

Sheet 7. SDEs.

Notation: (B_t) is a standard Brownian motion defined on $(\Omega, \mathcal{F}_t, \mathbb{P})$.

Exercise 1 1. Justify that for all $T < 1$ and $x \in \mathbb{R}$, there exists almost surely a solution to the SDE:

$$X_t^x = x + B_t - \int_0^t \frac{X_s^x}{1-s} ds, \quad t \in [0, T].$$

2. By applying the Itô formula to $(\frac{X_t^0}{1-t})_{t \in [0, T]}$, find an explicit formula for $(X_t^0, t \in [0, T])$.
3. Show that $X_t^x = X_t^0 + x(1-t)$, $t \in [0, T]$ and determine its distribution.
4. Show that X^x can be extended to a continuous process on $[0, 1]$.
5. What does $(X_t^x, t \in [0, 1])$ represent?

Exercise 2 Solve the stochastic differential equation

$$dX_t = |X_t| dB_t + \frac{X_t}{2} dt, \quad X_0 = x \in \mathbb{R}.$$

Exercise 3 1. Let H, Z, X be continuous semimartingales such that

$$X_t = H_t + \int_0^t X_s dZ_s.$$

Express X as a function of H, Z , and $\mathcal{E}[Z]$. *Hint:* Start with $H \equiv 1$ and use a variation of constants method as in classical ODEs.

2. Solve

$$X_t = x + B_t - \beta \int_0^t X_s ds,$$

where $x \in \mathbb{R}$ and $\beta \geq 0$ are constants. The process X is called the *Ornstein–Uhlenbeck process*.

Exercise 4 Let $M_t := \frac{1}{2}(B_t^2 - t)$, $t \geq 0$.

1. Justify that M is a martingale and express M_t as a stochastic integral.
2. Show that for all $b \geq 0$, the exponential local martingale $\mathcal{E}(-bM)$ is a (true) martingale. For all $T > 0$, justify that $\mathbb{Q} := \mathcal{E}(-bM)_T \cdot \mathbb{P}$ defines a probability measure.
3. Determine the SDE satisfied by $(B_t, t \in [0, T])$ on $\mathbb{Q}$. Deduce the distribution of B_t , $t \in [0, T]$ on $\mathbb{Q}$.
4. Deduce that for all $a, b \geq 0$,

$$\mathbb{E} \left[\exp \left\{ -aB_t^2 - \frac{b^2}{2} \int_0^t B_s^2 ds \right\} \right] = \left(\frac{b}{b \cosh(bt) + 2a \sinh(bt)} \right)^{1/2}.$$

5. $(\star)$ Using that for all $\alpha, \beta > 0$ and $s \geq 0$,

$$\frac{1}{\Gamma(\alpha)} \int_0^\infty \beta^\alpha x^{\alpha-1} e^{-(\beta+s)x} dx = \left(\frac{\beta}{s+\beta} \right)^\alpha,$$

compute

$$\mathbb{E} \left[\exp \left\{ -\frac{b^2}{2} \int_0^t B_s^2 ds \right\} \middle| B_t = y \right], \quad b > 0, y \in \mathbb{R}.$$

Exercise 5 (Brownian motion on a circle) Prove that $X_t = (\cos B_t, \sin B_t)$, $t \geq 0$, is the unique solution of the SDE:

$$dX_t = -\frac{1}{2}X_t dt + RX_t dB_t, \quad X_0 = (1, 0),$$

where R is (the matrix of) the rotation with angle $\pi/2$.

Exercise 6 Consider the solution X of the SDE $E(\sigma, b)$ in dimension $d = 1$.

1. Let $s : \mathbb{R} \rightarrow \mathbb{R}$ be of class C^2 such that

$$\frac{1}{2}s''\sigma^2 + s'b = 0.$$

Show that $(s(X_t), t \geq 0)$ is a continuous local martingale. The function s is called the *scale function* of X .

2. Assume that σ is continuous and that s' and σ never vanish. Let $a < x < b$ be real numbers, and set

$$T_{a,b} = \inf\{t \geq 0 : X_t \notin (a, b)\} \quad (\text{with } \inf \emptyset := \infty).$$

Show that $T_{a,b} < \infty$ almost surely, and that

$$\mathbb{P}(X_{T_{a,b}} = a) = \frac{s(b) - s(x)}{s(b) - s(a)}.$$

Exercise 7 1. Let $X = (X_t, t \geq 0)$ be the solution to $E(\sigma, b)$ with values in an open set $D \subset \mathbb{R}^d$. Let $\lambda \in \mathbb{R}$ and $f : D \rightarrow \mathbb{R}$ be of class C^2 such that $Lf = \lambda f$, where

$$(Lf)(x) := \frac{1}{2} \sum_{i,j=1}^d (\sigma\sigma^*)_{ij}(x) \frac{\partial^2 f}{\partial x_i \partial x_j}(x) + \sum_{i=1}^d b_i(x) \frac{\partial f}{\partial x_i}(x).$$

Show that $(f(X_t)e^{-\lambda t}, t \geq 0)$ is a continuous local martingale.

2. Let $B = (B_1, B_2, B_3)$ be a Brownian motion in $\mathbb{R}^3$, and $B_0 = a \in \mathbb{R}^3 \setminus \{0\}$. Let $X = |B|^2$, where $|B|$ is the Euclidean norm of B . Show that X satisfies some SDE $E(\sigma, b)$, and find the coefficients σ and b .
3. Assume now that $\lambda \geq 0$. Show that

$$2tf''(t) + 3f'(t) = \lambda f(t), \quad t > 0,$$

$$\text{for } f(t) = \frac{\sinh(\sqrt{2\lambda t})}{\sqrt{2\lambda t}}.$$

4. Let $x > |a|^2$ and $T_x = \inf\{t \geq 0 : X_t = x\}$. Show that for every $\lambda \geq 0$,

$$\mathbb{E}[e^{-\lambda T_x}] = \frac{\sinh(\sqrt{2\lambda}|a|^2)}{\sqrt{2\lambda x} \sqrt{2\lambda}|a|^2 \sinh(\sqrt{2\lambda x})}.$$

Exercise 8 Let $X = X_0 + M + V$ be a continuous semimartingale. Assume that there exists $K > 0$ such that

$$\int_0^\infty |dV_t| \leq K \quad \text{and} \quad \sup_{t \geq 0} |M_t| \leq K.$$

Set, for each $x \in \mathbb{R}$,

$$Y(x) = \int_0^\infty \text{sgn}(X_s - x) dM_s,$$

where $\text{sgn}(x) = 1$ if $x > 0$ and -1 otherwise. Let $p > 0$ and $x < y$.

1. Justify that $Y(x)$ is well-defined.

2. Show that

$$\mathbb{E}[|Y(x) - Y(y)|^p] \leq C_p \mathbb{E} \left[\left(\int_0^\infty \mathbf{1}_{\{x < X_s \leq y\}} d\langle M \rangle_s \right)^{p/2} \right],$$

where $C_p > 0$ depends only on p .

3. Let $g : \mathbb{R} \rightarrow [0, 2]$ be a C^1 function such that

$$g'(u) \geq \mathbf{1}_{(0,1]}(u), \quad \forall u \in \mathbb{R}.$$

Define

$$G(u) = \int_0^u g(v) dv, \quad \varphi(u) = G\left(\frac{u-x}{y-x}\right), \quad u \in \mathbb{R}.$$

Prove that

$$\frac{1}{2(y-x)^2} \int_0^\infty \mathbf{1}_{x < X_s \leq y} d\langle M \rangle_s \leq |\varphi(X_\infty) - \varphi(X_0)| + \left| \int_0^\infty \varphi'(X_s) dM_s \right| + \int_0^\infty |\varphi'(X_s)| |dV_s|.$$

4. Deduce that

$$\int_0^\infty \mathbf{1}_{\{x < X_s \leq y\}} d\langle M \rangle_s \leq 12K(y-x) + 4(y-x) \left| \int_0^\infty g\left(\frac{X_s-x}{y-x}\right) dM_s \right|.$$

5. Deduce the existence of a constant $C_{K,p}$ depending only on K and p , such that

$$\mathbb{E}[|Y(x) - Y(y)|^p] \leq C_{K,p}(y-x)^{p/2}.$$

6. Does the random process $(Y(x), x \in \mathbb{R})$ admit a continuous modification?

Hints.

- For Exercise 2, first consider the equation without the absolute value.
- For Exercise 4, Question 3, solve the SDE by variation of constants.
- For Exercise 4, Question 5, think of $|B_t = y|$ as $|B_t \in dy|$ and find the density of B_t^2 on the measure $\mathbb{P}^G(\cdot) = \frac{\mathbb{E}[G \cdot]}{\mathbb{E}[G]}$ where $G = e^{-\frac{b^2}{2} \int_0^t B_s^2 ds}$.