Medial Representations
Mathematics, Algorithms and Applications

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Motivation
Fig. 29. Equivalent objects in some simple A-morphologies. In the upper half, the object sym-axes have the same topology. In the lower half, the object and ground have the same directed graph.
Blum's A-Morphologies: 3D

Fig. 42. Sym-axes of some three-dimensional objects. The ellipsoids at the top show that the sym-ax can now be both arcs and surfaces. The bottom shows the sym-ax for a rectangular solid and for a general spherical envelope.
Blum's Grassfire Machine

"Figure 19 shows my first physical embodiment of the process. It uses a movie projector and camera with high contrast film. These are symmetrically driven apart from the lens in such a way as to keep a one to one magnification, but to increase the circle of confusion (defocussing). Corner detection is done by a separate process. I am presently building a closed loop electronic system to do both the wave generation and corner detection."

[A transformation for Extracting New Descriptors of Shape, 1967.]
Mathematics
The Rowboat Analogy

\textit{Figure 1.7.} Local medial geometry. a. Local geometric properties of a medial point and its boundary pre-image. b. The rowboat analogy for medial points.
Contact Classification

Theorem 1 (Giblin and Kimia) The internal medial locus of a three-dimensional object $\Omega$ generically consists of

1. sheets (manifolds with boundary) of $A_1^2$ medial points;

2. curves of $A_1^3$ points, along which these sheets join, three at a time;

3. curves of $A_3$ points, which bound the free (unconnected) edges of the sheets and for which the corresponding boundary points fall on a crest;

4. points of type $A_1^4$, which occur when four $A_1^3$ curves meet;

5. points of type $A_1A_3$ (i.e., $A_1$ contact and $A_3$ contact at a distinct pair of points) which occur when an $A_3$ curve meets an $A_1^3$ curve.
Euclidean Distance Function
Gradient Vector Field
Outward Flux
Outward Flux
Outward Flux

**Definition 1** The outward flux of $\mathbf{q}$ through $\partial R$ is given by

$$
\int_{\partial R} \langle \mathbf{q}, \mathbf{N} \rangle \, ds
$$

**Definition 2** The average outward flux of $\mathbf{q}$ through $\partial R$ is given by

$$
\frac{\int_{\partial R} \langle \mathbf{q}, \mathbf{N} \rangle \, ds}{\int_{\partial R} \, ds}
$$
Let \( S \) be a branch of the skeleton and let \( R = R_1 \cup R_2 \) be a path connected region which intersects it. Let \( \partial R = C_1 \cup C_2 \) and \( C_3 = S \cap R \). Let \( C'_3 \), \( C''_3 \) be parallel curves to \( C_3 \) which approach \( C_3 \) as \( t \to 0 \). Let \( R_{1t} \) and \( R_{2t} \) be the regions obtained from \( R_1 \) and \( R_2 \) by removing the region between the curves \( C'_3 \) and \( C''_3 \). Finally, let \( \dot{q}_+ \) denote \( \dot{q} \) above \( S \) and \( \dot{q}_- \) denote \( \dot{q} \) below \( S \).
The outward flux of $\dot{q}$ through $\partial R$ is given by

$$\int_{\partial R} \langle \dot{q}, \mathcal{N} \rangle \, ds = \int_{C_1} \langle \dot{q}, \mathcal{N} \rangle \, ds + \int_{C_2} \langle \dot{q}, \mathcal{N} \rangle \, ds.$$

Applying the divergence theorem to $R_{1t}$ and $R_{2t}$

$$\int_{R_{1t}} \text{div}(\dot{q}) \, dv = \int_{C_{1t}} \langle \dot{q}, \mathcal{N} \rangle \, ds + \int_{C_{3t}'} \langle \dot{q}, \mathcal{N} \rangle \, ds,$$

$$\int_{R_{2t}} \text{div}(\dot{q}) \, dv = \int_{C_{2t}} \langle \dot{q}, \mathcal{N} \rangle \, ds + \int_{-C_{3t}''} \langle \dot{q}, \mathcal{N} \rangle \, ds.$$
Adding the above two equations we have

\[
\int_{R_{1t}} \text{div}(\dot{\mathbf{q}}) \, dv + \int_{R_{2t}} \text{div}(\dot{\mathbf{q}}) \, dv = \\
\int_{C_{1t}} \langle \dot{\mathbf{q}}, \mathcal{N} \rangle \, ds + \int_{C_{2t}} \langle \dot{\mathbf{q}}, \mathcal{N} \rangle \, ds + \\
\int_{C'_{3t}} \langle \dot{\mathbf{q}}, \mathcal{N} \rangle \, ds + \int_{-C''_{3t}} \langle \dot{\mathbf{q}}, \mathcal{N} \rangle \, ds.
\]
It is a standard property that the tangent to the skeleton bisects the angle between $\dot{q}_+$ and $\dot{q}_-$ at a skeletal point (see Figure 2). Thus, on $C_3$ we have

$$\langle \dot{q}_+, N_+ \rangle = \langle \dot{q}_-, N_- \rangle,$$

where $N_+, N_-$ denote the normals to $C_3$ from above and from below, respectively. Thus, one can take the limit as $t \to 0$ of both sides of the above equation to obtain the following extension of the divergence theorem.
(extended) Divergence Theorem

**Theorem 1** For a path connected region $R$ which contains part of a skeletal curve $S$, the divergence of the vector field $\dot{q}$ is related to its flux through $\partial R$ by the following equation

$$\int_{R=R_1 \cup R_2} \text{div}(\dot{q}) \, dv =$$

$$\int_{\partial R} \langle \dot{q}, N \rangle \, ds + 2 \int_{C_3} \langle \dot{q}, N_{C_3} \rangle \, ds.$$ 

**Corollary.** The OF for a region shrinking to a skeleton point satisfies:

$$\lim_{R \to P} \text{OF}_R \to 2(\langle \nabla D(P), N \rangle) \text{length}(C_3)$$
Circular Neighborhoods

(Dimitrov, Damon, Siddiqi, CVPR’03)

The AOF for shrinking circular regions:

<table>
<thead>
<tr>
<th>$R \rightarrow P$</th>
<th>$\text{AOF}_{R \rightarrow x}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Regular Points</td>
<td>$-\frac{2}{\pi} \sin \alpha$</td>
</tr>
<tr>
<td>End-Points</td>
<td>$-\frac{1}{\pi} (\sin \alpha + \alpha)$</td>
</tr>
<tr>
<td>Junction Points</td>
<td>$-\frac{1}{\pi} \sum_{i=1}^{n} \sin \alpha_i$</td>
</tr>
<tr>
<td>Non-Skeletal Points</td>
<td>0</td>
</tr>
</tbody>
</table>
Average Outward Flux
Damon: Skeletal Structures

Figure 2. A Skeletal Structure $(M, U)$ defining a region $\Omega$ with smooth boundary $B$

**radial shape operator**

$$S_{rad}(v) = -\text{proj}_U \left( \frac{\partial U_1}{\partial v} \right)$$

**radial curvature**

$$K_{rad} = \det(S_{rad})$$
Damon: Radial Flow

Figure 3. a) Radial Flow and b) Grassfire Flow

- radial curvature condition + edge condition + compatibility condition ensure smoothness of boundary
- complete characterization of local and relative differential geometry of boundary in terms of radial shape operator on skeletal structure
(Rigorous) Divergence Theorem

**Theorem 9** (Modified Divergence Theorem). Let $\Omega$ be a region with smooth boundary $\mathcal{B}$ defined by the skeletal structure. Also, let $\Gamma$ be a region in $\Omega$ with regular piecewise smooth boundary. Suppose $F$ is a smooth vector field with discontinuities across $M$, then

\begin{equation}
\int_{\Gamma} \operatorname{div} F \, dV = \int_{\partial \Gamma} F \cdot n_{\Gamma} \, dS - \int_{\tilde{\Gamma}} c_F \, dM
\end{equation}

where $\tilde{\Gamma} = \tilde{M} \cap \pi^{-1}(M \cap \Gamma)$.

$\operatorname{proj}_{TM}(F) = c_F \cdot U_1$, where $\operatorname{proj}_{TM}$ denotes projection onto $U$ along $TM$.
Boundary Integrals as Medial Integrals

**Theorem 1.** Suppose \((M,U)\) is a skeletal structure defining a region with smooth boundary \(B\) and satisfying the partial Blum condition. Let \(g : B \rightarrow \mathbb{R}\) be Borel measurable and integrable with respect to the Riemannian volume measure. Then,

\[
\int_B g \, dV = \int_{\tilde{M}} \tilde{g} \cdot \det(I - rS_{rad}) \, dM
\]

where \(\tilde{g} = g \circ \psi_1\).
Algorithms
Algorithm 2: Topology Preserving Thinning.

**Data**: Object, Average Outward Flux Map.

**Result**: (2D or 3D) Skeleton.

**for** (each point $x$ on the boundary of the object) **do**

| if ($x$ is simple) **then**  
| insert($x$, maxHeap) with AOF($x$) as the sorting key for insertion;  

**while** (maxHeap.size > 0) **do**

| $x$ = HeapExtractMax(maxHeap);  
| if ($x$ is simple) **then**  
| if ($x$ is an end point) and (AOF($x$) < Thresh) **then**  
| mark $x$ as a medial surface (end) point;  
| else  
| Remove $x$;  
| for (all neighbors $y$ of $x$) **do**

| if ($y$ is simple) **then**  
| insert($y$, maxHeap) with AOF($y$) as the sorting key for insertion;  


To verify the theoretical results, boundary points corresponding to regular skeletal points are reconstructed according to: \[ Q_{1,2} = P + rR(\pm \alpha) t_P \]

**STEP 1.** Start with a binary shape.

**STEP 2.** Compute AOF of shape using circular regions.

**STEP 3.** Compute skeleton with algorithm presented in [3].

**STEP 4.** Using our results for shrinking circular regions, reconstruct boundary points from regular skeletal points.
The limiting average outward flux value determines the object angle, which in turn is used to recover the associated bi-tangent points, shown as filled circles.
Brain Ventrícles

Original

Medial Surface
Venus de Milo

Circa 100 BC
Applications
Virtual Endoscopy

Colon

Arteries
3D Medial Graph Matching
Medial Graph Matching

- Edit Distance Based Approaches
  (Sebastian, Kline, Kimia; Hancock, Torsello)
  - motivated by string edit distances
  - polynomial time algorithm for trees, (but need to define edit costs)

- Maximum Clique Approaches
  (Pelillo et al.)
  - subgraph isomorphism -> maximum clique in an association graph
  - discrete combinatorial problem -> continuous optimization

- Graph Spectra-Based Approaches
  (Shokoufandeh et al.)
  - eigenvalue analysis of adjacency matrix for DAGs
  - separation of "topology" and "geometry"
  - extension to handle indexing
A Topological Signature Vector

\[ \chi(V) = [S_1, S_2, ..., S_\Delta, 0, ..., 0] \]
\[ S_1 \geq S_2 \geq ... \geq S_\Delta \]

\[ S_i = |\lambda_1| + |\lambda_2| + ... + |\lambda_{k_i}| \]

- At node “a” compute the sum of the magnitudes of the “k” largest eigenvalues of the adjacency matrix of the subgraph rooted at “a”.
- Carry out this process recursively at all nodes.
- The sorted sums become the components of the “TSV” assigned to node V.
Matching Algorithm

(a) Two DAGs to be matched.

(b) A bi-partite graph is formed, spanning their nodes but excluding their edges. The edge weights $W(i,j)$ in the bi-partite graph encode node similarity as well as TSV similarity. The two most similar nodes are found, and are added to the solution set of correspondences.

(c) This process is applied, recursively, to the subgraphs of the two most similar nodes. This ensures that the search for corresponding nodes is focused in corresponding subgraphs, in a top-down manner.
Medial Surfaces to DAGs

Figure 2: A voxelized human form and chair ... set of connected parts:
1. Identify all manifolds comprised of 26-connected surface
points and border points.
2
3D Object Models: The McGill Shape Benchmark

- 420 models reflecting 18 object classes

- **Severe Articulation:** hands, humans, teddy-bears, eyeglasses, pliers, snakes, crabs, ants, spiders, octopuses

- **Moderate or No Articulation:** planes, birds, chairs, tables, cups, dolphins, four-limbed animals, fish

- **Precision Vs Recall Experiments:** shape distributions of Osada et al. (SD), harmonic spheres of Kazhdhan et al. (HS) and medial surfaces (MS).
Spiders

Teddy-bears
Summary
Medial Representations
Mathematics, Algorithms and Applications

Kaleem Siddiqi and Stephen M. Pizer
Springer (in press, 2006)

• Chapter 1: Pizer, Siddiqi, Yushkevich: “Introduction”

• PART 1 - MATHEMATICS

• Chapter 2: Giblin, Kimia: “Local Forms and Transitions of the Medial Axis”

• Chapter 3: Damon: “Geometry and Medial Structure”

• PART 2 - ALGORITHMS

• Chapter 4: Siddiqi, Bouix, Shah: “Skeletons Via Shocks of Boundary Evolution”

• Chapter 5: Borgefors, Nystrom, Sanniti di Baja: “Discrete Skeletons from Distance Transforms.”
Part 3 - Applications

• Chapter 9: Pizer et al.: “Statistical Applications with Deformable M-Reps”

• Chapter 10: Siddiqi et al: “3D Model Retrieval Using Medial Surfaces”

• Chapter 11: Leymarie, Kimia: “From the Infinitely Large to the Infinitely Small”
Selected References

- Bouix, Siddiqi, “Optics, Mechanics and Hamilton-Jacobi Skeletons” [Advances in Imaging and Electron Physics, 2005]
- Damon, “Global Geometry of Regions and Boundaries via Skeletal and Medial Integrals” [preprint, 2003]
- Dimitrov, Damon, Siddiqi, “Flux Invariants for Shape” [CVPR’03]
Selected References

- **Sebastian, Klein, Kimia**, “Recognition of Shapes By Editing their Shock Graphs” [ICCV’01, PAMI’04]

- **Shokoufandeh, Macrini, Dickinson, Siddiqi, Zucker**, “Indexing Hierarchical Structures Using Graph Spectra” [CVPR’99, PAMI’05]

- **Siddiqi, Bouix, Tannenbaum, Zucker**, “Hamilton–Jacobi Skeletons” [ICCV’99, IJCV’02]

- **Siddiqi, Shokoufandeh, Dickinson, Zucker**, “Shock Graphs and Shape Matching” [ICCV’98, IJCV’99]

- **Stolpner, Siddiqi** “Revealing Significant Medial Structure in Polyhedral Messhes” [3DPVT’06]