

Sheet 3. Brownian motion, martingales. Solutions.

Unless stated otherwise, we denote by $B = (B_t)_{t \geq 0}$ a Brownian motion with respect to some filtration $(\mathcal{F}_t)_{t \geq 0}$ (i.e. B is a Brownian motion and for each $t \geq 0$, the process $(B_{t+s} - B_s, s > 0)$ is independent of $\mathcal{F}_t$). We also set $\tau_x = \inf\{t \geq 0 : B_t = x\}$ for all $x \in \mathbb{R}$.

Exercise 1 1. Show that $(B_t^2 - t)_{t \geq 0}$ is a martingale.

2. Construct a martingale from $(B_t^3)_{t \geq 0}$. Same question with B_t^4 .

3. Prove that $(e^{\lambda B_t - \frac{\lambda^2}{2}t})_{t \geq 0}$ is a martingale for all $\lambda \in \mathbb{R}$.

4. Construct a martingale from $(\cosh(\lambda B_t))_{t \geq 0}$.

Solution. 2. We have: $B_t^3 = (B_t - B_s + B_s)^3 = B_s^3 + 3B_s^2(B_t - B_s) + 3B_s(B_t - B_s)^2 + (B_t - B_s)^3$. Hence $\mathbb{E}[B_t^3 | \mathcal{F}_s] = B_s^3 + 3B_s(t - s)$. But (B_t) is a martingale, so $\mathbb{E}[B_t^3 - 3tB_t | \mathcal{F}_s] = B_s^3 - 3sB_s$ ie $(B_t^3 - 3tB_t)$ is a martingale. Similarly, $\mathbb{E}[B_t^4 | \mathcal{F}_s] = B_s^4 + 6B_s^2(t - s) + 3(t - s)^2$. Then, using that $B_t^2 - t$ is a martingale, we have $\mathbb{E}[B_t^2 | \mathcal{F}_s] = B_s^2 + (t - s)$, so

$$\begin{aligned} \mathbb{E}[B_t^4 - 6tB_t^2 | \mathcal{F}_s] &= B_s^4 + 6B_s^2(t - s) + 3(t - s)^2 - 6t(B_s^2 + (t - s)) \\ &= B_s^4 - 6sB_s^2 - 3t^2 + 3s^2. \end{aligned}$$

Hence $(B_t^4 - 6tB_t^2 + 3t^2)_{t \geq 0}$ is a martingale.

Exercise 2 Let $a > 0$.

- Using the martingale $(B_t^2 - t)_{t \geq 0}$, compute the expectation of $T_a^* = \inf\{t \geq 0 : |B_t| = a\}$.
- Using a well-chosen martingale, compute the variance of T_a^* and its Laplace transform.
- Compute the Laplace transform of τ_a and find that it has the same law as $(a/B_1)^2$. What is the value of $\mathbb{E}[\tau_a]$?

Solution. 1. The stopped process $B_{n \wedge T_a^*}^2 - n \wedge T_a^*$ is a martingale so that $\mathbb{E}[B_{T_a^* \wedge n}^2] = \mathbb{E}[T_a^* \wedge n]$. Since $|B_{T_a^* \wedge n}| \leq a$ and $B_{T_a^*}^2 = a^2$ a.s., dominated and monotone convergence yield $\mathbb{E}[T_a^*] = a^2$.

2. Take the martingales $B_t^4 - 6tB_t^2 + 3t^2$ and $\cosh(\lambda B_t)$.

Exercise 3 1. Let M be a martingale with continuous trajectory such that $M_0 = x \in \mathbb{R}_+$. Suppose that $M_t \geq 0$ for all $t \geq 0$ and that $M_t \rightarrow 0$ as $t \rightarrow \infty$, a.s. Show that, for all $y > x$,

$$\mathbb{P}\left(\sup_{t \geq 0} M_t \geq y\right) = \frac{x}{y}.$$

2. Deduce the law of

$$\sup_{t \leq T_0} B_t,$$

when B is a Brownian motion started at $x > 0$ and $T_0 = \inf\{t \geq 0 : B_t = 0\}$.

3. Suppose now that B is started at 0, and let $\mu > 0$. Using a well-chosen exponential martingale, prove that

$$\sup_{t \geq 0} (B_t - \mu t)$$

follows the exponential distribution of parameter 2μ .

Solution.

1. Since M is continuous, we have $P(\sup_t M_t \geq y) = P(\tau_y < \infty)$, where $\tau_y = \inf\{t \geq 0 : M_t = y\}$. Consider $N = M^{\tau_y}$ the stopped martingale. N is a martingale so $E[N_t] = E[N_0] = x$. Then, N_t converges almost surely as $t \rightarrow \infty$ towards $y\mathbf{1}_{\tau_y < \infty} + 0\mathbf{1}_{\tau_y = \infty}$. Furthermore, $|N_t| \leq y$ a.s. By dominated convergence, $E[N_t] \rightarrow yP(\tau_y < \infty)$.
2. Set $M_t = B_{t \wedge T_0}$ and apply question 1 to find that for all $y > x$, $P(\sup_{t \leq T_0} B_t \geq y) = P(\sup_{t \geq 0} M_t \geq y) = x/y$.
3. Set $\lambda = 2\mu$ and $M_t = \exp\{\lambda B_t - \frac{\lambda^2}{2}t\}$. Then $P(\sup_{t \geq 0} (B_t - \mu t) \geq y)$ writes

$$P\left(\sup_{t \geq 0} (2\mu B_t - 2\mu^2 t) \geq 2\mu y\right) = P\left(\sup_{t \geq 0} (2\mu B_t - 2\mu^2 t) \geq 2\mu y\right) = P\left(\sup_{t \geq 0} M_t \geq e^{2\mu y}\right),$$

since $2\mu^2 = 2\lambda^2/4 = \lambda^2/2$. By question 1, the last probability is equal to $e^{-2\mu y}$.

Exercise 4 Let $a < 0 < b$ and set $T = \tau_a \wedge \tau_b$.

1. Prove that, for all $\lambda > 0$,

$$E[\exp(-\lambda T)] = \frac{\cosh\left(\frac{b+a}{2}\sqrt{2\lambda}\right)}{\cosh\left(\frac{b-a}{2}\sqrt{2\lambda}\right)}.$$

(Indication: introduce the martingale

$$M_t = \exp\left(\sqrt{2\lambda}(B_t - \alpha) - \lambda t\right) + \exp\left(-\sqrt{2\lambda}(B_t - \alpha) - \lambda t\right),$$

with a well-chosen α .)

2. Similarly, prove that, for all $\lambda > 0$,

$$E[\exp(-\lambda T)\mathbf{1}_{T=\tau_a}] = \frac{\sinh(b\sqrt{2\lambda})}{\sinh((b-a)\sqrt{2\lambda})}.$$

3. With the help of question 2, give an expression of $P(T_a < T_b)$.

Exercise 5 If X is a continuous process vanishing at 0, such that, for every real α , the process $M_t^\alpha = \exp\left\{\alpha X_t - \frac{\alpha^2 t}{2}\right\}$ is a martingale with respect to $(\mathcal{F}_t)$, prove that X is a $(\mathcal{F}_t)$ -Brownian motion. (*Hint:* use that if X is a r.v. and $\mathcal{G}$ is a sub- σ -algebra such that

$$E[e^{\lambda X}|\mathcal{B}] = E[e^{\lambda X}] < \infty,$$

for λ in a neighborhood of 0, then X and $\mathcal{B}$ are independent.)

Solution. Let $t \geq s$. We have $E[e^{\alpha(X_t - X_s) - \frac{\alpha^2(t-s)}{2}}|\mathcal{F}_s] = E[e^{\alpha X_t - \frac{\alpha^2 t}{2}}|\mathcal{F}_s]e^{-\alpha X_s + \frac{\alpha^2 s}{2}} = 1$. Therefore, $(X_t - X_s)$ is independent from $\mathcal{F}_s$ and $X_t - X_s \sim \mathcal{N}(0, t-s)$. It follows directly that for $t_1 \leq \dots \leq t_n$, the vector $(X_{t_n} - X_{t_{n-1}}, \dots, X_{t_2} - X_{t_1}, X_{t_1})$ is Gaussian, and thus $(X_{t_1}, \dots, X_{t_n})$ is also Gaussian by linear transformation. This combined with continuity entails that X is a BM.

Exercise 6 (Wald's identity). Let T be an integrable stopping time. Show that $E[B_T] = 0$ and that $E[B_T^2] = E[T]$.

Solution. Using that $B_t^2 - t$ is a martingale, we obtain that $(B_{n \wedge T})_n$ is a martingale bounded in L^2 . Therefore it converges in L^2 .