# Tubular Structure Segmentation Based on Minimal Path Method and Anisotropic Enhancement 

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#### Abstract

We present a new interactive method for tubular structure extraction. The main application and motivation for this work is vessel tracking in 2D and 3D images. The basic tools are minimal paths solved using the fast marching algorithm. This allows interactive tools for the physician by clicking on a small number of points in order to obtain a minimal path between two points or a set of paths in the case of a tree structure. Our method is based on a variant of the minimal path method that models the vessel as a centerline and surface. This is done by adding one dimension for the local radius around the centerline. The crucial step of our method is the definition of the local metrics to minimize. We have chosen to exploit the tubular structure of the vessels one wants to extract to built an anisotropic metric. The designed metric is well oriented along the direction of the vessel, admits higher velocity on the centerline, and provides a good estimate of the vessel radius. Based on the optimally oriented flux this measure is required to be robust against the disturbance introduced by noise or adjacent structures with intensity similar to the target vessel. We obtain promising results on noisy synthetic and real 2D and 3D images and we present a clinical validation.


Keywords Vessel segmentation • Minimal path method • Fast marching algorithm $\cdot$ Anisotropy $\cdot$ Enhancement $\cdot$ Multi-scale

[^0]
## 1 Introduction

In this paper we deal with the problem of finding a complete segmentation of tubular structures like vessels. The main objective is to extract at the same time the centerline of the tubular structure and its boundary. Such information is useful to diagnose stenoses (on coronary arteries or on carotid for example) using non invasive medical modalities like CT. During the last two decades, the extraction of vascular objects such as the blood vessels, coronary arteries, or other tube-like structures has attracted the attention of more and more researchers. Various methods such as vascular image enhancement methods (Sato et al. 1998; Krissian 2002; Frangi et al. 1998), or others were proposed, see Kirbas and Quek (2004), Lesage et al. (2009b) for a complete survey. We will first give a non-exhaustive overview of existing variational models for vessels boundary segmentation. Then, some existing methods for boundary and centerline extraction are presented.

### 1.1 Variational Models for Vessel Boundary Segmentation

One of the first variational model for tubular structures segmentation has been introduced by Davatzikos and Prince (1995). They proposed a modified active contour model called ribbon active contour for finding and mapping the outer cortex in 2D brain images, which is a tube-like structure in 2D. Even though this model has been applied only for 2D cortex mapping and suffers from different drawbacks, it was a precursor for different variational model for tube-like structure segmentation.

Siddiqi and Vasilevskiy (2001) proposed a boundary vessel segmentation based on flux maximizing flows. The essential idea is to evolve a curve (in 2 D ) or a surface (3D)
so that it clings to the features of interest in an intensity image. Since then, Descoteaux et al. (2008) improved the previous method by choosing a more appropriate vector field that drives the shape evolution. Instead of using the gradient of the image itself, they proposed to use the gradient of Frangi's et al. (1998) vesselness measure. Indeed, this vector field is more appropriate to estimate perpendicular directions of vessels.

Holtzman-Gazit et al. (2006) presented a segmentation method for extracting thin structures in 3D medical images. Their method is based on variational principles. They demonstrated the importance of the edge alignment and homogeneity terms in the segmentation of blood vessels and vascular trees. For that goal, they combined the Chan-Vese minimal variance method (Chan and Vese 2001) with the boundary alignment (Kimmel and Bruckstein 2003) and the geodesic active surface (Caselles et al. 1995, 1997) models.

Gooya et al. (2008a, 2008b, 2008c) have made a significant improvement of previous edge detection energy for vessel segmentation. More precisely, authors of Gooya et al. (2008c) propose to generalize energy edge detector functional on a Riemannian manifold that describes the local intrinsic orientation of the vessel, using structure tensor. The reason is that the edge detector energy has an isotropic behavior, meaning that it is equally sensitive to image gradients in all directions. Therefore, in a contour propagation scenario, those noisy image gradients parallel to the main orientation prevent further propagation. In fact, sensitivity is mainly needed across the planes normal to the vessel local orientation. The crucial point here is to define a well-behaved Riemannian metric that describes the local orientation of the vessel. Gooya et al. (2008c) utilized the structural tensor (Weickert 1999) to define their metric. The most significant improvement done by Gooya et al. (2008a, 2008b, 2008c) is the inclusion of vessel orientation in the model by considering a Riemannian edge detector energy.

Here, we recalled some relevant variational methods for vessel segmentation. The main drawback of these method is the time required for fixing parameters and more significantly computation time. Obviously, the listing given here is non exhaustive and it is important to point out recent relevant works along the same research line, like Law and Chung (2006, 2007), where they present a vessel segmentation method based on weighted local variance and active contour model. Also, other interesting methods have been presented in Manniesing et al. (2006, 2007), Nemitz et al. (2007), Krissian et al. (1997), Hernández Hoyos et al. (2006), Orkisz et al. (2008), Lesage et al. (2009a), Nain et al. (2004).

### 1.2 Vessel Boundary and Centerline Extraction

Deschamps and Cohen $(2001,2002,2007)$ proposed to use the minimal path method to segment tubular structures. The minimal path technique introduced by Cohen and Kimmel (1997) captures the global minimum curve ${ }^{1}$ between two points given by the user. This leads to the global minimum of an active contour energy. Unfortunately, despite their numerous advantages, classical minimal path techniques exhibit some disadvantages. First, vessel boundary extraction can be very difficult, even in 2D where the vessel's boundary can be completely described by two curves. Second, the path given by the minimal path technique does not always correspond to the centerline of the vessel. A readjustment step is required to obtain a central trajectory. Third, the minimal path technique provides only a trajectory and does not give information about the vessel boundary and local width.

In Deschamps and Cohen (2001, 2002), it was demonstrated that the front propagation could be stopped on the basis of a distance traveled by the front corresponding to the known length of the minimal path to the starting point while the front is propagated. When the propagation speed is almost constant inside the tubular shape, this length is almost the geodesic distance to the starting point inside the tubular object. However, classical segmentation problems do not provide an excellent contrast like air-filled colon on CT scanner, and the front usually flows over the boundaries of longer and thinner objects when propagating. Therefore, Cohen and Deschamps (2007) showed how the Fast Marching surface segmentation can be specifically optimized for this target. Indeed, it appears inefficient to use all points of the propagating front in the computation of the arrival time in the Eikonal equation. The idea presented in Deschamps and Cohen (2002), Cohen and Deschamps (2007) is that some of them located in the tail of the propagating front could be considered as walls, thus blocking the leakage that occurs. This Freezing process can be done by setting their speed to zero. Freezing points during propagation means we consider the front has reached the object boundaries when it visits them. Two Freezing criterion have been proposed in cited papers. The first one is time-based and the second one is distance-based. Once the contour of a vessel is obtained, authors of Deschamps and Cohen (2001) proposed a method to extract a centerline inside it.

Li and Yezzi $(2006,2007)$ proposed a new variant of the classical, purely spatial, minimal path technique by incorporating an extra non-spatial dimension into the search space. Each point of the 4D path (after adding the extra dimension for the 3D image) consists of three spatial coordinates plus a fourth coordinate that describes the vessel thickness at the corresponding 3D point, see Fig. 1 as a 2D example. Thus,

[^1]

Fig. 1 A tubular shape is presented as the envelope of a family of spheres or disks with continuously changing center points and radii
each 4 D point represents a sphere in 3 D space, and the vessel is obtained by taking the envelope of these spheres as we move along the 3D curve. A crucial step of this method is to build an adequate potential that drives the propagation. Li and Yezzi $(2006,2007)$ proposed different isotropic potentials. Since potentials proposed by Li and Yezzi are very parameter dependent, an important perspective of their work was the construction of a more appropriate potential. Moreover, by considering only isotropic metrics, Li and Yezzi did not take into account the vessel orientation.

Along the same research line, Mohan et al. (2008) used the 4D minimal path model of Li and Yezzi $(2006,2007)$ for segmenting the Cingulum Bundle from DW-MRI. After adding the extra non-spatial dimension (the radius dimension), they associated to the path an anisotropic potential related to the Finsler metric. Let us denote $\tilde{\gamma}(u)=$ $(\gamma(u), r(u))$ the 4D path with arc length parameterization (i.e $\left\|\tilde{\gamma}^{\prime}\right\|_{2}=1$ ) and $\mathbb{S}^{2} \subset \mathbb{R}^{3}$ the 2D sphere. The considered energy is:
$\mathcal{E}(\tilde{\gamma})=\int_{\tilde{\gamma}} \mathcal{P}\left(\tilde{\gamma}(s), \frac{\gamma^{\prime}(s)}{\left\|\gamma^{\prime}(s)\right\|}\right) \mathrm{d} s$,
where this time, ${ }^{2}$ the potential is $\mathcal{P}: \mathbb{R}^{4} \times \mathbb{S}^{2} \rightarrow \mathbb{R}_{+}^{*}$. Authors of Mohan et al. (2008) gave two different choices of $\mathcal{P}$ that are meaningful for extracting the Cingulum Bundle from DW-MRI, both based on local region statistics. In order to optimize energy functional (1), they used gradient descent method where the gradient direction is computed with respect to a geometrized Sobolev metric (Sundaramoorthi et al. 2009) instead of the classical $\mathbb{L}^{2}$ metric which is very unstable according to Melonakos et al. (2008). An important contribution of Mohan et al. (2008) is that they considered orientation of the vessel in their energy model, which is reasonable for tubular structures. But the proposed potentials were defined in an $a d$ hoc manner. Moreover, in the energy functional given by (1) the derivative of the non spatial dimension $r^{\prime}(s)$ is not taken into account. That makes their

[^2]model very insensitive to the radius dimension and inappropriate for tubular structures with varying radii, which is not the case of Cingulum Bundle.

### 1.3 Contributions

In this paper we propose a vessel segmentation method based on a minimal path formulation and anisotropic enhancement. Both vessel orientation and width are taken into account. Our contribution is threefold:

- First, inspired from Li and Yezzi (2006, 2007), we introduce a minimal path model, that takes into account the vessel width, by adding an extra dimension, and the vessel orientation by considering anisotropic metrics.
- Next, we propose a well motivated metric constructor for the anisotropic metric based on the Optimally Oriented Flux (OOF), introduced by Law and Chung (2008), by using its scalar function as well as its orientation (while authors of Law and Chung 2008 used only the scalars, that is the eigenvalues). That makes the propagation faster along the vessel's centerline and for exact associated scale. This means that the path location, orientation and scale (radius) have to be coherent with the local geometry of the image extracted by the OOF.
- Finally, we make a deep analysis of the chosen enhancer by establishing a link between the OOF and the widely used Hessian-based tubular structure enhancers like Frangi's et al. (1998), and by demonstrating an interpretation of the OOF in terms of steerable filters (Freeman and Adelson 1991; Jacob and Unser 2004). Some contributions have been added regarding the conference paper (Benmansour and Cohen 2009).


### 1.4 Paper Outline

In Sect. 2, we give some background on the minimal path method and Anisotropic Fast Marching. In Sect. 3 the Optimally Oriented Flux descriptor is presented, a link between the OOF and the Hessian-based vesselness measures is established, Oriented Flux filter is interpreted in terms of steerable filters, and the metric design is shown. In Sect. 4, first, we show advantages of anisotropy. Second, results on synthetic and real data are shown. Third, validation results are given. Finally, conclusions and perspectives follow in Sect. 5.

## 2 Background on the Minimal Path Method

### 2.1 Formalism

A geodesic is a path, linking two points, that globally minimizes an energy functional weighted by an image potential. Hence, a geodesic is also called minimal path. Without


Fig. 2 Minimal path examples on an isotropic case on the left image. On the middle, visualization by small ellipses of eigenvalues of a metric constant on each half side of the image. On the right, the minimal action map associated to the source point $\mathbf{p}_{1}$ with the minimal path $\mathcal{C}_{\mathbf{p}_{1}, \mathbf{p}_{2}}$
loss of generality and in order to simplify notations, we will assume hereinafter that a path $\gamma$ is parametrized along its length (i.e. $\left\|\gamma^{\prime}\right\|=1$ ). The energy minimized by a geodesic is under the form:
$E(\gamma)=\int_{\gamma} \mathcal{P}\left(\gamma(s), \gamma^{\prime}(s)\right) \mathrm{d} s$.
Such formulation was first presented to the computer vision community by Cohen and Kimmel (1997) in the particular isotropic case, when $\mathcal{P}>0$ does not depend on the orientation of the path.

This energy corresponds to the length of a curve in a Finsler ${ }^{3}$ manifold (Melonakos et al. 2008). In this paper, we are interested only in the particular case of Riemannian manifold, which is a potential under the form $\mathcal{P}\left(\gamma(),. \gamma^{\prime}().\right)=$ $\sqrt{\gamma^{\prime}(.)^{T} \mathcal{M}(\gamma(.)) \gamma^{\prime}(.)}$ describing an infinitesimal distance along the pathway $\gamma$ relative to the metric tensor $\mathcal{M}$ (symmetric definite positive). Thus, we are considering only the case of an elliptic ${ }^{4}$ medium. In the isotropic case $\mathcal{M}()=$. $\mathcal{P}^{2}()$.$I , where I$ is the identity matrix. A curve connecting a point $\mathbf{p}_{1}$ to $\mathbf{p}_{2}$ that globally minimizes the above energy (2) is a minimal path between $\mathbf{p}_{1}$ and $\mathbf{p}_{2}$, noted $\mathcal{C}_{\mathbf{p}_{1}, \mathbf{p}_{2}}$ and also called a geodesic.

The solution of this minimization problem is obtained through the computation of the minimal action map $\mathcal{U}$ : $\Omega \rightarrow \mathbb{R}^{+}$associated to $\mathbf{p}_{1}$ on the domain $\Omega \subset \mathbb{R}^{d}$ (in this paper $d=2,3$ or 4 ). The minimal action is the minimal energy integrated along a path between $\mathbf{p}_{1}$ and any point $\mathbf{x}$ of the domain $\Omega$ :
$\forall \mathbf{x} \in \Omega, \quad \mathcal{U}(\mathbf{x})=\min _{\gamma \in \mathcal{A}_{\mathbf{p}_{1}, \mathbf{x}}}\left\{\int_{\gamma} \mathcal{P}\left(\gamma(s), \gamma^{\prime}(s)\right) \mathrm{d} s\right\}$,
where $\mathcal{A}_{\mathbf{p}_{1}, \mathbf{x}}$ is the set of paths linking $\mathbf{x}$ to $\mathbf{p}_{1}$. The values of $\mathcal{U}$ may be regarded as the arrival times of a front propagating from the source $\mathbf{p}_{1}$ with oriented velocity related to the metric tensor $\mathcal{M}^{-1} . \mathcal{U}$ satisfies the Eikonal equation

$$
\begin{equation*}
\|\nabla \mathcal{U}(\mathbf{x})\|_{\mathcal{M}^{-1}(\mathbf{x})}=1 \quad \text { for } \mathbf{x} \in \Omega, \quad \text { and } \quad \mathcal{U}\left(\mathbf{p}_{1}\right)=0 \tag{4}
\end{equation*}
$$

[^3]where $\|\mathbf{v}\|_{M}=\sqrt{\mathbf{v}^{T} M \mathbf{v}}$. The map $\mathcal{U}$ has only one local minimum, the source point $\mathbf{p}_{1}$, and its flow lines satisfy the Euler-Lagrange equation of functional (2). Thus, the minimal path $\mathcal{C}_{\mathbf{p}_{1}, \mathbf{p}_{2}}$ can be retrieved with a simple gradient descent on $\mathcal{U}$ from $\mathbf{p}_{2}$ to $\mathbf{p}_{1}$ (see Fig. 2), solving the following ordinary differential equation with standard numerical methods like Heun's or Runge-Kutta's:
\[

$$
\begin{align*}
& \frac{\mathrm{d} \mathcal{C}_{\mathbf{p}_{1}, \mathbf{p}_{2}}}{\mathrm{~d} s}(s) \propto-\mathcal{M}^{-1}\left(\mathcal{C}_{\mathbf{p}_{1}, \mathbf{p}_{2}}(s)\right) \nabla \mathcal{U}\left(\mathcal{C}_{\mathbf{p}_{1}, \mathbf{p}_{2}}(s)\right) \\
& \quad \text { with } \mathcal{C}_{\mathbf{p}_{1}, \mathbf{p}_{2}}(0)=\mathbf{p}_{2} \tag{5}
\end{align*}
$$
\]

Equations (4) and (5) have been presented under their isotropic version in Cohen and Kimmel (1997). The Eikonal equation (4) belongs to the class of static Hamilton-Jacobi equations. The associated unique viscosity solution (Lions 1982) $\mathcal{U}$ satisfies the Hopf-Lax formula which is given by (3) in our case, ${ }^{5}$ see for example (Bornemann and Rasch 2006; Tavares and Jorge 2009; Benmansour 2009) for more details. The proof of (4) and (5), in the general case, can be found in Lions (1982). This general proof requires notions from calculus of variations (Evans 1998).

In Fig. 2, we show some examples of the minimal path method on an isotropic case and an anisotropic one. In the first image of Fig. 2, the metric is isotropic and the potential $\mathcal{P}$ in the grey region is half of the white one. Isolevel sets of the minimal action map associated to the source point $\mathbf{p}_{1}$ are displayed together with a minimal path $\mathcal{C}_{\mathbf{p}_{1}, \mathbf{p}_{2}}$. The second image represents a metric $\mathcal{M}\left(\mathcal{M}^{-1}\right.$ is displayed). We took two constant metrics in each half side of the image with different orientations. In the last image, the minimal action $\operatorname{map} \mathcal{U}$ associated to the metric $\mathcal{M}$ and to the source point $\mathbf{p}_{1}$ is shown. The minimal path $\mathcal{C}_{\mathbf{p}_{1}, \mathbf{p}_{2}}$ is found by solving (5). We can see that the minimal path tries to agree as much as possible with the orientation of the ellipses.

### 2.2 Eikonal Solvers

### 2.2.1 Short Overview

In this section we are interested in solving numerically the Eikonal equation (4). In the last two decades, considerable

[^4]efforts have been made to develop numerical methods for solving static Hamilton-Jacobi equation (which includes the Eikonal equation) on regular or unstructured meshes. The reader is referred to the literature for rigorous and more precise statements.

One of the first consistent discretization on a regular grid of (4), in its isotropic version, has been proposed by Rouy and Tourin (1992). They proposed an upwind finite difference discretization of the Eikonal equation $\|\nabla \mathcal{U}\|=P$ on a regular grid. They showed that the associated discrete solution converges toward the viscosity solution, ${ }^{6}$ and an iterative algorithm based on a nonlinear variant of the GaussSeidel iteration has been proposed to compute the discrete values. A few years later, Sethian (1996) proposed a singlepass algorithm to compute the solution of the upwind discretization. The algorithm proposed by Sethian, called Fast Marching is based on Dijkstra's algorithm for computing shortest paths on graphs (Dijkstra 1959). Independently, Tsitsiklis (1995) proposed a similar Dijkstra-like algorithm to solve numerically the isotropic Eikonal equation. However, the used discretization is not based on the Eikonal equation itself but on the Hopf-Lax formula (given in a particular case by (3)).

Both Sethian's and Tsitsiklis's methods (Sethian 1996; Tsitsiklis 1995) are suitable for isotropic metrics, but they fail for anisotropic metrics according to Chopp (2001). To deal with anisotropy, three different improvements of the classical fast marching method have been proposed:

1. Change the local approximation scheme as done by Lenglet et al. (2009),
2. Introduce a recursive approach as done by Konukoglu et al. (2007), they called the proposed method Recursive Fast Marching,
3. Extend the neighborhoods, depending on the local anisotropy as done by Sethian and Vladimirsky (2000), or in a fixed manner as done by Lin (2003) and used by Jbabdi et al. (2008).

Convergence toward the viscosity solution of the anisotropic Eikonal equation (4) has been proved only for the Ordered Upwind Method (OUM), using an extended neighborhood, in Sethian and Vladimirsky (2000). The main drawback of the OUM is that, for high anisotropy, the neighborhood needed for the local update is extremely vast.

Other iterative approaches to solve the anisotropic Eikonal equation have been proposed. Bornemann and Rasch (2006) presented a linear finite-element discretization for static Hamilton-Jacobi equations on unstructured triangulations. Similarly to Tsitsiklis (1995), their discretization is based on the Hopf-Lax formula. Bornemann and Rasch proposed

[^5]to use a tricky adaptive Gauss-Seidel iteration to solve the underlying system of non-linear equations. Convergence results have been shown in Bornemann and Rasch (2006) under some assumptions that are easy to satisfy in our Riemannian case.

In this paper we used Lin's scheme to approximate the solution of the anisotropic Eikonal equation, since it is much faster (than OUM and the iterative algorithm proposed in Bornemann and Rasch 2006) and, as long as the anisotropy remains reasonable, the introduced errors do not affect much the extracted geodesics.

### 2.2.2 Fast Marching Algorithm

The FMM is a front propagation approach that computes the values of $\mathcal{U}$ in increasing order, and the structure of the algorithm is almost identical to Dijkstra's algorithm for computing shortest paths on graphs (Dijkstra 1959). The main difference is the expression of the local contribution to the weighted distance. In the course of the algorithm, each grid point is tagged as either Alive (point for which $\mathcal{U}$ has been computed and frozen), Trial (point for which $\mathcal{U}$ has been estimated but not frozen) or Far (point for which $\mathcal{U}$ is unknown). The set of Trial points forms an interface between the set of grid points for which $\mathcal{U}$ has been frozen (the Alive points) and the set of other grid points (the Far points). This interface may be regarded as a front expanding from the source until every grid point has been reached. Let us denote by $\mathcal{N}_{\mathrm{M}}(\mathbf{x})$ the set of M neighbors of a grid point $\mathbf{x}$, where $\mathrm{m}=3^{d}-1$ if the dimension of $\Omega$ is equal to $d$. Initially, all grid points are tagged as Far, except the source point $\mathbf{p}_{1}$ that is tagged as Trial. At each iteration of the FMM one chooses the Trial point with the smallest $\mathcal{U}$ value, denoted by $\mathbf{x}_{\text {min }}$. Then, $\mathbf{x}_{\text {min }}$ is tagged as Alive and the value of $\mathcal{U}$ is updated for each point of the set $\mathcal{N}_{\mathrm{M}}\left(\mathbf{x}_{\text {min }}\right)$ which is either Trial or Far. In order to satisfy a causality condition, the way $\mathcal{U}$ is updated in the vicinity of $\mathbf{x}_{\text {min }}$ requires special care. The iteration ends by tagging every Far point of the set $\mathcal{N}_{\mathrm{M}}\left(\mathbf{x}_{\min }\right)$ as Trial. The algorithm automatically stops when all grid points are Alive. The key to the speed of the FMM is the use of a priority queue to quickly find the Trial point with the smallest $\mathcal{U}$ value. If Trial points are ordered in a min-heap data structure, the computational complexity of the FMM is $\mathcal{O}(N \log N)$, where $N$ is the total number of grid points.

A crucial step of the Fast Marching algorithm is the computation of the weighted distance between the front and the neighboring voxels in the Trial set. Here, we present a way to estimate this weighted distance in the anisotropic case and only in 3D. It is straightforward to extend it to 4D. Since the distance is anisotropic, we cannot use the standard methods, because they rely on the fact that the geodesics are perpendicular to the level sets of $\mathcal{U}$. To take into account the anisotropy (recall that the proposed solution here


Fig. 3 On the left position of the optimal point on a simplex such as to minimize the geodesic distance to $\mathbf{x}$. On the right the considered simplexes
does not converge toward the viscosity solution of (4)), and without using a vast neighborhood, Lin (2003) and Jbabdi et al. (2008) considered a set of simplexes that cover the whole neighborhood around a voxel of the front. The definition of a simplex neighboring a point $\mathbf{x}$ is simply a set of three points $\left(\mathbf{x}_{1}, \mathbf{x}_{2}, \mathbf{x}_{3}\right)$ that are among the 26 neighbors of $\mathbf{x}$, such that $\mathbf{x}_{1}$ is a 6-connectivity neighbor, $\mathbf{x}_{2}$ is a 18connectivity (and not 6-connectivity) neighbor, and $\mathbf{x}_{3}$ is a 26-connectivity (and not 18-connectivity) neighbor. This set of points defines a triangle that we denote $\overline{\mathbf{x}_{1} \mathbf{x}_{2} \mathbf{x}_{3}}$. There are 48 such triangles around $\mathbf{x}$ for the 26 connectivivy, see Fig. 3.

If the geodesic passing by $\mathbf{x}_{\mathrm{m}}$ comes from a triangle $\overline{\mathbf{x}_{1} \mathbf{x}_{2} \mathbf{x}_{3}}$ then the time of arrival is given by:
$\mathcal{U}\left(\mathbf{x}_{\mathrm{m}}\right)=\min _{\mathbf{x} \in \overline{\mathbf{x}_{1} \mathbf{x}_{2} \mathbf{x}_{3}}}\left\{\mathcal{U}(\mathbf{x})+\int_{\mathbf{x}}^{\mathbf{x}_{\mathrm{m}}} P\left(\gamma, \gamma^{\prime}\right)\right\}$,
where point $\mathbf{x}$ can be interpreted as a virtual source ${ }^{7}$ on the triangle. To estimate $\mathcal{U}\left(\mathbf{x}_{\mathrm{m}}\right)$, where $\mathbf{x}_{\mathrm{m}}$ is a neighbor of the last trial point $\mathbf{x}_{\text {min }}$, we make two approximations on the term one wants to minimize. First, we assume the path between $\mathbf{x}$ and $\mathbf{x}_{\mathrm{m}}$ to be a segment and the metric to be constant along this segment, equal to its value at point $\mathbf{x}_{\mathrm{m}}$. Hence, we want to minimize:
$F(\mathbf{x})=\mathcal{U}(\mathbf{x})+\left\|\mathbf{x}-\mathbf{x}_{\mathrm{m}}\right\|_{\mathcal{M}\left(\mathbf{x}_{\mathrm{m}}\right)}$.
Then, since the point $\mathbf{x}$ is in the triangle $\overline{\mathbf{x}_{1} \mathbf{x}_{2} \mathbf{x}_{3}}$, we can replace $\mathbf{x}$ by its barycentric expression $\mathbf{x}=\sum_{i=1}^{3} \alpha_{i} \mathbf{x}_{i}$, where $\alpha=\left(\alpha_{1}, \alpha_{2}, \alpha_{3}\right)$ with $\sum_{i} \alpha_{i}=1$ and $\alpha_{i} \geq 0$, and make a linear approximation of its $\mathcal{U}$-value. We then have to minimize:

$$
\begin{equation*}
f(\alpha)=\sum_{i=1}^{3} \alpha_{i} \mathcal{U}\left(\mathbf{x}_{i}\right)+\left\|\mathbf{x}_{\mathrm{m}}-\sum_{i=1}^{3} \alpha_{i} \mathbf{x}_{i}\right\|_{\mathcal{M}\left(\mathbf{x}_{\mathrm{m}}\right)} \tag{8}
\end{equation*}
$$

[^6]This equation follows Tsitsiklis's approximation (Tsitsiklis 1995). The function $f$ is convex and the constraints on $\alpha$, i.e $\sum_{i=1}^{3} \alpha_{i}=1$ and $\alpha_{i} \geq 0$, define a convex subset. Thus the minimization of $f$ can be done using classical optimization tools. See Appendix B of Jbabdi et al. (2008) for more details.

Solving this minimization for each triangle, we get a value $\mathcal{U}_{\overline{\mathbf{x}_{1} \mathbf{x}_{2} \mathbf{x}_{3}}}$. Finally, we choose the triangle giving the smallest value. Note that in order to approximate $\nabla \mathcal{U}$, computing the derivatives of $\mathcal{U}$ in the triangle using the estimate $\mathcal{U}\left(\mathbf{x}_{\mathrm{m}}\right)$ gives a consistent approximation of $\nabla \mathcal{U}\left(\mathbf{x}_{\mathrm{m}}\right)$ by the following:
$\nabla \mathcal{U}\left(\mathbf{x}_{\mathrm{m}}\right)=\left(\mathcal{U}\left(\mathbf{x}_{\mathrm{m}}\right)-\mathcal{U}\left(\mathbf{x}^{\star}\right)\right) \frac{\mathbf{x}_{\mathrm{m}}-\mathbf{x}^{\star}}{\left\|\mathbf{x}_{\mathrm{m}}-\mathbf{x}^{\star}\right\|}$,
where $\mathbf{x}^{\star}$ is the minimizer of function $f$ (see Fig. 3 left) and $\|$.$\| is the Euclidean norm. The computation of the gradient$ is very useful since it is used to solve the gradient descent described by (5).

### 2.3 Comparison and Discussion

In this section, we are going to compare numerical results and computation time of Lin's algorithm and the iterative algorithm of Bornemann and Rasch on a relevant example. For this purpose, we designed an anisotropic metric from the left image of Fig. 4.

The designed metric has high anisotropy along the spiral and is isotropic almost every where else with low propagation velocity. Such metric is constructed using methods presented later (Sect. 3.4), namely (19) at the associated fixed scale ( $r=3$ pixels) of the spiral. Indeed we took a maximal anisotropy ratio equal to 100 . Geometrically the maximal anisotropy ratio corresponds to the maximal ratio between the major and minor axis of ellipsoids displaying the metric (see Fig. 2 middle for example). On the second image of Fig. 4, the computed minimal action map $\mathcal{U}$, using the Fast Marching algorithm presented in Sect. 2.2.2, from the source point $\mathbf{p}_{1}$ and the minimal path $\mathcal{C}_{\mathbf{p}_{2}, \mathbf{p}_{1}}$ are displayed.

One can see that a shortcut occurs. Actually this is due not to the metric but to the introduced errors by the chosen Eikonal solver (Lin's Fast Marching algorithm). Indeed, we implemented Bornemann and Rasch algorithm (Bornemann and Rasch 2006) to solve the same problem, and obtained the result shown in the third image of Fig. 4. It has been shown in Bornemann and Rasch (2006), and numerically verified by our care, that the proposed iterative algorithm converges to the viscosity solution. However, the main drawback of Bornemann and Rasch algorithm is the computation time. On the right image of Fig. 4, one can see the number of iterations needed in order to mend the minimal action map values. Some points need to be updated up to 500 times. The computation time of the iterative algorithm for


Fig. 4 From image on the left, an anisotropic metric with high anisotropy ratio $(\mu=100)$ along the spiral is constructed. Second image, geodesic distance from source point $\mathbf{p}_{1}$ computed using Lin's algorithm and a geodesic $\mathcal{C}_{\mathbf{p}_{1}, \mathbf{p}_{2}}$. On third image, the geodesic distance
this spiral image of size $548 \times 600$ is about 2 minutes, while the computation time of the Fast Marching algorithm is less than a second. For higher dimensions, one can imagine that the computation time of the iterative algorithm (Bornemann and Rasch 2006) is heavily long.

As a matter of fact, there is a tradeoff between efficiency and accuracy. On the one hand, using the Fast Marching algorithm provides reasonable results as long as the anisotropy ratio remains small, disabling us for taking complete advantage of the Tubular Anisotropy method presented in this paper. On the other hand, one can increase the anisotropy ratio and use Bornemann and Rasch iterative algorithm, but this choice considerably penalizes the efficiency ${ }^{8}$ of the whole algorithm.

Unlike the Fast Marching algorithm, the iterative algorithm is parallelizable. In Weber et al. (2008), authors propose an efficient implementation on parallel architectures to compute geodesic distances on geometry images. An important perspective is to make a similar parallelized implementation of Bornemann and Rasch' algorithm in order to genuinely take advantage of the Tubular Anisotropy method.

## 3 Optimally Oriented Flux: An Anisotropy Descriptor

We are interested in the construction of a metric that extracts from the image the geometric information leading to reconstruction of vessels. This means that we wish to find an estimate for the local orientation and scale and a criterion on the local geometry to distinguish the presence of vessels from the background. First, we tried some existing vessel enhancers like the Hessian-based vesselness measures

[^7]
is computed using Bornemann and Rasch algorithm. The last image displays the number of iterations for Bornemann and Rasch algorithm at each point in the image domain, until the convergence criterion is satisfied
(Frangi et al. 1998; Sato et al. 1998; Lorenz et al. 1997; Lindeberg 1998). The main drawback of these enhancers is that they include adjacent features. The anisotropy descriptor presented here is inspired from the optimally oriented flux (OOF) recently introduced by Law and Chung (2008). Its main advantage is that adjacent objects are not taken into account. Moreover, this descriptor provides intuitively a good estimate of vessel direction enabling us to design a suited anisotropic metric for the oriented minimal path model presented in the previous section. Also, we establish a link between the OOF and the Hessian-based enhancers that justifies our choice.

### 3.1 Definitions

At the position $\mathbf{x}$ in an image $I$, the oriented flux is the amount of the image gradient projected along the axis $\mathbf{v}$ flowing out from a 3D local sphere ${ }^{9}$ (or a 2D circle) $S_{r}$. It is measured as in Law and Chung (2008),
$f(\mathbf{x}, \mathbf{v} ; r)=\int_{\partial S_{r}}((\nabla(G * I)(\mathbf{x}+\mathbf{h}) \cdot \mathbf{v}) \mathbf{v}) \cdot \mathbf{n} \mathrm{d} a$,
where $G$ is a Gaussian function with a scale factor of 1 pixel, $r$ is the radius of the sphere (or circle), $\mathbf{h}=r \mathbf{n}$ is the relative position vector along $\partial S_{r}$, with $\mathbf{n}$ the outward unit normal of $\partial S_{r}$, and $\mathrm{d} a$ is the infinitesimal area (or length) on $\partial S_{r}$. Function $f$ is the flux of the smoothed image gradient $\nabla(G * I)$ projected along direction $\mathbf{v}$ toward the sphere $\partial S_{r}$. To detect vessels having higher intensity than the background region, one would be interested in finding the vessel direction which minimizes $f(\mathbf{x}, \mathbf{v} ; r)$, i.e. we are looking for $\arg \min _{\mathbf{v}} f(\mathbf{x}, \mathbf{v} ; r)$. Using the divergence theorem, it can be shown that $f(\mathbf{x}, \mathbf{v} ; r)$ is a quadratic form on $\mathbf{v}$ and its associated matrix can be calculated using a simple convolu-

[^8]Fig. 5 The plots of the values of $f(\mathbf{x}, \mathbf{v} ; r)$ obtained from the synthetic image shown in the top, at four different positions with various radii and projection axes. (a) Four interesting positions, denoted as $\mathbf{x}_{1}, \mathbf{x}_{2}, \mathbf{x}_{3}$ and $\mathbf{x}_{4}$ are shown along with the original synthetic image. (b) An illustration regarding the polar coordinate system used in (c)-(f). (c)-(f) The plots of the values of $f(\cdot)$ and the corresponding eigenvectors, computed at the four different positions shown in (a), using various values of $r$ and different projection axes $(\cos \theta \sin \theta)^{T}$

tion,
$f(\mathbf{x}, \mathbf{v} ; r)=\mathbf{v}^{T}\left\{I *\left(\partial_{i, j} G\right) * \mathbb{1}_{S_{r}}(\mathbf{x})\right\} \mathbf{v}:=\mathbf{v}^{T}\left\{I * \mathbf{F}_{r}(\mathbf{x})\right\} \mathbf{v}$,
where $\left(\partial_{i, j} G\right)$ is the Hessian matrix of function $G$ and $\mathbb{1}_{S_{r}}$ is the indicator function inside the sphere (or circle) $S_{r} . \mathbf{F}_{r}$ is called the oriented flux filter. By differentiating the above equation with respect to $\mathbf{v}$, minimization of function $f$ is in turn acquired as solving a generalized eigenvalue decomposition problem. Solving the aforementioned generalized eigen decomposition problem gives $d$ eigenvalues (where $d=2$ or 3 is the dimension of the image), $\lambda_{1}(\cdot) \leq \cdots \leq \lambda_{d}(\cdot)$ and $d$ eigenvectors $\mathbf{v}_{i}(\cdot)$, i.e. $\lambda_{i}(\mathbf{x} ; r)=f\left(\mathbf{x}, \mathbf{v}_{i}(\mathbf{x} ; r) ; r\right)$ for $i=1, \ldots, d$. To handle the vessels having various radii, a multi-scale approach should be used along with the OOF method. Law and Chung (2008) have proposed to normalize the OOF's eigenvalues by the sphere surface area when the OOF method is incorporated in a multi-scale approach for 3D image volumes. In the 2D case the eigenvalues are normalized by the circle perimeter $2 \pi r$. In the 3 D case the eigenvalues are normalized by the sphere area $4 \pi r^{2}$.

In the 2D case (see Fig. 5), for a point on the centerline and if $r$ is equal to the radius of the vessel, the first eigenvector $\mathbf{v}_{1}$ represents the direction orthogonal to the vessel. $\mathbf{v}_{2}$ is the estimated direction of the vessel. If the point is inside the vessel but not on the centerline (for example, point $x_{2}$ in Fig. 5(a)), the OOF response provides two local minima, when $\mathbf{v}$ is parallel to the vessel centerline and for the two possible scales, see Fig. 5(d). Finally, if the point is outside the vessel and near its boundary, the estimated vessel
direction, i.e. $\mathbf{v}_{2}$, which is the eigenvector associated to the largest eigenvalue is perpendicular to the vessel direction.

In the 3D case, if the point is on the centerline, the two eigenvectors associated to the first eigenvalues $\left(\lambda_{1}, \lambda_{2}\right)$ represent the directions orthogonal to the vessel direction. $\mathbf{v}_{3}$ goes along the vessel, see Fig. 6. On the same figure, one can see that if the point $\mathbf{x}$ is on the centerline, the minimal response of the function $f$ is obtained when the radius $r$ is equal to the exact radius of the tube. If the point is inside the tube but not on the centerline, $\mathbf{v}_{\mathbf{3}}$ is parallel to the tube orientation, and the other eigenvectors depend on the scale $r$. If the point is outside the tube (last line), then the vector $\mathbf{v}_{3}$, corresponding to the red area, is oriented toward the centerline.

### 3.2 OOF as a Steerable Filter

In this section, we will demonstrate that the Optimally Oriented Flux function $f$ can be interpreted in terms of steerable filters (Freeman and Adelson 1991). Steerable filters are useful to detect edges or ridges (vessel-like structures) at any orientation. Steerable filters are filters for which the response at different orientations can be synthesized from linear combination of rotated versions of itself (this is called steerability relation). More precisly a filter $\mathcal{F}: \mathbb{R}^{2} \rightarrow \mathbb{R}$ is said to be steerable if
$\mathcal{F}\left(\mathbf{R}_{\theta} \mathbf{x}\right)=\sum_{i=1}^{M} k_{i}(\theta) \mathcal{F}\left(\mathbf{R}_{\theta_{i}} \mathbf{x}\right)$
where $\mathbf{R}_{\theta}$ is the rotation matrix:
$\mathbf{R}_{\theta}=\left(\begin{array}{cc}\cos (\theta) & \sin (\theta) \\ -\sin (\theta) & \cos (\theta)\end{array}\right)$

Fig. 6 Plot of $f(\mathbf{x}, \mathbf{v} ; r)$ superimposed on the original 3D synthetic image for three different points (on each line) and different values of the radius: $r=4, \ldots, 7$ from left to right. The radius of the tube on the top half side image is equal to 4 , and equal to 6 on the bottom half side. Similarly to Fig. 5, the visualization of the normalized flux function is done using a spherical coordinate system (instead of the polar one used in 2D). The first point is on the centerline of the tube. The second point is inside the tube but not on the centerline. The third point is outside the tube. The reader should zoom on each image. Notice that the colormaps are different for each point

and $\mathbf{x}=(x, y)$ is a point in $\mathbb{R}^{2}$. Inspired by the steerability property of the Hessian filter presented in Jacob and Unser (2004), we will show that the oriented flux filter is steerable as well. We will compare the associated template features of the Hessian and the oriented flux in order to justify qualitatively our choice.

Jacob and Unser (2004) showed that the popular ridge filter based on the Hessian matrix (Frangi et al. 1998; Sato et al. 1998; Lorenz et al. 1997; Lindeberg 1998) can be interpreted as a steerable filter and that this filter can be synthesized by a template which is $\partial_{x x} g_{r}$ (the second derivative along $x$ direction of a Gaussian $g_{r}$ at scale $r$ ). The steerability relation can be expressed in a matrix form as follows:

$$
\begin{align*}
\partial_{x x} g_{r}\left(\mathbf{R}_{\theta} \mathbf{x}\right)= & \mathbf{u}_{\theta}^{T}\left(\begin{array}{cc}
\partial_{x x} g_{r}(\mathbf{x}) & \partial_{x y} g_{r}(\mathbf{x}) \\
\partial_{y x} g_{r}(\mathbf{x}) & \partial_{y y} g_{r}(\mathbf{x})
\end{array}\right) \mathbf{u}_{\theta} \\
= & \cos ^{2}(\theta) \partial_{x x} g_{r}(\mathbf{x})+\sin (2 \theta) \partial_{x y} g_{r}(\mathbf{x}) \\
& +\sin ^{2}(\theta) \partial_{y y} g_{r}(\mathbf{x}), \tag{13}
\end{align*}
$$

where $\mathbf{u}_{\theta}=(\cos (\theta), \sin (\theta))^{T}$. Hence three representative directions of the Hessian filter are $\left(\theta_{1}=0, \theta_{1}=\frac{\pi}{4}, \theta_{2}=\right.$ $\left.\frac{\pi}{2}\right)$ associated to the function basis: $\left(\partial_{x x} g_{r}, \partial_{x y} g_{r}, \partial_{y y} g_{r}\right)$ (see (12)).

It has been shown by Law and Chung (2008), and rewritten by (11) that the response given by the optimally oriented flux can be expressed as a convolution of the image $I$ with $\mathbf{v}^{T} \mathbf{F}_{r} \mathbf{v}$, called here oriented flux filter. At a given orientation


Fig. 7 (Color online) Feature templates associated to the Hessian (on the left) and to the oriented flux (on the right) at the same scale $r$. The yellow dotted line simulates the vessel direction. $\sigma$ is the variance of $G$
$\mathbf{v}=\mathbf{u}_{\theta}$, the oriented flux filter can be synthesized as follows:
$\mathbf{u}_{\theta}^{T} \mathbf{F}_{r}(\mathbf{x}) \mathbf{u}_{\theta}=\left\{\partial_{x x} G * \mathbb{1}_{r}\right\}\left(\mathbf{R}_{\theta} \mathbf{x}\right)$.
We conclude that the oriented flux filter is a steerable filter and that an associated template is $\partial_{x x} G * \mathbb{1}_{r}$. This result is straightforwardly extensible to higher dimensions.

In Fig. 7, the templates associated to both the Hessian filter and the oriented flux filter are shown. Hence, we can qualitatively justify that the OOF is more suited for vessel enhancement and orientation detection. Indeed, unlike the Hessian template, the oriented flux template profile is more localized around the vessel boundaries. Thus, adjacent features to the vessel, like heart chambers or other organs, will most likely not be included by the OOF. In the next section,
we will highlight a link between the OOF and the Hessianbased filters in order to justify again the choice of the OOF for our medical application.

### 3.3 Unifying Flux Based and Hessian Based Vessel Enhancement

Let us recall the optimally oriented flux, and the Hessian formulations. It has been shown that the flux of the image gradient projected along a direction $\mathbf{v}$ through a sphere of radius $r$ can be written as follows,

$$
\begin{aligned}
f(\mathbf{x}, \mathbf{v} ; r) & =\int_{\partial S_{r}}((\nabla(G * I)(\mathbf{x}+\mathbf{h}) \cdot \mathbf{v}) \mathbf{v}) \cdot \frac{\mathbf{h}}{|\mathbf{h}|} \mathrm{d} a \\
& =\mathbf{v}^{T} \mathbf{Q}(\mathbf{x} ; r) \mathbf{v}
\end{aligned}
$$

where
$\mathbf{Q}(\mathbf{x} ; r)=\left(\left(\partial_{i, j} G\right)_{i, j} * \mathbb{1}_{S_{r}} * I\right)(\mathbf{x})$
where $\left(\partial_{i, j} G\right)$ is the Hessian matrix of function $G$ and $\mathbb{1}_{S_{r}}$ is the indicator function inside the sphere (or circle) $S_{r}$, and $G$ is a Gaussian with a scale $\sigma$ equal to the size of a voxel (or a pixel). The vesselness measure proposed by Law and Chung (2008) is based on the eigenvalues of $\mathbf{Q}$.

The Hessian-based vesselness measures proposed in Frangi et al. (1998), Sato et al. (1998), Lorenz et al. (1997), Lindeberg (1998) are based on the eigenvalues of the scaled Hessian of the image:
$\mathbf{H}(\mathbf{x} ; r)=\left(\left(\partial_{i, j} G_{r}\right)_{i, j} * I\right)(\mathbf{x})$,
where $\left(\partial_{i, j} G_{r}\right)_{i, j}$ is the Hessian of the Gaussian function $G_{r}$ with scale $r$. Finding the eigenvalues of $\mathbf{H}$ is similar to find optimal directions of the associated quadratic form. For a vector $\mathbf{v}$, the quadratic form associated to the Hessian matrix is:

$$
\begin{aligned}
h(\mathbf{x}, \mathbf{v} ; r) & =\mathbf{v}^{T} \mathbf{H}(\mathbf{x} ; r) \mathbf{v} \\
& =\mathbf{v}^{T}\left\{\left(\left(\partial_{i, j} G_{r}\right)_{i, j} * I\right)(\mathbf{x})\right\} \mathbf{v} \\
& =\mathbf{v}^{T}\left\{\int_{\mathbb{R}^{d}} \partial_{i, j} G_{r}(\mathbf{h}) I(\mathbf{x}+\mathbf{h}) \mathrm{d} \mathbf{h}\right\}_{i, j} \mathbf{v}
\end{aligned}
$$

Using integration by parts, one can show that:

$$
\int_{\mathbb{R}^{d}} \partial_{i, j} G_{r}(\mathbf{h}) I(\mathbf{x}+\mathbf{h}) \mathrm{d} \mathbf{h}=-\int_{\mathbb{R}^{d}} \frac{\partial G_{r}}{\partial x_{j}}(\mathbf{h}) \frac{\partial I}{\partial x_{i}}(\mathbf{x}+\mathbf{h}) \mathrm{d} \mathbf{h} .
$$

Hence,

$$
\begin{aligned}
h(\mathbf{x}, \mathbf{v} ; r) & =\mathbf{v}^{T} \mathbf{H}(\mathbf{x} ; r) \mathbf{v} \\
& =-\int_{\mathbb{R}^{d}}((\nabla I(\mathbf{x}+\mathbf{h}) . \mathbf{v}) \mathbf{v}) . \nabla G_{r}(\mathbf{h}) \mathrm{d} \mathbf{h} .
\end{aligned}
$$



Fig. 8 Profile of the normalized flux weighting function $\frac{g_{r}^{\prime}(R)}{g_{r}^{\prime}(r)}=$ $\frac{\operatorname{Rg}_{r}(R)}{r g_{r}(r)}$ associated to the Hessian

Recall that the Gaussian function is spherically symmetric. Thus, one can write $G_{r}(\mathbf{h})=g_{r}(\|\mathbf{h}\|)=g_{r}(R)=$ $\frac{1}{(r \sqrt{2 \pi})^{d}} \exp \left(-\frac{R^{2}}{2 r^{2}}\right)$, leading to $\nabla G_{r}(\mathbf{h})=g_{r}^{\prime}(\|\mathbf{h}\|) \mathbf{n}$, where $\mathbf{n}$ is the outward normal to the sphere of radius $R$. Therefore, using reduction formulae, one can show that:

$$
\begin{align*}
h(\mathbf{x}, \mathbf{v} ; r)= & -\int_{0}^{+\infty} g_{r}^{\prime}(R) \\
& \times \underbrace{\left(\int_{\partial S_{R}}((\nabla I(\mathbf{x}+R \mathbf{n}) \cdot \mathbf{v}) \mathbf{v}) \cdot \mathbf{n} \mathrm{d} a\right)}_{(\mathrm{I})} \mathrm{d} R \tag{16}
\end{align*}
$$

where $\mathrm{d} a$ in term (I) is the infinitesimal area (or length) on the surface sphere $\partial S_{R}$. One can recognize the term (I) of (16) as the oriented flux introduced by Law and Chung (2008) (without the smoothing term). As a matter of fact, one can interpret function $h$ as a pondered flux of the projected image gradient along direction $\mathbf{v}$ toward spheres. In Fig. 8 the normalized weight function associated to the Hessian is presented. One can see that contrary to the Hessian, the OOF weight function (which is the indicator function for $S_{r}$ ) takes into account only the values of the image gradient on the circle of the desired scale. That explains the behavior of the Hessian based vesselness measure, as shown in Fig. 4 of Law and Chung (2008), which misses vessels of weak intensity especially when they are adjacent to other structures (arrows 5 and 6). Indeed, if one considers that the influence of adjacent objects is significant until some threshold $T$ of the weight function (for example the red line in Fig. 8 corresponds to $T=0.5$ ), then all objects in the vicinity of the desired point with distance to the desired point in $[\beta r, \alpha r]$ are taken into consideration. This is the weak point of the Hessian based vesselness measures, that is, adjacent structures are taken into consideration, and hence distort the measure. As an example, for $T=0.5$, we have $\alpha \approx 0.32$ and $\beta \approx 1.9$. One can conclude that the Hessian based vesselness measure is not an appro-
priate choice when the desired vessels are adjacent to other features in the image, which usually happens in nature.

We showed here, that the Hessian based vesselness measures may include adjacent structures yielding to bad results. The point is that the flux weighting function (see Fig. 8) associated to the Hessian matrix is too much extended around the desired scale, i.e. when $R / r=1$. Meanwhile, the weighting function associated to the optimally oriented flux is the indicator of $R=r$, hence it assumes that vessels response on the image is perfectly contrasted and with uniform intensity. On natural images, this assumption is obviously not satisfied.

In order to improve the vesselness measure response, one has to choose a flux weighting function that is in between the very discontinuous indicator function and the very extended profile given by the derivative of the Gaussian (see Fig. 8). Obviously, flux weighting function has to depend on the response of tubular structures on the image which depends on many parameters like image modality, type of vessels (coronary, pulmonary or other arteries), and the used contrast agents. An important question to raise here is the following: Is there an optimal way to define a weighting function for some desired arteries on a given image? And therefore, what are these meaningful optimality conditions one can impose on the weighting function? These questions remain open. One perspective of this work is to define a set of meaningful optimality conditions and to find the optimal weighting function associated to them as a learning step.

### 3.4 How to Design the Metric

Our main contribution is to impove Li and Yezzi method ( Li and Yezzi 2006, 2007) (see Sect. 1.2 above) by adding to it an anisotropic formulation, and the anisotropic metric is constructed by extension of the OOF descriptor presented by Law and Chung (2008).

We consider $(d+1) \mathrm{D}$ minimal path that minimizes the following energy:
$\int_{\gamma}\left\{\sqrt{\gamma^{\prime}(s)^{T} \mathcal{M}(\gamma(s)) \gamma^{\prime}(s)}\right\} \mathrm{d} s$,
where $\mathcal{M}$ is the $(d+1) \mathrm{D}$ anisotropic metric we want to construct. It is not natural to consider orientations on the $(d+1)$ th dimension, i.e. the radius dimension. Thus we propose to decompose by blocks the matrix $\mathcal{M}$ as follows:
$\mathcal{M}(\mathbf{x}, r)=\left(\begin{array}{cc}\tilde{\mathcal{M}}(\mathbf{x}, r) & 0 \\ 0 & \mathcal{P}_{\text {radius }}(\mathbf{x}, r)\end{array}\right)$
where $\tilde{\mathcal{M}}(\mathbf{x}, r)$ is a $d \times d$ symmetric definite positive matrix giving the spatial anisotropy and $\mathcal{P}_{\text {radius }}(\mathbf{x}, r)$ is the radius potential (also strictly positive).

Since the result given by the anisotropic minimal path method is very dependent on the metric, results inherit advantages and drawbacks of the constructed metric, and we should be very careful with its construction. First, let us fix conditions on the desired metric. The spatial metric $\tilde{\mathcal{M}}$ has to be well oriented along the vessel centerline. And the radius potential $\mathcal{P}_{\text {radius }}$ has to be small for the adequate scale for any point of the image. $\sqrt{\mathcal{P}_{\text {radius }}}$ corresponds to the inverse velocity along the radius dimension. Since $\tilde{\mathcal{M}}$ is symmetric definite positive, one can decompose it as follows: $\tilde{\mathcal{M}}()=.\sum_{i=1}^{d} m_{i}(.) \mathbf{u}_{i}(.) \mathbf{u}_{i}(.)^{T}$, where $0<m_{1} \leq \cdots \leq m_{d}$ are the eigenvalues and $\mathbf{u}_{i}$ are the associated eigenvectors. The velocity of the propagating front along direction $\mathbf{u}_{i}$ is equal to $1 / \sqrt{m_{i}}$. In order to design the metric, let us recall that the oriented flux matrix is symmetric (but not necessarily positive), and it can be decomposed as follows: $\mathbf{Q}()=.\sum_{i=1}^{d} \lambda_{i}(.) \mathbf{v}_{i}(.) \mathbf{v}_{i}(.)^{T}$. The metric $\mathcal{M}$ has to be definite positive, and we want to make the propagation faster along the estimated vessel direction, i.e. $\mathbf{v}_{d}$. Therefore, we combined the OOF's eigenvalues and eigenvectors to construct the metric as follows:

$$
\left\{\begin{array}{l}
\tilde{\mathcal{M}}(.)=\sum_{i=1}^{d} \exp \left(\alpha \frac{\sum_{j \neq i} \lambda_{j}(.)}{d-1}\right) \mathbf{v}_{i}(.) \mathbf{v}_{i}(.)^{T}  \tag{19}\\
\mathcal{P}_{\text {radius }}(.)=\beta \exp \left(\alpha \frac{\sum_{i=1}^{d} \lambda_{i}(.)}{d}\right)
\end{array}\right.
$$

The choice of the spatial metric $\tilde{\mathcal{M}}$ is motivated by the fact that the propagation velocity along direction $\mathbf{v}_{i}$ is given by $\nu_{i}:=\exp \left(-\frac{\alpha}{2} \frac{\sum_{j \neq \neq} \lambda_{j}(.)}{d-1}\right)$ and therefore ${ }^{10} \nu_{d} \geq \cdots \geq v_{1}$. The radius potential $\mathcal{P}_{\text {radius }}$ is based on the OOF vesselness, proposed in the original paper Law and Chung (2008), which is the trace of the oriented flux matrix: $\operatorname{Tr}(\mathbf{Q})=\sum_{i} \lambda_{i}$. We proposed a formula coherent with the spatial metric $\tilde{\mathcal{M}}$ and we added parameter $\beta>0$ in order to control the radius propagation velocity, i.e. $v_{\text {radius }}:=1 / \sqrt{\mathcal{P}_{\text {radius }}}$, according to the spatial velocities $v_{i}$.

Here, we introduced two parameters: $\alpha$ and $\beta . \alpha$ is controlled by an intuitive parameter, which is the maximal spatial anisotropy ratio, noted $\mu$. It corresponds to the maximal spatial speed ratio one wants to impose. In the 2D case,

$$
\begin{aligned}
\mu & =\max _{\mathbf{x}, r} \sqrt{\frac{\exp \left(\alpha \lambda_{2}(\mathbf{x}, r)\right)}{\exp \left(\alpha \lambda_{1}(\mathbf{x}, r)\right)}} \\
& =\exp \left(\frac{1}{2} \alpha\left\{\max _{\mathbf{x}, r}\left(\lambda_{2}(\cdot)-\lambda_{1}(\cdot)\right)\right\}\right)
\end{aligned}
$$

[^9]Fig. 9 The constructed metric for different scales $r=5,10,15$ from left to right. The original image is shown in Fig. 5(a), the radius of the structure is equal to 10 . We used the same color range for all images, so one can see that the optimal anisotropy is obtained along the centerline of the tubular structure when the scale $r$ is equal to the exact radius of the tube. On the top, we show a display of $\tilde{\mathcal{M}}(\mathbf{x}, r)^{-1}$. On the bottom, responses of $\mathcal{P}_{\text {radius }}$ are shown

and in 3D,

$$
\begin{aligned}
\mu & =\max _{\mathbf{x}, r} \sqrt{\frac{\exp \left(\alpha \frac{\lambda_{2}(\mathbf{x}, r)+\lambda_{3}(\mathbf{x}, r)}{2}\right)}{\exp \left(\alpha \frac{\lambda_{1}(\mathbf{x}, r)+\lambda_{2}(\mathbf{x}, r)}{2}\right)}} \\
& =\exp \left(\frac{1}{4} \alpha\left\{\max _{\mathbf{x}, r}\left(\lambda_{3}(\cdot)-\lambda_{1}(\cdot)\right)\right\}\right)
\end{aligned}
$$

Geometrically, $\mu$ corresponds to the maximal ratio between the major and minor axis of ellipsoids displaying $\tilde{\mathcal{M}}$ as in Fig. 9.

By choosing the maximal spatial anisotropy ratio $\mu$, the constant $\alpha$ is fixed. And by doing so, the anisotropy descriptor $\mathcal{M}$ becomes affine contrast invariant because the OOF is linear on the image. The parameter $\beta$ controls the radius speed. If $v_{\text {radius }} \geq v_{d}$ then the Fast Marching propagation is faster along the radius direction than the spatial directions. If $v_{\text {radius }} \leq \nu_{1}$ then the propagation is slower. One can tune parameter $\beta$ depending on the tubular structure one wants to extract. If its radius changes a lot then $\beta$ should be chosen such that the propagation on the radius dimension is faster. Otherwise, $\beta$ is chosen such that the propagation is less sensitive on the radius dimension.

In Fig. 9, the constructed metric of Fig. 5 is shown at different scales. Since we chose the same color range for the visualization, we can see that the directions are well detected, and that the optimal values are obtained along the centerline of the tube when the scale is equal to the tube radius. For our experiments, we took $\mu=5$ and $\beta$ such that the radius speed is at least twice as high as the best spatial propagation speed. We did so, because we wanted our algorithm to be sensitive to the radius dimension.

## 4 Experimental Results

### 4.1 Advantages of Anisotropy

First, let us compare the results obtained by the anisotropic model we propose and the original isotropic model proposed by Li and Yezzi (2006, 2007). Again, the choice of the isotropic potential is crucial. As shown in the previous section, the proposed radius potential $\mathcal{P}_{\text {radius }}$ given by (19) has the good property of taking its smallest values along the centerline of a tubular shape when the scale $r$ is equal to its radius. Therefore, a reasonable choice of a potential that drives the propagation along the centerline for the right radius is $\mathcal{P}_{\text {radius }}$ itself. Nevertheless, in order to make a fair comparison between the isotropic and anisotropic model the potential $\mathcal{P}$ should be $\mathcal{P}=\sqrt{\mathcal{P}_{\text {radius }}}$. Indeed, according to notations of Sect. 2, the metric associated to the isotropic case is $\mathcal{M}=\mathcal{P}^{2} I d$, where $I d$ is the identity matrix.

For the example in Fig. 10, we took $\mu=6$. As expected, the anisotropy model is more robust than the isotropic one by avoiding the shortcut. We also tried the isotropic method using the two other components of the anisotropic metric $\mathcal{M}$, namely $\exp \left(\alpha \lambda_{1}\right)$ and $\exp \left(\alpha \lambda_{2}\right)$, and we obtained a similar shortcut.

### 4.2 Results on Synthetic Data

Our method is minimally interactive. First, the user has to decide if the desired vessels are darker or brighter than the background. So, we can consider different criteria on the signs of the eigenvalues. Then the scale range $\left[r_{\text {min }}, r_{\text {max }}\right]$,

Fig. 10 Comparing segmentation results, on a synthetic "U" shape, when using isotropic and anisotropic models. Here, we took $\mu=6$


Fig. 11 (Color online) The red cross points are source points given by the user, and the blue ones are endpoints. On each case the segmented centerlines are displayed as well as the envelope of the moving discs. The associated minimal action map $\mathcal{U}$ as well as the 3 D minimal path between the two selected points are shown (transparent visualization)

Fig. 12 On the left, original synthetic tube with changing radii. We added $10 \%$ of salt and pepper noise. On the middle, selected source and destination are well centered. On the right, decentered initialization

which corresponds to the range of radii of the vessel one wants to extract, is given by the user. In practice, the user has to provide only the minimal radius value $r_{\min }$ and the maximal one $r_{\text {max }}$. Using the spatial image spacing, i.e. $h_{x}, h_{y}$ and $h_{z}$, we chose a radius spacing equal to $\frac{1}{2} \min \left(h_{x}, h_{y}, h_{z}\right)$ in order to respect the Nyquist sampling rate. Finally, few points are required as source points or end points of the Fast Marching algorithm. We used the metric described in the previous section to find the minimal anisotropic path (as described in Sect. 2) between two or more selected points (see Fig. 11). For any selected point, the associated initial radius is equal to the minimal radius $r_{\text {min }}$ given by the user.

In Fig. 11, segmentation results on synthetic 2D images are shown. In the first synthetic image, the source point and destination are selected on the centerline. The obtained tube is perfectly detected as well as the centerline. In the second image, the initial points are not centered. But the centerline obtained by our algorithm goes back fast to the real centerline. This makes our algorithm robust to initialization. The third synthetic image shows that our approach is robust to scale changing.

In Fig. 12, we defined a cylinder in a 3D volume, with two different radii ( 4 and 6 voxels) on each half side of


Fig. 13 Evolution of the radius along the obtained minimal path of Fig. 12 middle, when the points are well centered along the desired tube


Fig. 14 Influence of the maximal anisotropy ratio $\mu$ on the obtained minimal tube. For $\mu=3$ a shortcut occurs
the volume. First, we tried our method with a reasonable anisotropy ratio, $\mu=3$, and by choosing the source and destination on the centerline of the desired tube. One can see that the desired tube is well segmented. Moreover, in Fig. 13 the evolution of the radius along the obtained minimal path shows that the radii are well detected. Also, we gave a bad initialization of the source and the destination for the second example of Fig. 12, and obtained a reasonable result.

In order to challenge our method, we initialized the source and destination far from the desired tubular shape, in a situation for which a shortcut likely occurs, see Fig. 14. The original image used here is the synthetic volume of Fig. 12. One can see that for a reasonably small maximal anisotropy ratio $\mu$, a shortcut occurs. While choosing a slightly higher value of $\mu$ allows us to overcome this problem.

Finally, in the example of Fig. 15 we designed a synthetic helix with linearly increasing radius (from 3 to 5 voxels). Similarly to the example of Fig. 14, we increased the value of the maximal anisotropy ratio $\mu$ to 5 , in order to overcome the shortcut issue. The chosen radii range is [1,6]. One can point out that the obtained centerline (the blue one) is slightly different than the theoretic one specifically near the source and the destination points. This is due to a boundary effect. Indeed, the source and destination are on the boundary of the volume and the computation of the OOF (on spatial or even on the Fourier domain) requires a special care for points near the boundary. For the spatial implementation of the OOF, when an image value for a point outside the domain is needed, we choose the nearest value on the domain. This choice has no effect on the orientation given by the OOF in the image of Fig. 12, while it can change considerably the estimation of the vessel orientation for points in the vicinity of the domain boundary in general and particularly for the example given in Fig. 15. Also, in the same figure one can see the


Fig. 15 (Color online) On the left, original synthetic helix with linearly changing radii. We added $10 \%$ of salt/pepper noise. On the middle, obtained segmentation. The blue line is the obtained centerline
and the red one is the theoretic centerline. On the right, evolution of the radius along the obtained minimal path

Fig. 16 (Color online) On the left, original images. On the right, the red cross points are source points given by the user, and the blue ones are endpoints. On each case the segmented centerlines are displayed as well as the envelope of the moving discs


Fig. 17 First line: RCA segmentation using the tubular anisotropy approach shown on the whole image and on the selected sub-volume. Second line: LAD segmentation shown on the whole image and on the selected sub-volume. Only few points are required (the extremities of the paths). The tubular anisotropy method provides the centerline as well as vessels boundaries

evolution of the radius along the obtained minimal path. Due to the voxelization, the radius evolution behaves almost like a piecewise constant function rather than a linear function.

### 4.3 Results on Real Images

In this section we are going to show some segmentation results on real images obtained by our method. For 2D images, see Fig. 16, we took a maximal anisotropy ratio $\mu=5$, and as said before, $\beta$ is chosen such that the propagation speed along the radius dimension is at least twice the best propagation spatial speed. One can claim that our algorithm is robust to highly curved tubular structures, noise and bifurcations.

In Fig. 17, segmentation results are shown on real medical images. First, right coronary arteries (RCA) are segmented. Second, left anterior descending (LAD) arteries are segmented. One can see that the obtained radius on the principal coronary branches are larger than those of the secondary. Thus, our approach is robust to scale changing and bifurcations. The results presented here are encouraging but obviously not sufficient. A clinical validation is presented in the next section.

### 4.4 Clinical Validation

In order to make a genuine validation, we participated to "The Carotid Bifurcation Algorithm Evaluation Frame-
work ${ }^{111}$ which took place during a workshop of the MICCAI'09 conference (Hameeteman et al. 2009). The goal of this framework is to evaluate and compare different algorithms for carotid bifurcation, lumen segmentation and stenosis grading from CTA data.

The tubular anisotropy method proposed in this paper, represents vessels with circular cross sections. This may yield various inaccuracies in the provided datasets, which contain vessels exhibiting non circular lumen and strong lumen occlusions. Therefore, we proposed to refine the segmentation using a region based level set model, see Mille et al. (2009) for more details.

The competition data consists of CT angiography images acquired at the Erasmus MC, University Medical Center Rotterdam, The Netherlands (36 datasets), Hôpital Louis Pradel, Bron, France (10 datasets) and the Hadassah Hebrew University Medical Centre, Jerusalem, Israel (10 datasets). The datasets have been selected such that they contain a large range of stenoses, from fully open to severe stenotic. Each hospital provided three points on the main branches of the carotid, for the whole data sets, and a manually drawn lumen segmentation for the training data sets, see Hameeteman et al. (2009) for more details.

Organizers of the competition proposed four performance measures:

[^10]Table 1 Summary lumen (Mille et al. 2009)

| Measure | \%/mm |  |  | rank |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  | min. | max. | avg. | min. | max. | avg. |
| L_dice | 43.3\% | 93.6\% | 83.6\% | 3 | 4 | 3.97 |
| L_msd | 0.20 mm | 3.98 mm | 0.80 mm | 3 | 4 | 3.97 |
| L_rmssd | 0.29 mm | 6.50 mm | 1.57 mm | 3 | 4 | 3.97 |
| L_max | 1.11 mm | 17.50 mm | 6.47 mm | 4 | 4 | 4.00 |
| Total (lumen) |  |  |  | 3 | 4 | 3.98 |

Table 2 Averages lumen (Mille et al. 2009)

| Team name | Total success | dice |  | msd |  | rmssd |  | max |  | Total rank |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  |  | \% | rank | mm | rank | mm | rank | mm | rank |  |
| TubularAnisotropy | 31 | 83.6 | 4.0 | 0.80 | 4.0 | 1.57 | 4.0 | 6.47 | 4.0 | 4.0 |
| ObserverA | 31 | 95.4 | 1.5 | 0.10 | 1.5 | 0.13 | 1.6 | 0.56 | 1.9 | 1.6 |
| ObserverB | 31 | 94.8 | 2.4 | 0.11 | 2.4 | 0.15 | 2.3 | 0.59 | 1.8 | 2.2 |
| ObserverC | 31 | 94.7 | 2.2 | 0.11 | 2.2 | 0.15 | 2.1 | 0.71 | 2.3 | 2.2 |

1. The Dice similarity index $D_{\mathrm{si}}$
$D_{\mathrm{si}}=\frac{2 \times\left|p v_{r} \cap p v_{p}\right|}{\left|p v_{r}\right|+\left|p v_{p}\right|}$,
where $p v_{r}$ and $p v_{p}$ are the reference and a participants partial volumes, the intersection operation is the voxelwise minimum operation, and $|$.$| is the volume, i.e. the$ integration of the voxel values over the complete image.
2. The mean surface distance $D_{\text {msd }}$,
$D_{\mathrm{msd}}=\frac{1}{2} \times\left(\frac{\int_{S_{r}}\left|s d m_{p}\right| \mathrm{d} s}{\left|S_{r}\right|}+\frac{\int_{S_{p}}\left|s d m_{r}\right| \mathrm{d} s}{\left|S_{p}\right|}\right)$,
where $s d m_{r}$ and $s d m_{p}$ are the signed distance maps of the reference and a participants segmentation, and $S_{r}$ and $S_{p}$ are the lumen boundary surfaces (isosurfaces of the signed distance map at the value 0 ), and $\left|S_{i}\right|$ is the surface area of surface $S_{i}$, i.e. $\left|S_{i}\right|=\int_{S_{i}} \mathrm{~d} s$.
3. The root mean squared surface distance $D_{\mathrm{rmss}}$,

$$
D_{\mathrm{rmssd}}=\frac{1}{2} \times\left(\sqrt{\frac{\int_{S_{r}}\left|s d m_{p}\right|^{2} \mathrm{~d} s}{\left|S_{r}\right|}}+\sqrt{\frac{\int_{S_{p}}\left|s d m_{r}\right|^{2} \mathrm{~d} s}{\left|S_{p}\right|}}\right)
$$

4. Maximum surface distance $D_{\max }$

$$
D_{\max }=\frac{1}{2} \times\left(\max _{x \in S_{r}}\left|s d m_{p}(x)\right|+\max _{x \in S_{p}}\left|s d m_{r}(x)\right|\right)
$$

All distance measures are symmetric, and all these measures are only evaluated in the region of interest, see Hameeteman et al. (2009) for more details.

A summary of lumen segmentation scores is presented in Table 1. Averages lumen scores are presented in Table 2. The average processing time for a single data set was approximately 2 minutes. Our dice mean score is $83.6 \%$, showing that our method is relatively robust. Over nine teams that have participated to the contest, our method is the fourth. As said previously, this might be explained by the fact that the Tubular Anisotropy method proposed in this paper detects vessels with circular cross sections yielding to inaccuracies on strong lumen occlusions. Moreover, for numerical reasons, we did not take advantage of the tubular anisotropy method by increasing as much as possible the anisotropy ratio $\mu$. This point has been discussed in Sect. 2.3.

## 5 Conclusion

In this paper we have proposed a new general method for tubular structure extraction in 2D and 3D images. Our method exploits the orientation of the vessels by using the optimally oriented flux to construct a multi-resolution anisotropic metric that extracts from the image the local geometry and describes the vessels orientation and scales. Combining this metric with anisotropic minimal path technique, we were able to find a complete description of the tubular structure, i.e. the centerline as well as the boundary. To summarize, our method is minimally interactive, robust to initialization, scale variations, and bifurcations. Moreover, obtained clinical validation results are encouraging.

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[^1]:    ${ }^{1}$ A minimal path minimizes an energy weighted by an image potential.

[^2]:    ${ }^{2}$ In the classical Finsler formulation, the potential is under the form $\mathcal{P}: \mathbb{R}^{n} \times \mathbb{S}^{n-1} \rightarrow \mathbb{R}_{+}^{*}$.

[^3]:    ${ }^{3}$ Finsler geometry is just Riemannian geometry without the quadratic restriction (Chern 1996).
    ${ }^{4}$ A Riemannian metric is described by a symmetric definite positive tensor field: $\mathcal{M}$. Each such matrix can be represented by an ellipsoid.

[^4]:    ${ }^{5}$ Indeed, it is easy to show that, in our case, the required conditions for existence and uniqueness of the viscosity solution are satisfied.

[^5]:    ${ }^{6}$ Under some regularity assumptions on $P$, see Rouy and Tourin (1992) for more details.

[^6]:    ${ }^{7}$ This formulation is based on Huygens-Fresnel principle from geometric optics and problems of wave propagation.

[^7]:    ${ }^{8}$ The complexity of the algorithm proposed in Bornemann and Rasch (2006) is $\mathcal{O}\left(C_{\mu} N^{1+1 / d}\right)$, where $d$ is the dimension of the image and $N$ the total number of grid points, compared to $\mathcal{O}(N \log (N))$ for Lin's (2003). It has been shown empirically (by our care) that $C_{\mu}$ is an increasing function on $\mu$. Actually, $C_{\mu}$ depends also on the topology of the features in the image.

[^8]:    ${ }^{9}$ Centered at point $\mathbf{x}$

[^9]:    ${ }^{10}$ This can be easily verified using the fact that $\lambda_{1} \leq \lambda_{2} \leq \cdots \leq \lambda_{d}$.

[^10]:    ${ }^{11}$ http://cls2009.bigr.nl/

