Modeling textures with total variation minimization and oscillating patterns in image processing

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Let $f : I\!R^2 \rightarrow I\!R$ be a given image, $f \in L^2(I\!R^2)$

Standard degradation model: $f = u + v$
u is the true original image
v is additive noise (or texture), with $\int v = 0$

Rudin-Osher-Fatemi restoration model (ROF)
(total variation minimization, 1992):

$$\inf_{u \in BV} F(u) = \int |\nabla u| + \lambda \int |f - u|^2 dxdy$$

$\lambda > 0$ is a parameter

Let $v := f - u$ or $f = u + v$. Then (ROF):

$$\inf_{u \in BV} \{F(u) = \int |\nabla u| + \lambda \|v\|_{L^2}^2, \ f = u + v\}$$

ROF model: very efficient to denoise images while preserving edges

Minimizer $u \in BV$ (allows for discontinuities along curves)

Def. $u \in BV(I\!R^2)$ iff $u \in L^1(I\!R^2)$ and

$$TV(u) = \sup_{\vec{g}} \{ \int u \mathrm{div} \vec{g} : \vec{g} \in C_c^1(I\!R^2; I\!R^2), |\vec{g}| \leq 1 \} < \infty.$$

The minimizer u in the (ROF) model is a cartoon or sketchy approximation of f

Residual $v := f - u$ (noise or texture):

$$\inf_u F(u) \Rightarrow \ v := f - u = -\frac{1}{2\lambda} \mathrm{div}(\frac{\nabla u}{|\nabla u|})$$

- The ROF model (and others like Mumford-Shah, Perona-Malik) cannot represent well textures.
- The texture is treated like noise and removed from f.
- Only a cartoon representation u of f is kept.
- If $f = \chi_{disk}$, u is not always f (Y. Meyer)

Framework to extract both $u \in BV$ and v as an oscillating function (texture or noise) from f (following Yves Meyer, AMS 2002)

$v \in$ dual of BV in L^2, with a better norm:

Def. G = *Banach space of functions*

$$v(x,y) = \frac{\partial}{\partial x} g_1(x,y) + \frac{\partial}{\partial y} g_2(x,y),\ g_1, g_2 \in L^\infty(\mathbb{R}^2)$$

Norm:

$$\|v\|_* = \inf_{\vec{g}=(g_1,g_2)\in(L^\infty)^2,\ v=\text{div}\vec{g}} \|\sqrt{g_1^2 + g_2^2}\|_{L^\infty}$$

(Y. Meyer) If v = texture or noise, then $v \in G$

$$\inf_u \{E(u) = \int |\nabla u| + \lambda \|v\|_*,\ f = u + v\}$$

$\|v\|_*$ distinguishes between textures (very important; other techniques using Gabor transform are computationally more expensive).

Our contribution: propose a simple practical algorithm for solving this new model.

Using:

$$\|\sqrt{g_1^2+g_2^2}\|_{L^\infty} = \lim_{p\to\infty} \|\sqrt{g_1^2+g_2^2}\|_{L^p}$$

The proposed algorithm:

$$\inf_{u,\vec{g}} G_p(u,\vec{g}) = \int |\nabla u| + \lambda \int |f-u-\text{div}\vec{g}|^2 dxdy$$
$$+\mu\Big[\int \Big(\sqrt{g_1^2+g_2^2}\Big)^p dxdy\Big]^{\frac{1}{p}}$$

$\lambda, \mu > 0$ are tuning parameters, and $p \to \infty$.

- First term insures that $u \in BV(\mathbb{R}^2)$
- Second term insures that $f \approx u + \text{div} g$
- Third term is a penalty on $\|v\|_*$

If $\lambda \to \infty$ and $p \to \infty$, this model formally approximates the model proposed by Y. Meyer.

The Euler-Lagrange equations:

Formally minimizing the energy with respect to u and $\vec{g} = (g_1, g_2)$:

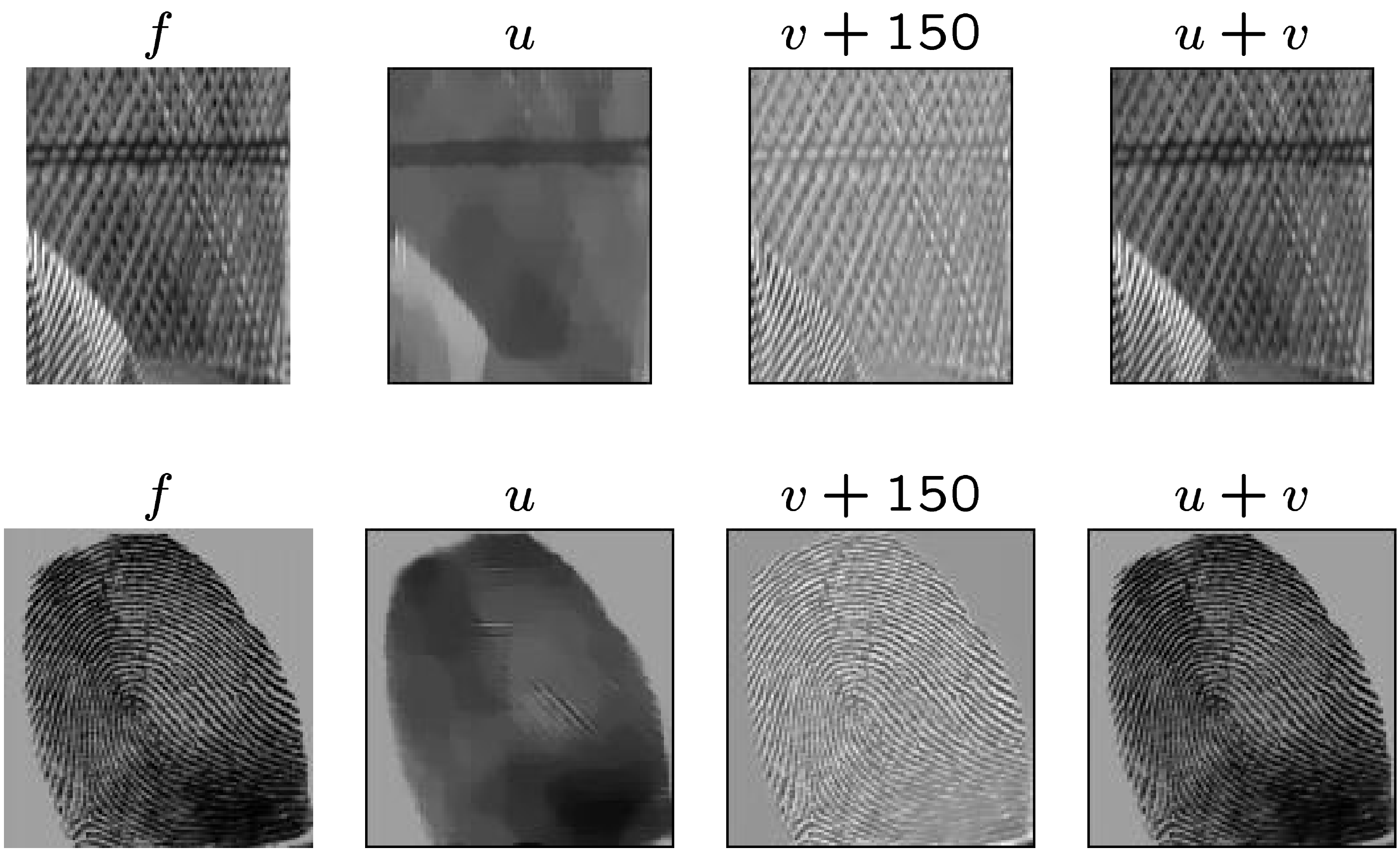
$p = 1$:

$$\begin{aligned} u &= f - \partial_x g_1 - \partial_y g_2 + \frac{1}{2\lambda}\text{div}(\frac{\nabla u}{|\nabla u|}) \\ \mu\frac{g_1}{|\vec{g}|} &= 2\lambda[\frac{\partial}{\partial x}(u - f) + \partial^2_{xx} g_1 + \partial^2_{xy} g_2] \\ \mu\frac{g_2}{|\vec{g}|} &= 2\lambda[\frac{\partial}{\partial y}(u - f) + \partial^2_{xy} g_1 + \partial^2_{yy} g_2] \end{aligned}$$

$p > 1$:

$$\begin{aligned} u &= f - \partial_x g_1 - \partial_y g_2 + \frac{1}{2\lambda}\text{div}(\frac{\nabla u}{|\nabla u|}) \\ \mu\frac{g_1|\vec{g}|^{p-2}}{\||\vec{g}|\|_p^{p-1}} &= 2\lambda[\frac{\partial}{\partial x}(u - f) + \partial^2_{xx} g_1 + \partial^2_{xy} g_2] \\ \mu\frac{g_2|\vec{g}|^{p-2}}{\||\vec{g}|\|_p^{p-1}} &= 2\lambda[\frac{\partial}{\partial y}(u - f) + \partial^2_{xy} g_1 + \partial^2_{yy} g_2] \end{aligned}$$

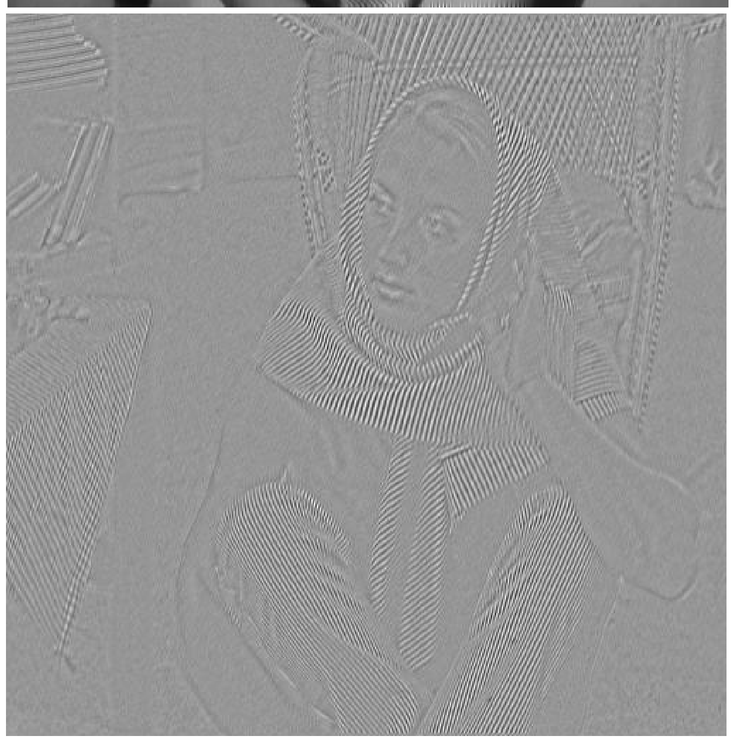
Numerical Results



f

$u + v$

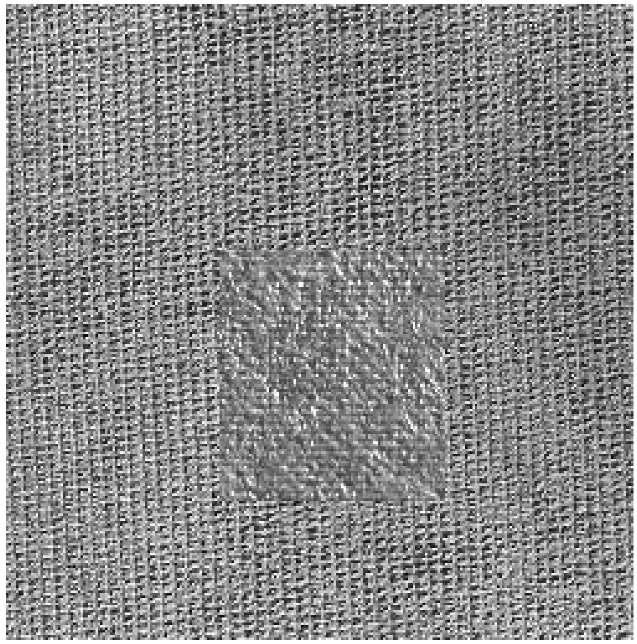
u



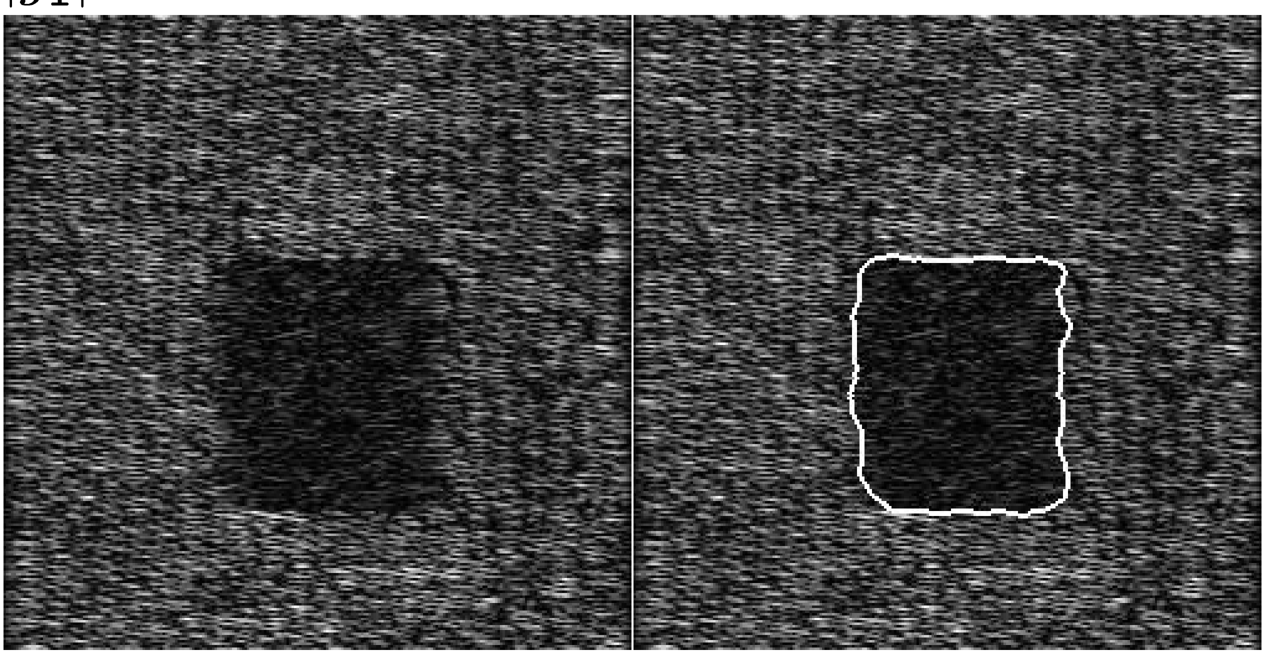
$v + 150$

Texture discrimination:

u

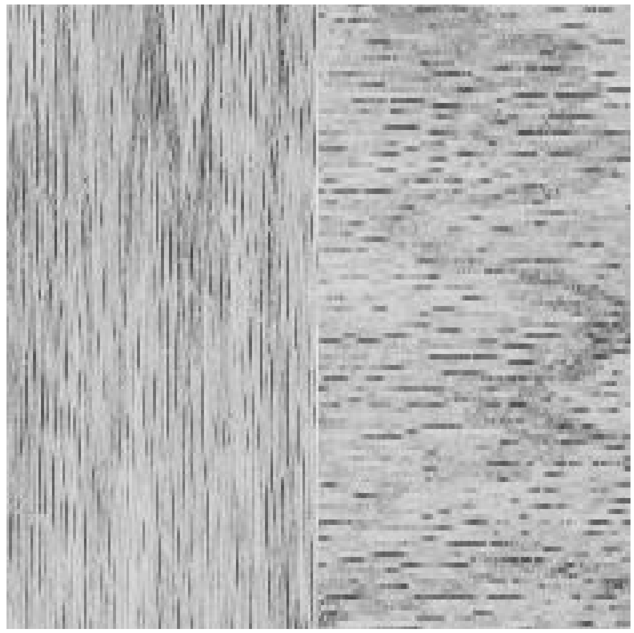


$|g_1|$

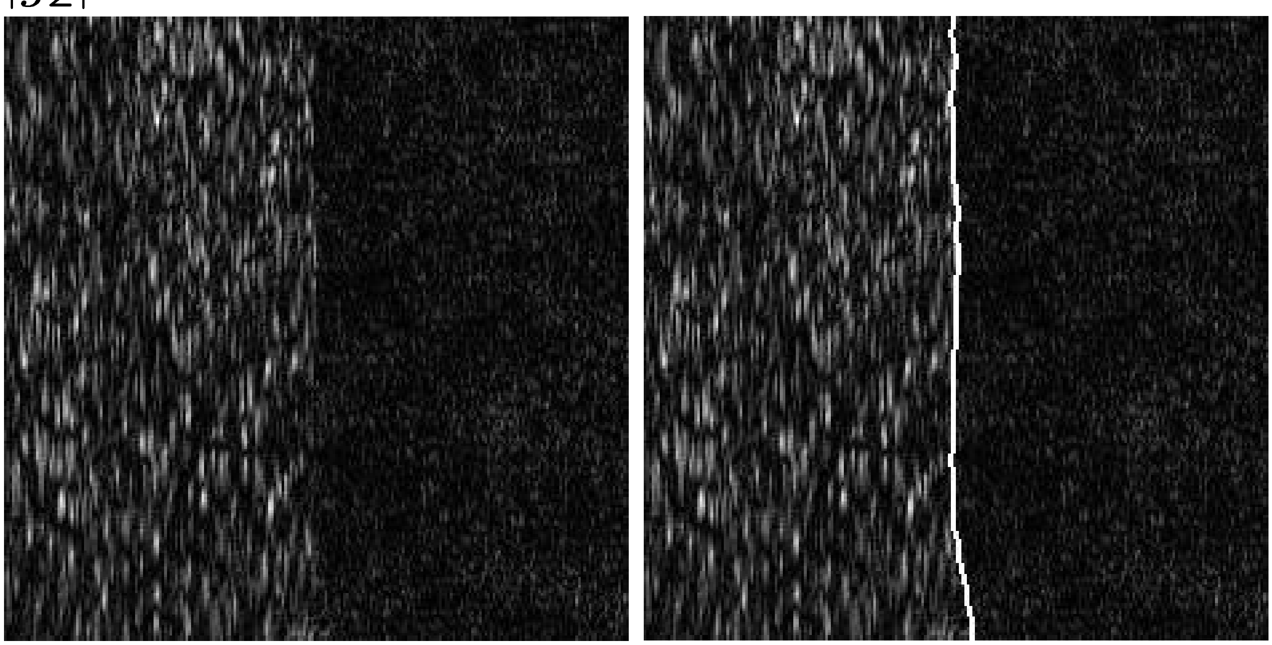


Texture discrimination:

u



$|g_2|$



Future directions:

• Apply the same technique to an image segmentation model (Mumford and Shah)

• Use the function $\vec{g} = (g_1, g_2)$ for texture classification, discrimination and segmentation

• Applications to image inpainting (M. Bertalmio, G. Sapiro, et al.)

• Other formulations of the model can be proposed (work in progress):

1) Exact decomposition $f = u + v$ (energy function of $\vec{g}$ only, but yields a fourth order PDE)

2) Write: $\vec{g} = \nabla P + \vec{Q}$, with $\text{div}\vec{Q} = 0$
$\Rightarrow v = \text{div}\vec{g} = \triangle P$ (with Osher and Solé)