Sheet 7. SDEs.

Notation: (B_t) is a standard Brownian motion defined on $(\Omega, \mathcal{F}_t, \mathbb{P})$.

Exercise 1 1. Justify that for all $T < 1$ and $x \in \mathbb{R}$, there exists almost surely a solution to the SDE:

$$X_t^x = x + B_t - \int_0^t \frac{X_s^x}{1-s}\mathrm{d}s, \quad t \in [0,T].$$

2. By applying the Itô formula to $(\frac{X_t^0}{1-t})_{t\in[0,T]}$, find an explicit formula for $(X_t^0, t \in [0,T])$.
3. Show that $X_t^x = X_t^0 + x(1-t)$, $t \in [0,T]$ and determine its distribution.
4. Show that X^x can be extended to a continuous process on $[0,1]$.
5. What does $(X_t^x, t \in [0,1])$ represent?

Solution.

1. $\sigma(t,x) = 1$ and $b(t,x) = x/(1-t)$ satisfies $|b(t,x) - b(t,y)| \leq |x-y|/(1-T)$ for all $t \leq T$. Hence there exists of a strong solution and there is pathwise unicity.
2. By the Itô formula,

$$\frac{X_t^0}{1-t} = \int_0^t \frac{1}{1-s}\mathrm{d}X_s^0 + \int_0^t \frac{X_s^0}{(1-s)^2}\mathrm{d}s = \int_0^t \frac{1}{1-s}\mathrm{d}B_s.$$

   Hence $X_t^0 = \int_0^t \frac{1-t}{1-s}\mathrm{d}B_s$.
3. Let $Y_t = X_t^0 + x(1-t)$. Then $Y_0 = x$ and

$$\mathrm{d}Y_t = -x\mathrm{d}t + \mathrm{d}B_t - \frac{X_t^0}{1-t}\mathrm{d}t = \mathrm{d}B_t - \frac{Y_t}{1-t}.$$

   Therefore Y_t is a (strong) solution of the SDE satisfied by X_t^x, and thus $Y = X^x$ by pathwise unicity. By the previous question, $(X_t^0)_{t\leq T}$ is a centered Gaussian process of covariance (let $s < t$)

$$\mathbb{E}[X_t^0 X_s^0] = (1-t)(1-s)\int_0^s \frac{1}{(1-u)^2}\mathrm{d}u = (1-t) - (1-t)(1-s) = s(1-t).$$

   Thus, X^x is a Gaussian process with the above covariance and mean $x(1-t)$.
4. By question 3, it is enough to show it for X_t^0. Observe that by the formula of question 2, X_t^0 is well-defined as a continuous Gaussian process on $[0,1)$. One can check continuity at $t = 1$ (with $X_1^0 = 0$) using Kolmogorov's continuity theorem. Alternatively, one can use that X_t^0 has the same distribution as the Brownian bridge as continuous processes on $[0,1)$. Therefore, $\mathbb{P}(\lim_{t\to 1} X_t = 0) = 1$ as the same holds for Brownian bridge.
5. X_t^x is a Brownian bridge from x to 0, since they are both continuous Gaussian processes on $[0,1]$ with the same mean and covariance.

Exercise 2 Solve the stochastic differential equation

$$dX_t = |X_t|\, dB_t + \frac{X_t}{2}\, dt, \quad X_0 = x \in \mathbb{R}.$$

Solution. $\sigma(t,x) = |x|$ and $b(t,x) = |x|$ are Lipschitz. Hence there is pathwise uniqueness of the solution. Moreover, if $x \geq 0$, then $X_t = x\exp\{B_t\}$ is a solution. If $x \leq 0$, then $X_t = x\exp\{-B_t\}$ is a solution.

Exercise 3 1. Let H, Z, X be continuous semimartingales such that

$$X_t = H_t + \int_0^t X_s \, dZ_s.$$

Express X as a function of H, Z, and $\mathcal{E}[Z]$. *Hint:* Start with $H \equiv 1$ and use a variation of constants method as in classical ODEs.

2. Solve

$$X_t = x + B_t - \beta \int_0^t X_s \, ds,$$

where $x \in \mathbb{R}$ and $\beta \geq 0$ are constants. The process X is called the *Ornstein–Uhlenbeck process.*

Solution. **(i)** Par définition, $\mathrm{d}\mathscr{E}(Z)_t = \mathscr{E}(Z)_t \, \mathrm{d}Z_t$. Comme $\mathscr{E}(Z)$ est à valeurs dans $\mathbb{R}_+^*$, on applique la formule d'Itô pour voir que

$$\mathrm{d}\Big(\frac{1}{\mathscr{E}(Z)_t}\Big) = -\frac{\mathrm{d}\mathscr{E}(Z)_t}{\mathscr{E}(Z)_t^2} + \frac{\mathrm{d}\langle\mathscr{E}(Z)\rangle_t}{\mathscr{E}(Z)_t^3} = -\frac{\mathrm{d}Z_t}{\mathscr{E}(Z)_t} + \frac{\mathrm{d}\langle Z\rangle_t}{\mathscr{E}(Z)_t}.$$

Let $Y_t = \frac{X_t}{\mathscr{E}(Z)_t}$. Par intégration par parties,

$$\begin{aligned}
\mathrm{d}Y_t &= X_t \, \mathrm{d}\Big(\frac{1}{\mathscr{E}(Z)_t}\Big) + \frac{\mathrm{d}X_t}{\mathscr{E}(Z)_t} + \mathrm{d}\Big\langle X, \frac{1}{\mathscr{E}(Z)}\Big\rangle_t \\
&= -\frac{X_t \, \mathrm{d}Z_t}{\mathscr{E}(Z)_t} + \frac{X_t \, \mathrm{d}\langle Z\rangle_t}{\mathscr{E}(Z)_t} + \frac{\mathrm{d}H_t}{\mathscr{E}(Z)_t} + \frac{X_t \, \mathrm{d}Z_t}{\mathscr{E}(Z)_t} - \frac{\mathrm{d}\langle H, Z\rangle_t}{\mathscr{E}(Z)_t} - \frac{X_t \, \mathrm{d}\langle Z\rangle_t}{\mathscr{E}(Z)_t} \\
&= \frac{\mathrm{d}H_t - \mathrm{d}\langle H, Z\rangle_t}{\mathscr{E}(Z)_t}.
\end{aligned}$$

D'où :

$$X_t = \mathscr{E}(Z)_t Y_t = \mathscr{E}(Z)_t \Big(H_0 \mathrm{e}^{-Z_0} + \int_0^t \frac{\mathrm{d}H_s}{\mathscr{E}(Z)_s} - \int_0^t \frac{\mathrm{d}\langle H, Z\rangle_s}{\mathscr{E}(Z)_s} \Big).$$

(ii) Comme les coefficients sont lipschitziens, on sait qu'il y a unicité trajectorielle. Par (i) (avec $H_t := x + B_t$ et $Z_t := -\beta t$), on voit que quel que soit l'espace filtré $(\Omega, \mathscr{F}, (\mathscr{F}_t), \mathbb{P})$ (vérifiant les conditions habituelles) et quel que soit le $(\mathscr{F}_t)$-mouvement brownien B, l'unique solution (qui est une solution forte) de l'EDS est $X_t = \mathrm{e}^{-\beta t}(x + \int_0^t \mathrm{e}^{\beta s} \, \mathrm{d}B_s)$. □

Exercise 4 Let $M_t := \frac{1}{2}(B_t^2 - t), t \geq 0$.

1. Justify that M is a martingale and express M_t as a stochastic integral.
2. Show that for all $b \geq 0$, the exponential local martingale $\mathscr{E}(-bM)$ is a (true) martingale. For all $T > 0$, justify that $\mathbb{Q} := \mathscr{E}(-bM)_T \cdot \mathbb{P}$ defines a probability measure.
3. Determine the SDE satisfied by $(B_t, t \in [0,T])$ on $\mathbb{Q}$. Deduce the distribution of B_t, $t \in [0,T]$ on $\mathbb{Q}$.
4. Deduce that for all $a, b \geq 0$,

$$\mathbb{E}\left[\exp\left\{-aB_t^2 - \frac{b^2}{2}\int_0^t B_s^2 \mathrm{d}s\right\}\right] = \left(\frac{b}{b\cosh(bt) + 2a\sinh(bt)}\right)^{1/2}.$$

5. ($\star$) Using that for all $\alpha, \beta > 0$ and $s \geq 0$,

$$\frac{1}{\Gamma(\alpha)}\int_0^\infty \beta^\alpha x^{\alpha-1} e^{-(\beta+s)x} \mathrm{d}x = \left(\frac{\beta}{s+\beta}\right)^\alpha,$$

compute

$$\mathbb{E}\left[\exp\left\{-\frac{b^2}{2}\int_0^t B_s^2 \mathrm{d}s\right\} \middle| B_t = y\right], \quad b > 0, y \in \mathbb{R}.$$

Solution.

1. We have $\mathrm{d}M_t = B_t \mathrm{d}B_t$ where $B \in L^2(B)$ and B is a martingale hence M is a martingale.

2. We have $\mathscr{E}(-bM)_t = \exp\{-bM_t - \frac{b^2}{2}\langle M\rangle_t\}$ where $\langle M\rangle_t = \int_0^t B_s^2 \mathrm{d}s$. It holds that $\mathscr{E}(-bM)_t \leq \exp\{bt/2\}$, hence $\mathbb{E}[\sup_{s\leq t} \mathscr{E}(-bM)_s] < \infty$ for all $t \geq 0$, implying that the (non-negative) local martingale $\mathscr{E}(-bM)$ is a (true) martingale.

3. By Girsanov's theorem, $(B_t - \langle B, -bM\rangle_t)_{t\leq T}$ is a Brownian motion on $\mathbb{Q}$, where $\mathrm{d}\langle B, M\rangle_t = B_t \mathrm{d}t$. Hence, on $\mathbb{Q}$, B satisfies the SDE
$$\mathrm{d}B_t = \mathrm{d}\tilde{B}_t - bB_t \mathrm{d}t,$$
where $\tilde{B}$ is a Brownian motion. By Exercise 3, $B_t = e^{-bt}\int_0^t e^{bs}\mathrm{d}\tilde{B}_s$. (This is an Ornstein-Uhlenbeck process.) Under $\mathbb{Q}$, B is a Gaussian process with distribution $B_t \sim \mathcal{N}(0, \frac{1-e^{-2bt}}{2b})$.

4. We have,
$$\mathbb{E}\left[\exp\left\{-aB_t^2 - \frac{b^2}{2}\int_0^t B_s^2 \mathrm{d}s\right\}\right] = e^{-bt/2}\mathbb{E}_{\mathbb{Q}}\left[\exp\left\{(b/2-a)B_t^2\right\}\right]$$
Using that for $X \sim \mathcal{N}(0,1)$, $\mathbb{E}[e^{\lambda X^2}] = (1-2\lambda)^{-1/2}$ when $\lambda < 1/2$, we get from the previous question that
$$e^{-bt/2}\mathbb{E}_{\mathbb{Q}}\left[\exp\left\{(b/2-a)B_t^2\right\}\right] = e^{-bt/2}(1-(b/2-a)(1-e^{-2bt})/b)^{-1/2}.$$
This is equal to
$$\left(\frac{b}{be^{bt} - (b/2-a)(e^{bt}-e^{-bt})}\right)^{1/2}.$$

Exercise 5 (Brownian motion on a circle) Prove that $X_t = (\cos B_t, \sin B_t)$, $t \geq 0$, is the unique solution of the SDE:
$$\mathrm{d}X_t = -\frac{1}{2}X_t \mathrm{d}t + RX_t \mathrm{d}B_t, \qquad X_0 = (1,0),$$
where R is (the matrix of) the rotation with angle $\pi/2$.

Exercise 6 Consider the (well-posed) solution X of the SDE $E(\sigma, b)$ in dimension $d = 1$.

1. Let $s : \mathbb{R} \to \mathbb{R}$ be of class C^2 such that
$$\frac{1}{2}s''\sigma^2 + s'b = 0.$$
Show that $(s(X_t), t \geq 0)$ is a continuous local martingale. The function s is called the *scale function* of X.

2. Assume that $\sigma = \sigma(x)$ is continuous and that s' and σ never vanish. Let $a < x < d$ be real numbers, and set
$$T_{a,d} = \inf\{t \geq 0 : X_t \notin (a,d)\} \quad (\text{with } \inf\varnothing := \infty).$$
Show that $T_{a,d} < \infty$ almost surely, and that
$$\mathbb{P}(X_{T_{a,d}} = a) = \frac{s(d)-s(x)}{s(d)-s(a)}.$$

Exercise 7 1. Let $X = (X_t, t \geq 0)$ be the solution to $E(\sigma, b)$ with values in an open set $D \subset \mathbb{R}^d$. Let $\lambda \in \mathbb{R}$ and $f : D \to \mathbb{R}$ be of class C^2 such that $Lf = \lambda f$, where
$$(Lf)(x) := \frac{1}{2}\sum_{i,j=1}^d (\sigma\sigma^*)_{ij}(x)\frac{\partial^2 f}{\partial x_i \partial x_j}(x) + \sum_{i=1}^d b_i(x)\frac{\partial f}{\partial x_i}(x).$$
Show that $(f(X_t)e^{-\lambda t}, t \geq 0)$ is a continuous local martingale.

2. Let $B = (B_1, B_2, B_3)$ be a Brownian motion in $\mathbb{R}^3$, and $B_0 = a \in \mathbb{R}^3 \setminus \{0\}$. Let $X = |B|^2$, where $|B|$ is the Euclidean norm of B. Show that X satisfies some SDE $E(\sigma, b)$, and find the coefficients σ and b.

3. Assume now that $\lambda \geq 0$. Show that

$$2tf''(t) + 3f'(t) = \lambda f(t), \quad t > 0,$$

for $f(t) = \dfrac{\sinh(\sqrt{2\lambda t})}{\sqrt{2\lambda t}}$.

4. Let $x > |a|^2$ and $T_x = \inf\{t \geq 0 : X_t = x\}$. Show that for every $\lambda \geq 0$,

$$\mathbb{E}[e^{-\lambda T_x}] = \frac{\sinh(\sqrt{2\lambda}|a|^2)}{\sqrt{2\lambda x}\,\sqrt{2\lambda}|a|^2\,\sinh(\sqrt{2\lambda}x)}.$$

Exercise 8 Let $X = X_0 + M + V$ be a continuous semimartingale. Assume that there exists $K > 0$ such that

$$\int_0^\infty |dV_t| \leq K \quad \text{and} \quad \sup_{t\geq 0} |M_t| \leq K.$$

Set, for each $x \in \mathbb{R}$,

$$Y(x) = \int_0^\infty \operatorname{sgn}(X_s - x)\, dM_s,$$

where $\operatorname{sgn}(x) = 1$ if $x > 0$ and -1 otherwise. Let $p > 0$ and $x < y$.

1. Show that $Y(x)$ is well-defined.

2. Show that

$$\mathbb{E}[|Y(x) - Y(y)|^p] \leq C_p\, \mathbb{E}\left[\left(\int_0^\infty \mathbf{1}_{\{x<X_s\leq y\}}\, d\langle M\rangle_s\right)^{p/2}\right],$$

where $C_p > 0$ depends only on p.

3. Let $g : \mathbb{R} \to [0, 2]$ be a C^1 function such that

$$g'(u) \geq \mathbf{1}_{(0,1]}(u), \quad \forall u \in \mathbb{R}.$$

Define

$$G(u) = \int_0^u g(v)\, dv, \qquad \varphi(u) = G\left(\frac{u-x}{y-x}\right), \quad u \in \mathbb{R}.$$

Show that

$$\frac{1}{2(y-x)^2}\int_0^\infty \mathbf{1}_{x<X_s\leq y} \mathrm{d}\langle M\rangle_s \leq |\varphi(X_\infty) - \varphi(X_0)| + \left|\int_0^\infty \varphi'(X_s)\mathrm{d}M_s\right| + \int_0^\infty |\varphi'(X_s)||\mathrm{d}V_s|.$$

4. Show that

$$\int_0^\infty \mathbf{1}_{\{x<X_s\leq y\}}\, d\langle M\rangle_s \leq 12K(y-x) + 4(y-x)\left|\int_0^\infty g\left(\frac{X_s - x}{y-x}\right) dM_s\right|.$$

5. Show that for every $p > 0$,

$$\mathbb{E}[|Y(x) - Y(y)|^p] \leq C_{K,p}(y-x)^{p/2},$$

where $C_{K,p}$ depends only on K and p.

6. Does the random process $(Y(x), x \in \mathbb{R})$ admit a continuous modification?

Solution.

1. Let $H_s = \operatorname{sgn}(X_s - x)$ which is progressive. Since $H_s^2 = 1$, we have

$$\mathbb{E}\left[\int_0^\infty H_s^2 \mathrm{d}\langle M\rangle_s\right] = \mathbb{E}\left[\langle M\rangle_\infty\right] = \lim_{t\to\infty} \mathbb{E}\left[\langle M\rangle_t\right] = \lim_{t\to\infty} \mathbb{E}\left[M_t^2\right] \leq K^2 < \infty,$$

where the second equality is by monotone convergence and the third equality uses that M is a (true) martingale as it is a bounded local martingale. Therefore $H \in L^2(M)$ and so Y is well defined.

2. Let $Y_t = \int_0^t (\mathrm{sgn}(X_s - x) - \mathrm{sgn}(X_s - y))\mathrm{d}M_s$ which defines a martingale since $Y \in L^2(M)$ by the previous question. We have $|Y(x) - Y(y)| \leq Y_\infty^\star$ where $Y_t^\star = \sup_{s \leq t} |Y_s|$, hence

$$\mathbb{E}\left[|Y(x) - Y(y)|^p\right] \leq \mathbb{E}\left[|Y_\infty^\star|^p\right] \leq C_p \mathbb{E}\left[\langle Y \rangle_\infty^{p/2}\right],$$

by the Burkholder-Davis-Gundy inequality. Moreover,

$$\begin{aligned}\langle Y \rangle_\infty &= \int_0^\infty (\mathrm{sgn}(X_s - x) - \mathrm{sgn}(X_s - y))^2 \mathrm{d}\langle M \rangle_s \\ &= 2\int_0^\infty (1 - \mathrm{sgn}(X_s - x)\mathrm{sgn}(X_s - y))\mathrm{d}\langle M \rangle_s = 4\int_0^\infty \mathbf{1}_{\{x < X_s \leq y\}} \mathrm{d}\langle M \rangle_s.\end{aligned}$$

3. By hypothesis $\int \mathrm{d}V_t$ converges absolutely hence V_t converges almost surely to a limit V_∞. Moreover, we saw that M is a bounded martingale which thus converges almost surely to a limit M_∞. Hence X_t converges almost surely to a limit X_∞. Since φ is continuous (actually $\mathcal{C}^2$), we obtain that $\varphi(X_t) \to \varphi(X_\infty)$ almost surely. Furthermore, $\int_0^\infty \varphi'(X_s)\mathrm{d}M_s$ is well defined as the limit of the martingale $\int_0^t \varphi'(X_s)\mathrm{d}M_s$ which is bounded in L^2, which can be seen by using that φ' is bounded. By Itô's formula,

$$\varphi(X_t) - \varphi(X_0) = \int_0^t \varphi'(X_s)\mathrm{d}M_s + \int_0^t \varphi'(X_s)\mathrm{d}V_s + \frac{1}{2}\int_0^t \varphi''(X_s)\mathrm{d}\langle M \rangle_s.$$

Observe that $\varphi''(X_s) = (y - x)^{-2} g'(\frac{X_s - x}{y - x}) \geq (y - x)^{-2}\mathbf{1}_{x < X_s \leq y}$ by hypothesis. Hence,

$$\frac{1}{2(y-x)^2}\int_0^t \mathbf{1}_{x < X_s \leq y}\mathrm{d}\langle M \rangle_s \leq |\varphi(X_t) - \varphi(X_0)| + \left|\int_0^\infty \varphi'(X_s)\mathrm{d}M_s\right| + \int_0^\infty |\varphi'(X_s)||\mathrm{d}V_s|.$$

Take $t \to \infty$ to obtain the result by monotone convergence.

4. Since G is 2-lipschitz, φ is $2/(y - x)$-Lipschitz and we get

$$\frac{1}{2(y-x)^2}\int_0^\infty \mathbf{1}_{x < X_s \leq y}\mathrm{d}\langle M \rangle_s \leq \frac{2}{y - x}|X_\infty - X_0| + \left|\int_0^\infty \varphi'(X_s)\mathrm{d}M_s\right| + \int_0^\infty |\varphi'(X_s)||\mathrm{d}V_s|.$$

We conclude using $\varphi'(u) = \frac{1}{y-x} g(\frac{u-x}{y-x}) \leq \frac{2}{y-x}$, $|X_\infty - X_0| \leq 2K$ (by hypothesis) and $\int_0^\infty |\mathrm{d}V_t| \leq K$.

5. Let $M_t^\star = \sup_{s \leq t} |M_s|$. Recall that $g \leq 2$. By the Burkholder-Davis-Gundy inequality (two times),

$$\mathbb{E}\left[\left|\int_0^\infty g\left(\frac{X_s - x}{y - x}\right) dM_s\right|^{p/2}\right] \leq C_p 2^{p/2} \mathbb{E}\left[\langle M \rangle_\infty^{p/4}\right] \leq C_p'\left[(M_\infty^\star)^{p/2}\right] \leq C_p' K^{p/2}.$$

We conclude combining the results of the previous questions.