
Sheet 6. Stochastic Calculus.

Unless otherwise stated, we denote by $B = (B_t)_{t \geq 0}$ a Brownian motion with respect to some filtration $(\mathcal{F}_t)_{t \geq 0}$. We write $\psi = (\psi_s)_{s \geq 0} \in L^p_{loc}(B)$ if ψ is a progressive process and if

$$\forall t \geq 0, \quad \int_0^t \psi_s^p ds < \infty, \quad a.s.$$

We write $\psi \in L^2(B)$ if ψ is progressive and for all $t \geq 0$, $\mathbb{E} \int_0^t \psi_s^2 ds < \infty$.

Exercise 1 1. Let $\phi \in L^2_{loc}(B)$ and $\psi \in L^1_{loc}(B)$ and consider

$$dX_t = \phi_t dB_t + \psi_t dt, \quad X_0 = 0,$$

that is

$$X_t = \int_0^t \phi_u dB_u + \int_0^t \psi_u du, \quad t \geq 0.$$

Prove that X is a square-integrable martingale if and only if $\psi = 0$ and $\phi \in L^2(B)$.

2. Let $f: \mathbb{R} \mapsto \mathbb{R}$ be continuously twice-differentiable. Show that the process

$$X_t := f(B_t) - \frac{1}{2} \int_0^t f''(B_s) ds, \quad t \geq 0,$$

is a continuous local martingale. Give a sufficient condition for X to be a martingale.

3. In each of the following cases, show that M_t is a martingale by computing its stochastic differential:

- (a) $M_t := B_t^2 - t$.
- (b) $M_t := B_t^5 - 10tB_t^3 + 15t^2B_t$.
- (c) $M_t := \exp\left(\theta B_t - \frac{\theta^2}{2}t\right)$, with $\theta \in \mathbb{R}$.
- (d) $M_t := \cos(\theta B_t)e^{\frac{1}{2}\theta^2 t}$, with $\theta \in \mathbb{R}$.
- (e) $M_t := f(t, B_t)$, where $f \in C^2(\mathbb{R}^2)$ satisfies an appropriate condition.

Exercise 2 Let X and Y be two independent 1-dimensional Brownian motions and let H be a progressive process. Set

$$\begin{aligned} \beta_t &= \int_0^t \cos(H_s) dX_s + \int_0^t \sin(H_s) dY_s, \\ \gamma_t &= \int_0^t \sin(H_s) dX_s - \int_0^t \cos(H_s) dY_s. \end{aligned}$$

Show that β and γ are two independent 1-dimensional Brownian motions.

Exercise 3 In this exercise, $B = (B^{(1)}, B^{(2)}, B^{(3)})$ is a three-dimensional Brownian motion starting from $x_0 \in \mathbb{R}^3 \setminus \{0\}$. Consider the Green function

$$G(x) := \frac{1}{\|x\|}, \quad x \in \mathbb{R}^3 \setminus \{0\},$$

where $\|\cdot\|$ is the Euclidian norm and

$$T_a = \inf\{t \geq 0: \|B_t\| = a\}, \quad a \geq 0.$$

1. Check that $\langle B^{(i)}, B^{(j)} \rangle \equiv 0$ whenever $i \neq j$, where $i, j \in \{1, 2, 3\}$.

2. Check that $\Delta G(x) = 0$ for all $x \neq 0$.
3. Assume $0 < r < \|x_0\|$. Using Itô's formula, prove that $\{G(B_{t \wedge T_r})\}_{t \geq 0}$ is a martingale.
4. Assume $0 < r < \|x_0\| < R < \infty$. Show that $T_R < \infty$ a.s. and

$$P(T_r < T_R) = \frac{R^{-1} - \|x_0\|^{-1}}{R^{-1} - r^{-1}}.$$

5. Deduce from the previous formula that a.s. $B_t \neq 0$ for all $t \geq 0$.
6. Prove that $\{G(B_t)\}_{t \geq 0}$ is a continuous local martingale.
7. Check that $\{G(B_t)\}_{t \geq 0}$ is a non-negative super-martingale and deduce thereof that $\|B_t\|$ a.s. converges to $+\infty$, as $t \rightarrow \infty$.
8. Prove that $\{G(B_t)\}_{t \geq 0}$ is bounded in L^2 but is not a martingale.

Exercise 4 Let $b \in \mathbb{R}$, $a > 0$ and set $X_t = B_t - bt$. Let $T = \inf\{t \geq 0 : X_t = a\}$.

1. Let $t \geq 0$. Find a probability measure $\mathbb{Q}$ on $\mathcal{F}_\infty$ under which $(X_s)_{s \in [0, t]}$ is a Brownian motion.
2. Deduce the values of $\mathbb{P}(T \leq t)$, $\mathbb{P}(T < \infty)$ and the distribution of $Z = \sup_{t \geq 0} X_t$ under $\mathbb{P}$.

(We recall that $\mathbb{E}[\exp\{-u\tau_a\}] = \exp\{-a\sqrt{2u}\}$ for all $u \geq 0$, where $\tau_a = \inf\{t \geq 0 : B_t = a\}$.)

Exercise 5 Let ξ be a bounded progressive process.

1. Suppose that M is a continuous L^p -integrable martingale for some $1 < p < \infty$. Show that $\xi \cdot M$ is also an L^p -integrable martingale.
2. Suppose that M is a continuous local martingale such that $M^* = (\sup_{s \leq t} |M_s|)_{t \geq 0}$ is an integrable process. What can be said of $\xi \cdot M$?

Exercise 6 Let $b: \mathbb{R}_+ \times \mathbb{R} \rightarrow \mathbb{R}$ be a bounded measurable function. Assume that there exists $g \in L^2(\mathbb{R}_+)$ such that

$$|b(t, x)| \leq g(t), \quad \forall (t, x) \in \mathbb{R}_+ \times \mathbb{R}.$$

Using Girsanov's theorem, find a probability measure $\mathbb{Q}$ on $\mathcal{F}_\infty$ under which the process defined by

$$B_t - \int_0^t b(s, B_s) ds, \quad t \geq 0,$$

is a Brownian motion. Interpret this result in terms of an SDE property.

Exercise 7 (Cameron-Martin formula) Let B be a Brownian motion and let $h \in L^2([0, T], dt)$. Show that for any measurable function (positive or bounded) $\Phi: C([0, T], \mathbb{R}) \rightarrow \mathbb{R}$, we have:

$$\mathbb{E} \left[\Phi \left(\left(B_t + \int_0^t h(s) ds \right)_{t \in [0, T]} \right) \right] = \mathbb{E} \left[\Phi \left((B_t)_{t \in [0, T]} \right) \cdot \exp \left(\int_0^T h(s) dB_s - \frac{1}{2} \int_0^T h^2(s) ds \right) \right].$$