
Sheet 4. Local martingales. Quadratic variation.

We denote by $B = (B_t)_{t \geq 0}$ a Brownian motion with respect to some filtration $(\mathcal{F}_t)_{t \geq 0}$.

Exercise 1 Let M be a continuous local martingale. Prove that there exists a sequence of stopping times $\tau_n \nearrow \infty$ such that for all n , $M^{\tau_n} - M_0$ is a continuous, bounded local martingale.

Exercise 2 Let M be a continuous adapted process. Assume that there exists $\tau_n \nearrow \infty$ such that M^{τ_n} is a local martingale. Show that M is a local martingale.

Exercise 3 Let U be a real random variable that is $\mathcal{F}_0$ -measurable, and let M be a local martingale. Show that the process $N_t = UM_t$ is still a local martingale.

Exercise 4 If M is a continuous martingale and a Gaussian process, prove that $\langle M, M \rangle$ is deterministic, i.e. there is a function f on $\mathbb{R}_+$, such that $\langle M, M \rangle_t = f(t)$ a.s.

Exercise 5 Let M, N be a continuous local martingales and τ a stopping time. Show that $\langle M^\tau, N \rangle = \langle N^\tau, M^\tau \rangle = \langle N, M \rangle^\tau$ and $\langle M - M^\tau \rangle = \langle M \rangle - \langle M \rangle^\tau$.

Exercise 6 1. Let M and N be two continuous local martingales. Prove that if M and N are independent, then they are orthogonal (that is $\langle M, N \rangle = 0$). Deduce the value of $\langle B^i, B^j \rangle_t$ when $(B^1, \dots, B^d)$ is a d -dimensional Brownian motion.

2. Show that the converse is wrong. (*Hint: Consider M^τ and $M - M^\tau$, or consider (X, Y) a 2-dimensional Brownian motion and X^T, Y^T for a stopping time T .)*

Exercise 7 Show that a continuous local martingale M is a martingale bounded in L^2 if and only if $M_0 \in L^2$ and $\langle M \rangle_\infty \in L^1$.

Exercise 8 Let M be a continuous local martingale such that $M_0 = 0$.

- Pour all $n \geq 1$, let $T_n = \inf\{t \geq 0 : |M_t| = n\}$. Show that a.s.

$$\left\{ \lim_{t \rightarrow \infty} M_t \text{ exists and is finite} \right\} = \cup_{n=1}^{\infty} \{T_n = \infty\} \subset \{\langle M, M \rangle_\infty < \infty\}.$$

- Let $S_n = \inf\{t \geq 0 : \langle M, M \rangle_t = n\}$ pour all $n \geq 1$. Show that a.s.

$$\{\langle M, M \rangle_\infty < \infty\} = \cup_{n=1}^{\infty} \{S_n = \infty\} \subset \left\{ \lim_{t \rightarrow \infty} M_t \text{ exists and is finite} \right\}$$

and conclude that

$$\{\langle M, M \rangle_\infty < \infty\} = \left\{ \lim_{t \rightarrow \infty} M_t \text{ exists and is finite} \right\}.$$

Exercise 9 Let M and N be continuous local martingales and H a measurable process such that for all $t \geq 0$, $\int_0^t H_s^2 d\langle M \rangle_s < \infty$ and $\int_0^t H_s^2 d\langle N \rangle_s < \infty$ a.s. Show that $t \geq 0$, $\int_0^t H_s^2 d\langle M + N \rangle_s < \infty$ a.s.

Exercise 10 Let M be a continuous local martingale such that $M_0 = 0$ a.s.

1. Show that for all a.s. finite stopping time τ , we have $EM_\tau^2 \leq E[\langle M \rangle_\tau]$
2. Let $a > 0$ and set $\tau_a = \inf\{t \geq 0 : |M_t| \geq a\}$. Justify that, for all $t > 0$, $E[\langle M \rangle_{\tau_a \wedge t}] \geq a^2 P(\tau_a \leq t)$.
3. Show that $P(\sup_{s \in [0, t]} |M_s| \geq a) \leq a^{-2} E[\langle M \rangle_t]$.