Sheet 3. Brownian motion, martingales.

Unless stated otherwise, we denote by $B = (B_t)_{t\geq 0}$ a Brownian motion with respect to some filtration $(\mathcal{F}_t)_{t\geq 0}$ (i.e. B is a Brownian motion and for each $t \geq 0$, the process $(B_{t+s} - B_s, s > 0)$ is independent of $\mathcal{F}_t$). We also set $\tau_x = \inf\{t \geq 0 : B_t = x\}$ for all $x \in \mathbb{R}$.

Exercise 1

1. Show that $(B_t^2 - t)_{t\geq 0}$ is a martingale.
2. Construct a martingale from $(B_t^3)_{t\geq 0}$. Same question with B_t^4.
3. Prove that $(e^{\lambda B_t - \frac{\lambda^2}{2}t})_{t\geq 0}$ is a martingale for all $\lambda \in \mathbb{R}$.
4. Construct a martingale from $\{\cosh(\lambda B_t)\}_{t\geq 0}$.

Exercise 2 Let $a > 0$.

1. Using the martingale $(B_t^2 - t)_{t\geq 0}$, compute the expectation of $T_a^\star = \inf\{t \geq 0 : |B_t| = a\}$.
2. Using a well-chosen martingale, compute the variance of $T_a^\star$ and its Laplace transform.
3. Compute the Laplace transform of τ_a and find that it has the same law as $(a/B_1)^2$. What is the value of $\mathrm{E}[\tau_a]$?

Exercise 3

1. Let M be a martingale with continuous trajectory such that $M_0 = x \in \mathbb{R}_+$. Suppose that $M_t \geq 0$ for all $t \geq 0$ and that $M_t \to 0$ as $t \to \infty$, a.s. Show that, for all $y > x$,
$$\mathrm{P}\left(\sup_{t\geq 0} M_t \geq y\right) = \frac{x}{y}.$$
2. Deduce the law of
$$\sup_{t\leq T_0} B_t,$$
when B is a Brownian motion started at $x > 0$ and $T_0 = \inf\{t \geq 0 : B_t = 0\}$.
3. Suppose now that B is started at 0, and let $\mu > 0$. Using a well-chosen exponential martingale, prove that
$$\sup_{t\geq 0}(B_t - \mu t)$$
follows the exponential distribution of parameter 2μ.

Exercise 4 Let $a < 0 < b$ and set $T = \tau_a \wedge \tau_b$.

1. Prove that, for all $\lambda > 0$,
$$\mathrm{E}\left[\exp(-\lambda T)\right] = \frac{\cosh\left(\frac{b+a}{2}\sqrt{2\lambda}\right)}{\cosh\left(\frac{b-a}{2}\sqrt{2\lambda}\right)}.$$
(Indication: introduce the martingale
$$M_t = \exp\left(\sqrt{2\lambda}(B_t - \alpha) - \lambda t\right) + \exp\left(-\sqrt{2\lambda}(B_t - \alpha) - \lambda t\right),$$
with a well-chosen α.)
2. Similarly, prove that, for all $\lambda > 0$,
$$\mathrm{E}\left[\exp(-\lambda T)\mathbf{1}_{T=\tau_a}\right] = \frac{\sinh(b\sqrt{2\lambda})}{\sinh((b-a)\sqrt{2\lambda})}.$$

3. With the help of question 2, give an expression of $\mathrm{P}(T_a < T_b)$.

Exercise 5 If X is a continuous process vanishing at 0, such that, for every real α, the process $M_t^\alpha = \exp\left\{\alpha X_t - \frac{\alpha^2 t}{2}\right\}$ is a martingale with respect to $(\mathcal{F}_t)$, prove that X is a $(\mathcal{F}_t)$-Brownian motion. (*Hint:* use that if X is a r.v. and $\mathcal{B}$ is a sub-σ-algebra such that

$$\mathrm{E}[e^{\lambda X}|\mathcal{B}] = \mathrm{E}[e^{\lambda X}] < \infty,$$

for λ in a neighborhood of 0, then X and $\mathcal{B}$ are independent.)

Exercise 6 (Wald's identity). Let T be an integrable stopping time. Show that $\mathrm{E}[B_T] = 0$ and that $\mathrm{E}[B_T^2] = \mathrm{E}[T]$.

Exercise 7 Let $M = (M_t, t \geq 0)$ be a sub-martingale. Let $(\mathcal{G}_t)$ be a sub-filtration of $(\mathcal{F}_t)$. Prove that $N_t = \mathrm{E}[M_t|\mathcal{G}_t]$ is a $(\mathcal{G}_t)$-sub-martingale.

Exercise 8 Let $\sigma \leq \tau$ be two bounded stopping times. Show that $\mathrm{E}[(B_\tau - B_\sigma)^2] = \mathrm{E}[B_\tau^2] - \mathrm{E}[B_\sigma^2] = \mathrm{E}[\tau - \sigma]$.

Exercise 9 Let $(M_t, t \in [0,1])$ be a sub-martingale such that $\mathrm{E}[M_0] = \mathrm{E}[M_1]$. Prove that M is a martingale.

Exercise 10 Let M be a càd martingale. Let $t \geq 0$. Prove that $M_{t+\varepsilon} \to M_t$ when $\varepsilon \to 0$ in L^1.