JOURNAL OF FUNCTIONAL ANALYSIS 71, 142–174 (1987)

Complete Blow-Up after $T_{\max}$ for the Solution of a Semilinear Heat Equation

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Communicated by H. Brezis

Received June 1985; revised December 1985

INTRODUCTION

Let Ω be a bounded open subset of $\mathbb{R}^N$ with a smooth boundary $\partial\Omega$. Consider the problem

$$\begin{aligned} u_t - \Delta u &= f(u) && \text{in} \quad \Omega \times (0, T), \\ u &= 0 && \text{on} \quad \partial\Omega \times (0, T), \qquad\qquad (P) \\ u(x, 0) &= u_0(x) && \text{for all } x \in \Omega, \end{aligned}$$

where $f: \mathbb{R}^+ \to \mathbb{R}^+$ is locally Lipschitz, nondecreasing and $f(0) = 0$. If u_0 is a continuous function on $\bar{\Omega}$, there exists a unique classical solution u of (P) defined on $[0, T_{\max})$ and such that $u \in \mathscr{C}^{2,1}(\bar{\Omega} \times (0, T_{\max})) \cap \mathscr{C}(\Omega \times [0, T_{\max}))$ with $\lim_{t \to T_{\max}} \|u\|_\infty = \infty$ if $T_{\max} < \infty$. A well-known result asserts that if u is large enough and $f(u) = u^p$, $p > 1$, for example, then $T_{\max} < \infty$ (this is the case when $1/2\, |\nabla u_0|^2 - 1/(p+1) \int_\Omega |u_0|^{p+1} < 0$ see, for example, [1] or Corollary 2.2). In what follows, we suppose that $T_{\max} < +\infty$.

Assume $f_n: \mathbb{R}^+ \to \mathbb{R}^+$ is a sequence of functions such that

(a) for each $n, u \to f_n(u)$ is globally Lipschitz, non decreasing, $f_n(0) = 0$,

(b) for each $u, n \to f_n(u)$ is increasing and converges to $f(u)$.

0022-1236/87 $3.00

Let u_n be the unique global classical solution of

$$\begin{aligned} u_{nt} - \Delta u_n &= f_n(u_n) && \text{in} \quad \Omega \times (0, +\infty), \\ u_n &= 0 && \text{on} \quad \partial\Omega \times (0, +\infty), \\ u_n(x, 0) &= u_0(x) && \text{for all } x \in \Omega. \end{aligned} \tag{P$_n$}$$

We say that f satisfies (h) if:

$$f \text{ is convex and } \exists \gamma > 1, a \geqslant 0 \text{ such that } u \to f(u)/u^\gamma \text{ is nondecreasing on } (a, +\infty). \tag{h}$$

Our main result is

THEOREM 1. *Let $u_0 \in L^\infty(\Omega)$, $u_0 \geqslant 0$. Suppose that one of the following hypotheses holds:*

(H1) *Ω convex and if $N \geqslant 2$, there exists $p \in (1, N/(N-2))$ and $c > 0$ such that $0 \leqslant f'(u) \leqslant C(u^{p-1} + 1)$ for all $u \geqslant 0$. $u_0 \in W_0^{1,1}(\Omega)$, $\Delta u_0 + f(u_0) \geqslant 0$ in $\mathscr{D}'(\Omega)$. (No hypothesis on f for $N = 1$.)*

(H2) *f satisfies* (h) *and $u_0 \in W_0^{1,1}(\Omega)$, $\Delta u_0 + f(u_0) \geqslant 0$ in $\mathscr{D}'(\Omega)$.*

(H3) *f is convex and there exists $p \in (1, (N+2)/(N-2))$ such that $0 \leqslant \lim_{u \to \infty} (f(u)/u^p) < \infty$.*

Then

(i) $\lim_{n \to \infty} u_n(x, t) = u(x, t)$ *for all* $(x, t) \in \Omega \times [0, T_{\max})$,

(ii) $\lim_{n \to \infty} u_n(x, t) = \infty$ *for all* $(x, t) \in \Omega \times (T_{\max}, \infty)$.

We will see that Theorem 1 proves, in some appropriate sense that u cannot be extended beyond $T_{\max}$ and blows up everywhere on $\Omega \times (T_{\max}, \infty)$ which is a conjecture of H. Brezis.

In this paper, we consider also the notion of an integral solution of (P) which is, in some sense the weakest definition of a positive solution and we prove that it cannot be extended beyond $T_{\max}$. Let us be more precise. Let u_0 be a nonnegative measure on Ω. We say that V is an *integral solution of* (P) if $V(x, t)$: $\Omega \times (0, +\infty) \to [0, +\infty]$ is a nonnegative measurable function such that

$$V(x, t) = \int_\Omega G(t, x, y)\, u_0(y)\, dy + \int_0^t \int_\Omega G(t-s, x, y) f(V(y, s))\, dy\, ds \tag{1}$$

for a.e. (x, t) in $\Omega \times (0, \infty)$, where $G(t-s, x, y)$ denotes the Green function of the heat equation with Dirichlet boundary condition. Given an integral solution V we define its *true Time of existence*

$$T^*(V) = \sup\{T;\ V \text{ is finite a.e. on } \Omega \times (0, T)\}.$$

Using the properties of G, we easily see that $V \equiv +\infty$ on $\Omega \times (T^*(V), \infty)$ and if u_0 belongs to $\mathscr{C}(\bar{\Omega})$, we have $u \leqslant V$ on $\Omega \times (0, T_{\max})$ where u is the classical solution of (P) (see Proposition 2.1). In this framework, our main result becomes:

THEOREM 2. *Under the assumptions of Theorem* 1, *let V be any integral solution of* (P) *then* $T^*(V) \leqslant T_{\max}$.

Let us explain why Theorems 1 and 2 are equivalent. The integral solutions of (P) are not in general unique (see [2, 11]). Among all these solutions, there exists a minimal element U which is the least integral solution of (P) and it is easy to see that we have

$$\lim_n u_n(x, t) = U(x, t) \qquad \text{for all } (x, t) \text{ in } \Omega \times (0, \infty)$$

and

$$\lim_n u_n(x, t) = u(x, t) \qquad \text{for all } (x, t) \text{ in } \Omega \times (0, T_{\max})$$

(see Proposition 2.1).

If we suppose that Theorem 1 is false, U would be a continuation of u behond $T_{\max}$ and we would have $T^*(U) > T_{\max}$ which contradicts Theorem 2. Thus Theorem 2 implies Theorem 1. For the converse, observe that Theorem 1 means $T^*(U) = T_{\max}$ and we have for all integral solution V of (P) $T^*(V) \leqslant T^*(U)$ (see Proposition 2.1).

To prove these theorems, we use different techniques. In the first part, we prove directly Theorem 1 under (H1) by resuming some techniques of [4, 8].

In the second part, we prove Theorem 2 under (H2). The method is quite different because no usual a priori estimates on u hold in this case, but we know a necessary and sufficient condition on u_0 and T to get an integral solution U of (P) such that $T^*(U) \geqslant T$ (see [5]). The first step of the proof is Lemma 2.1 where we establish in some sense that if (P) has an integral solution U when the initial data is u_0, the least integral solution of (P) when the initial data is λu_0, is a classical solution on $(0, T^*(U))$ for all λ in (0, 1). (Remark 2.1). The second step is to prove that there exists a function $\xi^* \not\equiv 0$ which realizes the equality in the criterion given in [5]. Recall that this criterion can be written

$$\int_\Omega \xi(0)\, u_0 \leqslant \int_0^T \int_\Omega f^*((-\xi_t - \Delta\xi)/\xi)\, \xi \, dx\, dt$$

for all suitable test function ξ. Here f^* is the conjugate function of f. We prove that ξ^* is a solution of

$$-\xi^*t - \Delta\xi^* = f'(u)\,\xi^* > 0 \qquad \text{on} \quad \Omega \times (0, T_{max}),$$
$$\xi^* = 0 \qquad \text{on} \quad \partial\Omega \times (0, T_{max}),$$
$$\xi^*(x, T_{max}) = 0 \qquad \text{for} \quad \text{a.e. } x \in \Omega,$$

$f'(u)\,\xi^*$ and $uf'(u)\,\xi^*$ belong to $L^1(\Omega \times (0, T_{max}))$ (see Theorem 2.2), Theorem 2 is then a corollary of this result.

In the third part, we prove Theorem 2 under (H3). Using the techniques of [6], we begin by proving that without any restriction on the growth of f (other than (h)), the least integral solution of (P) satisfies

$$U \in L^4_{loc}((0, T^*), H^1_0(\Omega)), dU/dt \in L^2_{loc}((0, T^*), L^2(\Omega)),$$
$$Uf(U) \in L^2_{loc}((0, T^*), L^1(\Omega)),$$

where $T^* = T^*(U)$. Theorem 2 is then a consequence of a result of Giga (see [9]). As corollary of Theorem 2, we prove that under (H3), $u_0 \to T_{max}$ is continuous on $L^\infty(\Omega)$.

Finally, note that many authors have studied the behavior of u near T_{max} (cf. [7, 10, 12, 17, 18]). Especially, Weissler proves in [18] that for suitable u_0 and f, $\lim_{t \uparrow T_{max}} u(x, t) < \infty$ except at one point. Friedman and B. MacLeod [7] obtain under some specific assumptions that $\lim_{t \uparrow T_{max}} \|u(\cdot, t)\|_q < \infty$ when $q < N(p-1)/2$ and $f(u) = u^p$.

We would like to thank F. B. Weissler for his helpful suggestions which lead to an improved version of this paper.

1. Proof of Theorem 1 under Hypothesis ($H1$)

We recall u and u_n are, respectively, the classical solutions of (P) and (P_n).

We shall first derive some properties of u and u_n. By applying the maximum principle (see [15]), we have that u_n and u are positive for $x \in \Omega$ and $t > 0$. Since (f_n) is nondecreasing in n, so is (u_n). Therefore we can define:

$$U(x, t) = \lim_{n \to \infty} u_n(x, t) \qquad \text{for all } (x, t) \text{ in } \Omega \times (0, \infty). \tag{1.1}$$

Note that $U(x, t) \in \mathbb{R}^+ \cup \{\infty\}$ and

$$U(x, t) \leqslant u(x, t) \qquad \text{for all } (x, t) \text{ in } \Omega \times [0, T_{max}). \tag{1.2}$$

Moreover, the following lemma shows that $u(x, t)$ is nondecreasing in t for all x in Ω.

LEMMA 1.1. *If $u_0 \in L^\infty(\Omega) \cap W_0^{1,1}(\Omega)$ and $\Delta u_0 + f(u_0) \geqslant 0$ in $\mathscr{D}'(\Omega)$, then the solution u of* (P) *on* $[0, T_{\max}]$ *is nondecreasing in t for all x in Ω.*

Proof. Let ω_n be the solution of

$$\begin{aligned} -\Delta\omega_n + n\omega_n &= f(u_0) + nu_0 \qquad \text{on} \quad \Omega, \\ \omega_n &= 0 \qquad \text{on} \quad \Omega. \end{aligned} \tag{Q_n}$$

Problem (Q_n) has a unique solution in $W^{2,p}(\Omega) \cap H_0^1(\Omega)$ for all finite p. It follows that

$$\begin{aligned} -\Delta(\omega_n - u_0) + n(\omega_n - u_0) &= f(u_0) + \Delta u_0 \geqslant 0 \qquad \text{in} \quad \mathscr{D}'(\Omega). \\ (\omega_n - u_0) &\in W_0^{1,1}(\Omega). \end{aligned} \tag{1.3}$$

Therefore, for all $n \in \mathbb{N}$, $\omega_n - u_0 \geqslant 0$, and thus

$$\begin{aligned} -\Delta(\omega_n - \omega_{n+1}) + n(\omega_n - \omega_{n+1}) &= \omega_{n+1} - u_0 \qquad \text{on} \quad \Omega, \\ \omega_n - \omega_{n+1} &\in W_0^{1,1}(\Omega). \end{aligned} \tag{1.4}$$

Hence, (ω_n) is a nonincreasing sequence and, by (1.3), its limit is necessarily u_0.

Moreover, $\Delta\omega_n + f(\omega_n) = n(\omega_n - u_0) + f(\omega_n) - f(u_0) \geqslant 0$ on Ω, and the following problem has a unique solution on $[0, T_n)$:

$$\begin{aligned} W_{n_t} - \Delta W_n &= f(W_n) \qquad \text{on} \quad \Omega \times [0, T_n), \\ W_n(0) &= \omega_n \qquad \text{on} \quad \Omega, \\ W_n &= 0 \qquad \text{on} \quad \partial\Omega \times [0, T_n), \end{aligned} \tag{1.5}$$

with $\lim_{t \uparrow T_n} \|W_n(\cdot, t)\|_\infty = \infty$. Applying the maximum principle to W_{n_t}, the solution of:

$$\begin{aligned} d(W_{n_t})/dt - \Delta W_{n_t} &= f'(W_n)\, W_{n_t} \qquad \text{on} \quad \Omega \times [0, T_n), \\ W_{n_t}(0) &= \Delta\omega_n + f(\omega_n) \geqslant 0 \qquad \text{on} \quad \Omega, \\ W_{n_t} &= 0 \qquad \text{on} \quad \partial\Omega \times [0, T_n), \end{aligned} \tag{1.6}$$

we find that W_n is nondecreasing in t. Applying the maximum principle again we have that for all $n \in \mathbb{N}$:

$$T_n \leqslant T_{n+1} \leqslant T_{\max} \qquad \text{and} \qquad W_n \geqslant W_{n+1} \geqslant u \qquad \text{on} \quad [0, T_n). \tag{1.7}$$

We can therefore set:

$$T' = \lim_{n \to \infty} T_n \quad \text{and} \quad W = \lim_{n \to \infty} W_n \quad \text{on} \quad [0, T'). \tag{1.8}$$

If $t \in (0, T')$, W is bounded on $[0, t]$ and W is an integral solution of (P) with initial data u_0. This can be seen by passing to the monotone limit in the equation satisfied by W_n. So W is classical on $[0, t]$ an then equal to u by uniqueness. W_n is nondecreasing in t for all n and so is $u = W$ on $[0, T']$.

We now show that indeed $T' = T_{\max}$. We can assume that $u_0 \in C(\bar{\Omega})$ by working with the initial value $u(x, t_0)$ for some $t_0 \in (0, T')$. First, ω_n and u_0 are continuous on Ω, a compact set. By Dini's Theorem, (ω_n) converges uniformly to u_0 on Ω. Let $T \in (0, T_{\max})$, $M = \|u\|_{L^\infty(\Omega \times (0,T))} < \infty$, and C the lipschitz constant for f on $[0, M+1]$. Then there exists $M_0 \in \mathbb{N}$ such that

$$\text{for all } n \geqslant N_0, \qquad \|\omega_n - u_0\| \, e^{CT\max} < 1. \tag{1.9}$$

If $n \geqslant N_0$ set $A_n = \{t \in [0, T_n) \mid \forall \tau \in [0, t], \ \|(W_n - u)(\tau)\|_\infty \leqslant 1\}$. A_n can be written $A_n = [0, T'_n)$ with $0 < T'_n \leqslant T_n$. If $T_n < T$, then $T'_n < T_n$ as $\lim_{t \uparrow T_n} \|W_n(\cdot, t)\|_\infty = \infty$. For $t \in [0, T'_n]$, we have

$$\|(w_n - u)(t)\|_\infty \leqslant \|\omega_n - u_0\|_\infty + \int_0^t C \, \|(W_n - u)(\tau)\| \, d\tau. \tag{1.10}$$

An application of Gronwall's Lemma gives:

$$\|(W_n - u)(\cdot, T'_n)\|_\infty \leqslant \|\omega_n - u_0\| \, e^{CT'_n} < 1. \tag{1.11}$$

This contradicts the definition of T'_n. Thus $T_n \geqslant T$ for $n \geqslant N_0$. We have also shown that $T' = T_{\max}$ and that W_n converges uniformly to u on $\bar{\Omega} \times [0, T]$ for all $T \in [0, T_{\max})$. We deduce then that u is nondecreasing in t. So we can define $\bar{u}(x) = \lim_{t \uparrow T_{\max}} u(x, t)$ in $\mathbb{R} \cup \{\infty\}$.

Define $[f(u)]_n = \mathrm{Inf}(f(u), n)$. For the moment, we will assume that $f_n(u) = [f(u)]_n$. This assumption will be removed later. For $n \geqslant \|u_0\|_\infty$, we have $\Delta u_0 + f_n(u_0) = \Delta u_0 + f(u_0) \geqslant 0$, and by applying Lemma 1.1 we see that u_n is nondecreasing in t.

To show (i) and (ii), we use the three following lemmas:

LEMMA 1.2. *For all $\varepsilon > 0$ and all $T > T_{\max} + \varepsilon$ and all $\Omega' \Subset \Omega$, there exists a constant $C > 0$ such that*:

$$u_n(x, t) \geqslant C(t - T_{\max} - \varepsilon) \int_{C'} f_n(u_n(y, T_{\max})) \, dy$$

$$\textit{for all } (x, t) \quad \textit{in} \quad \Omega' \times [T_{\max} + \varepsilon, T]. \tag{1.12}$$

Proof We use here an idea of Baras and Goldstein [4]. If $\varphi \in L^{\infty}(\Omega)$, we can write

$$(S(t)\,\varphi)(x) = \int_{\Omega} (S(t)\,\varphi)(y)\,\delta_x(y)\,dy = \int_{\Omega} (S(t)\,\delta_x)(y)\,\varphi(y)\,dy, \quad (1.13)$$

where δ_x is Dirac mass at point x. By the maximum principle we have

$$C = \operatorname{Inf}\{(S(t)\,\delta_x)(y),\ (t, x, y) \in [\varepsilon, T] \times \Omega' \times \Omega'\} > 0. \quad (1.14)$$

By writing the integral formulation of (P_n) and using (1.13) and (1.14), we have

$$u_n(x, t) = (S(t)\,u_0)(x) + \int_0^t S(t-s) f_n(u_n(x, s))\,ds. \quad (1.15)$$

Thus if $(x, t) \in \Omega' \times [T_{\max} + \varepsilon, T]$, we have

$$u_n(x, t) \geq \int_0^{t-\varepsilon} \int_{\Omega'} f_n(u_n(y, s))\,dy\,ds \quad (1.16)$$

and

$$u_n(x, t) \geq C \int_{T_{\max}}^{t-\varepsilon} \int_{\Omega'} f_n(u_n(y, T_{\max}))\,ds. \quad (1.16')$$

That s, (1.12).

LEMMA 1.3. Assume (H1). Then,

$$\lim_{t \uparrow T_{\max}} \|f(u(\cdot, t))\|_1 = \infty. \quad (1.17)$$

Proof We recall that if $1 \leq p \leq q \leq \infty$, $S(t)$: $L^p(\Omega) \to L^q(\Omega)$ is bounded and

$$\|S(t)\,\varphi\|_q \leq \frac{1}{(4\Pi t)^{N(1/p - 1/q)/2}} \|\varphi\|_p \qquad \text{for all } \varphi \in L^p(\Omega) \quad \text{and} \quad t > 0 \quad (1.18)$$

Use of (1.18) in the integral equation satisfied by u yields the inequality:

$$\|u(\cdot, t)\|_q \leq \|u_0\|_q + \int_0^t \frac{1}{(4\Pi(t-s))^{N(1-1/q)/2}} \|f(u(\cdot, s))\|_1\,ds \qquad \text{for all } t \text{ in } [0, T_{\max}). \quad (1.19)$$

We remark that $\|f(u(\cdot,t))\|_1$ is nondecreasing in t and so have a limit as $t\uparrow T_{\max}$. It is sufficient to show that it is not bounded. If $N=1$, $\lim_{t\uparrow T_{\max}}\|u(\cdot,t)\|_\infty=\infty$ and (1.17) follows by taking $q=\infty$.

In the same way, if $N\geqslant 2$, $\lim\|u(\cdot,t)\|_q=\infty$ for $q>N(p-1)/2$ (see [16] and [5]), and (H1) permits us to find $q>N(p-1)/2$ such that $N(1-1/q)/2<1$.

LEMMA 1.4. *We recall* $\bar{u}(x)=\lim_{t\uparrow T_{\max}}u(x,t)$. *If* $\|f(\bar{u})\|_1=\infty$, *then there exists* $\Omega_1\Subset\Omega$ *such that*

$$\int_{\Omega_1} f(\bar{u}(x))\,dx=\lim_{t\uparrow T_{\max}}\int_{\Omega_1} f(u(x,t))\,dx=\infty. \tag{1.20}$$

Proof. We use the same definitions as in Gidas–Ni–Nirenberg [8] as applied in Ni–Sacks–Tavantzis [14].

Recall that Ω is a bounded convex open set with smooth boundary. If $x\in\partial\Omega$, we denote by ν the outward unit normal vector at x. We then define the hyperplanes $T(\lambda,x)=\{y\in\mathbb{R}^n, y\cdot\nu=\lambda\}$. Ω is bounded, so for λ large enough, $\Omega\cap T(\lambda,x)=\emptyset$. If $\lambda_x=x\cdot\nu$, $T(\lambda_x,x)$ is the tangent hyperplane to Ω at x, and if $\lambda>\lambda_x$ then $T(\lambda,x)\cap\Omega=\emptyset$ and $T(\lambda_x,x)\cap\Omega\ni x$. For $\lambda<\lambda_x$ we set

$$\sum(\lambda,x)=\{y\in\Omega, \lambda<y\cdot\nu<\lambda_x\}$$

and

$$\sum{}'(\lambda,x)=\Pi_{\lambda,x}\left(\sum(\lambda,x)\right),$$

where $\Pi_{\lambda,x}$ is the reflection across $T(\lambda,x)$. For $\lambda_x-\lambda$ small enough $\sum'(\lambda,x)\subset\Omega$.

By the strong maximum principle (see [15]), $\nabla u(x,t_0)\cdot\nu<0$, for all $t_0>0$.

Let $t_0\in(0,T_{\max})$ then we can find a neighborhood of x such that $\nabla u(y,t_0)\cdot\nu<0$ on this neighborhood.

We can choose local coordinates at x defined by $(x, T(\lambda,x),\nu)$, if $y\in\mathbb{R}^N$ it can be written $y=(y',y_N)$. A neighborhood of x can be choosen of the shape $C_\varepsilon=\{y\in\mathbb{R}^n, |y'|<\varepsilon_1, |y_N|<\varepsilon_2\}\cap\Omega$ with $\varepsilon=\{\varepsilon_1,\varepsilon_2\}$. We can make this construction at every point x of $\partial\Omega$.

Let $x_0\in\partial\Omega$ and $K_{x_0}=T(\lambda_{x_0},x_0)\cap\Omega$. K_{x_0} is compact convex set which contains x_0, moreover

$$K_{x_0}=\bigcap_{\lambda<\lambda_{x_0}}\sum(\lambda,x_0).$$

For all $x \in K_{x0}$, ν is the same exterior normal and we can define an open neighborhood O_x of x on which $\nabla u(y, t_0) \cdot \nu < 0$ and of the shape of C_ε. $K_{x0} \subset \bigcup O_x$ so we can extract a finite cover of K_{x0} by $O_{x_i} = C(x_i, \varepsilon_i)$ for $1 \leqslant i \leqslant n_{x0} \cdot U_{x0} = \bigcup O_{x_i}$ is an open set containing K_{x0} and so there exists a $\lambda < \lambda_{x0}$ such that $\Sigma(\lambda, x_0) \subset U_{x0}$. Set $\mu_{x0} = (\lambda + \lambda_{x0})/2$ we have $\Sigma'(\mu_{x0}, x_0) \subset \Omega$ and $\Sigma'(\mu_{x0}, x_0) \cup \Sigma'(\mu_{x0}, x_0) \subset U_{x0}$.

Note that if, for instance, Ω is strictly convex, $K_{x0} = \{x_0\}$ and what we have just done is unnecessary.

We set $v(x, \ ) = u(\Pi_{\mu_{x0}, x0}(x), t)$ on $\Sigma(\pi_{x0}, x_0)$. We have then

$$u_t - \Delta u = f(u)$$

and

$$\begin{aligned} v_t - \Delta v &= f(v) && \text{on} \quad \Sigma(\mu_{x0}, x_0) \times (0, T_{\max}), \\ v \geqslant 0 &= u && \text{on} \quad \partial\Sigma_1 = \partial\Omega \cap \Sigma(\mu_{x0}, x_0) \times (0, T_{\max}) \\ v &= u && \text{on} \quad \partial\Sigma_2 = \Omega \cap T(\mu_{x0}, x_0) \times (0, T_{\max}). \end{aligned}$$

Since $\nabla u \cdot \nu < 0$ on $U_{x0} \times \{t_0\}$ and $\Sigma \cup \Sigma'(\mu_{x0}, x_0) \subset U_{x0}$ we have

$$v(x, t_0) \geqslant u(x, t_0) \qquad \text{on} \quad \Sigma(\mu_{x0}, x_0). \tag{1.21}$$

We have then

$$u_t - \Delta u = f(u)$$

and

$$\begin{aligned} v_t - \Delta v &= f(v) \quad \text{on} \quad \Sigma(\mu_{x0}, x_0) \times (0, T_{\max}), \\ & v \geqslant u \quad \text{on} \quad \partial\Sigma \times (0, T_{\max}), \\ & v(x, t_0) \geqslant u(x, t_0) \quad \text{on} \quad \Sigma(\mu_{x0}, x_0). \end{aligned}$$

By the maximum principle,

$$v(x, t) \geqslant u(x, t) \qquad \text{for all } (x, t) \in \Sigma(\mu_{x0}, x_0) \times (t_0, T_{\max}).$$

$\Sigma(\mu_{x0}, x_0)$ contains an open set of the type $C_\varepsilon \cap \Omega$ where $C_\varepsilon = \{y \in \mathbb{R}^n, |y_1| < \varepsilon_1, |y_N| < \varepsilon_2\}$ with coordinates in $(x_0, T(\lambda_{x0}, x_0), \nu)$. If we choose $\varepsilon_2 < \lambda_{x0} - \mu_{x0}$ then the reflection of $C_\varepsilon \cap \Omega$ across $T(\mu_{x0}, x_0)$ has compact closure in Ω. These neighborhoods C_ε form an open cover of $\partial\Omega$. Therefore we can extract a finite subcover denoted $O_{x_1}, ..., O_{x_p}$.

We set $\Omega' = \Omega / \bigcup_{i=1}^{p} O_{x_i}$. Ω' is open and has compact closure in Ω. If $t > t_0$:

$$\begin{aligned} \int_\Omega f(u(x, t))\, dx &\leqslant \sum_{i=1}^{p} \int_{O_{x_i} \cap \Omega} f(u(x, t))\, dx + \int_{\Omega'} f(u(x, t))\, dx \\ &\leqslant \sum_{i=1}^{p} \int_{\Pi_{\mu_{x_i}, x_i}(O_{x_i} \cap \Omega)} f(u(x, t))\, dx + \int_{\Omega'} f(u(x, t))\, dx \end{aligned}$$

$\Pi_{\mu_{x_i}, x_i}(O_{x_i} \cap \Omega)$ has compact closure in Ω. If we set $\Omega_1 = \Omega' \cup \bigcup_{i=1} \Pi_{\mu_{x_i}, x_i}(O_{x_i} \cap \Omega)$, Ω_1 is compact in Ω and

$$\int_{\Omega} f(u(x, t))\, dx \leqslant \left(\int_{\Omega_1} f(u(x, t))\, dx \right)(p+1) \qquad \text{for all } t > t_0.$$

By taking the limit as $t \uparrow T_{\max}$, we have

$$\int_{\Omega_1} f(\bar{u}(x))\, dx = \infty.$$

Remark. We can see from the above proof that for the case $N = 1$ and $\Omega = (0, 1)$, if u_0 is increasing in x on $[0, a]$, then u remains increasing in x and on the half interval $[0, a/2]$ for all $t \in (0, T_{\max})$.

Proof of (i) *and* (ii). We deduce from (1.2) that U is an integral solution of (P) on $[0, T_{\max})$ and is bounded on $\Omega \times [0, T]$ for $T < T_{\max}$. Then $U = u$ on $\Omega \times [0, T]$ by uniqueness of bounded solution of (P) on $[0, T]$. So we have (i).

To prove (ii), we chose $(x, t) \in \Omega \times (T_{\max}, \infty)$. We define Ω_1 by applying Lemmas 1 and 1.4, and $\Omega' \Subset \Omega$ which contains Ω_1 and x. Choose $\varepsilon > 0$ such that $t > T_{\max} + \varepsilon$. Then, Lemma 1.2 implies

$$U(x, t) \geqslant C(t - T_{\max} - \varepsilon) \int_{\Omega'} f(\bar{u}(x))\, dx = \infty. \tag{1.22}$$

which is (ii).

Observe that under (H1) or (H2), we have

PROPOSITION 1.5. $\lim_{t \to T_{\max}} u(x, t) = \lim_{n \to \infty} u_n(x, T_{\max})$ *for all* x *in* Ω.

Proof. U is increasing in t, since u_n is for n large enough. Therefore if $t \in [0, T_{\max})$, $u(x, t) = U(x, t) \leqslant U(x, T_{\max})$ for all $x \in \Omega$. Taking the limit as $t \uparrow T_{\max}$, we have $\bar{u}(x) \leqslant U(x, T_{\max})$, for all x in Ω.

To prove the other inequality, we write

$$u_n(x, t) \leqslant u(x, t) \leqslant \bar{u}(x) \qquad \text{for all } n \in \mathbb{N}, \qquad \text{and all } (x, t) \in \Omega \times [0, T_{\max}), \tag{1.23}$$

We then take the limits $t \uparrow T_{\max}$ followed by $n \to \infty$.

The theorem is now proved for $f_n = [f]_n$ which gives a solution u_n. If z_n is the solution for f_n in the general case, we have:

$$u_p \leqslant \lim_{n \to \infty} z_n \leqslant u \qquad \text{for all } p \in \mathbb{N}. \tag{1.24}$$

Indeed, (f_n) is increasing in u and converges uniformly to f on every compact set by Dini's Theorem. Therefore for $\varepsilon > 0$, there exists n_0 such that if $n \geqslant n_0$ and p is fixed:

$$f_n(u) \geqslant [f(u)]_p - \varepsilon. \tag{1.25}$$

Since u_p and z_n are solutions of (P_p) and (P_n) with f_p replaced by $[f]_p$ for u_p, we have

$$(u_p - z_n)_t - \Delta(u_p - z_n) = [f(u_p)]_p - f_n(z_n). \tag{1.26}$$

Multiplication by $(u_p - z_n)^+$ and integration over Ω yields

$$d/dt \, \|(u_p - z_n)^+(t)\|_2^2 \leqslant K_p \, \|(u_p - z_n)^+(t)\|_2^2 + C\varepsilon \, \|(u_p - z_n)^+(t)\|_2. \tag{1.27}$$

Where C is a constant depending only on Ω. By Gronwall's Lemma,

$$\|(u_p - z_n)^+(t)\|_2^2 \leqslant C\varepsilon T \, \|(u_p - z_n)^+\|_{L^\infty(O,T;L^2(\Omega))} e^{K_p T} \quad \text{for all } t \in [0, T]. \tag{1.28}$$

and hence

$$\begin{aligned} &\|(u_p - z_n)^+\|_{L^\infty(0,T;L^2(\Omega))} \leqslant C\varepsilon T e^{K_p T}, \\ &\lim_{n \to \infty} \|(u_p - z_n)^+(t)\|_2 = 0 \qquad \text{uniformly for } t \in [0, T]. \end{aligned} \tag{1.29}$$

Therefore,

$$\lim_{n \to \infty} z_n \geqslant u_p \qquad \text{for all } p \in \mathbb{N}, \quad t \in [0, \infty). \tag{1.30}$$

That is, (1.24). We then deduce (i) and (ii) for z_n. For Proposition 1.5, we only have the inequality obtained by (1.23).

Remark 1.2. If (f_n) is no longer assumed to be increasing in n, then the preceding proof remains valid with $\lim_{n \to \infty} z_n(x, t) = \infty$, for all (x, t) in $\Omega \times (T_{\max}, \infty)$.

Remark 1.3. A result of Weissler [19] permits us to extend hypothesis (H1) to $f(u) = u^{N/(N-2)}$ for $N \geqslant 3$, which is the limit power in (H1).

II. Case (H2)

In this part, we do not need Ω bounded except for Theorem 2.2 and our results hold for more general elliptic operators than Δ satisfying maximum principle.

PROPOSITION 2.1. *Let u_0 be a nonnegative measure on Ω then*

(i) *there exists a least integral solution U of* (P) *that is whenever V is an integral solution of* (P) *we have*

$$V \geqslant U \quad \text{a.e. on } \Omega \times (0, \infty) \quad \text{and so} \quad T^*(V) \leqslant T^*(U).$$

(ii) *If V is an integral solution of* (P) *then, $V \equiv +\infty$ on $\Omega \times (T^*(V), \infty)$.*

(iii) *If $u_0 \in L^\infty(\Omega)$, $\lim_{n \to \infty} u_n(x, t) = U(x, t)$ a.e. on $\Omega \times (0, \infty)$*

$$u(x, t) = U(x, t) \qquad \text{on} \quad \Omega \times [0, T_{\max}).$$

where u_n is the solution of (P_n) *and u the classical solution of* (P).

Proof. Let V be an integral solution of (P) and $(u_n^k)_{n,k \in \mathbb{N}}$ the sequence defined by:

$$\begin{aligned} u_{nt}^k - \Delta u_n^k &= f_n(u_n^{k-1}) && \text{on} \quad \Omega \times (0, \infty), \\ u_n^k &= 0 && \text{on} \quad \partial\Omega \times (0, \infty), \\ u_n^k(x, 0) &= u_0(x) && \text{for a.e. } x \text{ in } \Omega. \end{aligned}$$

and $u_n^0 \equiv 0$ on $\Omega \times (0, \infty)$. We see by recurrence:

$$u_n^{k-1} \leqslant u_n^k \leqslant u_{n+1}^k \leqslant V \qquad \text{on} \quad \Omega \times (0, \infty).$$

The uniqueness of the solution for (P_n) implies $\lim_{k \to \infty} u_n^k = u_n$ thus $u_n \leqslant V$ on $\Omega \times (0, \infty)$. Taking the limit in n, we obtain $\lim_{n \to \infty} u_n(x, t) \leqslant V(x, t)$ for a.e. (x, t) in $\Omega \times (0, \infty)$.

On the other hand, u_n satisfies

$$u_n(x, t) = \int_\Omega G(t, x, y)\, u_0(y)\, dy + \int_0^t \int_\Omega G(t-s, x, y) f_n(u_n(y, s))\, dy\, ds.$$

By monotone convergence theorem, we deduce that $U = \lim_n u_n$ is an integral solution of (P) which satisfies $U \leqslant V$ whatever V.

To prove the second point, let t_0 be such that there exists x_0 in Ω with $V(x_0, t_0) < +\infty$. The definition of an integral solution then implies

$$\int_0^{t_0} \int_\Omega G(t_0 - s, x_0, y) f(V(y, s))\, dy\, ds < \infty$$

from which we deduce

$$\int_0^t \int_\Omega G(t-s, x, y) f(V(y, s))\, dy\, ds < \infty \qquad \text{for a.e. } (x, t) \quad \text{in} \quad \Omega \times [0, t_0),$$

so V is finite a.e. on $\Omega \times [0, t_0)$, and $T^*(V) \geqslant t_0$ which proves (ii). The third point is immediate.

Consider the problem

$$\begin{aligned} u_{\lambda t} - \Delta u_\lambda &= f(u_\lambda) && \text{on} \quad \Omega \times (0, T), \\ u_\lambda &= 0 && \text{on} \quad \partial\Omega \times (0\ T), \\ u_\lambda(x, 0) &= \lambda u_0(x) && \text{for all } x \text{ in } \Omega. \end{aligned} \tag{P_λ}$$

Let U_λ be the least integral solution of (P_λ) and $T^*(\lambda) = T^*(U_\lambda)$.

LEMMA 2.1. Suppose (h). Let u_0 be a nonnegative bounded measure on Ω and suppose there exists $\lambda > 1$ such that $T^*(\lambda) > 0$ then:

$$U(x, t) \leqslant (\lambda/(\lambda^{\gamma-1} - 1)^{1/(\gamma-1)})(S(t)\, u_0(x) + a)$$
$$\textit{for all } (x, t) \textit{ in } \Omega \times (0, T^*(\lambda)), \tag{2.1}$$

where U is the least integral solution of $(P) = (P_1)$.

Recall that γ and a are the constants given in the hypothesis (h) and that $S(t)\, u_0$ denotes the unique solution of:

$$\begin{aligned} V_t - \Delta V &= 0 && \text{on} \quad \Omega \times (0, \infty)\ v \in \mathscr{C}^{2,1}(\Omega \times (0, T)), \\ V &= 0 && \text{on} \quad \partial\Omega \times (0, \infty), \\ \lim_{t \to 0} \int_\Omega V(t)\Phi &= \int_\Omega \Phi u_0(dx) && \text{for all } \Phi \text{ in } \mathscr{C}(\bar{\Omega}). \end{aligned}$$

Proof. First, suppose $u_0 \in \mathscr{C}(\bar{\Omega})$. Let u_λ^n be the sequence given by

$$u_\lambda^0 \equiv 0 \quad \text{on} \quad \Omega \times [0, T],$$
$$u_\lambda^n \in \mathscr{C}^{2,1}(\bar{\Omega} \times (0, T)) \cap \mathscr{C}(\Omega \times [0, T]),$$

$$\begin{aligned} u_{\lambda,t}^n - \Delta u_\lambda^n &= f(u_\lambda^{n-1}) && \text{on} \quad \Omega \times (0, T), \\ u_\lambda^n &= 0 && \text{on} \quad \partial\Omega \times (0, T), \\ u_\lambda^n(x, 0) &= \lambda u_0(x) && \text{for all } x \text{ in } \Omega, \end{aligned} \tag{2.2}$$

where $T = T^*(\lambda)$. We see by recurrence,

$$\begin{aligned} &0 \leqslant u_\lambda^n \leqslant u_\lambda^{n+1} \leqslant U_\lambda && \text{on} \quad \Omega \times (0, T), \\ &\lambda u_1^n \leqslant u_\lambda^n && \text{on} \quad \Omega \times (0, T) \quad \text{for all } \lambda \geqslant 1. \end{aligned} \tag{2.3}$$

For $m \in \mathbb{N}$ and $\mu \geqslant 1$, we define

$$E_\mu^m = \{(x, t) \in \Omega \times (0, T);\ u_1^m(x, t) > \mu\Phi(x, t)\},$$

where $\Phi(x, t) = S(t)\, u_0(x) + a$, and

$$g_n^m(\mu) = \inf_{(x,t) \in E_\mu^m} \frac{u_\lambda^n(x, t)}{u_1^m(x, t)},$$

$$w(x, t) = u_\lambda^{n+1}(x, t) - g_n^m(\mu)^\gamma u_1^m(x, t) + \lambda(g_n^m(\mu)^\gamma - g_{n+1}^m(\mu))\, \Phi(x, t);$$

w belongs to $\mathscr{C}(\Omega \times [0, T]) \cap \mathscr{C}^{2,1}(\bar{\Omega} \times (0, T])$ and for $n \geqslant m > 1$, $\lambda > 0$ we have

$$w_t - \Delta w = f(u_\lambda^m) - g_n^m(\mu)^\gamma f(u_1^{m-1}) \qquad \text{on} \quad E_\mu^m,$$

$$w \geqslant g_{n+1}^m(\mu)\, u_1^m - g_n^m(\mu)^\gamma\, u_1^m + \lambda(g_n^m(\mu)^\gamma - g_{n+1}^m(\mu))\, \Phi \qquad \text{in} \quad E_\mu^m$$

we deduce from (2.3),

$$g_n^m(\mu) \geqslant \lambda > 1 \qquad \text{for all } \mu \geqslant 1 \tag{2.4}$$

and from (h):

$$f(u_\lambda^n) \geqslant f(g_n^m(\mu)\, u_1^m) \geqslant g_n^m(\mu)^\gamma f(u_1^m) \qquad \text{in} \quad E_\mu^m.$$

We obtain with (2.3)

$$w_t - \Delta w \geqslant 0 \qquad \text{on} \quad E_\mu^m.$$

Since $u_1^m = \mu\Phi$ on $\partial E_\mu^m \backslash (\Omega \times \{T\})$, we have

$$w \geqslant 0 \qquad \text{on} \quad \partial E_\mu^m \backslash (\Omega \times \{T\}).$$

we deduce from the maximum principle that $w \geqslant 0$ in E_μ^m.

For $\mu' \geqslant \mu$ we have $E_{\mu'}^m \subset E_\mu^m$ and

$$\Phi(x, t) < (1/\mu')\, u_1^m(x, t) \qquad \text{for all } (x, t) \quad \text{in } E_{\mu'}^m.$$

$w \geqslant 0$ on E_μ^m then implies:

$$g_{n+1}^m(\mu') \geqslant g_n^m(\mu)^\gamma - (g_n^m(\mu)^\gamma - g_{n+1}^m(\mu))\, \mu/\mu'.$$

For all μ, m such that $E_\mu^m \neq \varnothing$, $\{g_n^m(\mu)\}_{n \in \mathbb{N}}$ is a nondecreasing sequence bounded by $\inf_{E_\mu^m} U_\lambda / u_\lambda^m$ which is finite because $T = T^*(\lambda)$. Its limit $g^m(\mu)$ satisfies:

$$g^m(\mu') \geqslant g^m(\mu)^\gamma - (g^m(\mu)^\gamma - g^m(\mu))\, \mu/\mu',$$

hence,

$$\frac{g^m(\mu') - g^m(\mu)}{g^m(\mu)^\gamma - g^m(\mu)} \geqslant \frac{\mu' - \mu}{\mu'} \qquad \text{for all } \mu' \geqslant \mu > 1,$$

so

$$\mathrm{Log}(\mu/\mu_0) \leqslant \int_{g(\mu_0)}^{\infty} \frac{d\sigma}{\sigma^{\gamma} - \sigma} \qquad \text{for all } \mu \geqslant \mu_0 > 1.$$

Since (2.4) implies $g^m(\mu_0) \geqslant \lambda$, we obtain that if μ satisfies

$$\mu > \frac{\lambda}{(\lambda^{\gamma-1} - 1)^{1/(\gamma-1)}},$$

then necessarily $E^m(\mu) = \varnothing$. Thus

$$u_1^m(x, t) \leqslant \frac{\lambda}{(\lambda^{\gamma-1} - 1)^{1/(\gamma-1)}} \Phi(x, t) \qquad \text{for all } (x, t) \quad \text{in} \quad \Omega \times (0, T).$$

Taking the limit in (2.3), we obtain

$$\lim_{m \to \infty} u_\lambda^m \leqslant U_\lambda$$

However, by using the monotone convergence Theorem, $\lim_m u_\lambda^m$ is an integral solution of (P_λ), so we have $U_\lambda = \lim u_\lambda^m$ whatever $\lambda \geqslant 1$ (Proposition 2.1 (i)). We deduce $U = \lim_{m \to \infty} u_1^m$, we have proved (2.1) when $u_0 \in \mathscr{C}(\bar{\Omega})$.

When u_0 is a nonnegative bounded measure on Ω, we easily verify that (2.3) holds. Taking the limit in the second inequality, we obtain: $\lambda U \leqslant U_\lambda$ on $\Omega \times (0, T)$ for $\lambda \geqslant 1$.

Take the origin at $t = \varepsilon > 0$ and the initial data u_ε equal to $S(\varepsilon) u_0$. We have $u_\varepsilon \in \mathscr{C}(\bar{\Omega})$ and we deduce from the above inequality that

$$\begin{aligned} u_t - \Delta u &= f(u) && \text{on} \quad \Omega \times (\varepsilon, T) \\ u &= 0 && \text{on} \quad \partial\Omega \times (\varepsilon, T) \\ u(x, \varepsilon) &= \lambda u_\varepsilon(x) && \text{for all } x \in \Omega \end{aligned} \qquad (P_{\varepsilon,\lambda})$$

has an integral solution $U_\lambda^\varepsilon \leqslant U_\lambda$. By applying the lemma we obtain:

$$U^\varepsilon(x, t) \leqslant \frac{\lambda}{(\lambda^{\gamma-1} - 1)^{1/(\gamma-1)}} (S(t - \varepsilon)(S(\varepsilon) u_0)(x) + a)$$

$$\text{for all } (x, t) \quad \text{in } \Omega \times (\varepsilon, T^*(\lambda)),$$

where U^ε is the least integral solution of $(P_{\varepsilon,1})$. Observe that

$$U^\varepsilon(x, \varepsilon' - \varepsilon) \geqslant (S(\varepsilon') u_0)(x) \qquad \text{for} \quad \varepsilon' > \varepsilon \qquad \text{and} \qquad x \text{ in } \Omega.$$

Thus

$$U^{\varepsilon}(x,t) \geqslant U^{\varepsilon'}(x,t) \qquad \text{for} \quad \varepsilon' > \varepsilon \qquad \text{and} \qquad (x,t) \quad \text{in}\, \Omega \times (\varepsilon', \infty).$$

We deduce that $\lim_{\varepsilon \downarrow 0} U^{\varepsilon} = U$ and, taking the limit, we obtain Lemma 2.1.

$T^*(\lambda)$ is a nonincreasing function. Let $T^*(\lambda^+)$ (resp. $T^*(\lambda^-)$) be the right limit (resp. left limit) of $T^*(\lambda)$. We can easily see that

$$T^*(\lambda) = T^*(\lambda^-) \geqslant T^*(\lambda^+).$$

When u_0 belongs to $L^{\infty}(\Omega)$, we define $T_{\max}(\lambda)$ as the maximal time of existence of the classical[1] solution of (P_λ). We have $T_{\max}(\lambda) \leqslant T^*(\lambda)$. We see later that $T_{\max}(\lambda) = T^*(\lambda^+)$. However, we can already deduce from Lemma 2.1 the following remarks:

Remark 2.1. Suppose u_0 is a bounded nonnegative measure then

$$U \in \mathscr{C}^{2,1}(\bar{\Omega} \times (0, T^*(1^+))).$$

Proof. For every $T < T^*(1^+)$, we can find λ_0 such that $T^*(\lambda_0) \in (T, T^*(1^+))$ and $\lambda_0 > 1$. By applying (2.1), we deduce that U belongs to $L^{\infty}_{\text{loc}}((0, T) \times L^{\infty}(\Omega))$ and so with standard bootstrap argument, U belongs to $\mathscr{C}^{2,1}(\bar{\Omega} \times (0, T^*(1^+)))$.

Remark 2.2. Suppose u_0 in $L^{\infty}(\Omega)$ $u_0 \geqslant 0$, we deduce from Remark 2.1 that $T_{\max} = T_{\max}(1) \geqslant T^*(1^+)$ and that U, the least integral solution, is the classical solution on $(0, T^*(1^+))$ (see Proposition 2.1 (iii) before).

Remark 2.3. Suppose u_0 in $L^1_{\text{loc}}(\Omega)$, $u_0 \geqslant 0$. Then U is the limit of an increasing sequence of classical solutions on $(0, T^*(1))$ of the problem (P).

Proof. Take $u_{0_n} = (1 - 1/n) \inf(u_0, n)$ and call U_n the least integral solution of (P) with initial data u_{0_n}. We deduce from Lemma 2.1 and a bootstrap argument as in Remark 2.1 that U_n is a classical solution on $(0, T^*(1))$. We deduce from Proposition 2.1 that $U(x,t) = \lim_{n \to \infty} U_n(x,t)$ for all (x,t) in $\Omega \times [0, T^*(1)]$.

Remark 2.4. Suppose that (P) has a global solution for all nonnegative function u_0 such that $\sup(|u_0|_{\infty}, |u_0|_1)$ is small enough and $a = 0$ in (h) if Ω is unbounded. Then if a bounded nonnegative measure u_0 is such that (P) has a local integral solution (i.e., $T^*(1) > 0$) there exists $\lambda > 0$ such that (P_λ) has a global solution.

Proof. Choose $\lambda_0 < 1$, we deduce from Lemma 2.1 that $U_{\lambda_0}(x,t) \leqslant (1/(1 - \lambda_0^{\gamma-1}))^{1/(\gamma-1)}((S(t)\, u_0)(x) + a)$ for all (x,t) in $\Omega \times (0, T^*(1))$. By

[1] When $u_0 \in L^{\infty}(\Omega)$, u is the classical solution of (P) on $(0, T)$ if $u \in \mathscr{C}^{2,1}(\bar{\Omega} \times (0, T)) \cap L^{\infty}(\Omega \times (0, T))$ and $\lim_{t \to 0} u(x,t) = u_0(x)$ for a.e. $x \in \Omega$.

using the same construction as in (2.3) of Lemma 2.1, we see that for $\lambda \leqslant \lambda_0$, $U_\lambda \leqslant (\lambda/\lambda_0)\, U_{\lambda_0}$ on $\Omega \times (0, \infty)$. Choose $t_0 \in (0, T^*(1))$, we deduce from these two inequalities the existence of $\lambda > 0$ such that $\sup(|U_\lambda(\cdot, t_0)|_{+\infty}, |U_\lambda(\cdot, t_0)|_1)$ is small enough.

Remark 2.5. Lemma 2.1 is valid for all u_0 nonnegative measure (not necessarily bounded). Indeed, if K_n is a sequence of compact subsets of Ω increasing to Ω, we can apply (2.1) with $u_0 \chi_{K_n}$ and take the limit.

Let f^* be the conjugate function of f, that is

$$f^*(r) = \sup_{\alpha \geqslant 0} (r\alpha - f(\alpha)).$$

We can improve the necessary condition for the existence of an integral solution of (P) given in [5].

LEMMA 2.2. *Suppose that f satisfies* (h). *Let u_0 be a nonnegative measure on Ω. If* (P) *has an integral solution U such that $T^*(U) \geqslant T$ then*

$$\int_\Omega \xi(0)\, u_0 \leqslant \int_{\Omega \times (0, T)} f^*(h/\xi)\, \xi \chi_{(h>0)}\, dx\, dt \tag{2.5}$$

for all (h, ξ) such that

$$\begin{aligned} h \in L^1(\Omega \times (0, T)) \qquad & h \geqslant 0 \quad \text{on} \quad \Omega \times (0, T), \\ -\xi_t - \Delta\xi = h \qquad & \text{in} \quad \Omega \times (0, T), \\ \xi = 0 \qquad & \text{on} \quad \partial\Omega \times (0, T), \\ \xi(T) = 0 \qquad & \text{in} \quad \Omega, \end{aligned} \tag{2.6}$$

where $\chi_E(x, t) = 0$ if $(x, t) \notin E$, $\chi_E(x, t) = 1$ if $(x, t) \in E$. Let us recall that (2.6) *is equivalent to*

$$\xi(x, t) = \int_{\Omega \times (0,T)} G(s - t, y, x)\, h(y, s)\, dy\, ds. \tag{2.6 bis}$$

Proof. Suppose first u_0 in $L^1_{\text{loc}}(\Omega)$. Let u_n be the sequence given by Remark 2.3. Multiply by ξ and integrate the equation satisfied by u_n, we obtain

$$\int_{\Omega \times (0,T)} u_n h = \int_{\Omega \times (0,T)} f(u_n)\, \xi + \int_\Omega u_{0n} \xi(0).$$

Hence

$$\int_\Omega u_{0n}\xi(0)\leqslant\int_{\Omega\times(0,T)}(u_n(h/\xi)-f(u_n))\,\xi\chi_{\{h>0\}}$$

$$\leqslant\int_{\Omega\times(0,T)}f^*(h/\xi)\,\xi\chi_{\{h>0\}}.$$

Take the limit to obtain (2.5).

If u_0 is a nonnegative measure, for $\varepsilon>0$, $U(\cdot,\varepsilon)$ belongs to $L^1_{\text{loc}}(\Omega)$ and we can apply (2.5) on $\Omega\times(\varepsilon,T)$:

$$\int_\Omega \xi(x,\varepsilon)\,U(x,\varepsilon)\,dx\leqslant\int_{\Omega\times(\varepsilon,T)}f^*(h/\xi)\,\xi\chi_{\{h>0\}},$$

but

$$\int_\Omega U(x,\varepsilon)\,\xi(x,\varepsilon)\,dx=\int_\Omega U(x,\varepsilon)\left(\int_\varepsilon^T S(s-\varepsilon)\,h(s)\,ds\right)(x)\,dx$$

$$\geqslant\int_\Omega (S(\varepsilon)\,u_0(x)\left(\int_\varepsilon S(s-\varepsilon)\,h(s)\,ds\right)(x)\,dx$$

$$\geqslant\int_\varepsilon\int_\Omega\left(S(s)\,u_0)(x)\,h(x,s)\,dx\,ds.\right.$$

We then deduce, taking the limit:

$$\int_{\Omega\times(0,T)}h(x,s)(S(s)\,u_0)(x)\,dx\,ds\leqslant\int_{\Omega\times(0,T)}f^*(h/\xi)\,\xi\chi_{\{h>0\}}$$

which is equivalent to (2.5).

The necessary condition (2.5) leads us to define:

$$X=\{h\in L^1(\Omega\times(0,T)),\,h\geqslant 0, f^*(h/\xi)\,\xi\chi_{\{h>0\}}\in L^1(\Omega\times(0,T)),$$
$$\text{where } \xi \text{ is given by (2.6)}\}.$$

and for a nonnegative measure u_0:

$$|u_0|_T=\sup_{\substack{h\in X\\ h\not\equiv 0}}\left\{\int_\Omega \xi(0)\,u_0\Big/\int_{\Omega\times(0,T)}f^*(h/\xi)\,\xi\chi_{\{h>0\}}\right\}.$$

Equation (2.5) becomes $|u_0|_T\leqslant 1$. It is also a sufficient condition which ensures the existence of an integral solution U of (P) such that $T^*(U)\geqslant T$. Indeed, we have

THEOREM 2.1. *Suppose that f satisfies* (h). *Let u_0 be a nonnegative measure on Ω and $T>0$.* (P) *has an integral solution U such that $T^*(U)\geqslant T$ if and only if*

$$|u_0|_T\leqslant 1 \tag{2.7}$$

Proof. We have proved the necessity in Lemma 2.2. First, observe that (h) implies the existence of two constants c_1, c_2 such that

$$f^*(r)\leqslant c_1 r^{\gamma'}+c_2 r, \qquad \forall r\geqslant 0, \tag{2.8}$$

where $1/\gamma'+1/\gamma=1$.

We apply Theorem 2.1 of [5]. Equation (2.5) implies the condition (11) of [5], so we have to prove that the solution provided by this Theorem is an integral solution U of (P) such that $T^*(U)\geqslant T$. To do this, we deduce from (2.8) as in Section III (2°) of [5] that whatever $K\Subset\Omega$ and $0<T_1<T$, the space $\hat{X}$ of [5] contains a function which is positive on $K\times(0,T)$. We know that $U\cdot h\in L^1(\Omega\times(0,T))$ for all h in $\hat{X}$, so we obtain that U belongs to $L^1_{\text{loc}}(\Omega\times(0,T))$, hence $T^*(U)\geqslant T$.

COROLLARY 2.1. *Suppose that f satisfies* (h). *Let u_0 be a nonnegative bounded measure on Ω, then*

(i) *$T\to|u_0|_T$ is a nondecreasing continuous function,*

(ii) $|u_0|_T=1\Leftrightarrow T\in[T^*(1^+), T^*(1^-)]$.

Proof. (i) $T\to|u_0|_T$ nondecreasing is a consequence of the definition of $|u_0|_T$. Let $\lambda_\pm$ be defined by

$$1/\lambda_+=\lim_{T\downarrow T_0}|u_0|_T, \qquad 1/\lambda_-=\lim_{T\uparrow T_0}|u_0|_T,$$

we have to prove that $\lambda_+=\lambda_-$.

First, observe that $\lambda_-|u_0|_T\leqslant 1$ for all $T<T_0$, Theorem 2.1 implies that (P_{λ_-}) has an integral solution U_{λ_-} such that $T^*(U_{\lambda_-})\geqslant T$ for all $T<T_0$, so we have $T^*(U_{\lambda_-})\geqslant T_0$. Thus, there exists an integral solution U_λ such that $T^*(U_\lambda)\geqslant T_0$ for all $\lambda\in[\lambda_+,\lambda_-]$. Suppose $\lambda_+<\lambda_-$ and let λ_0 be such that $\lambda_+<\lambda_0<\lambda_-$. We deduce from Lemma 2.1 that U_{λ_0} belongs to $L^\infty(\Omega\times(T_0/2,T_0))$ and so can be extended on $(0,T_1)$ for some $T_1>T_0$ hence $|\lambda_0u_0|_{T_1}\leqslant 1$, but

$$|\lambda_0u_0|_{T_1}\geqslant\lambda_0\lim_{T\downarrow T_0}|u_0|_T\geqslant\lambda_0/\lambda_+>1$$

we obtain a contradiction.

To prove (ii), first observe that

$$T \leqslant T^*(1^-) = T^*(1) \Leftrightarrow |u_0|_T \leqslant 1.$$

Let λ, T be such that $\lambda > 1$, $T > T^*(\lambda)$, we obtain

$$|\lambda u_0|_T > 1$$

and thus, for $T > T^*(1^+)$ we have $|u_0|_T \geqslant 1$; and so, for $T > T^*(1^+)$ we have $|u_0|_T \geqslant 1$.

We obtain that $T \in (T^*(1^+), T^*(1^-))$ implies $|u_0|_T = 1$. By using the continuity of $T \to |u_0|_T$, we obtain that $T \in [T^*(1^+), T^*(1^-)]$ implies $|u_0|_T = 1$.

Suppose now $|u_0|_T = 1$. We have immediately $T \leqslant T^*(1)$ and for $\lambda > 1$ $|u_0|_{T^*(\lambda)} = 1/\lambda < 1$ implies $T > T^*(\lambda)$ and thus $T \geqslant T^*(1^+)$.

COROLLARY 2.2. (i) *Let u_0 be a nonnegative bounded measure such that $u_0 \not\equiv 0$, and $T > 0$, then there exists $\lambda > 0$ such that $T^*(\lambda) \leqslant T$.*

(ii) *Let $u_0 \in L^\infty(\Omega)$, $u_0 \geqslant 0$ and $T > 0$, then there exists $\lambda > 0$ such that $T_{\max}(\lambda) \leqslant T$.*

This corollary implies that there does not exist any nonnegative initial data such that (P_λ) has a global solution for all $\lambda > 0$ (classical or integral).

Proof. (i) implies (ii) because $T_{\max}(\lambda) \leqslant T^*(\lambda)$. We have $|\lambda u_0|_T = \lambda\,|u_0|_T$ and so for $\lambda > 1/|u_0|_T$, Theorem 2.1 implies that $T^*(\lambda) < T$.

If $|u_0|_T = 1$, the question which arises is does there exist a $h \not\equiv 0$ which realizes the equality in (2.5)? We have the following result:

THEOREM 2.2. *Suppose* (h), *Ω bounded. Let $u_0 \in L^\infty(\Omega)$, $u_0 \geqslant 0$, and T be such that $|u_0|_T = 1$. Let U be the least integral solution of* (P). *There exists ξ^* such that:*

(i) *$\xi^* \geqslant 0$ in $\Omega \times (0, T)$, $\xi^* \not\equiv 0$,*

(ii) *$f'(U)\,\xi^*$ and $Uf'(U)\,\xi^*$ belong to $L^1(\Omega \times (0, T))$,*

(iii)

$$\begin{aligned} -\xi^*_t - \Delta\xi^* &= f'(U)\,\xi^* && \text{on} \quad \Omega \times (0, T), \\ \xi^* &= 0 && \text{on} \quad \partial\Omega \times (0, T), \\ \xi^*(T) &= 0 && \text{on} \quad \Omega, \end{aligned} \tag{2.9}$$

and

(iv) $\int_\Omega u_0 \xi^*(0) = \int_{\Omega \times (0,T)} f^*(h^*/\xi^*)\,\xi^* \chi_{\{h^* > 0\}} = 1,$

where $h^* = f'(U)\,\xi^*$.

To prove this theorem, we need some lemmas. For $g \in L^1_{\text{loc}}(\Omega \times (0, T))$, $g \geqslant 0$, we set

$$\|g\|_T = \sup_{\substack{h \in X \\ h \neq 0 \\ (\xi, h) \text{ verifying } (2.6)}} \left(\int_{\Omega \times (0,T)} gh \right) \Big/ \int_{\Omega \times (0,T)} f^*(h/\xi)\, \xi \chi_{\{h>0\}} \Big),$$

hence $\|S(t)\, u_0\|_T = |u_0|_T$.

LEMMA 2.3. *For $T > 0$, we have*

(i) *for all $\varepsilon > 0$, there exists $\eta > 0$ such that*

$$\left. \begin{array}{l} A \text{ measurable subset of } \Omega \times (0, T) \\ \text{meas}(A) < \eta \end{array} \right\} \Rightarrow \|\chi_A\|_T < \varepsilon$$

(ii) $\|\chi_{\Omega \times (0,T)}\|_T < \infty$.

Proof. Let A be a measurable subset of $\Omega \times (0, T)$. For $\varepsilon > 0$, consider the problem

$$u(x, t) = \int_0^t G(t-s, x, y) f(u(y, s))\, dy\, ds + 1/\varepsilon \chi_A(x, t). \tag{P$_\varepsilon$}$$

A necessary condition for the existence of a nonnegative bounded solution of (P$_\varepsilon$) is

$$\|\chi_A\|_T \leqslant \varepsilon. \tag{2.10}$$

Indeed, let h be a function of X and ξ the solution of (2.6). Multiply (P$_\varepsilon$) by h and integrate, we obtain

$$\int_{\Omega \times (0,T)} uh = \int_{\Omega \times (0,T)} h(x, t) \int_0^t \int_\Omega G(t-s, x, y) f(u(y, s))\, dy\, ds\, dx\, dt$$

$$+ 1/\varepsilon \int_{\Omega \times (0,T)} \chi_A h.$$

Hence by Fubini Theorem, we have

$$\int_{\Omega \times (0,T)} h(x, t) \int_0 G(t-s, x, y) f(u(y, s))\, dy\, ds\, dx\, dt$$

$$= \int_{\Omega \times (0,T)} f(u)(y, s)\, \xi(y, s)\, dy\, ds$$

and so

$$(1/\varepsilon)\int_{\Omega\times(0,T)}\chi_A h=\int_{\Omega\times(0,T)} uh-f(u)\,\xi\leqslant\int_{\Omega\times(0,T)} f^*(h/\xi)\,\xi\chi_{\{h>0\}}.$$

To prove that (2.10) holds as soon as A is suitable, we show that (P_ε) has a bounded solution. To do this, it is sufficient to find a bounded upper solution of (P_ε) on $\Omega\times(0,T)$.

Let C_0 be such that

$$C_0>0,\qquad \int_{2C_0}^{+\infty}(1/f(\sigma))\,d\sigma>T.$$

Let $C(t)$ be the solution of:

$$C'(t)=1/2f(2C(t)),\qquad C(0)=C_0. \tag{2.11}$$

Verify that $w(x,t)=(1/\varepsilon)\,\chi_A(x,t)+C(t)$ is a bounded upper solution of (P_ε). Since F is convex, we have

$$\begin{aligned}\int_0^t\int_\Omega & G(t-s,x,y)f(w(y,s))\,dy\,ds+(1/\varepsilon)\,\chi_A\\ &\leqslant 1/2\int_0^t\int_\Omega G(t-s,x,y)f((2/\varepsilon)\,\chi_A(y,s))\,dy\,ds\\ &\quad+1/2\int_0^t\int_\Omega G(t-s,x,y)f(2C(s))\,ds\,dy+w(x,t)-C(t).\end{aligned}$$

We have

$$\int_0^t\int_\Omega G(t-s,x,y)f(2C(s))\,ds\,dy\leqslant\int_0^t f(2C(s))\,ds$$

and

$$\begin{aligned}\int_0^t\int_\Omega & G(t-s,x,y)f((2/\varepsilon)\,\chi_A(y,s))\,dy\,ds\\ &=f(2/\varepsilon)\int_0^t\int_\Omega G(t-s,x,y)\,\chi_A(y,s))\,dy\,ds.\end{aligned}$$

w is an upper solution if

$$1/2f(2/\varepsilon)\int_0^t\int_\Omega G(t-s,x,y)\,\chi_A(y,s))\,dy\,ds+1/2\int_0^t f(2C(s))\,ds\leqslant C(t)$$

with (2. 1), we obtain

$$\int_0^t \int_\Omega G(t-s, x, y)\, \chi_A(y, s))\, dy\, ds \leqslant (2C_0/f(2/\varepsilon)). \tag{2.12}$$

To prove (ii), choose $A = \Omega \times (0, T)$. The left-hand side of (2.12) is bounded on $\Omega \times (0, T)$ and for ε large enough, (2.12) will be satisfied on $\Omega \times (0, T)$.

To prove (i), observe that (2.12) is equivalent to

$$\int_A \left(\int_s^T \int_\Omega G(t-s, x, y)\, \varphi(x, t)\, dx\, dt \right) dy\, ds \leqslant (2C_0/f(2/\varepsilon))$$

for all $\varphi \in L^1(\Omega \times (0, T))$, $\varphi \geqslant 0$ and $\int_{\Omega \times (0,T)} \varphi(x, t)\, dx\, dt = 1$.

By Dunford–Pettis Theorem, (i) is then equivalent to the relative weak compactness in $L^1(\Omega \times (0, T))$ of the subset:

$$\left\{ g(y, s) = \int_s^T \int_\Omega G(t-s, x, y)\, \varphi(x, t)\, dx\, dt,\ \varphi \in L^1(\Omega \times (0, T)), \right.$$

$$\left. \varphi \geqslant 0 \text{ and } \int_{\Omega \times (0,T)} \varphi = 1 \right\}$$

which is a consequence of the compactness of the operator $\varphi \to g$,

where g is the solution of
$$\begin{aligned} -g_t - \Delta g &= \varphi && \text{on } \Omega \times (0, T), \\ g &= 0 && \text{on } \partial\Omega \times (0, T), \\ g(x, T) &= 0 && \text{for all } x \text{ in } \Omega, \end{aligned}$$

from $L^1(\Omega \times (0, T))$ to $L^1(\Omega \times (0, T))$.

LEMMA 2.4. (i) *$h \to (f^*(h/\xi)\, \xi\chi_{\{h>0\}}(x, t)$ where (ξ, h) satisfies (2.6) is a convex function from X to $\mathbb{R}^+$ for a.e. (x, t) in $\Omega \times (0, T)$.*

(ii) *$h \to \int_{\Omega \times (0,T)} f^*(h/\xi)\, \xi\chi_{\{h>0\}}(x, t)\, dx\, dt$*

is a lower semi continuous function on X with L^1 norm.

Proof. Let (h_i/ξ_i) $i = 1, 2$ be two pairs of function satisfying (2.6). Using the convexity of f^*, we obtain

$$\begin{aligned} &f^*((h_1+h_2)/(\xi_1+\xi_2))(\xi_1+\xi_2) \\ &\quad = f^*((h_1/\xi_1)(\xi_1/(\xi_1+\xi_2)) + f^*(h_2/\xi_2)(\xi_2/(\xi_1+\xi_2)))(\xi_1+\xi_2) \\ &\quad \leqslant f^*(h_1/\xi_1)\, \xi_1 + f^*(h_2/\xi_2)\, \xi_2 \end{aligned}$$

and, for $\theta > 0$ and (h, ξ) satisfying (2.6), we have

$$f^*(\theta h/\theta\xi)\, \theta\xi = \theta(f^*(h/\xi)\, \xi)$$

the convexity is established.

We deduce the lower semicontinuity from Lebesgue and Fatou Theorems.

LEMMA 2.5. *Suppose* (h). *Then there exists* $K > 0$, $c > 1$ *and* $b \geqslant 0$ *such that*

$$f^*(cr) \leqslant Kf^*(r) \qquad \textit{for all } r \geqslant b. \tag{2.13}$$

Proof. Observe that (h) implies for all $x > 1$ and $\alpha \geqslant a$,

$$f(x\alpha) \geqslant x^{\gamma} f(\alpha)$$

we deduce

$$\text{for all } r \geqslant 0,\ \alpha \geqslant a, \qquad x^{\gamma} r\alpha - f(x\alpha) \leqslant x^{\gamma}(r\alpha - f(\alpha)).$$

Put $c = x^{\gamma - 1}$, we obtain

$$f^*(cr) \leqslant c^{\gamma/(\gamma - 1)} f^*(r)$$

for all r such that $f^*(r) = r\alpha f(\alpha)$ for some $\alpha \geqslant a$ that is $r \geqslant f'(a)$. (2.13) holds with $b = f(a)$, $K = c^{\gamma/(\gamma - 1)}$.

Proof of Theorem 2.2. Recall that we suppose $|u_0|_T = 1$ and $u_0 \in L^{\infty}(\Omega)$, $u_0 \geqslant 0$. Because $|u_0|_T = \|S(t)\, u_0\|_T = 1$, we can find a sequence $\{(h_n, \xi_n)\}_{n \in \mathbb{N}}$ such that (h_n, ξ_n) satisfies (2.6) and

$$\int_{\Omega \times (0, T)} f^*(h_n/\xi_n)\, \xi_n \chi_{\{h_n > 0\}} = 1,$$

$$\int_{\Omega \times (0, T)} S(t)\, u_0(x)\, h_n(x, t)\, dx\, dt \to 1 \qquad \text{when} \quad n \to \infty.$$

We deduce from Lemma 2.3(i) that for all $\varepsilon > 0$, there exists $\eta > 0$ such that meas $(U) < \eta$ implies $\int_U h_n < \varepsilon$ and from Lemma 2.3(ii).

$$\int_{\Omega \times (0, T)} h_n \leqslant \|\chi_{\Omega \times (0, T)}\|_T < \infty.$$

Since Ω is bounded, we may conclude from Dunford–Pettis Theorem that $\{h_n\}_{n \in \mathbb{N}}$ is weakly relatively compact in $L^1(\Omega \times (0, T))$. Let h^* be a weak limit of a subsequence.

Using the fact that u_0 is bounded, we have

$$\int_{\Omega \times (0,T)} S(t)\, u_0(x)\, h^*(x,t)\, dx\, dt = 1$$

which implies $h^* \not\equiv 0$. Lemma 2.4 then implies that $h^* \in X$ and

$$\int_{\Omega \times (0,T)} f^*(h^*/\xi^*)\, \xi^* \chi_{\{h^*>0\}} \leqslant 1,$$

where (ξ^*, h^*) satisfies (2.6). We deduce from $\|S(t)\, u_0\|_T = 1$ that

$$\int_{\Omega \times (0,T)} S(t)\, u_0(x)\, h^*(x,t)\, dx\, dt = \int_{\Omega \times (0,T)} f^*(h^*/\xi^*)\, \xi^* \chi_{\{h>0\}} = 1. \tag{2.14}$$

Let U be the least integral solution of (P). We claim that Uh belongs to $L^1(\Omega \times (0,T))$ for all h in X. Indeed, we deduce from (2.13) that $f^*(c(h/\xi))\, \xi\chi_{\{h>0\}}$ belongs to $L^1(\Omega \times (0,T))$ for all $h \in X$ and for some $c > 1$. Take U_n given by Remark 2.3, we have $U_n \in L^\infty(\Omega \times (0,T))$ and so for all (h, ξ) satisfying (2.6),

$$\int_{\Omega \times (0,T)} U_n h = \int_{\Omega \times (0,T)} f(U_n)\, \xi + \int_{\Omega \times (0,T)} S(t)\, u_{0n}(x)\, h(x,t)\, dx\, dt < \infty.$$

Thus

$$\int_{\Omega \times (0,T)} S(t)\, u_{0n}(x)\, h(x,t)\, dx\, dt + (c-1) \int_{\Omega \times (0,T)} U_n h = \int_{\Omega \times (0,T)} (U_n c(h/\xi) - f(U_n))\, \xi \leqslant \int_{\Omega \times (0,T)} f^*(ch/\xi)\, \xi\chi_{\{h>0\}}.$$

The right-hand side is finite as soon as $h \in X$. Taking the limit we obtain our assertion. In particular, we have $Uh^* \in L^1(\Omega \times (0,T))$ and we can rewrite (2.14).

$$\int_{\Omega \times (0,T)} Uh^* - f(U)\, \xi^* - f^*(h^*/\xi^*)\, \xi^* \chi_{\{h^*>0\}} = 0.$$

which proves that $Uh^* = f(U)\, \xi^* + f^*(h^*/\xi^*)\, \xi^*$ a.e. on $\Omega \times (0,T)$ from which we deduce

$$(h^*/\xi^*)(x,t) = f'(U(x,t)) \qquad \text{for a.e.} (x,t) \quad \text{in} \quad \{\xi^* > 0\},$$

and we have established (iii). $h^* \in X$ and $h^* \not\equiv 0$ imply (i), (ii) is an immediate consequence of h^* and $Uh^* \in L^1(\Omega \times (0, T))$ and (iv) is (2.14).

COROLLARY 2.3. *Under the hypotheses of Theorem* 2.2,

(i) $T_{\max} = T^*(1^+)$.

(ii) *There exists* ξ^* *satisfying* (i)–(iv) *of Theorem* 2.2 *and such that*

$$\operatorname{supp} \xi^* = \Omega \times [0, T_{\max}].$$

Proof. We know that $T_{\max} \geqslant T^*(1^+)$ and that U is equal to the classical solution on $\Omega \times (0, T^*(1^+))$ (Remark 2.2). Suppose $T_{\max} > T^*(1^+)$. U is then bounded on $\Omega \times (0, T^*(1^+))$. Corollary 2.1 implies that we may apply Theorem 2.2 with $T = T^*(1^+)$. We obtain the existence of a nontrivial and nonnegative solution of (2.9) with $f'(U)\,\xi^* \in L^1(\Omega \times (0, T^*(1^+))$ which is impossible because $f'(U) \in L^\infty(\Omega \times (0, T^*(1^+))$.

By using the same argument, we see that it is impossible that $\operatorname{supp} \xi^* \subset \Omega \times (0, T')$ with $T' < T_{\max}$ when ξ^* is a solution of (2.9).

Now, we can prove Theorem 2 under the assumption (H2).

Proof of Theorem 2 *under* (H2). Recall that we have to prove that whatever an integral solution V of (P), we have $T^*(V) \leqslant T_{\max}$. Using Proposition 2.1, it is sufficient to show that $T^*(U) \leqslant T_{\max}$ which means with our notation $T^*(1) \leqslant T_{\max}$. From Corollary 2.3, we see that Theorem 2.2 is then equivalent to $T^*(1^+) = T^*(1^-)$. Suppose not and choose $\varepsilon > 0$ such that $\varepsilon < T_{\max}$ and $\varepsilon < T^*(1^-) - T^*(1^+)$. We deduce from Lemma 1.1,

$$\int_\Omega u_0(x)\,\xi^*(x, 0)\,dx < \int_\Omega U(X, \varepsilon)\,\xi^*(x, 0)\,dx$$

(if the equality holds, u_0 would be a stationary solution and $T_{\max} = +\infty$) where ξ^* is a solution of (i)–(iv) and $\operatorname{supp} \xi^* = \Omega \times [0, T_{\max}]$. We deduce from the point (iv),

$$1 = |u_0|_{T\max} < |U(\varepsilon)|_{T\max}$$

But $U(t+\varepsilon)$ is an integral solution of (P) with initial data $U(\varepsilon)$ and the time of existence of this solution is more than $T_{\max}$. Thus, we deduce from Theorem 2.1, $|U(\varepsilon)|_{T\max} \leqslant 1$. We obtain a contradiction.

We can deduce from Theorem 2.2 the following uniqueness result.

COROLLARY 2.4. *Suppose* (h), Ω *bounded and* $u_0 \in L^\infty(\Omega)$, $u_0 \geqslant 0$. *Then for all integral solution of* (P) *such that* $T^*(V) \geqslant T_{\max}$, *we have* $V = u$ *on* $\Omega \times (0, T_{\max})$.

Proof. Let ξ^* be given by Corollary 2.3(ii). We easily see that for all t in $(0, T_{max})$,

$$\int_\Omega u(x, t)\,\xi^*(x, t)\,dx = \int_t^{T_{max}} \int_\Omega f^*(h^*/\xi^*)\,\xi^*\,dx\,dt.$$

For al integral solution V of (P), we have (Proposition 2.1),

$$u \leqslant V \qquad \text{on} \quad \Omega \times (0, T_{max}).$$

and if $T^*(V) \geqslant T_{max}$ we have $|V(\cdot, t)|_{T_{max}-t} \leqslant 1$ which implies

$$\int_\Omega V(x, t)\,\xi^*(x, t)\,dx \leqslant \int_t^{T_{max}} \int_\Omega f^*(h^*/\xi^*)\,\xi^*\,dx\,dt$$

hence $u = V$ a.e. on $\Omega \times (0, T_{max})$.

III. Case (H3)

We begin to prove the following result where we do not suppose (H3).

THEOREM 3.1. *Suppose* (h), Ω *bounded and* $u_0 \in L^\infty(\Omega)$, $u_0 \geqslant 0$. *Let U be the least integral solution of* (P) *then*:

(i) $Uf(U) \in L^2_{loc}([0, T^*); L^1(\Omega))$,

(ii) $U \in L^4_{loc}((0, T^*); H^1_0(\Omega)) \cap L^\infty_{loc}([0, T^*); L^2(\Omega))$,

(iii) $(dU/dt) \in L^2_{loc}((0, T^*); L^2(\Omega))$,

where $T^* = T^*(U)$.

Proof. First, we establish some a priori estimates on a classical solution of (P) on $(0, T)$. We have (see [1]):

$$\tfrac{1}{2}(d/dt)\,|u|_2^2 + |\nabla u|_2^2 = \int_\Omega f(u)\,u \tag{3.1}$$

and

$$|(du/dt)|_2^2 + (d/dt)\,E(u) = 0, \tag{3.2}$$

where $E(u) = 1/2\,|\nabla u|_2^2 - \int_\Omega F(u)$, $F(r) = \int_0^r f(s)\,ds$ and $|\ \ |_2$ is the norm of $L^2(\Omega)$.

(h) implies the existence of two constants c and a such that:

$$c>2, \quad a \geqslant 0, \quad \text{and} \qquad f(r)\, r \geqslant cF(r)-a \qquad \text{for all } r \geqslant 0.$$

Then, we deduce from (3.1)

$$\tfrac{1}{2}(d/dt)\, |u|_2^2 + 2E(u) \geqslant (c-2) \int_\Omega F(u) - a\, |\Omega|. \tag{3.3}$$

We deduce from the convexity of f that $\Phi(r) = F(\sqrt{r})$ is a convex function. Using (h), we see that

$$\Phi(r) \geqslant c_1 r^{((\gamma+1)/2)} \qquad \text{for all } r \geqslant r_0 \geqslant 0,$$

where c_1 and r_0 suitable positive constants. We deduce from Jensen inequality,

$$\left(\int_\Omega u^2\right)^{((\gamma+1)/2)} \leqslant [(|\Omega|^{((\gamma+1)/2)})/c_1]\, \Phi\left[(1/|\Omega|) \int_\Omega u^2\right]$$

$$\leqslant [(|\Omega|^{((\gamma-1)/2)})/c_1] \int_\Omega F(u)$$

for all t such that $\int_\Omega u^2 \geqslant r_0\, |\Omega|$. Then (3.3) implies

$$\tfrac{1}{2}(d/dt)\, |u|_2^2 \geqslant a\, |u|_2^{(\gamma+1)} - (2E(u) + a\, |\Omega|). \tag{3.4}$$

for all t such that: $\int_\Omega u^2 \geqslant r_0\, |\Omega|$ and for $a = (c-2)\, c_1/|\Omega|^{((\gamma-1)/2} > 0$.

Suppose

$$\begin{aligned} |u(t_0)|_2^2 &\geqslant r_0\, |\Omega|, \\ 2E(u(t_0)) + a\, |\Omega| &\leqslant 0. \end{aligned} \tag{3.5}$$

We deduce from (3.2) and (3.4) that (3.5) remains true for all $t \geqslant t_0$. Put $h(t) = |u|_2^2\, (t)$. Then (3.4) implies

$$h' \geqslant 2ah^{((\gamma+1)/2)}, \qquad h(t_0) = |u(t_0)|_2^2$$

and because we suppose that (P) has a classical solution on $(0, T)$, we obtain,

$$|u(t_0)|_2^2 \leqslant \{a(\gamma-1)(T-t_0)\}^{(-2/(\gamma-1))} = f_T(t_0).$$

Suppose now $|u(t_0)|_2^2 \geqslant \operatorname{Sup}(f_T(t_0), r_0\, |\Omega|)$, (3.5) is false, so we have

$$E(u(t_0)) \geqslant -(a/2)\, |\Omega|.$$

But we ceduce from (3.2),

$$|u(t_0)|_2 < |u(t)|_2 + (t_0 - t)^{1/2}(E(u(t)) - E(u(t_0)))^{1/2} \qquad \text{for all } t \leqslant t_0.$$

Hence:

$$|u(t_0)|_2^2 < 2\,|u(t)|_2^2 + (t_0 - t)(2E(u(t)) + a\,|\Omega|).$$

If we define

$$\psi(t_0) = \inf_{0 \leqslant t \leqslant t_0} (2\,|u(t)|_2^2 + (t_0 - t)(2E(u(t)) + a\,|\Omega|).$$

We have proved

$$|u(t_0)|_2^2 \leqslant A_T(t_0) = \operatorname{Max}\{r_0\,|\Omega|, f_T(t_0), \psi(t_0)\} \qquad \text{for all } t_0 \in (0, T). \quad (3.6)$$

By integration, we deduce from (3.3),

$$\tfrac{1}{2}\,|u(t_0)|_2^2 + \int_{t_1}^{t_0} (a\,|\Omega| + 2E(u)(s)\,ds \geqslant (c-2) \int_{t_1}^{t_0} \int_\Omega F(u)(s)\,ds$$

for all $T > t_0 > t_1 > 0$. Thus,

$$\int_{t_1}^{t_0} \int_\Omega F(u)(s)\,ds \leqslant (1/(c-2))(\tfrac{1}{2}A_T(t_0) + (t_0 - t_1)(a\,|\Omega|\;2E(u(t_1))))$$

$$\text{for all } t_1 \leqslant t_0 < t. \qquad (3.7)$$

Multiply (3.2) by $(t_0 - t)$ and integrate by parts. This yields

$$\int_{t_1}^{t_0} (t_0 - t)\,|du/dt|_2^2\,(t)\,dt = (t_0 - t_1)\,E(u(t_1)) - \int_{t_1}^{t_0} E(u(s))\,ds,$$

so, we have

$$\int_{t_1}^{t_0} (t_0 - t)\,|du/dt|_2^2\,(t)\,dt \leqslant (t_0 - t_1)\,E(u(t_1)) + \int_{t_1}^{t_0} \int_\Omega F(u)(s)\,ds,$$

$$\text{for all } t_1 \leqslant t_0 < t.$$

Using (3.7), we obtain

$$\int_{t_1}^{t_0} (t_0 - t)\,|du/dt|_2^2\,(t)\,dt$$

$$\leqslant (t_0 - t_1)\,a\,|\Omega| + (c/(c-2))\,E(u(t_1)) + (1/2(c-2))\,A_T(t_0). \quad (3.8)$$

We deduce from (3.3),

$$(c-2)\int_{\Omega} F(u) \leqslant a\,|\Omega| + 2E(u) + |u|_2\,|du/dt|_2.$$

and

$$(c-2)^2\int_{t_1}^{t_0}(t_0-t)\left(\int_{\Omega} F(u)(t)\right)^2 dt$$
$$\leqslant (t_0-t_1)(a\,|\Omega| + 2E(u(t_1)))^2$$
$$+\,2\sup_{(t_1,t_0)}(|u|_2^2)\int_{t_1}^{t_0}(t_0-t)\,|du/dt|_2^2\,(t)\,dt. \tag{3.9}$$

At last, we deduce from (3.2):

$$\tfrac{1}{2}\,|\nabla u(t)|_2^2 \leqslant \int_{\Omega}(F(u)(t) + E(u(t_1))) \qquad \text{for all } t \geqslant t_1 \geqslant 0. \tag{3.10}$$

Choose $t_1 \in (0, T)$ and $\varepsilon > 0$. We deduce from (3.6) and the definition of ψ,

$$|u|_2(t) \leqslant C_1(|u(t_1)|_2, E(u(t_1)), \varepsilon) \qquad \text{for all } t \in (t_1, T-\varepsilon).$$

Applying (3.8) with $t_0 = T-(\varepsilon/2)$, we obtain

$$\int_{t_1}^{T-\varepsilon} |du/dt|_2^2 \leqslant C_2(|u(t_1)|_2, E(u(t_1)), \varepsilon).$$

Then (3.9) and (3.10) implies

$$\int_{t_1}^{T-\varepsilon}\left(\int_{\Omega} F(u)(s)\right)^2 ds \leqslant C_3(|u(t_1)|_2, E(u(t_1)), \varepsilon),$$

$$\int_{t_1}^{T-\varepsilon} |\nabla u(s)|_2^4\, ds \leqslant C_4(|u(t_1)|_2, E(u(t_1)), \varepsilon),$$

where the C_i, $i = 1, 2, 3, 4$ are continuous functions depending on $|u(t_1)|_2$, $E(u(t_1))$ and $\varepsilon > 0$.

Let $u_0 \in L^{\infty}(\Omega)$ and U be the least integral solution of (P). We can apply the above inequalities to each term of the sequence U_n constructed in Remark 2.3. We take $t_1 \in (0, T_{\max})$ in such a way that the right handsides of the inequalities remain bounded when n tends to infinity. Because U_n is a classical solution on $(0, T^*(U))$, we may apply these estimates with $T = T^*(U)$. Theorem 3.1 follows. ((i) in a consequence of (3.1) and the above estimates).

Remark 3.1. The assumption $u_0 \in L^{\infty}(\Omega)$ can be weakened. Indeed, we use only the existence of t_1 such that $|u_n(t_1)|_2$ and $E(u_n(t_1))$ remain bounded to obtain the estimates on $[t_1, T^* - \varepsilon)$. We need only the following assumption: u_0 is a nonnegative measure on Ω and there exists $T_0 > 0$ such that $|u_0|_{T_0} < 1$. Indeed, we deduce that (P) has an integral solution with a true time of existence bigger that T_0 when the initial data is $(u_0/|u_0|_{T_0})$. We deduce that $T^*(1^+) \geqslant T_0$ and we apply Remark 2.5 and Remark 2.1 to obtain the existence of $t_1 \in (0, T_0)$. Under this hypothesis, the behavior of U near $t = 0$ can be deduced from the estimate (2.1) which holds with $l = (1/|u_0|_{T_0})$ and $T^*(l) = T_0$.

COROLLARY 3.1. *We make the hypotheses of Theorem* 3.1. *Then*

$$\lim_{t \to T_{\max}} E(u(t)) = -\infty \qquad \textit{implies } T_{\max} = T^*(1).$$

Proof. Suppose $T_{\max} < T^*(1)$. We deduce from Theorem (3.1) that there exists $t_0 \in (T_{\max}, T^*(1))$ such that $F(U(t_0)) \in L^1(\Omega)$. Let U_n be the sequence of Remark 2.3. We deduce from (3.2),

$$E(U_n(t)) \geqslant E(U_n(t_0)) \geqslant -\int_\Omega F(U_n(t_0)) \geqslant \int_\Omega F(U(t_0))$$

for all $t < t_0$. For $t \in (0, T_{\max})$, we have

$$\lim_{n \to \infty} E(U_n(t)) = E(u(t))$$

so, for all $t \in (0, T_{\max})$, $E(u(t)) \geqslant -\int_\Omega F(U(t_0))$.
This proves the corollary.

Proof of Theorem 2 *under* (H3). It is proved by Y. Giga in [9] that when f satisfies (H3), we have $\lim_{t \to T_{\max}} E(u(t)) = -\infty$. Observe that (H3) implies (h). Thus we can apply Corollary 3.1. We obtain $T_{\max} = T^*(1)$ which is equivalent to Theorem 2 as we have already seen.

COROLLARY 3.2. *Suppose* (H3). $u_0 \to T_{\max}(u_0)$ *is a continuous function from* $L^\infty(\Omega)^+$ *to* $\mathbb{R}^+$.

Proof. We easily deduce from the definition of $|u_0|_T$ and from the Lemma 2.4 (i) that $|\ |_T$ defines a norm on $L^\infty(\Omega)$ and for $u_0 \geqslant 0$ and $T > 0$, we have

$$|u_0|_T \leqslant |\chi_\Omega|_T \, |u_0|_\infty. \tag{3.11}$$

$|\chi_\Omega|_T$ is finite for all $T > 0$ because (P) has a classical solution on $(0, T)$ when the initial data is a positive constant small enough (the best constant is just $(1/|\chi_\Omega|_T)$, see Theorem 2.1).

Suppose that u_{0_n} is a sequence such that:

$$T_{max}(u_{0_n}) \to T \neq T_{max}(u_0) \qquad \text{and} \qquad u_{0_n} \to u_0 \quad \text{in} \quad L^\infty(\Omega).$$

First suppose $T > T_{max}(u_0)$ and choose $T_0 \in (T_{max}(u_0), T)$, we deduce from Theorem 2.1 that for n big enough we have $|u_{0_n}|_{T_0} \leqslant 1$. Inequality (3.10) implies that $u_0 \to |u_0|_{T_0}$ is continuous from $L^\infty(\Omega)$, hence $|u_0|_{T_0} \leqslant 1$. We obtain that $T^*(u_0) \geqslant T_0$ (see Theorem 2.1). But Theorem 2 implies $T^*(u_0) = T_{max}(u_0)$, we have a contradiction.

Suppose now $T < T_{max}(u_0)$ and choose $T_0 \in (T, T_{max}(u_0))$, using the same arguments, we obtain $|u_0|_{T_0} \geqslant 1$ and so $|u_0|_{T_0} = |u_0|_{T_{max}} = 1$. Then we deduce from the Corollary 2.1(ii) that $T_0 \in [T^*(1^+), T^*(1^-)]$. $T_{max} > T_0$ contradicts the Corollary 2.3.

Remark 3.1. Observe that to prove Corollary 3.2, we use only Corollaries 2.1 and 2.3 and $T^*(1^+) = T^*(1^-)$. Thus, we have under the hypothesis (h) and for $u_0 \in L^\infty(\Omega)$, $u_0 \geqslant 0$:

$u_0 \to T_{max}(u_0)$ continuous on $L^\infty(\Omega)$ at point u_0 is equivalent to $T_{max}(u_0) = T^*(1)$.

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