

Sheet 2. Brownian motion. Markovian properties. Some solutions.

Exercise 1 Let $\tau = \inf\{t \geq 0 : B_t = 1\}$ and $\eta = \inf\{t \geq \tau : B_t = 0\}$.

1. Is η a stopping time?
2. Determine the distribution of η .

Solution. 1. We have

$$\begin{aligned} \{\eta \leq t\} &= \{\tau \leq t\} \cap \left\{ \inf_{u \in [\tau, t]} B_u \leq 0 \right\} \\ &= \{\tau \leq t\} \bigcap \bigcup_{n \geq 1} \bigcup_{q \in \mathbb{Q} \cap [0, t]} \left\{ \tau \leq q, |B_q| < \frac{1}{n} \right\}, \end{aligned}$$

by continuity of the path (we disregard $\{\tau = t\} \subset \{B_t = 0\}$ which is $\mathcal{F}_t$ -measurable and of null probability). Since τ is a stopping time, all these events belong to $\mathcal{F}_t$, and hence η is a stopping time.

2. Let $\tilde{B}_t = B_{t+\tau} - B_\tau$ and $\tilde{\tau}_{-1} = \inf\{t \geq 0 : \tilde{B}_t = -1\}$. By definition, we have $\eta = \tau + \tilde{\tau}_{-1}$. By the Strong Markov property, $\tilde{B}$ has the law of a Brownian motion and is independent of $\mathcal{F}_\tau$. This implies that $\tilde{\tau}_{-1}$ has the same law as τ_{-1} and is independent of τ . Moreover, $\tau_1 \stackrel{d}{=} \tau_{-1} \stackrel{d}{=} 1/B_1^2$ where the last equality in law is by the reflection principle (cf. the course). Hence $\eta \stackrel{d}{=} B_1^{-2} + \tilde{B}_1^{-2}$ where B and $\tilde{B}$ are two independent Brownian motions.

Exercise 2 Let $S_t = \sup_{s \in [0, t]} B_s$. Prove that $S_2 - S_1$ is distributed as $\max\{|N| - |N'|, 0\}$, where N and N' are independent Gaussian $\mathcal{N}(0, 1)$ random variables.

Solution. Recall that $S_1 \stackrel{d}{=} |N|$ by the reflection principle. Write $S_2 - S_1 = \max\{0, \sup_{1 \leq t \leq 2} B_t - S_1\}$. This is $\max\{0, S'_1 - (S_1 - B_1)\}$, where $S'_1 = \sup_{0 \leq t \leq 1} (B_{1+t} - B_1)$ is independent of $\mathcal{F}_1$ and has law $|N|$ by Markov's property. By time reversal, $S_1 - B_1$ has the same law as S_1 , hence as $|N'|$. Independence of S'_1 from $S_1 - B_1$ gives the result.

Exercise 3 Prove that as $t \rightarrow \infty$, $(\int_0^t e^{B_s} ds)^{1/t^{1/2}} \rightarrow e^{|N|}$ in distribution, where N is a Gaussian $\mathcal{N}(0, 1)$ random variable.

Solution. Take the logarithm. By the change of variable $s = tu$ and Brownian scaling $(B_{tu})_u \stackrel{d}{=} (\sqrt{t}B_u)_u$,

$$\frac{1}{\sqrt{t}} \log \int_0^t e^{B_s} ds \stackrel{d}{=} \frac{\log t}{\sqrt{t}} + \frac{1}{\sqrt{t}} \log \int_0^1 e^{\sqrt{t}B_u} du.$$

It is not hard to check that for any continuous function f on $[0, 1]$, one has

$$\frac{1}{t} \log \int_0^1 e^{tf(u)} du \rightarrow \sup_{u \in [0, 1]} f(u), \quad t \rightarrow \infty.$$

(An upper bound is clear. For a lower bound, consider an interval where the function is at distance at most ε from the maximum of f .) Hence by continuity of B , we have $\frac{1}{\sqrt{t}} \log \int_0^1 e^{\sqrt{t}B_u} du \rightarrow \sup_{u \leq 1} B_u$ almost surely. The supremum has law $|N|$ by the reflection principle, and $\log t / \sqrt{t} \rightarrow 0$. Exponentiating yields the claimed convergence in distribution.

Exercise 4 Prove that $\int_0^\infty e^{-|B_t|^2} dt = \infty$ a.s.

Solution. Define iteratively $\sigma_0 = 0$, $T_i = \inf\{t \geq \sigma_i : B_t = 1\}$ and $\sigma_{i+1} = \inf\{t \geq T_i : B_t = 0\}$. These are stopping times which are almost surely finite and strictly positive. By the strong Markov property, $X_i = \int_{\sigma_i}^{T_i} e^{-|B_t|^2} dt$ are independent, positive (non-zero) random variables. Hence $X = \sum_i X_i = \infty$ almost surely. The inequality $\int_0^\infty e^{-|B_t|^2} dt \geq X$ concludes.

Exercise 5 (Zeros of Brownian motion) Let $H := \{t \in [0, 1] : B_t = 0\}$. Using the strong Markov property, show that H is a.s. a compact set without any isolated point and of zero Lebesgue measure.

Solution. By continuity of the path, $H = B^{-1}(\{0\}) \cap [0, 1]$ is almost surely a closed and bounded set, hence a compact set. Its Lebesgue measure satisfies $\mathbb{E}[\text{Leb}(H)] = \int_0^1 \mathbb{E}[\mathbf{1}_{B_t=0}] dt = 0$ by Fubini-Tonelli and $\mathbb{P}(B_t = 0) = 0$. By positivity, $\text{Leb}(H) = 0$ almost surely.

We now show that the zeros are not isolated. We start by an observation. Let $p \in [0, 1]$ be a rational and define $\tau_p = \inf\{s \geq p : B_s = 0\}$. By the strong Markov property, $\tilde{B}_t = B_{t+\tau_p} - B_{\tau_p} = B_{t+\tau_p}$ is a Brownian motion started at zero. Therefore, there exists another zero of $\tilde{B}$ arbitrarily close to 0 (to justify this, consider a (random) sequence $t_i > 0$, $t_i \rightarrow 0$ such that $\tilde{B}_{t_i} > 0$ infinitely often and $\tilde{B}_{t_i} < 0$ infinitely often, which exists almost surely by the course. The continuity of $\tilde{B}$ concludes for the existence). This implies that τ_p is not an isolated zero of B almost surely. This extends almost surely to all rationals p as $\mathbb{Q}$ is countable.

Now suppose that $t \in H$ is an isolated zero. Then there exists two rationals p and q in $[0, 1]$ such that $p < t < q$ and $[p, q]$ contains only one zero of B which is t . Therefore $\tau_p = t$ by continuity of the path. However τ_p is not an isolated zero which provides a contradiction.