malgouyres
malgouy@math.univ-paris13.fr
http://www.zeus.math.univ-paris13.fr/∼malgouy

Context

We look for $v \in L^2(\mathbb{T})$ from

$$u_{m,n} = s * v(m,n) + b_{m,n}$$

- The interpolation

  - Ways: total variation minimization.

- Deconvolution: We neglect aliasing and look for the low frequencies of v.

  - Ways: Wavelet packets and total variation minimization.

- Denoising: we look for $s * v(m,n)$

  - Ways: wavelet (and other bases) approaches and total variation minimization.

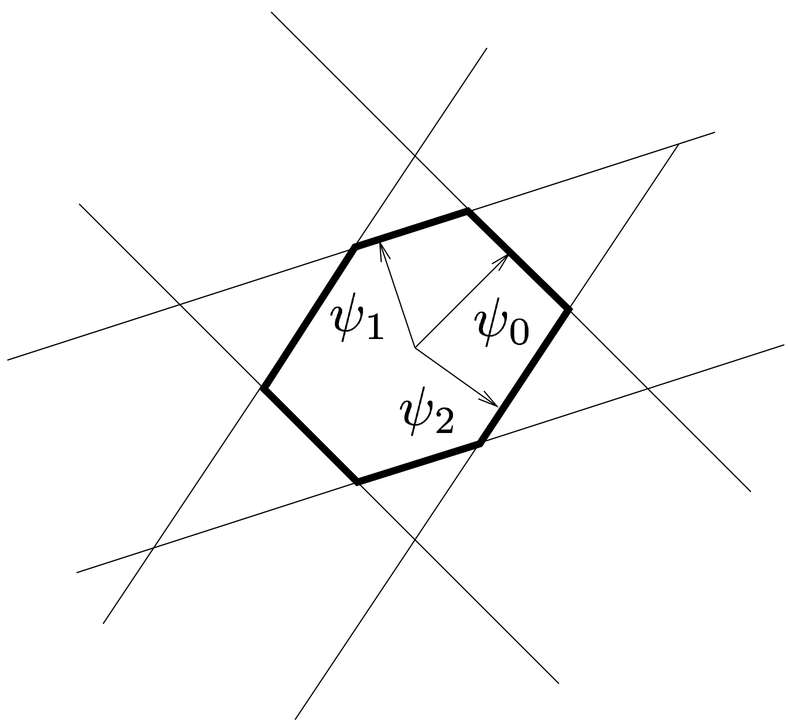
A single framework for both wavelet and variational approaches

Let us consider the minimization of an energy $E(w)$, under the constraint

$$s*w-u \in \mathcal{N}_{\mathcal{D},\tau} = \{b' \in L^2(\mathbb{T}), \forall \Psi \in \mathcal{D}, |\langle b', \Psi\rangle| \leq \tau\},$$

for a dictionary

$$\mathcal{D} = \{\Psi_l\}_{l\in I}$$



Wavelet soft-thresholding

Let us consider $\mathcal{D} = (\Psi_l)_l$, a wavelet basis, and any energy of the kind

$$E(w) = \sum_l f_l \left(|\langle w, \Psi_l \rangle| \right)$$

where f_l is an increasing function.

Advantages:

- The threshold τ is relatively small ($\sigma\sqrt{4 \ln N}$ for an image of size $N \times N$)
- Fast
- For any l, $|\langle s * w - u, \Psi_l \rangle| \leq \tau$

Drawbacks:

- Local in the dictionary $\mathcal{D}$ (bad when some information is lost).
- The dictionary is a basis.

Rudin-Osher-Fatemi method

$$E(w) = \int_{\mathbb{T}} |\nabla w|$$

with

$$\mathcal{D} = \{\Psi \in L^2(\mathbb{T}), \|\Psi\|_2 = 1\}.$$

$(\|s * w - u\|_2 \leq \sigma)$

Advantage:

- Retrieves some lost information

Drawbacks:

- The same constraint whatever the direction (including $\frac{b}{\|b\|}$!)
- The "threshold" τ is large (σN for an image of size $N \times N$)

A combination of both approaches

$$\begin{cases} E(w) = \int_{\mathbb{T}} |\nabla w| \\ \mathcal{D} \text{ a dictionary to be determined} \end{cases} \quad (1)$$

- There exists a solution under the hypotheses

  – $BV(\mathbb{T}) \cap \{w \in L^2(\mathbb{T}), s * w - u \in \mathcal{N}_{\mathcal{D},\tau}\} \neq \emptyset$

  – $\exists C > 0, \forall w \in \{w \in L^2(\mathbb{T}), s*w-u \in \mathcal{N}_{\mathcal{D},\tau}\}$,

$$|\int_{\mathbb{T}} w| \leq C$$

- The solution is computed as the limit ($\epsilon \to 0$) of the minimum of

$$\int_{\mathbb{T}} |\nabla w| + \frac{1}{\epsilon} \sum_{\Psi \in \mathcal{D}} \sup(|\langle s * w - u, \Psi \rangle| - \tau, 0)^2$$

  under the hypotheses above and:

  - $\mathcal{D}$ is countable

  - There exists $\tau' < \tau$ such that

$$BV(\mathbb{T}) \cap \{w \in L^2(\mathbb{T}), s * w - u \in \mathcal{N}_{\mathcal{D},\tau'}\} \neq \emptyset$$

Numerical issues

We compute a solution with a steepest descent algorithm.

The difficulty comes from the fact that, if we denote by

$$J_\Psi(w) = \sup(|\langle s * w - u, \Psi\rangle| - \tau, 0)^2,$$

we have, for $(\varphi_j)_j$ an orthonormal basis,

$$\frac{\partial J_\Psi(w^n)}{\partial \varphi_j} = 0$$

if $|\langle s * w^n - u, \Psi\rangle| < \tau$, and

$$\frac{\partial J_\Psi(w^n)}{\partial \varphi_j} = 2sign\left(\langle s * w^n - u, \Psi\rangle\right) \\ \langle s * \varphi_j, \Psi\rangle \left(|\langle s * w^n - u, \Psi\rangle| - \tau\right),$$

otherwise.

Terms $\langle s * \varphi_j, \Psi\rangle$ are difficult to compute and store. **We restrict ourselves to elements Ψ such that $s * \Psi \simeq \lambda \Psi$.**

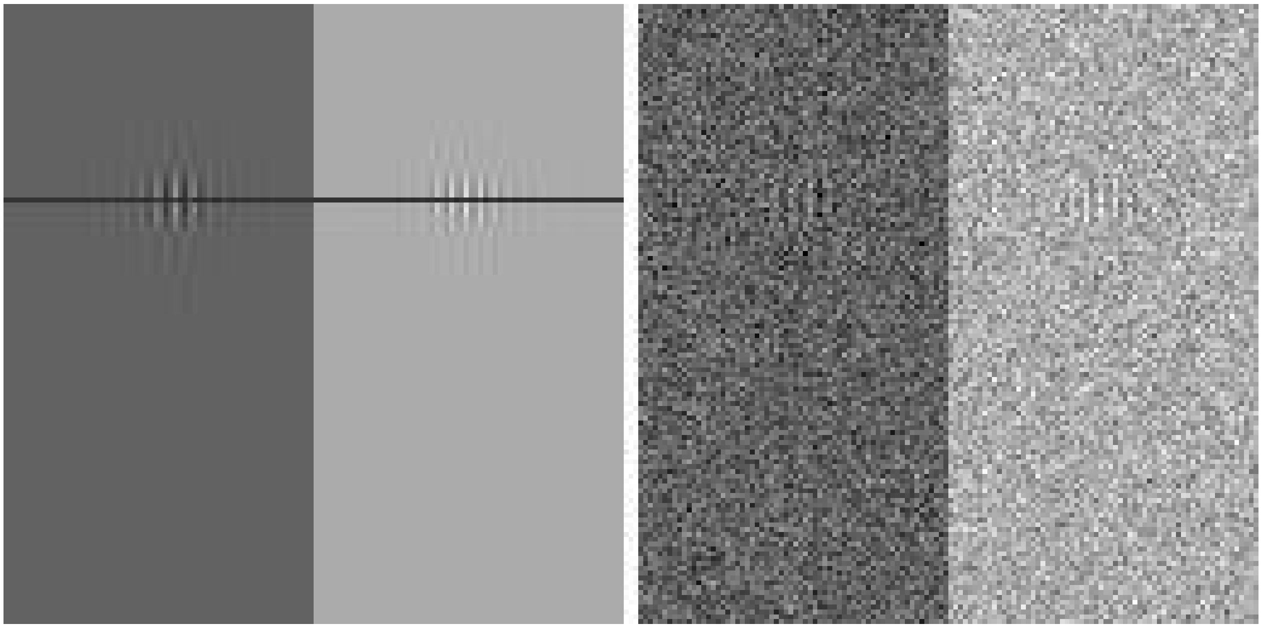


Figure 1: Left: the initial image (the black line is the one extracted for Figure 2 and 3). Right: The noisy image.

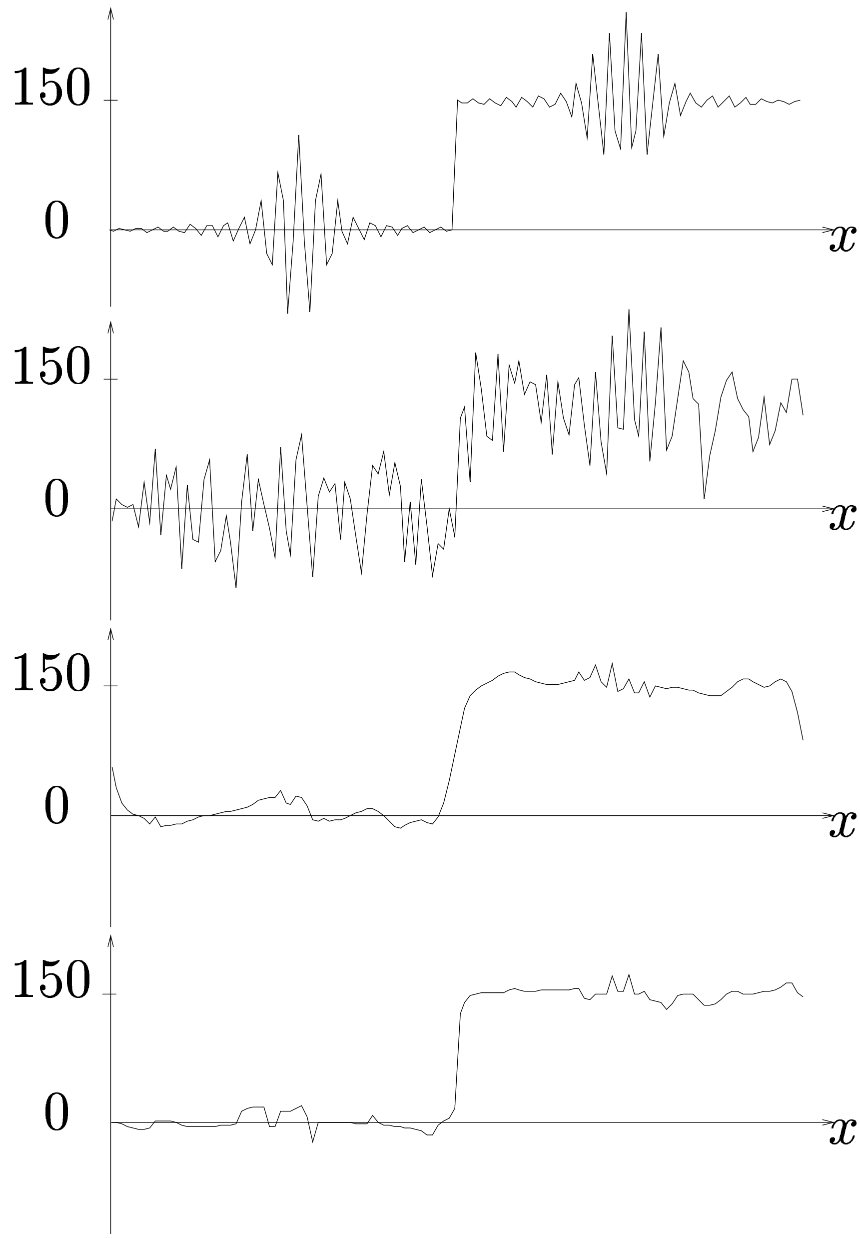


Figure 2: Extracted lines from images. From up to down: the initial image; the noisy image; the denoised image with a wavelet thresholding; a solution of (1) when the dictionary is made of one wavelet basis.

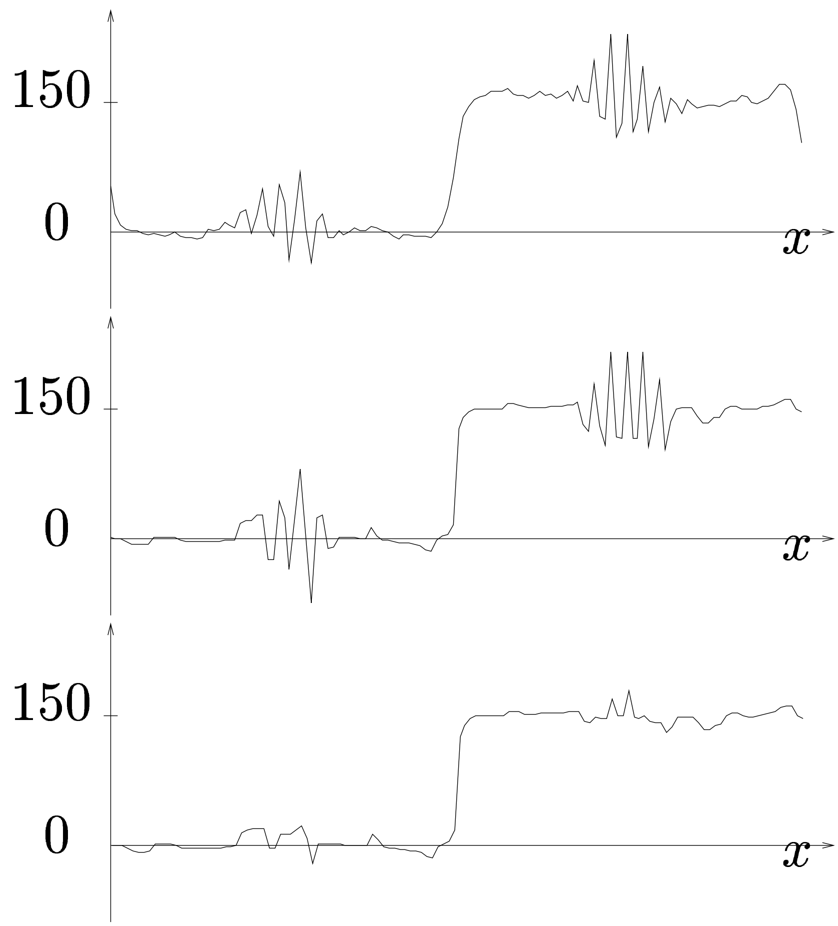


Figure 3: Extracted lines from images. From up to down: the noise selection approach in a wavelet packet dictionary; a solution of (1) with a wavelet packet dictionary; Rudin-Osher-Fatemi method.

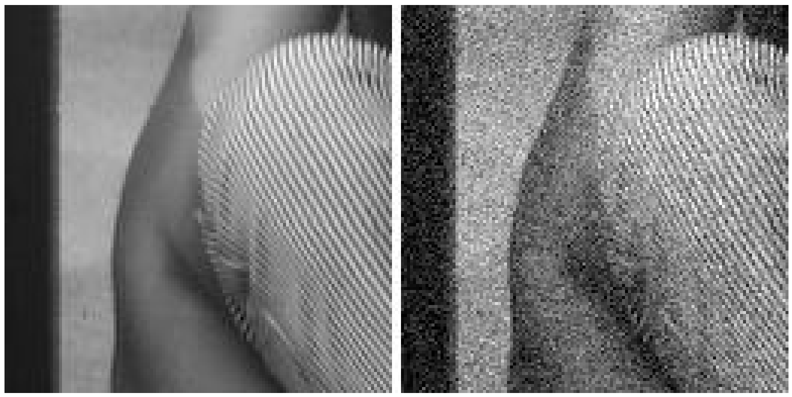


Figure 4: Left: original image; Right : noisy image ($\sigma = 20$).

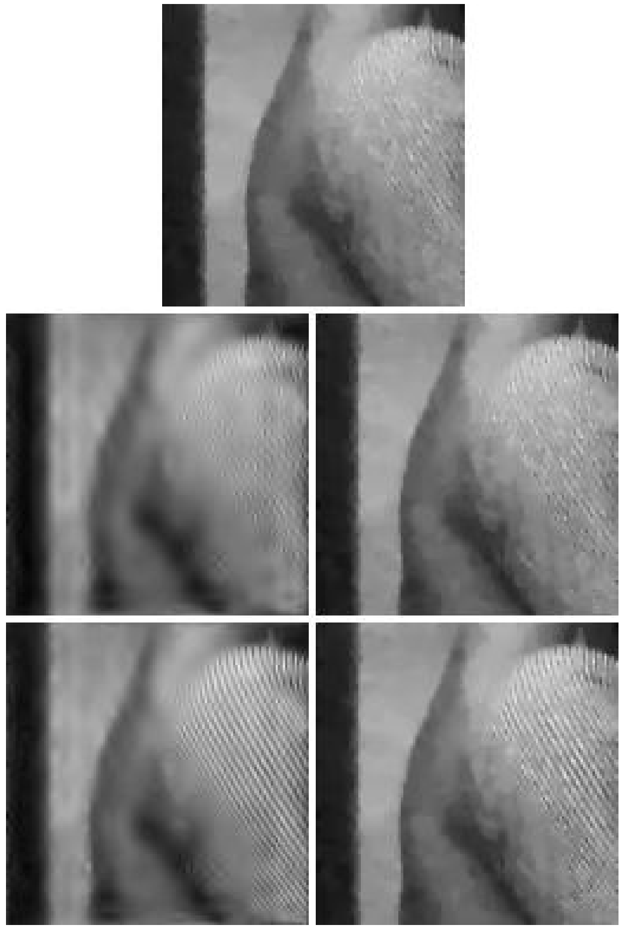
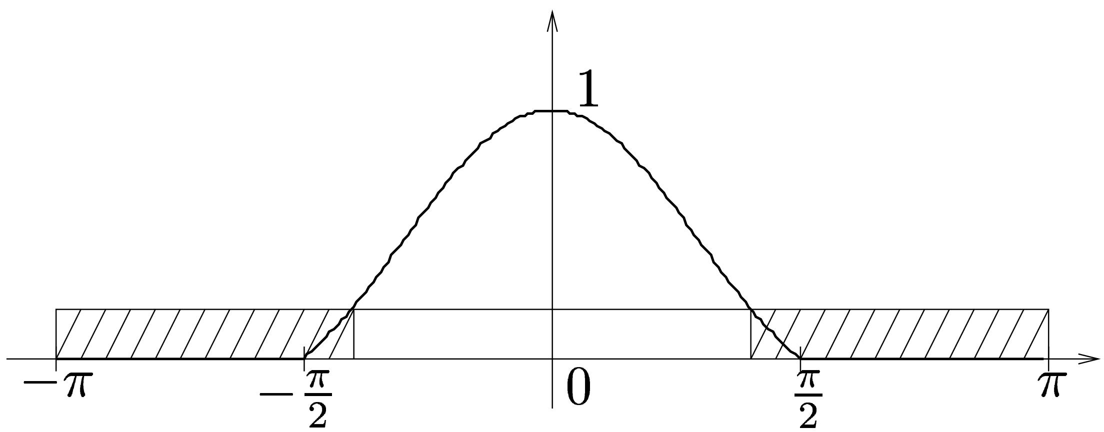


Figure 5: R.O.F.; Wavelet thresholding; Solution of (1) with a wavelet basis; Noise tracking in a wavelet packet dictionary; Solution of (1) with a wavelet packet dictionary.

Deconvolution: $\sigma = 2$



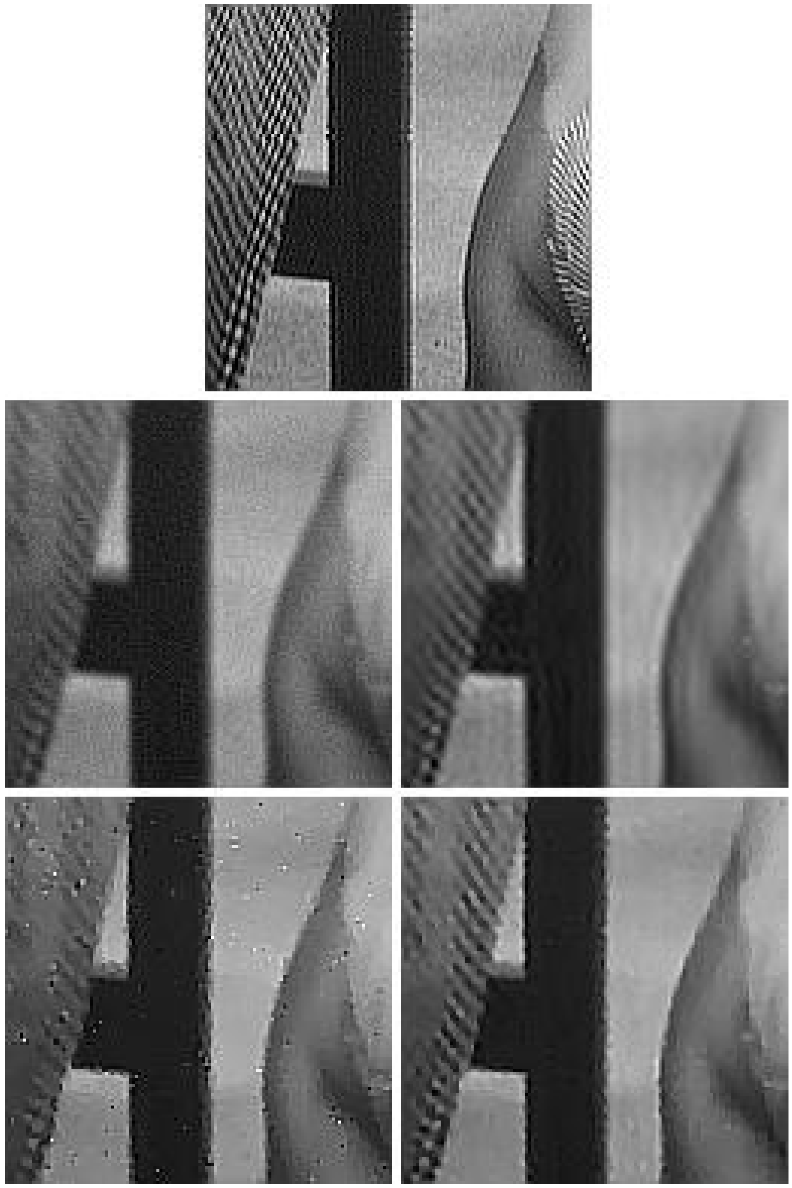


Figure 6: **Sharpened** images.
Original; blurred image; Noise tracking in a wavelet packet dictionary; R.O.F.; Solution of (1) with a wavelet packet dictionary.