

Sheet 5. Stochastic integral.

We denote by $B = (B_t)_{t \geq 0}$ a Brownian motion with respect to some filtration $(\mathcal{F}_t)_{t \geq 0}$ and by $\mathcal{M}_\infty^2$ the set of continuous martingales started at 0 and bounded in L^2 . For all $M \in \mathcal{M}_\infty^2$, we say that $H \in L_\infty^2(M)$ if it is progressive and if

$$\mathbb{E} \left[\int_0^\infty H_s^2 d\langle M \rangle_s \right] < \infty.$$

Exercise 1 Let M be a continuous martingale such that $M_0 = 0$, and H a progressive process that is locally bounded. Show that

$$H \cdot M \in \mathcal{M}_\infty^2 \quad \text{if and only if} \quad \mathbb{E} \left[\int_0^\infty H_s^2 d\langle M \rangle_s \right] < \infty.$$

Exercise 2 Let $M \in \mathcal{M}_\infty^2$ and $N \in \mathcal{M}_\infty^2$. Let $H \in L_\infty^2(M)$ and $K \in L_\infty^2(N)$. Let S and T be stopping times. Show that

$$\mathbb{E} \left[\left(\int_0^S H_s dM_s \right) \left(\int_0^T K_s dN_s \right) \right] = \mathbb{E} \left[\int_0^{S \wedge T} H_s K_s d\langle M, N \rangle_s \right].$$

Exercise 3 Let M be a continuous local martingale. Let $H \in L_{\text{loc}}^2(M)$, that is, H is a progressive process such that for all $t \geq 0$,

$$\int_0^t H_s^2 d\langle M \rangle_s < \infty \quad \text{a.s.}$$

1. Show that for all real numbers $t \geq 0$, $a > 0$, and $b > 0$,

$$\mathbb{P} \left(\int_0^t H_s dM_s \geq a \right) \leq \mathbb{P} \left(\int_0^t H_s^2 d\langle M \rangle_s \geq b \right) + \frac{b}{a^2}.$$

You may consider the local martingale $H \cdot M$ stopped at a suitably chosen stopping time.

2. Let (H^n) be a sequence in $L_{\text{loc}}^2(M)$ such that for all $t \geq 0$,

$$\int_0^t (H_s^n - H_s)^2 d\langle M \rangle_s \rightarrow 0 \quad \text{in probability.}$$

Show that for all $t \geq 0$,

$$\int_0^t H_s^n dM_s \rightarrow \int_0^t H_s dM_s \quad \text{in probability.}$$

Exercise 4 Determine the law of the process

$$X_t = \int_0^{\sqrt{t}} \sqrt{2u} dB_u, \quad t \geq 0.$$

Exercise 5 (Ornstein-Uhlenbeck process). Let V_0 be a real random variable independent of $(B_t)_{t \geq 0}$, and let $b, \sigma > 0$. We define a process $(V_t)_{t \geq 0}$ by:

$$V_t := e^{-bt} \left(V_0 + \sigma \int_0^t e^{bs} dB_s \right).$$

1. Show that V_t converges in distribution as $t \rightarrow \infty$, and determine the limit.
2. Now suppose that $V_0 \sim \mathcal{N} \left(0, \frac{\sigma^2}{2b} \right)$. Show that $(V_t)_{t \geq 0}$ is a stationary Gaussian process, and specify its parameters.

3. What can be said about the process $(W_t)_{t \in \mathbb{R}}$ defined below?

$$W_t := \frac{\sigma}{\sqrt{2b}} e^{-bt} B_{e^{2bt}}.$$

Exercise 6 Let $f \in \mathcal{C}^1(\mathbb{R}_+, \mathbb{R})$.

1. By using a limit of stochastic integrals of simple processes, show that for all $t \geq 0$, we have almost surely

$$\int_0^t f(s) dB_s + \int_0^t f'(s) B_s ds = f(t) B_t.$$

2. Under the additional assumptions that $f(t) \rightarrow 0$ as $t \rightarrow \infty$ and

$$\int_0^\infty |f'(t)| \sqrt{t} dt < \infty,$$

deduce that

$$\int_{\mathbb{R}_+} f(s) dB_s = - \int_{\mathbb{R}_+} f'(s) B_s ds.$$