# Introduction to statistical mechanics

## General framework (Static setting)

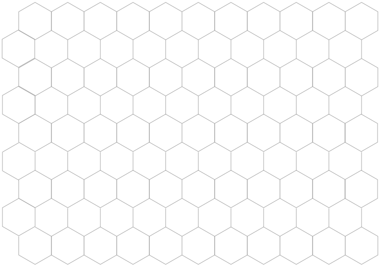
Dauphine, April 5, 2024

# STATISTICAL MECHANICS

*Understand the large scale behavior of a physics system whose interactions are described on the microscopic level*

Start from a model, whose general framework is the following.

- Structure of the physics system is represented by a graph $G = (V, E)$, finite.



# Statistical mechanics

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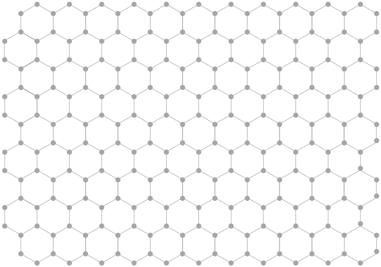
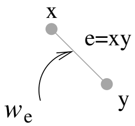


# Statistical mechanics

- Set of configurations on the graph $\mathsf{G}$: $\mathcal{C}(\mathsf{G})$,
  - vertex configurations,
  - edge configurations.
- Parameters:
  - intensity of interactions between microscopic components
  - external temperature.

$\Rightarrow$ Positive weight function $w = (w_{\mathsf{e}})_{\mathsf{e}\in\mathsf{E}}$ on the edges.



# Statistical mechanics

- To a configuration $\mathsf{C}$, one assigns an energy $\mathcal{E}_w(\mathsf{C})$.
- Boltzmann measure on configurations:

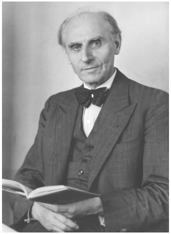
$$\forall\, \mathsf{C} \in \mathcal{C}(\mathsf{G}), \quad \mathbb{P}(\mathsf{C}) = \frac{e^{-\mathcal{E}_w(\mathsf{C})}}{Z(\mathsf{G}, w)},$$

where $Z(\mathsf{G}, w) = \sum_{\mathsf{C} \in \mathcal{C}(\mathsf{G})} e^{-\mathcal{E}_w(\mathsf{C})}$ is the partition function.

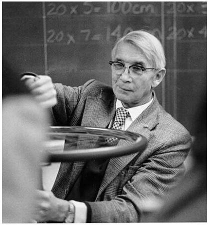
*Understand the behavior of configurations when the graph is large (infinite).*

# The Ising model

*Model of ferromagnetism - mixture of two materials*



Wilhelm Lenz (1888-1957)



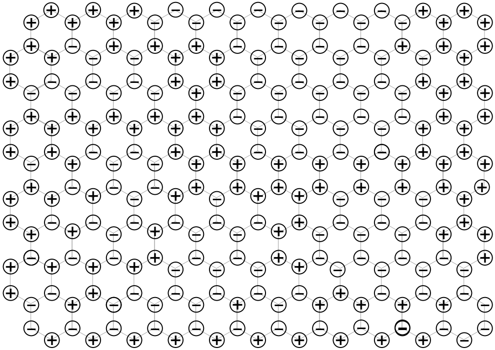

Ernst Ising (1900-1998)

- Graph $G = (V, E)$.
- A spin configuration $\sigma$ assigns a spin $\sigma_x \in \{-1, 1\}$ to a each vertex $x$ of the graph $G$.

$$\Rightarrow \mathcal{C}(G) = \{-1, 1\}^V = \text{set of spin configurations.}$$

# The Ising model

- A spin configuration

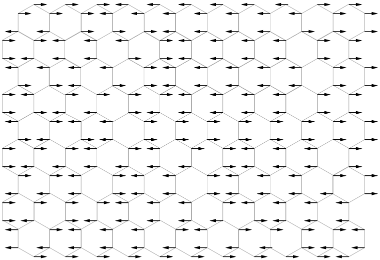


# The Ising model

- A spin configuration / two interpretations

Magnetic moments:

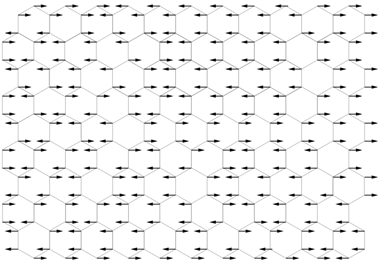
$+1/\rightarrow,\ -1/\leftarrow$



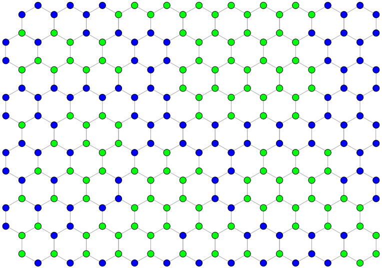
# The Ising model

- A spin configuration / two interpretations

Magnetic moments:
+1/→, −1/←



Mixture of two materials:
+1/●, -1/●.



# The Ising model

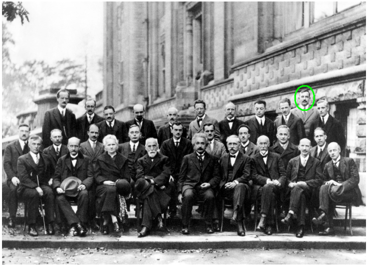
- Positive weight function: coupling constants $J = (J_e)_{e \in E}$.
- Energy of a spin configuration: $\mathcal{E}_J(\sigma) = -\sum_{e=xy \in E} J_{xy}\sigma_x\sigma_y$.
- Ising Boltzmann measure:

$$\forall\, \sigma \in \{-1,1\}^V, \quad \mathbb{P}_{\text{Ising}}(\sigma) = \frac{e^{-\mathcal{E}_J(\sigma)}}{Z_{\text{Ising}}(G, J)}.$$

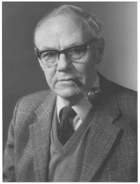
  - Two neighboring spins $\sigma_x, \sigma_y$ tend to align.
  - The higher the coupling $J_{xy}$, the higher this tendency.

# The dimer model

*Adsorption of di-atomic molecules on the surface of a cristal*



Sir Ralph H. Fowler (1889-1944)
Congrès Solvay 1927.



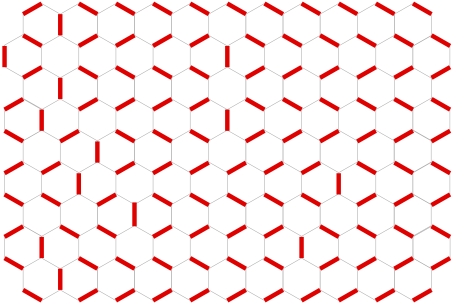
George S. Rushbrooke (1915-1995)

- Graph $\mathsf{G} = (\mathsf{V}, \mathsf{E})$.
- A dimer configuration or perfect matching: subset of edges such that each vertex touches exactly one edge of this subset.

$$\Rightarrow \mathcal{C}(\mathsf{G}) = \mathcal{M}(\mathsf{G}) = \text{set of dimer configurations.}$$

# The dimer model

- A dimer configuration.



# The dimer model

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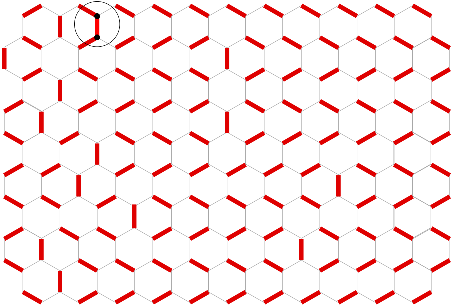

# The dimer model

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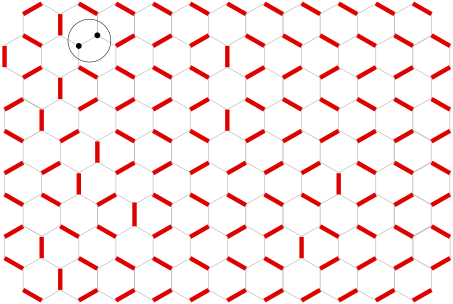
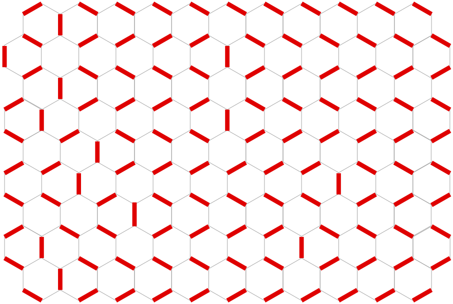


# The dimer model

- A dimer configuration.



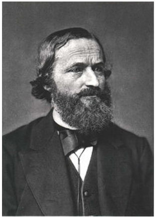
- Positive weight function: $\nu = (\nu_{\mathsf{e}})_{\mathsf{e} \in \mathsf{E}}$.
- Energy of a configuration $\mathsf{M}$: $\mathcal{E}_\nu(\mathsf{M}) = -\sum_{\mathsf{e} \in \mathsf{M}} \log \nu_{\mathsf{e}}$.
- Dimer Botzmann measure:

$$\forall\, \mathsf{M} \in \mathcal{M}(\mathsf{G}), \quad \mathbb{P}_{\mathrm{dimer}}(\mathsf{M}) = \frac{\prod_{\mathsf{e} \in \mathsf{M}} \nu_{\mathsf{e}}}{Z_{\mathrm{dimer}}(\mathsf{G}, \nu)}.$$

  - Edges with higher weights are more likely to occur.

# Spanning trees

*Related to electrical networks*



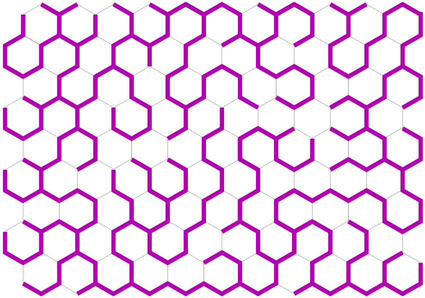
Gustav Kirchhoff (1824-1887)

- Graph $G = (V, E)$.
- A spanning tree: subset of edges covering all vertices of the graph, connected, with no cycle.

$$\Rightarrow \mathcal{C}(G) = \mathcal{T}(G) = \text{set of spanning trees.}$$

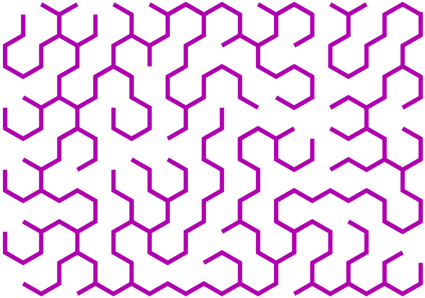
# Spanning trees

- A spanning tree



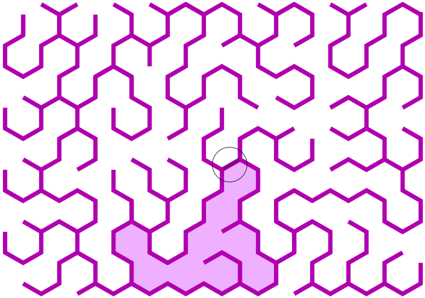
# Spanning trees

- A spanning tree



# Spanning trees

- A spanning tree



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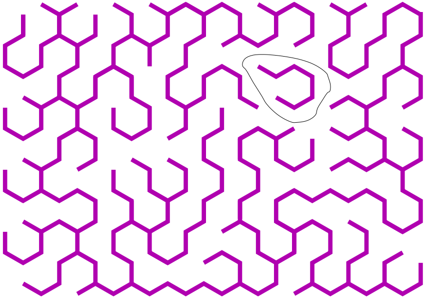
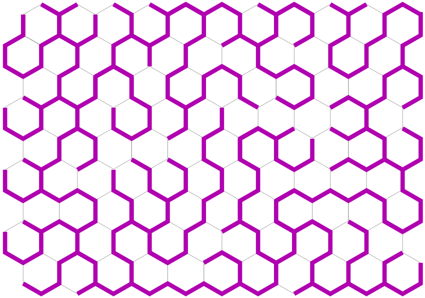
- A spanning tree



# Spanning trees

- A spanning tree



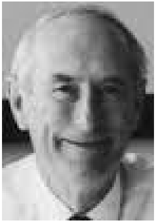
- Positive weight function: $\rho = (\rho_\mathsf{e})_{\mathsf{e}\in\mathsf{E}}$.
- Energy of a tree $\mathsf{T}$: $\mathcal{E}_\rho(\mathsf{T}) = -\sum_{\mathsf{e}\in\mathsf{T}} \log \rho_\mathsf{e}$.
- Tree Boltzmann measure:

$$\forall\, \mathsf{T} \in \mathcal{T}(\mathsf{G}), \quad \mathbb{P}_{\mathrm{tree}}(\mathsf{T}) = \frac{\prod_{\mathsf{e}\in\mathsf{T}} \rho_\mathsf{e}}{Z_{\mathrm{tree}}(\mathsf{G},\rho)}.$$

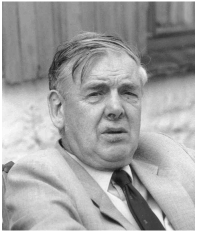
  - Edges with higher weights are more likely to occur.

# Percolation

*Flow of a liquid through a porous material*



Simon Broadbent (1928-2002)



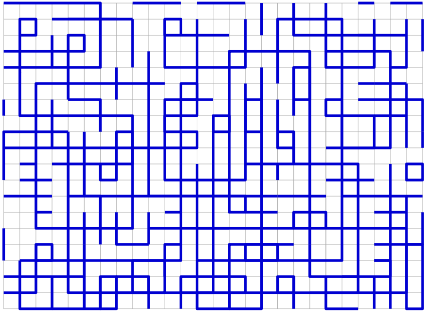
John Hammersley (1920-2004)

- Graph $G = (V, E)$.
- Configuration of open and closed edges: $\forall e \in E,\ \omega_e \in \{0, 1\}$.

$$\Rightarrow \quad \mathcal{C}(G) = \{0, 1\}^E.$$

# Percolation

- A percolation configuration



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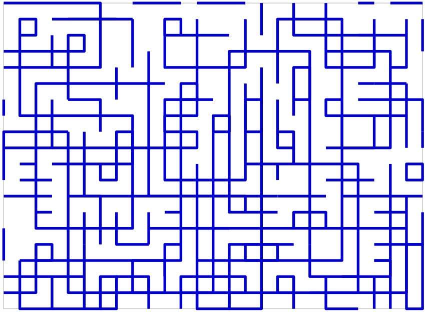
- A percolation configuration



- Let $p \in [0,1]$. Each edge is open/closed with probability $p$ / $1-p$, independently.
- Percolation measure:

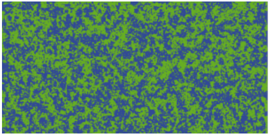
$$\forall\, \omega \in \{0,1\}^{\mathsf{E}}, \quad \mathbb{P}_{\text{perco}}(\omega) = p^{\sum_{e \in \mathsf{E}} \omega_e} (1-p)^{|\mathsf{E}| - \sum_{e \in \mathsf{E}} \omega_e}.$$

  - The higher $p$ is, the more open edges there are
  - For which values of $p$ do we percolate ?

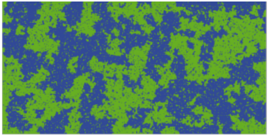
# Macroscopic behavior

*Let the edge-length tend to 0*
*Look at a "typical" configuration.*

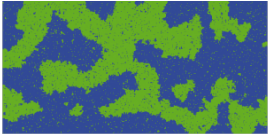
- Ising model (Illustrations of R. Cerf)



$J$ small



$J$ critical

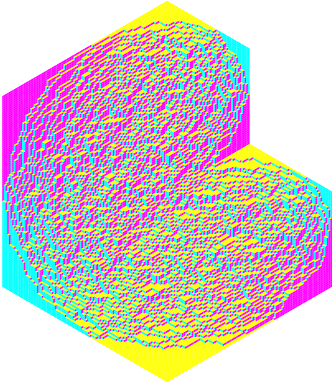


$J$ large

  - On $\mathbb{Z}^2$: $J_c \equiv \frac{1}{2}\log(1+\sqrt{2})$ [Kramers et Wannier]
  - Phase transition: studied through magnetization.

# Macroscopic behavior

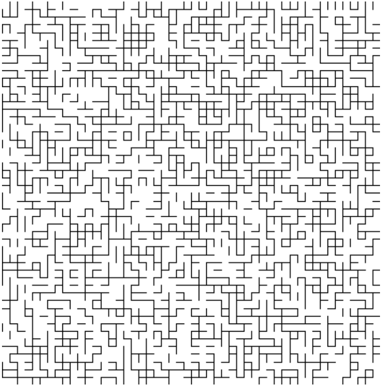
- Dimer model (Illustration of R. Kenyon)



- One sees two phases on the same figure.
- Phase transition studied through decay of correlations.

# Macroscopic behavior

- Percolation (Illustration of Erzbischof)



$p = 0.51$

- On $\mathbb{Z}^2$: $p_c = 0.5$ [Kesten]
- Phase transition: (non) existence of an infinite connected component.

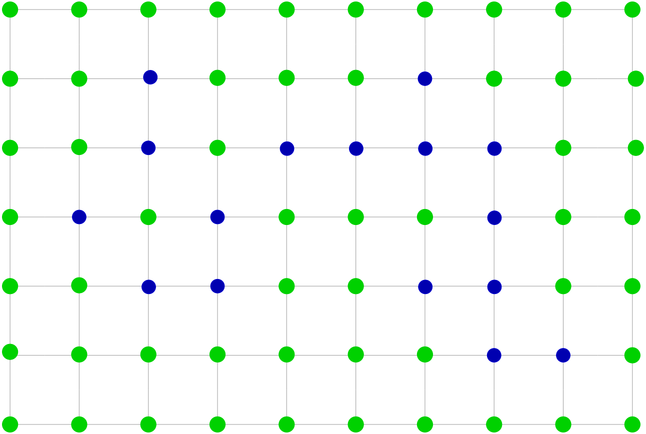
# Macroscopic behavior

- Identification of the phase transition.
- Understand the sub/super critical models.
- Understand the critical model (at the phase transition):
  - Universality and conformal invariance.
  - Conjectures: Nienhuis, Cardy, Duplantier ...

    Proofs: Lawler, Schramm, Werner (Fields 2006), D. Chelkak, S. Smirnov (Fields 2010), H. Duminil-Copin (Fields 2022), ...

# 2. Relations between models: Ising - dimers [Fisher'66]

- Ising model on G.



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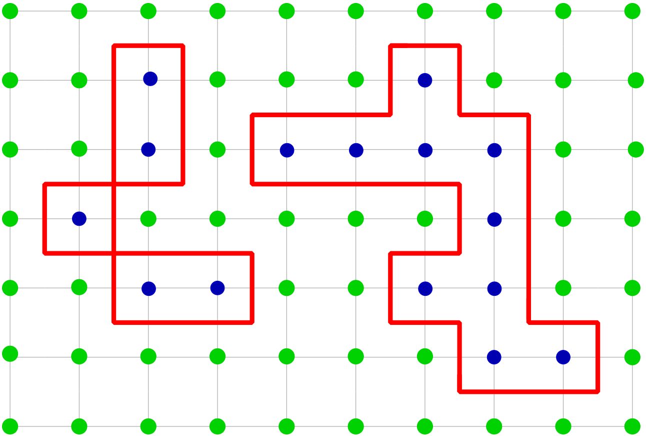
- Low temperature expansion [Kramers-Wannier].



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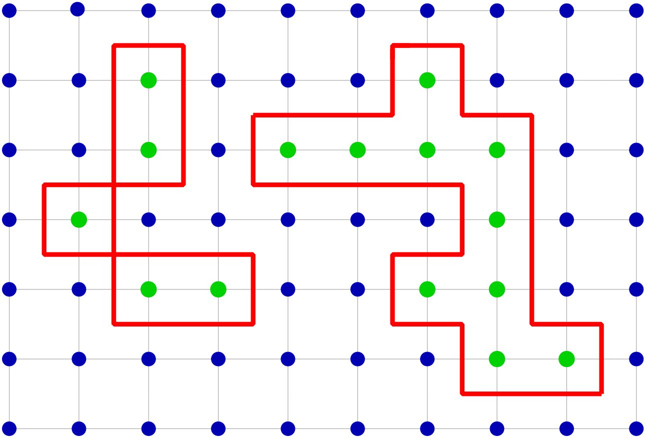
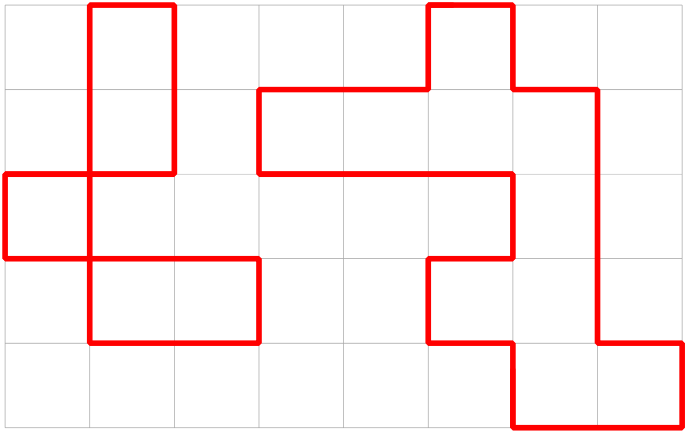


## 2. Relations between models: Ising - dimers [Fisher'66]

- Polygon contour configurations on G*.



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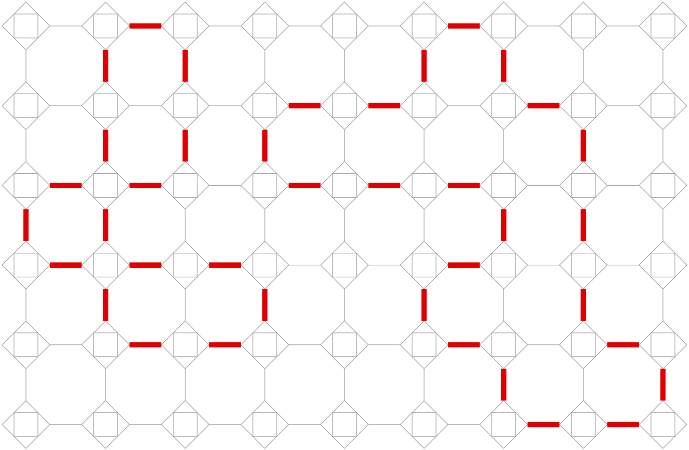
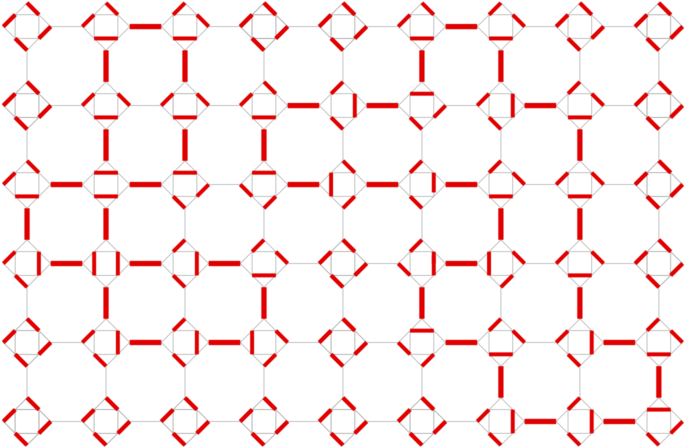
- Fisher's correspondence: exactly keep the polygon contour edges.



## 2. Relations between models: Ising - dimers [Fisher'66]

- Fill the decorations: $2^{|V^*|}$ possibilities.



# Exactly solvable models

- One of the tools to study the macroscopic behavior is the partition function:

$$Z(\mathsf{G}, w) = \sum_{\mathsf{C} \in \mathcal{C}(\mathsf{G})} e^{-\mathcal{E}_w(\mathsf{C})},$$

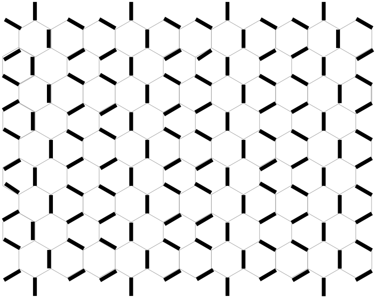
the normalizing constant in the Boltzmann measure.

$$\forall\, \mathsf{C} \in \mathcal{C}(\mathsf{G}), \quad \mathbb{P}(\mathsf{C}) = \frac{e^{-\mathcal{E}_w(\mathsf{C})}}{Z(\mathsf{G}, w)}.$$

- The model is exactly solvable if there exists an exact, explicit formula for the partition function
- Three exactly solvable models:
  - Ising-2d: Onsager (1944) - Fisher (1966).
  - Dimers-2d: Kasteleyn, Temperley-Fisher (1961).
  - Spanning trees: Kirchhoff (1848).

# Preliminaries

- Let $M_1$, $M_2$ be two dimer configurations of $G$, and $M_1 \cup M_2$ be their superposition.



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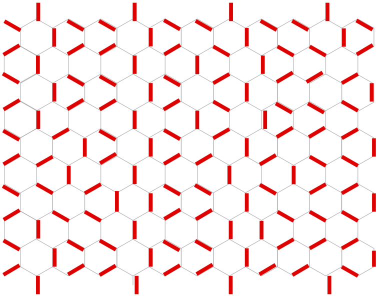

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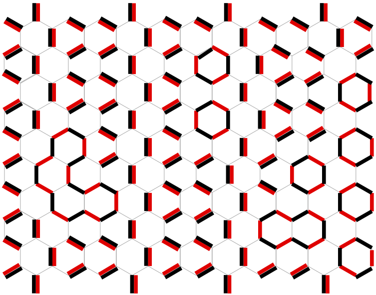

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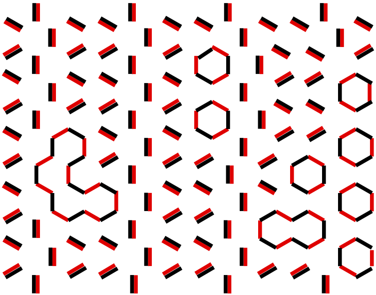

- $M_1 \cup M_2$ is a disjoint union of alternating cycles, where an alternating cycles has edges alternating between $M_1$ and $M_2$. Alternating cycles of length 2 are called doubled edges.