

Solutions for sheets 4 and 5.

Sheet 4, Exercise 6, Question 1. Let M and N be two continuous local martingales. Prove that if M and N are independent, then they are orthogonal (that is $\langle M, N \rangle = 0$).

Solution. Since the bracket does not depend on the initial value we may assume that $M_0 = N_0 = 0$. First, we suppose that M and N are bounded and thus (true) martingales. Then, consider the filtration $(\mathcal{F}_s^M)_{s \geq 0}$ (respectively $(\mathcal{F}_s^N)_{s \geq 0}$) generated by the path of M (respectively N). Let $\mathcal{F}_s^0 = \sigma(\mathcal{F}_s^M, \mathcal{F}_s^N)$ denote the filtration generated by $\mathcal{F}_s^M$ and $\mathcal{F}_s^N$. (Note that both M and N are adapted to $\mathcal{F}^0$.) Then, by the tower property $(\mathcal{F}_s^0 \subset \sigma(\mathcal{F}_t^M, \mathcal{F}_s^N))$,

$$\begin{aligned} \mathbb{E}[M_t N_t | \mathcal{F}_s^0] &= \mathbb{E}[M_t \mathbb{E}[N_t | \mathcal{F}_t^M, \mathcal{F}_s^N] | \mathcal{F}_s^0] \\ &= \mathbb{E}[M_t \mathbb{E}[N_t | \mathcal{F}_s^N] | \mathcal{F}_s^0] \\ &= N_s \mathbb{E}[M_t | \mathcal{F}_s^0] = M_s N_s. \end{aligned}$$

where the second equality uses the independence of M and N and last two equalities uses the martingale property of N and M respectively with $\mathcal{F}^N$ and $\mathcal{F}^0$. (Note that $M_t, N_t, M_t N_t \in L^1$ since we assume that M and N are bounded, hence the conditional expectations above make sense.)

We have shown that MN is a (true) martingale, which implies that $\langle M, N \rangle_t = 0$ for all $t \geq 0$ by unicity of the bracket ($MN - 0$ is a local martingale).

Finally, to remove the assumption that M and N are bounded, consider the localizing sequences $T_n = \inf\{t \geq 0 : |M_t| \geq n\}$ and $S_n = \inf\{t \geq 0 : |N_t| \geq n\}$ for the local martingales M and N . As T_n and S_n are independent by definition, it holds that M^{T_n} and N^{S_n} are independent, bounded (true) martingales. By what precedes,

$$\forall t \geq 0, \quad \langle M^{T_n}, N^{S_n} \rangle_t = 0.$$

We conclude using that $\langle M^{T_n}, N^{S_n} \rangle_t = \langle M, N \rangle_{t \wedge T_n \wedge S_n} \rightarrow \langle M, N \rangle_t$ as $n \rightarrow \infty$ since $S_n, T_n \rightarrow \infty$ a.s.

Sheet 5, Exercise 3, Question 1. Let M be a continuous local martingale. Let $H \in L_{\text{loc}}^2(M)$, that is, H is a progressive process such that for all $t \geq 0$,

$$\int_0^t H_s^2 d\langle M \rangle_s < \infty \quad \text{a.s.}$$

Show that for all real numbers $t \geq 0$, $a > 0$, and $b > 0$,

$$\mathbb{P}\left(\int_0^t H_s dM_s \geq a\right) \leq \mathbb{P}\left(\int_0^t H_s^2 d\langle M \rangle_s \geq b\right) + \frac{b}{a^2}.$$

Solution. Let $\tau = \inf\{t \geq 0 : \langle H \cdot M \rangle_t \geq b\}$ which is a local time. Decomposing on $\{\tau \leq t\}$ and $\{\tau > t\}$, we find

$$\mathbb{P}\left(\int_0^t H_s dM_s \geq a\right) \leq \mathbb{P}\left(\int_0^t H_s^2 d\langle M \rangle_s \geq b\right) + \mathbb{P}\left(\int_0^{t \wedge \tau} H_s dM_s \geq a\right).$$

Then, Markov's inequality yields

$$\mathbb{P}\left(\int_0^{t \wedge \tau} H_s dM_s \geq a\right) \leq \frac{1}{a^2} \mathbb{E}[(\langle H \cdot M \rangle_{t \wedge \tau})^2].$$

Finally, the stopped local martingale $(H \cdot M)^\tau$ is a local martingale whose quadratic variation satisfies $\langle (H \cdot M)^\tau \rangle_\infty = \langle H \cdot M \rangle_\tau \leq b$. Hence, $(H \cdot M)^\tau$ is a (true) martingale bounded in L^2 and

$$\mathbb{E}[(\langle H \cdot M \rangle_{t \wedge \tau})^2] = \mathbb{E}[\langle H \cdot M \rangle_{t \wedge \tau}] \leq b.$$

Sheet 5, Exercise 5 There was no issue with question 3. You should recover the law of V_t .