Université Paris-Dauphine

Introduction to the Ising model

Master 2, MATH

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Based on the lecture notes and book of

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CONTENTS

Chapter 1

Definitions and notation

1.1 The Ising model with free boundary conditions

We start by describing the context and defining the Ising model.

- The material (piece of metal, liquid) is represented by a connected subgraph $(\Lambda, E) \subset \mathbb{Z}^d$, for some fixed $d \geq 1$; that is, Λ is a subset of vertices of $\mathbb{Z}^d$, and E is the corresponding subset of edges:
$$E = \{xy : \ x \sim y \text{ in } \mathbb{Z}^d, \ x, y \in \Lambda\}.$$

- On Λ, we consider the set of *spin configurations*, denoted by Ω_Λ, defined as
$$\Omega_\Lambda = \{-1, 1\}^\Lambda = \{\sigma = (\sigma_x)_{x \in \Lambda} : \ \forall x \in \Lambda, \ \sigma_x \in \{-1, 1\}\}.$$

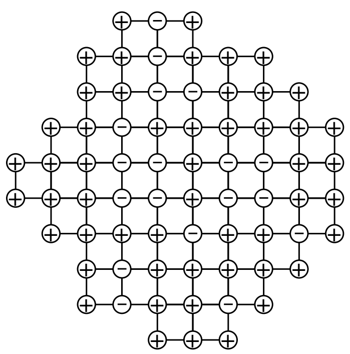


Figure 1.1: An example of subgraph (Λ, E) of $\mathbb{Z}^2$ together with a spin configuration σ.

- The *inverse external temperature* is represented by a parameter $\beta \geq 0$.

- For every $\sigma \in \Omega_\Lambda$, the *energy of the configuration* σ, denoted by $H_{\Lambda,\beta}(\sigma)$ is defined to be
$$H_{\Lambda,\beta}(\sigma) = -\beta \sum_{xy \in E} \sigma_x \sigma_y.$$

- The probability of occurrence of a spin configuration is given by the *Ising Boltzmann measure*, denoted by $\mu_{\Lambda,\beta}$, and defined by

$$\forall\ \sigma \in \Omega_\Lambda, \quad \mu_{\Lambda,\beta}(\sigma) = \frac{1}{Z_{\Lambda,\beta}} e^{-H_{\Lambda,\beta}(\sigma)},$$

where $Z_{\Lambda,\beta} = \sum_{\sigma\in\Omega_\Lambda} e^{-H_{\Lambda,\beta}(\sigma)}$ is the *partition function.* Note that this measure favors neighboring spins that agree, and that this tendency becomes greater as the temperature lowers (β increases).

1.2 Boundary conditions, magnetization, phase transition

Under the measure $\mu_{\Lambda,\beta}$, since for all $\sigma \in \Omega_\Lambda$, $\mu_\beta(\sigma) = \mu_\beta(-\sigma)$, the expected spin at 0 is

$$\langle \sigma_0 \rangle_{\Lambda,\beta} = \mu_{\Lambda,\beta}(\sigma_0 = +1) - \mu_{\Lambda,\beta}(\sigma_0 = -1) = 0.$$

In order to break this $\pm$ symmetry, one introduces more general boundary conditions.

1.2.1 Ising model with boundary conditions

Consider $(\Lambda, E) \subset \mathbb{Z}^d$ as before. Let us define

- $\partial\Lambda = \{x \in \mathbb{Z}^d \setminus \Lambda :\ \exists\, y \in \Lambda,\ x \sim y\}$,
- $\bar{\Lambda} := \Lambda \cup \partial\Lambda$,
- $\partial E = \{xy : x \sim y \text{ in } \mathbb{Z}^d,\ x \in \Lambda,\ y \in \partial\Lambda\}$,
- $\bar{E} = E \cup \partial E$.

A *boundary condition* for Λ is the data of $\omega \in \{-1, 0, 1\}^{\partial\Lambda}$.

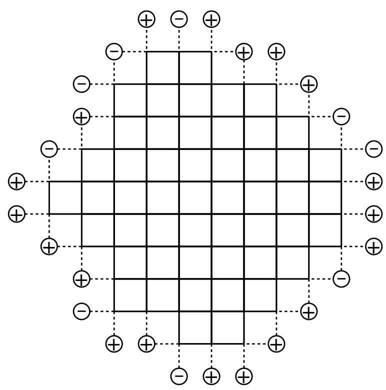


Figure 1.2: A subgraph (Λ, E) of $\mathbb{Z}^2$, its boundary vertices $\partial\Lambda$, the edges ∂E (dotted), and the edges $\bar{E}$ (plain and dotted); the vertices of $\partial\Lambda$ are assigned a boundary configuration $\omega \in \{-1, 1\}^{\partial\Lambda}$.

Given a boundary condition ω, the corresponding definitions of energy and Ising Boltzmann measure are:

- For every $\sigma \in \Omega_\Lambda$, the *energy of the configuration* σ is:

$$H^\omega_{\Lambda,\beta}(\sigma) = -\beta \sum_{xy \in E} \sigma_x \sigma_y - \beta \sum_{xy \in \partial E} \sigma_x \omega_y.$$

- The *Ising Boltzmann measure with* ω*-boundary conditions* is:

$$\forall\ \sigma \in \Omega_\Lambda, \quad \mu^\omega_{\Lambda,\beta}(\sigma) = \frac{1}{Z^\omega_{\Lambda,\beta}} e^{-H^\omega_{\Lambda,\beta}(\sigma)},$$

where $Z^\omega_{\Lambda,\beta} = \sum_{\sigma \in \Omega_\Lambda} e^{-H^\omega_{\Lambda,\beta}}$ is the corresponding *partition function.*

Notation. From now on, we try and simplify notation and depending on the context, we will use the notation:

$$H^\omega_{\Lambda,\beta} = H^\omega_\beta = H^\omega, \quad \mu^\omega_{\Lambda,\beta} = \mu^\omega_\beta = \mu^\omega,$$

For every function $f : \Omega_\Lambda \to \mathbb{R}$, $\quad \langle f \rangle^\omega_{\Lambda,\beta} = \langle f \rangle^\omega_\beta = \langle f \rangle^\omega = \int_{\Omega_\Lambda} f d\mu^\omega = \sum_{\sigma \in \Omega_\Lambda} f(\sigma)\mu^\omega(\sigma).$

Example 1.1. Here are some examples of boundary conditions (b.c.).

- $+$ *b.c.*: $\forall\ x \in \partial\Lambda$, $\omega_x = +1$, with the notation $H^+, \mu^+, \langle f \rangle^+$.
- $-$ *b.c.*: $\forall\ x \in \partial\Lambda$, $\omega_x = -1$, with the notation $H^-, \mu^-, \langle f \rangle^-$.
- *Free b.c.*: $\forall\ x \in \partial\Lambda$, $\omega_x = 0$, and we recover the Ising model with free boundary conditions of Section 1.1. The usual notation is $H^\emptyset = H, \mu^\emptyset = \mu, \langle f \rangle^\emptyset = \langle f \rangle$.

1.2.2 Magnetization and phase transition

Let us get ahead of ourselves and explain one of the main goals of these lectures. We want to prove that, when $d \geq 2$, the Ising model undergoes *a phase transition* between a low and a high temperature regime; in the first regime, neighboring spins are expected to influence each other, while in the second, the model is closer to independent. One way to study this is to see how boundary conditions in a box influence the value of the spin at the center of the box. This motivates the following definitions.

An *exhaustion* of $\mathbb{Z}^d$ is a sequence $(\Lambda_n)_{n\geq1}$ of increasing subsets of $\mathbb{Z}^d$, such that $\bigcup_{n\geq1} \Lambda_n = \mathbb{Z}^d$.

The *magnetization* is defined to be:

$$\langle \sigma_0 \rangle^+_\beta = \lim_{n\to\infty} \langle \sigma_0 \rangle^+_{\Lambda_n,\beta}.$$

Remark 1.2. As we go along, we will be proving that:

- this limit exists,
- for all $\beta \geq 0$, $\langle \sigma_0 \rangle^+_\beta \geq 0$,

- $\langle \sigma_0 \rangle_\beta^+$ is increasing as a function of β.

Assuming this for the moment, one defines the *critical inverse temperature* $\beta_c(d)$ as the largest β such that the magnetization is zero:

$$\beta_c(d) = \sup\{\beta \geq 0 : \ \langle \sigma_0 \rangle_\beta^+ = 0\}.$$

When $d \geq 2$, we will show that for β small enough $\langle \sigma_0 \rangle_\beta^+ = 0$, and that for β large enough $\langle \sigma_0 \rangle_\beta^+ > 0$.

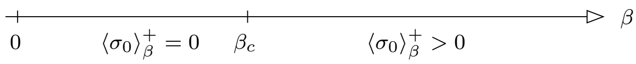


Figure 1.3: Pictorial view of the phase diagram of the Ising model.

As a consequence, the Ising model undergoes a *phase transition* between the high temperature regime where the magnetization behaves as if it had free boundary conditions (it does not feel the effect of the boundary), and a low temperature regime where the model feels the boundary (the $+$ on the boundary influence the spin at 0).

We will also prove that, when $d = 1$, for all $\beta \geq 0$, $\langle \sigma_0 \rangle_\beta^+ = 0$, meaning that there is no phase transition in this case; this result was originally established by Ising.

Another goal of these lectures is to prove an equivalent characterization of the phase transition using uniqueness/non-uniqueness of the infinite volume limits of the Boltzmann measures with different boundary conditions, which we will first need to construct.

Chapter 2

Domain Markov property, correlation inequalities

In order to study limits of Boltzmann measures and phase transition, we need to study the Ising model in infinite volume. This relies on a good understanding of the model defined on a finite subgraph of $\mathbb{Z}^d$. Our main target for this section is to state correlation inequalities and their consequences, but we start with another interesting property known as the domain Markov property.

In all this section (Λ, E) is a finite, connected subgraph of $\mathbb{Z}^d$, and the inverse temperature β is assumed to be fixed.

2.1 Domain Markov property

An interesting feature of the Ising model is that it satisfies a spatial Markov property, referred to as *domain Markov property*, as stated by the following, see also Figure 2.1.

Proposition 2.1. *Let $\Lambda' \subset \Lambda$ (where the inclusion is assumed to be strict), and let $\omega \in \{-1,1\}^{\partial\Lambda}$ be a boundary condition on Λ. Let $\bar{\sigma}$ be a fixed spin configuration on $\Lambda \setminus \Lambda'$ inducing a boundary condition ω' on $\partial\Lambda'$: $\forall\, x \in \partial\Lambda'$, $\omega'_x = \bar{\sigma}_x$. Let $C = \{\sigma \in \Omega_\Lambda :\ \forall\, x \in \Lambda \setminus \Lambda',\ \sigma_x = \bar{\sigma}_x\}$. Then,*

$$\mu_\Lambda^\omega(\,\cdot\,|C) = \mu_{\Lambda'}^{\omega'}(\,\cdot\,).$$

Proof. Note that there is a natural bijection between C and $\Omega_{\Lambda'}$. We need to show that, for all $\sigma \in C$, $\frac{\mu_\Lambda^\omega(\sigma)}{\mu_\Lambda^\omega(C)} = \mu_{\Lambda'}^{\omega'}(\sigma)$. We have,

$$\begin{aligned}
-H_\Lambda^\omega(\sigma) &= \beta \sum_{xy\in E} \sigma_x\sigma_y + \beta \sum_{xy\in\partial E} \sigma_x\omega_y \\
&= \beta \sum_{xy\in E'} \sigma_x\sigma_y + \beta \sum_{xy\in\partial E'} \sigma_x\omega'_y + \beta \sum_{xy\in E\setminus\bar{E}'} \bar{\sigma}_x\bar{\sigma}_y + \beta \sum_{xy\in\partial E} \bar{\sigma}_x\omega_y \\
&= -H_{\Lambda'}^{\omega'}(\sigma) + \mathcal{C}(\bar{\sigma}, \omega),
\end{aligned}$$

where $\mathcal{C}(\bar{\sigma},\omega) = \sum_{xy \in E \setminus \bar{E}'} \bar{\sigma}_x \bar{\sigma}_y + \beta \sum_{xy \in \partial E} \bar{\sigma}_x \omega_y$ does not depend on $\sigma \in C$. As a consequence,

$$\frac{\mu_\Lambda^\omega(\sigma)}{\mu_\Lambda^\omega(C)} = \frac{e^{-H_\Lambda^\omega(\sigma)}}{\sum_{\sigma \in C} e^{-H_\Lambda^\omega(\sigma)}} = \frac{e^{\mathcal{C}(\bar{\sigma},\omega)} e^{-H_{\Lambda'}^{\omega'}(\sigma)}}{e^{\mathcal{C}(\bar{\sigma},\omega)} \sum_{\sigma \in C} e^{-H_{\Lambda'}^{\omega'}(\sigma)}} = \mu_{\Lambda'}^{\omega'}(\sigma). \qquad \square$$

Remark 2.2. This implies that, for every function $f : \Omega_{\Lambda'} \to \mathbb{R}$,

$$\frac{\langle f\, \mathbb{I}_C \rangle_\Lambda^\omega}{\langle \mathbb{I}_C \rangle_\Lambda^\omega} = \langle f \rangle_{\Lambda'}^{\omega'}.$$

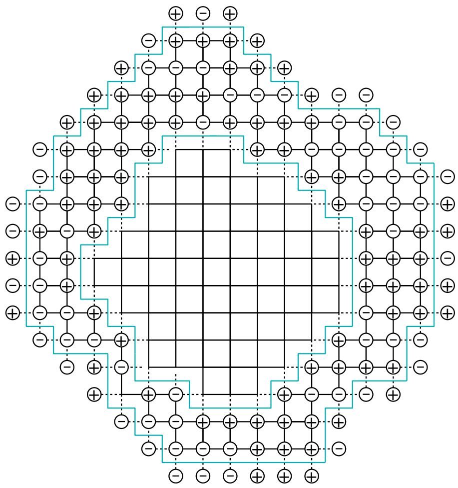


Figure 2.1: Illustration of the domain Markov property.

2.2 Increasing functions and local functions

We define two special kinds of functions that will play an important role in the sequel, namely *increasing*, resp. *local* functions. In all that follows, we let $\Omega = \{-1, 1\}^{\mathbb{Z}^d}$.

Definition 2.3. A function $f : \Omega \to \mathbb{R}$ is said to be *increasing* if

$$\sigma \leq \sigma' \quad \Rightarrow \quad f(\sigma) \leq f(\sigma'),$$

where $\sigma \leq \sigma'$ means that, for all $x \in \mathbb{Z}^d$, $\sigma_x \leq \sigma_{x'}$.

Example 2.4.

1. For every $x \in \mathbb{Z}^d$, the function $\tilde{\sigma} \mapsto \sigma_x(\tilde{\sigma}) = \tilde{\sigma}_x$ is increasing.
2. For every subset $A \subset \Lambda$, define the *multipoint spin function* σ_A to be

$$\tilde{\sigma} \mapsto \sigma_A(\tilde{\sigma}) = \prod_{x \in A} \sigma_x(\tilde{\sigma}) = \prod_{x \in A} \tilde{\sigma}_x.$$

This function is not increasing if $|A| \geq 2$. For example, take $A = \{x_1, x_2\}$ with $x_1 \neq x_2$, and

$$\tilde{\sigma} = (\tilde{\sigma}_x) \text{ with } \tilde{\sigma}_x = \begin{cases} +1 & \text{if } x \notin \{x_1, x_2\} \\ -1 & \text{if } x \in \{x_1, x_2\} \end{cases}, \quad \tilde{\sigma}' = (\tilde{\sigma}'_x) \text{ with } \tilde{\sigma}'_x = \begin{cases} +1 & \text{if } x \notin \{x_1, x_2\} \\ 1 & \text{if } x = x_1 \\ -1 & \text{if } x = x_2. \end{cases}$$

Then $\tilde{\sigma} \leq \tilde{\sigma}'$, and $\sigma_A(\tilde{\sigma}) > \sigma_A(\tilde{\sigma}')$.

3. For all $x \in \Lambda$, the function $n_x = \mathbb{I}_{\{\sigma_x = 1\}} : \tilde{\sigma} \mapsto \frac{\tilde{\sigma}_x + 1}{2} = \mathbb{I}_{\{\sigma_x=1\}}(\tilde{\sigma})$ is increasing.
4. For all $A \subset \Lambda$, the function $n_A = \prod_{x \in A} n_x = \mathbb{I}_{\{\forall x \in A,\ \sigma_x = 1\}}$ is increasing.
5. For all $A \subset \Lambda$, the function $(\sum_{x \in A} n_x) - n_A$ is increasing. Indeed,

$$\left(\sum_{x \in A} n_x\right) - n_A = |\{x \in A \,:\, \sigma_x = 1\}| - \mathbb{I}_{\{\forall x \in A \,:\, \sigma_x = 1\}}.$$

Consider $\tilde{\sigma} \leq \tilde{\sigma}'$. If both configurations are equal on A, it is trivial; else, if $\tilde{\sigma} < \tilde{\sigma}'$ on A,

$$|\{x \in A \,:\, \tilde{\sigma}'_x = 1\}| \geq |\{x \in A \,:\, \tilde{\sigma}_x = 1\}| + 1, \text{ and } \mathbb{I}_{\{\forall x \in A \,:\, \tilde{\sigma}'_x = 1\}} \leq \mathbb{I}_{\{\forall x \in A \,:\, \tilde{\sigma}_x = 1\}} + 1,$$

concluding the proof.

The fact that these functions are increasing will turn out to be crucial in forthcoming proofs.

Definition 2.5. A function $f : \Omega \to \mathbb{R}$ is said to be *local* if there exists $\Lambda \subset \mathbb{Z}^d$ such that f is Ω_Λ-measurable. Otherwise stated, a function f is *local* if it only depends on a finite number of spins. We denote by $\mathrm{supp}(f)$ the *support* of f, that is the smallest subset of $\mathbb{Z}^d$ whose values determine f.

EXAMPLE 2.6. The functions $\sigma_x, \sigma_A, n_x, n_A$ are all local functions, with respective support x, A, x, A.

The next lemma gives two useful basis of local functions. Consider $\Lambda \subset \mathbb{Z}^d$, and define $\mathcal{L}_\Lambda$ to be the set of local functions with support equal to Λ:

$$\mathcal{L}_\Lambda = \{f : \Omega \to \mathbb{R} :\ f \text{ is local and } \mathrm{supp}(f) = \Lambda\}.$$

This set has dimension $2^{|\Lambda|}$, the number of spin configurations in Ω_Λ. Indeed, any function f in $\mathcal{L}_\Lambda$ is Ω_Λ-measurable and can be written as $f = \sum_{\sigma \in \Omega_\Lambda} f(\sigma)\mathbb{I}_\sigma$. On $\mathcal{L}_\Lambda$, consider the inner product $(f, g) = \mathbb{E}[fg]$, where the expectation is taken with respect to the uniform measure on Ω_Λ.

Lemma 2.7. *We have that $(\sigma_A)_{A \subset \Lambda}$ is an orthonormal basis of $\mathcal{L}_\Lambda$ with respect to the scalar product $(\cdot, \cdot)$, and $(n_A)_{A \subset \Lambda}$ is a basis of $\mathcal{L}_\Lambda$.*

Proof. Let us prove that $(\sigma_A)_{A \subset \Lambda}$ are orthonormal. Let $A, B \subset \Lambda$. Then,

$$(\sigma_A, \sigma_B) = \mathbb{E}[\sigma_{A \setminus B}\sigma_{A \cap B}\sigma_{B \setminus A}\sigma_{B \cap A}] \overset{\text{sym. diff.}}{=} \mathbb{E}[\sigma_{A \Delta B}] \overset{\text{ind.}}{=} \prod_{x \in A \Delta B} \mathbb{E}[\sigma_x] = \begin{cases} 0 & \text{if } A \neq B \\ 1 & \text{otherwise.} \end{cases}$$

The proof is concluded by observing that $(\sigma_A)_{A \subset \Lambda}$ has $2^{|\Lambda|}$ elements, which is the dimension of $\mathcal{L}_\Lambda$.

By definition, for every $A' \subset \Lambda$, $\sigma_{A'} = \prod_{x \in A'}(2n_x - 1)$ is a linear combination of $(n_A)_{A \subset \Lambda}$ (with some coefficients equal to 0), implying that $(n_A)_{A \subset \Lambda}$ generates $\mathcal{L}_\Lambda$. Since it also has $2^{|\Lambda|}$ elements, it is a basis. $\square$

2.3 Correlation inequalities and consequences

The goal of this section is to state two fundamental correlation inequalities: the GKS and FKG inequalities; we refer to [FV17, Section 3.8] for their proofs.

Note that the coming assumption $\mathrm{supp}(f) \subset \Lambda$ on a local function $f : \Omega \to \mathbb{R}$ is there to ensure that f is Ω_Λ-measurable. This implies that f can be seen as a function $f : \Omega_\Lambda \to \mathbb{R}$, and that $\langle f \rangle_\Lambda$ is well defined.

Theorem 2.8 (Griffiths-Kelly-Sherman inequalities)**.** *Let $A, B \subset \Lambda$, then*

$$\begin{aligned} &i) \quad \langle \sigma_A \rangle_\Lambda^+ \geq 0 \\ &ii) \quad \langle \sigma_A \sigma_B \rangle_\Lambda^+ \geq \langle \sigma_A \rangle_\Lambda^+ \langle \sigma_B \rangle_\Lambda^+. \end{aligned}$$

Theorem 2.9 (Fortuin-Kasteleyn-Ginibre inequality)**.** *For every pair of increasing, local functions $f, g : \Omega \to \mathbb{R}$ such that $\mathrm{supp}(f), \mathrm{supp}(g) \subset \Lambda$, and for every boundary condition $\omega \in \{-1,1\}^{\partial\Lambda}$, we have*

$$\langle f\, g \rangle_\Lambda^\omega \geq \langle f \rangle_\Lambda^\omega \langle g \rangle_\Lambda^\omega.$$

Here are two consequences. The first allows to compare boundary conditions, while the second allows to compare nested domains.

Lemma 2.10. *For every increasing, local function $f : \Omega \to \mathbb{R}$ such that $\mathrm{supp}(f) \subset \Lambda$, and for every boundary condition $\omega \in \{-1,1\}^{\partial\Lambda}$, we have*

$$\langle f \rangle_\Lambda^- \leq \langle f \rangle_\Lambda^\omega \leq \langle f \rangle_\Lambda^+.$$

Proof. Define $I^\omega(\sigma) = e^{\beta \sum_{xy \in \partial E} \sigma_x (1-\omega_y)}$. Then, since $1 - \omega_y \geq 0$, the function I^ω is increasing. Moreover, for every $\sigma \in \Omega_\Lambda$,

$$\begin{aligned} -H^+(\sigma) &= \beta \sum_{xy \in E} \sigma_x \sigma_y + \beta \sum_{xy \in \partial E} \sigma_x \cdot 1 \\ &= \beta \sum_{xy \in E} \sigma_x \sigma_y + \beta \sum_{xy \in \partial E} \sigma_x \omega_y + \log\big(I^\omega(\sigma)\big) = -H^\omega(\sigma) + \log(I^\omega(\sigma)). \end{aligned}$$

As a consequence, for every $\sigma \in \Omega_\Lambda$ and every local function f,

$$e^{-H^+(\sigma)} = e^{-H^\omega(\sigma)} I^\omega(\sigma), \quad e^{-H^+(\sigma)} f(\sigma) = e^{-H^\omega(\sigma)} I^\omega(\sigma) f(\sigma).$$

Using that I^ω is increasing and the FKG inequality we deduce that, for every increasing, local function f,

$$\langle f \rangle_\Lambda^+ = \frac{\sum_{\sigma \in \Omega_\Lambda} e^{-H^+(\sigma)} f(\sigma)}{\sum_{\sigma \in \Omega_\Lambda} e^{-H^+(\sigma)}} = \frac{\sum_{\sigma \in \Omega_\Lambda} e^{-H^\omega(\sigma)} I^\omega(\sigma) f(\sigma)}{\sum_{\sigma \in \Omega_\Lambda} e^{-H^\omega(\sigma)} I^\omega(\sigma)} = \frac{\langle I^\omega f \rangle_\Lambda^\omega}{\langle I^\omega \rangle_\Lambda^\omega} \overset{\text{FKG}}{\geq} \langle f \rangle_\Lambda^\omega. \qquad \square$$

Lemma 2.11. *Let $\Lambda' \subset \Lambda$ (where the inclusion is assumed to be strict). Then for every increasing, local function $f : \Omega \to \mathbb{R}$ such that $\mathrm{supp}(f) \subset \Lambda'$, we have*

$$\langle f \rangle_{\Lambda'}^+ \geq \langle f \rangle_\Lambda^+.$$

Proof. Let $C = \{\sigma \in \Omega_\Lambda : \forall\, x \in \Lambda \setminus \Lambda',\, \sigma_x = 1\}$. Note that we have, $\mathbb{I}_C = n_{\Lambda\setminus\Lambda'}$ which is an increasing function. Let f be as in the statement, then by the domain Markov property (and Remark 2.2), and by the FKG inequality, we have

$$\langle f \rangle^+_{\Lambda'} \overset{\text{DMP}}{=} \frac{\langle f\, n_{\Lambda\setminus\Lambda'} \rangle^+_\Lambda}{\langle n_{\Lambda\setminus\Lambda'} \rangle^+_\Lambda} \overset{\text{FKG}}{\geq} \langle f \rangle^+_\Lambda. \qquad \square$$

Chapter 3

Thermodynamic limit

The goal of this section is to define infinite volume versions of the Ising Boltzmann measures. This poses some natural questions: how are such Ising probability measures defined ? Are there many or just one ? How do you construct such measures ?

Let us stay informal at this point. A natural definition uses the DLR (Dobrushin-Lanford-Ruelle) approach: an *infinite volume Ising Gibbs measure* is a probability measure on $(\Omega, \mathcal{F})$ (where $\mathcal{F}$ is the σ-field generated by cylinder sets) such that, when conditioning on a fixed spin configuration outside of a box, the conditional measure inside the box is the Ising Boltzmann measure with the induced boundary conditions. We refer to [FV17, Section 6.2.1] for more details on this definition.

An explicit approach for constructing infinite volume Gibbs measures consists in considering weak limits of the Boltzmann measures $\mu_{\Lambda_n}^{\omega_n}$, where $(\Lambda_n)_{n\geq 1}$ is an exhaustion of $\mathbb{Z}^d$, and $(\omega_n)_{n\geq 1}$ is a sequence of boundary conditions, and to use the Riesz representation theorem. *Infinite volume Gibbs measures* are then defined to be accumulation points of these weak limits. It can be shown that this definition is consistent with the one given by the DLR approach, see for example [FV17, Chapter 6, Equation (6.11)].

In this section we consider the second approach, and explicitly construct two Ising Gibbs measures as the weak limit of the Boltzmann measures with $\pm$ boundary conditions. We will also address the question of (non)-uniqueness.

3.1 Prerequisites

Recall that $\Omega = \{-1, 1\}^{\mathbb{Z}^d}$.

3.1.1 Structure of compact metric space on Ω

Consider Ω as a topological space, with the product topology arising from the discrete topology on $\{-1, 1\}$. Then, Ω is a compact topological space as the cartesian product of the compact topological space $\{-1, 1\}$.

The space Ω can also be seen as a metric space with distance function d defined by

$$\forall\, \sigma, \sigma' \in \Omega, \quad d(\sigma, \sigma') = \sum_{x\in\mathbb{Z}^d} \frac{1}{2^{\|x\|_\infty}} \mathbb{I}_{\{\sigma_x \neq \sigma'_x\}},$$

where $\|x\|_\infty = \max_{i\in\{1,\dots,d\}} |x_i|$. This metric induces the product topology on Ω. Intuitively, two spin configurations should be thought of as "close" if they coincide on a large ball surrounding the origin. The space Ω is thus now equipped with a structure of compact metric space.

Let $C(\Omega)$ be the set of continuous function on Ω. Note that since Ω is compact, it coincides with the set of uniformly continuous functions:

$$\{f : \Omega \to \mathbb{R} : \ \forall \varepsilon > 0, \exists \delta > 0, \text{ s.t. } \forall \sigma, \sigma', \ d(\sigma, \sigma') < \delta \Rightarrow |f(\sigma) - f(\sigma')| < \varepsilon\}.$$

One can check that continuous functions are bounded, and that the set of local functions of Definition 2.5 is dense in $C(\Omega)$. More details can be found in [FV17, Section 6.4.1].

3.1.2 Weak convergence and the Riesz-Markov-Kakutani theorem

Let us denote by $\mathcal{F}$ the *σ-field generated by cylinder sets* on Ω, that is, $\mathcal{F}$ is defined to be

$$\mathcal{F} = \sigma(\cup_{\Lambda \subset \mathbb{Z}^d} \mathcal{C}_\Lambda),$$

where $\mathcal{C}_\Lambda = \{\sigma \in \Omega : \ \sigma|_\Lambda \in A, A \in \mathcal{P}(\Omega_\Lambda)\}$ is the cylinder corresponding to the box Λ. Then, one can check that $\mathcal{F}$ coincides with the Borel σ-field on Ω, see [FV17, Exercice 6.10].

We can now introduce the notion of weak convergence for Boltzmann measures. A sequence of Boltzmann measures $(\mu_{\Lambda_n}^{\omega_n})_{n\geq 1}$ on $(\Omega, \mathcal{F})$ *converges weakly* to a probability measure μ on $(\Omega, \mathcal{F})$, if for every bounded continuous function on Ω,

$$\lim_{n\to\infty} \langle f \rangle_{\Lambda_n}^{\omega_n} = \langle f \rangle_\mu,$$

one then writes $\mu_{\Lambda_n}^{\omega_n} \Rightarrow \mu$.

Remark 3.1. In our case, continuous functions are bounded and local functions are dense in the set of continuous functions. One can check that $\mu_{\Lambda_n}^{\omega_n} \Rightarrow \mu$ if and only if, for every local function $f : \Omega \to \mathbb{R}$, $\lim_{n\to\infty} \langle f \rangle_{\Lambda_n}^{\omega_n} = \langle f \rangle_\mu$.

Suppose that we have proved:

$$\text{For every local function } f, \quad \lim_{n\to\infty} \langle f \rangle_{\Lambda_n}^{\omega_n} = \ell(f), \tag{3.1}$$

for some value $\ell(f)$. Then the functional $\ell : \{\text{Local functions}\} \to \mathbb{R}$, is positive, linear and satisfies $\ell(1) = 1$. Since local functions are dense in continuous functions, the functional ℓ can be extended to the set of continuous functions $C(\Omega)$.

Can we deduce the existence of a probability measure μ on $(\Omega, \mathcal{F})$ such that $\ell(f) = \langle f \rangle_\mu$? The answer is yes and is given by the following.

Theorem 3.2 (A version of the Riesz-Markov-Kakutani representation theorem)**.** *Let X be a compact metric space. If $\ell : C(X) \to \mathbb{R}$ is a positive linear functional on $C(X)$ such that $\ell(1) = 1$, then there exists a unique probability measure on $(X, \mathcal{F})$ such that*

$$\ell(f) = \langle f \rangle_\mu,$$

where $\mathcal{F}$ is the Borel σ-field of X.

Definition 3.3. An *infinite volume Ising Gibbs measure* is defined to be an accumulation point of the Boltzmann measures $(\mu_{\Lambda_n}^{\omega_n})_{n\geq 1}$, as described above.

The bottom line of this discussion is that as soon as we prove Condition (3.1) for a sequence of Boltzmann measures, this establishes weak convergence of this sequence to an infinite volume Gibbs measure.

3.2 Two infinite volume Gibbs measure

We now prove weak convergence of the Boltzmann measure with $\pm$ boundary conditions.

Theorem 3.4. *Let us fix $\beta \geq 0$. Then, for every exhaustion $(\Lambda_n)_{n\geq 1}$ of $\mathbb{Z}^d$, the sequence of measures $(\mu^+_{\Lambda_n})_{n\geq 1}$, resp. $(\mu^-_{\Lambda_n})_{n\geq 1}$, converges weakly to an infinite volume Ising Gibbs measure μ^+, resp μ^-, and this limit is independent of the choice of exhaustion. Moreover, the measures μ^+, μ^- are invariant by translation.*

Proof of Theorem 3.4. Let $(\Lambda_n)_{n\geq 1}$ be an exhaustion of $\mathbb{Z}^d$. By Section 3.1.2, we need to prove Condition (3.1), *i.e.*, that for every local function f, $(\langle f\rangle^+_{\Lambda_n})_{n\geq 1}$ converges; we also need to prove that this limit is independent of the choice of the exhaustion $(\Lambda_n)_{n\geq 1}$.

Let $A \subset \mathbb{Z}^d$, then n_A is a local increasing function. Let m be large enough so that $\Lambda_m \supset A$. Then, by Lemma 2.11, for every $n \geq m$, we have $\langle n_A\rangle^+_{\Lambda_{n+1}} \leq \langle n_A\rangle^+_{\Lambda_n}$. Moreover n_A is a bounded function, so that $(\langle n_A\rangle^+_{\Lambda_n})_{n\geq m}$ is a bounded decreasing sequence, thus convergent to a limit that we denote by $\langle n_A\rangle^+$.

Let f be a local function. Then by Lemma 2.7, we can write $f = \sum_{A\subset \mathrm{supp}(f)} \tilde{f}_A\, n_A$ and, since the sum is finite, we deduce from the above that

$$\lim_{n\to\infty} \langle f\rangle^+_{\Lambda_n} = \sum_{A\subset \mathrm{supp}(f)} \tilde{f}_A \langle n_A\rangle^+ = \langle f\rangle^+.$$

We now prove that the limit is independent of the exhaustion $(\Lambda_n)_{n\geq 1}$ of $\mathbb{Z}^d$.

Let $(\Lambda^1_n)_{n\geq 1}$, $(\Lambda^2_n)_{n\geq 1}$ be two exhaustions of $\mathbb{Z}^d$, and let μ^+_1, μ^+_2 be the two corresponding limiting Gibbs measures. Define a new exhaustion $(\Delta_n)_{n\geq 1}$ of $\mathbb{Z}^d$ as follows

$$\Delta_1 = \Lambda^1_1, \quad \forall\, k \geq 1,\ \Delta_{2k} = \bigcap_{n\geq 1} \{\Lambda^2_n :\ \Lambda^2_n \supset \Delta_{2k-1}\}, \quad \Delta_{2k+1} = \bigcap_{n\geq 1} \{\Lambda^1_n :\ \Lambda^1_n \supset \Delta_{2k}\}.$$

Since $(\Delta_n)_{n\geq 1}$ is an exhaustion, $\mu^+_{\Delta_n}$ converge weakly to a Gibbs measure μ^+. Since it converges, it does so on subsequences and the limit is the same. As a consequence, it converges to μ^+ on a subsequence of $(\Lambda^1_n)_{n\geq 1}$, resp. $(\Lambda^2_n)_{n\geq 1}$, implying that $\mu^+_1 = \mu^+_2$.

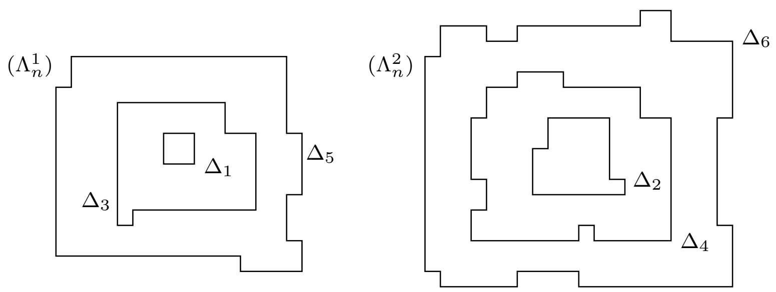


Figure 3.1: The two exhaustions $(\Lambda^1_n)_{n\geq 1}$, $(\Lambda^2_n)_{n\geq 1}$, and the exhaustion $(\Delta_n)_{n\geq 1}$.

Translation invariance follows from the definition of the Boltzmann measure. The same argument holds for $-$ boundary conditions. $\square$

3.3 A characterization of the phase transition

The goal of this section is to characterize the phase transition of the Ising model using the Gibbs measures $\mu^{\pm}$. But before that, recall that in Section 1.2.2 we went ahead of ourselves when defining the phase transition; we now have all the tools to do it rigorously, which is the purpose of the next sub-section.

3.3.1 Magnetization and phase transition, rigorous definitions

Since the value of the spin at the origin is a local function, by Theorem 3.4, we have that the magnetization

$$\langle \sigma_0 \rangle_\beta^+ = \lim_{n \to \infty} \langle \sigma_0 \rangle_{\Lambda_n, \beta}^+$$

is well defined and independent of the choice of exhaustion $(\Lambda_n)_{n \geq 1}$. Moreover, using the first GKS inequality (Theorem 2.8), we deduce that $\langle \sigma_0 \rangle_\beta^+ \geq 0$. Hence, in order to prove that the critical temperature is well defined, we are left with showing the following.

Lemma 3.5. *The function $\beta \mapsto \langle \sigma_0 \rangle_\beta^+$ is increasing.*

Notation. Before turning to the proof of this lemma, let us introduce the following useful notation for the Ising model with $\pm$ boundary conditions. Recall the notation $\Lambda, \partial\Lambda, \bar{\Lambda}, E, \partial E, \bar{E}$. We now introduce

$$\Omega_\Lambda^+ = \{\sigma \in \{-1, 1\}^{\bar{\Lambda}} : \forall x \in \partial\Lambda,\, \sigma_x = 1\},$$

which is in bijection with Ω_Λ. As a consequence,

$$\forall \sigma \in \Omega_\Lambda^+, \quad H_\beta^+(\sigma) = -\beta \sum_{xy \in \bar{E}} \sigma_x \sigma_y, \quad \mu_\beta^+(\sigma) = \frac{1}{Z_\beta^+} e^{-H_\beta^+(\sigma)}, \text{ where } Z_\beta^+ = \sum_{\sigma \in \Omega_\Lambda^+} e^{-H_\beta^+(\sigma)}.$$

Proof of Lemma 3.5. It suffices to prove that the result holds for all finite $\Lambda \subset \mathbb{Z}^d$. Note that for a fixed $\Lambda \subset \mathbb{Z}^d$, the function $\beta \mapsto \langle \sigma_0 \rangle_{\Lambda,\beta}^+$ is differentiable, and we have

$$\begin{aligned}
\frac{\partial}{\partial \beta} \langle \sigma_0 \rangle_{\Lambda,\beta}^+ &= \frac{\partial}{\partial \beta} \frac{\sum\limits_{\sigma \in \Omega_\Lambda^+} \sigma_0 \, e^{\beta \sum_{xy \in \bar{E}} \sigma_x \sigma_y}}{\sum\limits_{\sigma \in \Omega_\Lambda^+} e^{\beta \sum_{xy \in \bar{E}} \sigma_x \sigma_y}} \\
&= \frac{\sum\limits_{xy \in \bar{E}} \sum\limits_{\sigma \in \Omega_\Lambda^+} \sigma_x \sigma_y \sigma_0 e^{\beta \sum_{xy \in \bar{E}} \sigma_x \sigma_y}}{Z_{\Lambda,\beta}^+} - \frac{\sum\limits_{xy \in \bar{E}} \sum\limits_{\sigma \in \Omega_\Lambda^+} \sigma_x \sigma_y e^{\beta \sum_{xy \in \bar{E}} \sigma_x \sigma_y}}{Z_{\Lambda,\beta}^+} \frac{\sum\limits_{\sigma \in \Omega_\Lambda^+} \sigma_0 e^{\beta \sum_{xy \in \bar{E}} \sigma_x \sigma_y}}{Z_{\Lambda,\beta}^+} \\
&= \sum_{xy \in \bar{E}} [\langle \sigma_x \sigma_y \sigma_0 \rangle_{\Lambda,\beta}^+ - \langle \sigma_0 \rangle_{\Lambda,\beta}^+ \langle \sigma_x \sigma_y \rangle_{\Lambda,\beta}^+].
\end{aligned}$$

Now, by the second GKS inequality (Theorem 2.8), each term in the sum is non-negative, so that the derivative is. □

We are now ready for the following.

Definition 3.6. The *inverse critical temperature* $\beta_c(d)$ is defined to be

$$\beta_c(d) = \sup\{\beta \geq 0 : \ \langle \sigma_0 \rangle_\beta^+ = 0\}.$$

Remark 3.7. Since $\mu^+(\sigma_0 = \pm 1) = \mu^-(\sigma_0 = \mp 1)$, we have that $\langle \sigma_0 \rangle_\beta^+ = -\langle \sigma_0 \rangle_\beta^-$. This implies that

$$\langle \sigma_0 \rangle_\beta^+ > 0 \Leftrightarrow \langle \sigma_0 \rangle_\beta^+ \neq \langle \sigma_0 \rangle_\beta^-.$$

3.3.2 Characterization of the phase transition

The next result characterizes the phase transition through the Ising Gibbs measures.

Theorem 3.8. *Let $\beta \geq 0$. Then the following are equivalent.*

1. *There exists a unique infinite volume Gibbs measure.*
2. $\mu_\beta^+ = \mu_\beta^-$.
3. $\langle \sigma_0 \rangle_\beta^+ = \langle \sigma_0 \rangle_\beta^-$.

Proof. The implications $1 \Rightarrow 2 \Rightarrow 3$ are straightforward.

Consider an exhaustion $(\Lambda_n)_{n \geq 1}$ of $\mathbb{Z}^d$. In all that follows, we let $A \subset \mathbb{Z}^d$ be finite, and n be large enough so that $\Lambda_n \supset A$.

$\underline{2 \Rightarrow 1}$. As a consequence of Lemma 2.10, we have that for every boundary condition $\omega \in \{-1, 1\}^{\partial \Lambda_n}$,

$$\langle n_A \rangle_{\Lambda_n}^- \leq \langle n_A \rangle_{\Lambda_n}^\omega \leq \langle n_A \rangle_{\Lambda_n}^+.$$

By Point 2., this implies that the limit is unique (independently of the choice of boundary condition ω and of the exhaustion). The proof is concluded by using Lemma 2.7 stating that every local function can be decomposed in the basis $(n_A)_{A \subset \mathrm{supp}(f)}$.

$\underline{3 \Rightarrow 2}$. Since the function $\sum_{x \in A} n_x - n_A$ is increasing (see Example 2.4), as a consequence of Lemma 2.10, we have that

$$\Big\langle \sum_{x \in A} n_x - n_A \Big\rangle_{\Lambda_n}^- \leq \Big\langle \sum_{x \in A} n_x - n_A \Big\rangle_{\Lambda_n}^+.$$

Taking the limit $n \to \infty$ gives

$$\sum_{x \in A} \big(\langle n_x \rangle^+ - \langle n_x \rangle^- \big) \geq \langle n_A \rangle^+ - \langle n_A \rangle^- \geq 0,$$

where in the last inequality we used that n_A is increasing. By translation invariance of $\mu^\pm$, we have that

$$\forall\, x \in A, \quad \langle n_x \rangle^\pm = \langle n_0 \rangle^\pm.$$

Moreover by definition, $\langle n_0 \rangle^\pm = \frac{1}{2} \langle \sigma_0 \rangle^\pm + \frac{1}{2}$. As a consequence, if $\langle \sigma_0 \rangle^+ = \langle \sigma_0 \rangle^-$, then $\langle n_x \rangle^+ = \langle n_x \rangle^-$, implying that $\langle n_A \rangle^+ = \langle n_A \rangle^-$. The proof is again concluded by using Lemma 2.7. $\square$

Chapter 4

Magnetization and phase diagram

We now turn to establishing that the phase diagram of the Ising model for $d \geq 1$. Note that it undergoes a phase transition if $0 < \beta_c < \infty$.

4.1 Non-uniqueness at low temperature

The goal of this section is to prove the following.

Theorem 4.1. *For all $d \geq 2$, $\beta_c(d) < \infty$.*

This theorem implies non uniqueness of the Gibbs measures when $d \geq 2$ and β is large enough. The proof uses what is known as the *Peierl's argument* and relies on the low temperature expansion of Kramers and Wannier, which we now explain. In fact, we will prove this result for $d = 2$. A way to extend it to higher dimensions consists in proving that $\langle \sigma_0 \rangle^+_{\Lambda_n,\beta}$ is an increasing function of the dimension (when Λ_n is the box of size $2n$ around 0), see for example [Vel].

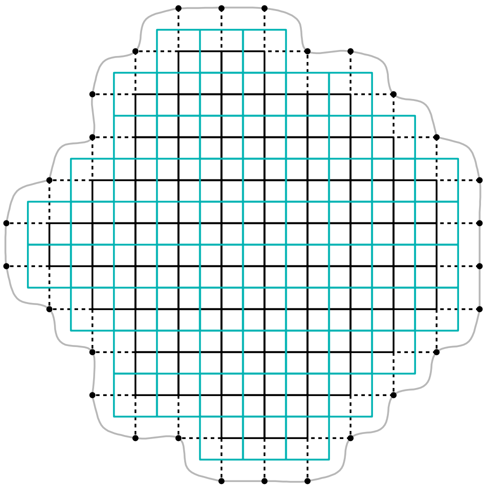
4.1.1 Low temperature expansion $d = 2$

Note that despite the name, this is an exact expansion of the partition function. We fix $\Lambda \subset \mathbb{Z}^2$, and consider the Ising model with $+$ boundary conditions. A slight adaptation of the argument allows to handle other boundary conditions.

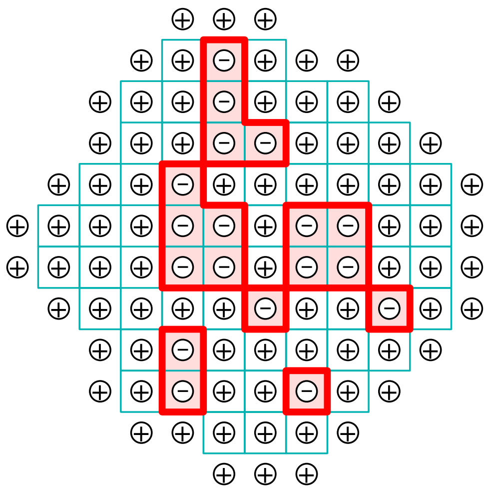
Consider the graph $(\bar{\Lambda}, \bar{E})$ where the boundary vertices $\partial\Lambda$ are merged into one vertex, and the corresponding graph is embedded in the sphere. Denote by (Λ^*, E^*) the *dual* of this graph, and observe that edges of E^* are in bijection with those of $\bar{E}$.

Let $\mathcal{P}(\Lambda^*)$ denote the set of *polygon configurations* of (Λ^*, E^*), otherwise stated:

$$\mathcal{P}(\Lambda^*) = \{P : \ P \text{ subgraph of } (\Lambda^*, E^*) \text{ such that all vertices of } \Lambda^* \text{ have even degree in } P\}.$$



(a) The graph $(\bar{\Lambda}, \bar{E})$ where boundary vertices are merged into one (represented by the grey line), thought of as embedded in the sphere. In green: the dual graph (Λ^*, E^*).



(b) A polygon configuration of $\mathcal{P}(\Lambda^*)$ and the corresponding spin configuration.

Note that, for all $\sigma \in \Omega_\Lambda^+$,

$$\beta \sum_{xy\in\bar{E}} \sigma_x\sigma_y = \beta|\bar{E}| - \beta \sum_{xy\in\bar{E}} (1-\sigma_x\sigma_y) = \beta|\bar{E}| - 2\beta|\{xy \in \bar{E} : \sigma_x \neq \sigma_y\}|.$$

To every spin configuration σ, assign the subgraph $P(\sigma)$ of (Λ^*, E^*) consisting of the dual edges of the edges xy such that $\sigma_x \neq \sigma_y$. Then, a simple case handling shows that every vertex of $P(\sigma)$ has even degree; it is thus a polygon configuration. The configuration $P(\sigma)$ consists of the contours separating the *clusters* of $\pm$ spins.

Conversely, to every $P \in \mathcal{P}(\Lambda^*)$, there corresponds a unique σ such that $P = P(\sigma)$: the spins of one cluster are fixed by the $+$ boundary conditions, and then the spins of clusters change each time a polygon edge is crossed. As a consequence on has,

$$\forall\,\sigma \in \Omega_\Lambda^+, \quad \beta \sum_{xy\in\bar{E}} \sigma_x\sigma_y = \beta|\bar{E}| - 2\beta|P(\sigma)|,$$

where $|P(\sigma)|$ is the number of edges of $P(\sigma)$.

The Ising Boltzmann measure with $+$ boundary conditions can thus be re-written as,

$$\forall\,\sigma \in \Omega_\Lambda^+, \quad \mu_\beta^+(\sigma) = \frac{e^{\beta|\bar{E}|}e^{-2\beta|P(\sigma)|}}{e^{\beta|\bar{E}|}\sum_{\sigma\in\Omega_\Lambda^+} e^{-2\beta|P(\sigma)|}} = \frac{e^{-2\beta|P(\sigma)|}}{\sum_{\sigma\in\Omega_\Lambda^+} e^{-2\beta|P(\sigma)|}}.$$

With the above bijection, the Ising Boltzmann measure μ^+ can be seen as a probability measure on polygon configurations $\mathcal{P}(\Lambda^*)$ where the probability of occurrence of a polygon configuration P is proportional to $\prod_{e^*\in P} e^{-2\beta}$.

4.1.2 Proof of Theorem 4.1

• We want to prove that $\langle\sigma_0\rangle_\beta^+ > 0$ for β sufficiently large. It thus suffices to prove that, uniformly in n,

$$\langle\sigma_0\rangle_{\Lambda_n,\beta}^+ > c,$$

for some $c > 0$. Since

$$\langle\sigma_0\rangle_{\Lambda_n,\beta}^+ = \mu_{\Lambda_n,\beta}^+(\sigma_0 = 1) - \mu_{\Lambda_n,\beta}^+(\sigma_0 = -1) = 1 - 2\mu_{\Lambda_n,\beta}^+(\sigma_0 = -1),$$

this is equivalent to showing that, for β sufficiently large, uniformly in n,

$$\mu_{\Lambda_n,\beta}^+(\sigma_0 = -1) < \frac{1}{2} - c'. \tag{4.1}$$

• Let Λ_n be the square of size $2n$ centered at the origin. We have the following inclusion:

$$\{\sigma_0 = -1\} \subset \{P \in \mathcal{P}(\Lambda_n^*) : P \text{ contains a contour } \gamma \text{ surrounding } 0 = (0,0)\}.$$

Indeed, take a path from 0 to the boundary $\partial\Lambda_n$, then along this path, the spin changes each time a contour is crossed. If P does not contain a contour γ surrounding 0, the number of contours crossed from 0 to the boundary is even, so that the spin at 0 and the boundary are the same thus proving the reverse inclusion for complementary events.

• If $P \in \mathcal{P}(\Lambda_n^*)$ and P contains a contour γ surrounding the origin, then $P \setminus \gamma \in \mathcal{P}(\Lambda_n^*)$. Given such a contour γ, define

$$\mathcal{P}_\gamma(\Lambda_n^*) = \{P \setminus \gamma : P \in \mathcal{P}(\Lambda_n^*) \text{ and } P \text{ contains } \gamma\} \subset \mathcal{P}(\Lambda_n^*).$$

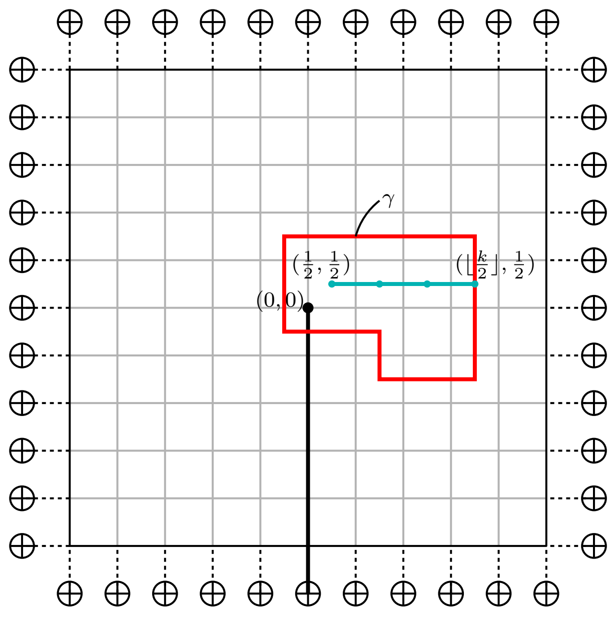


Figure 4.2: A contour γ surrounding the origin; the segment $\{(i-1/2, 1/2) : i \in \{1, \cdots, \lfloor k/2 \rfloor\}$.

• We are not ready to prove (4.1).

$$
\begin{aligned}
\mu^+_{\Lambda_n,\beta}(\sigma_0=-1) &\le \mu^+_{\Lambda_n,\beta}(\exists\,\text{contour }\gamma\text{ surrounding }0)\\
&\le \sum_{\{\gamma\text{ surr. }0\}} \frac{\sum_{P\in\mathcal{P}(\Lambda_n^*),\,P\supset\gamma} e^{-2\beta|P|}}{\sum_{P\in\mathcal{P}(\Lambda_n^*)} e^{-2\beta|P|}}\\
&= \sum_{\{\gamma\text{ surr. }0\}} \frac{e^{-2\beta|\gamma|}\sum_{P\in\mathcal{P}(\Lambda_n^*),\,P\supset\gamma} e^{-2\beta(|P|-|\gamma|)}}{\sum_{P\in\mathcal{P}(\Lambda_n^*)} e^{-2\beta|P|}} = \sum_{\{\gamma\text{ surr. }0\}} e^{-2\beta|\gamma|}\frac{\sum_{P\in\mathcal{P}_\gamma(\Lambda_n^*)} e^{-2\beta|P|}}{\sum_{P\in\mathcal{P}(\Lambda_n^*)} e^{-2\beta|P|}}\\
&\le \sum_{\{\gamma\text{ surr. }0\}} e^{-2\beta|\gamma|},\ \text{since } \mathcal{P}_\gamma(\Lambda_n^*)\subset\mathcal{P}(\Lambda_n^*)\\
&= \sum_{k\ge 4} e^{-2\beta k}|\{\gamma:\gamma\text{ contour in }\mathbb{Z}^2\text{ surrounding }\ 0,\ \text{and }\gamma\text{ has length }k\}|\\
&\le \sum_{k\ge 4} e^{-2\beta k}4\cdot 3^{k-1}\frac{k}{2}.
\end{aligned}
$$

Let us show the last inequality. A contour γ in $\mathbb{Z}^2$ of length k surrounding 0 necessarily contains a vertex of the segment $\{(i-1/2,1/2)\,:\,i\in\{1,\cdots,\lfloor k/2\rfloor\}\}$. Indeed, a contour containing 0 either crosses this segment, in which case we are OK, or it strictly contains this segment, in which case it has length $>k$ and this is not possible. We can thus choose the starting point of γ in this segment, and we have $\lfloor k/2\rfloor$ choices. Given such a starting point, a contour of length k containing 0 is obviously also a path of length k. For the first step of the path, there are 4 choices (since we are on $\mathbb{Z}^2$); then each subsequent step has 3 choices, since one has to choose a new edge, giving the contribution $4\cdot 3^{k-1}$.

The proof is concluded by noting that, uniformly in n, $\sum_{k\ge1}k(3e^{-2\beta})^k=\frac{3e^{-2\beta}}{(1-3e^{-2\beta})^2}\to 0$ as $\beta\to\infty$.

4.2 Uniqueness at high temperature

The goal of this section is to prove the following.

Theorem 4.2. *For all $d\ge1$, $\beta_c(d)>0$.*

This theorem implies uniqueness of the Gibbs measure when $d\ge1$ and β is small enough. The method of proof proposed here is also a *Peierl's type argument.* It relies on the high temperature expansion of Kramers and Wannier which we now turn to.

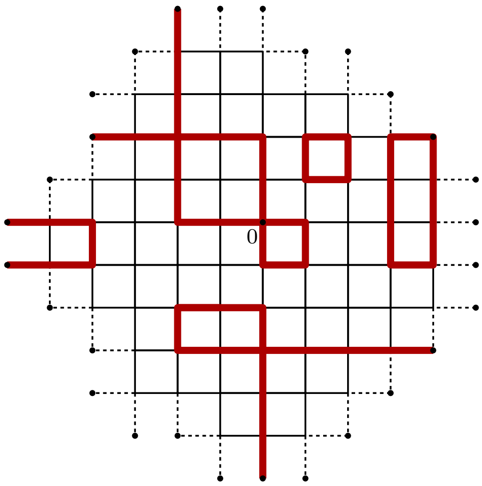
4.2.1 High temperature expansion

Again, the high temperature expansion is an exact expansion of the partition function using polygon configurations. This time the latter live on the graph itself and not the dual. As we will see, there is not bijection between spin and polygon configurations. This expansion is nevertheless very useful since it allows to express observables, as for example the partition function or the expected spin at 0, in terms of polygon configurations.

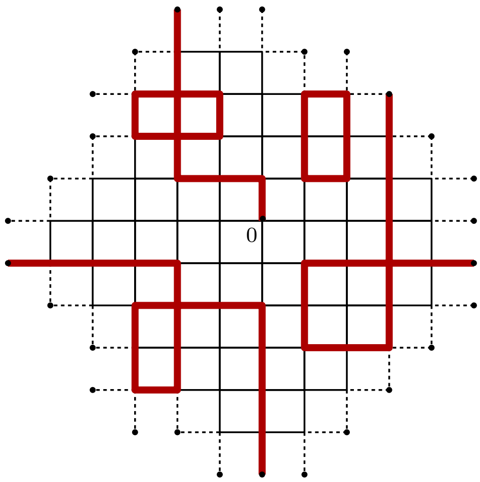
Fix $\Lambda\subset\mathbb{Z}^d$ such that $0\in\Lambda$, and consider the Ising model with + boundary conditions.

Similarly to the low temperature expansion (but this time on the primal graph) define,

$$\mathcal{P}(\bar{\Lambda}) = \{P : P \text{ subgraph of } (\bar{\Lambda}, \bar{E}) \text{ s.t. all vertices of } \Lambda \text{ have even degree in } P\}$$
$$\mathcal{P}_0(\bar{\Lambda}) = \{P : P \text{ subgraph of } (\bar{\Lambda}, \bar{E}) \text{ s.t. all vertices of } \Lambda \setminus \{0\} \text{ have even degree, } 0 \text{ has odd degree}\}.$$



(a) An example of polygon configuration of $\mathcal{P}(\bar{\Lambda})$.



(b) An example of polygon configuration of $\mathcal{P}_0(\bar{\Lambda})$.

The high temperature expansion relies on the following key identity.

$$\forall\, \sigma \in \Omega_\Lambda^+, \quad e^{\beta\sigma_x\sigma_y} = \cosh\beta + \sigma_x\sigma_y \sinh\beta = \cosh\beta(1 + \sigma_x\sigma_y \tanh\beta).$$

Using this, the partition function can be rewritten as follows

$$Z_\beta^+ = \sum_{\sigma\in\Omega_\Lambda^+} \prod_{xy\in\bar{E}} e^{\beta\sigma_x\sigma_y} = (\cosh\beta)^{|\bar{E}|} \sum_{\sigma\in\Omega_\Lambda^+} \prod_{xy\in\bar{E}} (1 + \sigma_x\sigma_y \tanh\beta).$$

Let us set $C := (\cosh\beta)^{|\bar{E}|}$, and observe that a subgraph of $\bar{\Lambda}$ can be seen as a subset of edges of $\bar{E}$. Then, expanding the product gives

$$\begin{aligned} Z_\beta^+ = \sum_{\sigma\in\Omega_\Lambda^+} \prod_{xy\in\bar{E}} e^{\beta\sigma_x\sigma_y} &= C \sum_{\sigma\in\Omega_\Lambda^+} \sum_{\{P:\ \text{subgraph of } (\bar{\Lambda},\bar{E})\}} \prod_{xy\in P} (\sigma_x\sigma_y \tanh\beta) \\ &= C \sum_{\sigma\in\Omega_\Lambda^+} \sum_{\{P:\ \text{subgraph of } (\bar{\Lambda},\bar{E})\}} (\tanh\beta)^{|P|} \prod_{x\in P} \sigma_x^{\deg_P(x)}, \end{aligned}$$

where $|P|$ denotes the number of edges of P, and $\deg_P(x)$ the degree of the vertex x in P. Note that we have

$$\prod_{x\in\Lambda\setminus P} \sigma_x^{\deg_P(x)} = 1, \text{ since all vertices of } \Lambda \setminus P \text{ have degree 0 in } P$$
$$\prod_{x\in\partial\Lambda\cap P} \sigma_x^{\deg_P(x)} = 1, \text{ since all boundary vertices have spins equal to } +1.$$

As a consequence, we have

$$Z^+_\beta = C \sum_{\{P\colon \text{subgraph of } (\bar\Lambda, \bar E)\}} (\tanh\beta)^{|P|} \sum_{\sigma\in\Omega_\Lambda} \prod_{x\in\Lambda} \sigma_x^{\deg_P(x)},$$

where we have replaces Ω^+_Λ by Ω_Λ since there is no dependence on the boundary spins. Now we have

$$\sum_{\sigma\in\Omega_\Lambda} \prod_{x\in\Lambda} \sigma_x^{\deg_P(x)} = \prod_{x\in\Lambda} \sum_{\sigma_x\in\{-1,1\}} \sigma_x^{\deg_P(x)}, \text{ and } \sum_{\sigma_x\in\{-1,1\}} \sigma_x^{\deg_P(x)} = 2\cdot \mathbb{I}_{\{\deg_P(x) \text{ is even}\}},$$

and we conclude that

$$Z^+_\beta = 2^{|\Lambda|} C \sum_{P\in\mathcal{P}(\bar\Lambda)} (\tanh\beta)^{|P|}.$$

Let us now turn to the expected spin at 0. Using a similar argument we obtain

$$\begin{aligned}
Z^+_\beta \langle\sigma_0\rangle^+_{\Lambda,\beta} &= \sum_{\sigma\in\Omega^+_\Lambda} \sigma_0 \prod_{xy\in\bar E} e^{\beta\sigma_x\sigma_y} \\
&= C \sum_{P\colon \text{subgraph of } (\bar\Lambda,\bar E)} (\tanh\beta)^{|P|} \sum_{\sigma\in\Omega_\Lambda} \sigma_0^{\deg_P(0)+1} \prod_{x\in\Lambda\setminus\{0\}} \sigma_x^{\deg_P(x)} \\
&= 2^{|\Lambda|} C \sum_{P\in\mathcal{P}_0(\bar\Lambda)} (\tanh\beta)^{|P|},
\end{aligned}$$

and we conclude that

$$\langle\sigma_0\rangle^+_{\Lambda,\beta} = \frac{\sum_{P\in\mathcal{P}_0(\bar\Lambda)} (\tanh\beta)^{|P|}}{\sum_{P\in\mathcal{P}(\bar\Lambda)} (\tanh\beta)^{|P|}}.$$

4.2.2 Proof of Theorem 4.2

Our goal is to prove that $\langle\sigma_0\rangle^+_\beta = 0$ for β small enough.

• Let Λ_n be the box of size $2n$ centered at the origin. We now use a rewriting of the high temperature expansion of $\langle\sigma_0\rangle^+_{\Lambda_n,\beta} = 0$. Given a subgraph P of $(\bar\Lambda, \bar E)$, denote by

$$\Delta(P) = \{\text{edges of } \bar\Lambda \text{ containing no vertices of } P\}.$$

As a consequence, every $P \in \mathcal{P}_0(\bar\Lambda)$ can be decomposed as $P = P_0 \sqcup P'$, where P_0 is the connected component containing the origin, $P' \subset \Delta(P_0)$ and $P' \in \mathcal{P}(\bar\Lambda)$ (all vertices of Λ have even degree in P').

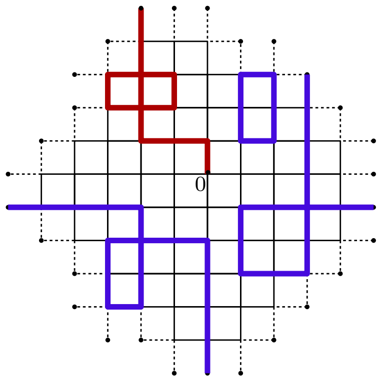


Figure 4.4: An example of decomposition of $P \in \mathcal{P}_0(\bar{\Lambda})$ as P_0 (red) and P' (blue).

Using this decomposition and the high temperature expansion, we have

$$\begin{aligned}
\langle \sigma_0 \rangle^+_{\Lambda_n,\beta} &= \sum_{\{P_0 \in \mathcal{P}_0(\bar{\Lambda}_n)\colon P_0 \text{ connected}\}} (\tanh\beta)^{|P_0|} \frac{\sum_{\{P' \in \mathcal{P}(\bar{\Lambda}_n)\colon P' \subset \Delta(P_0)\}} (\tanh\beta)^{|P'|}}{\sum_{\{P \in \mathcal{P}(\bar{\Lambda}_n)\}} (\tanh\beta)^{|P|}} \\
&\leq \sum_{\{P_0 \in \mathcal{P}_0(\bar{\Lambda}_n)\colon P_0 \text{ connected}\}} (\tanh\beta)^{|P_0|}, \text{ since the second term is } \leq 1 \\
&= \sum_{\ell \geq 1} \sum_{\{P_0 \in \mathcal{P}_0(\bar{\Lambda}_n)\colon P_0 \text{ connected},\ |P_0| = \ell\}} (\tanh\beta)^{\ell} \\
&= \sum_{\ell \geq 1} (\tanh\beta)^{\ell} |\{P_0 \in \mathcal{P}_0(\bar{\Lambda}_n) : \ P_0 \text{ connected},\ |P_0| = \ell\}|.
\end{aligned}$$

• Before proceeding with the proof, let us prove the following lemma.

Lemma 4.3. *Let G be a finite connected graph. Starting from an arbitrary vertex of G, there exists a closed path in G crossing each edge of G exactly twice.*

Proof. Let us show this by induction on the number m of edges of G. If $m = 1$, this is clear. We now use the following fact: an arbitrary connected graph can be constructed by adding one edge after the other while always staying connected. Suppose that G has m edges, and add an edge $x'y'$ so that it stays connected, yielding a graph G' with $m+1$ edges. Suppose without loss of generality that x' is a vertex of G. If one starts from a vertex x of G, we let γ_{2m} be such a closed path; when the path γ_{2m} reaches x', stop, explore the edge $x'y'$ back and forth and then proceed with γ_{2m}. This gives a closed path starting from x, of length $2(m+1)$ crossing each edge of G' twice. If one starts from the vertex $y' \in G' \setminus G$ (only possibility outside of G), explore (y', x'), then proceed with a closed path of length $2m$ starting from x', and finally explore (x', y'), giving a closed path of length $2(m+1)$ starting from y'. $\square$

• Let us now bound $|\{P_0 \in \mathcal{P}_0(\bar{\Lambda}_n) : \ P_0 \text{ connected},\ |P_0| = \ell\}|$.

First note that since each edge is incident to two vertices, we have

$$\sum_{x \in \bar{\Lambda}_n} \deg_{P_0}(x) = 2|P_0|.$$

Now since 0 has odd degree in P_0, and all other vertices of Λ_n have even degree, there is an odd number vertices of $\partial\Lambda_n$ having odd degree, in particular there is at least one. This means that P_0 contains a boundary vertex, and since it is connected it must have length at least n; hence $\ell \geq n$.

By Lemma 4.3, each P_0 in the set above gives a path starting from 0 of length 2ℓ, and two different polygon configurations yield different paths, so that

$$|\{P_0 \in \mathcal{P}_0(\bar{\Lambda}_n) : \ P_0 \text{ connected}, |P_0| = \ell\}| \subset |\{\text{Paths starting from 0 of length } 2\ell,\ \ell \geq n\}|.$$

For each ℓ, the number of such paths is smaller than $(2d)^{2\ell}$ since there is $2d$ choice for each of the 2ℓ edges of the path.

Wrapping up, we obtain

$$\begin{aligned}\langle \sigma_0 \rangle^+_{\Lambda_n,\beta} &\leq \sum_{\ell \geq n} (\tanh\beta)^\ell (2d)^{2\ell} = \sum_{\ell \geq n} \big(\tanh\beta(2d)^2\big)^\ell \\ &= \frac{(\tanh\beta(2d)^2)^n}{1-\tanh\beta(2d)^2}, \text{ if } \tanh\beta < \frac{1}{4d^2} \\ &\xrightarrow{n\to\infty} 0, \text{ if } \tanh\beta < \frac{1}{4d^2}.\end{aligned}$$

As a conclusion, we have that for all β such that $\tanh\beta < \frac{1}{4d^2}$, $\langle \sigma_0 \rangle^+_\beta = 0$ implying that $\beta_c(d) > 0$.

4.3 No phase transition in dimension 1

Using the high temperature expansion, let us prove that there is no phase transition in dimension 1, that is $\beta_c(1) = \infty$, or otherwise stated that

$$\forall\ \beta \geq 0, \quad \langle \sigma_0 \rangle^+_\beta = 0.$$

Let Λ_n be a segment of length $2n$ around 0. Recall that we have

$$\langle \sigma_0 \rangle^+_{\Lambda_n,\beta} = \frac{\sum_{P\in\mathcal{P}_0(\bar{\Lambda}_n)} (\tanh\beta)^{|P|}}{\sum_{P\in\mathcal{P}(\bar{\Lambda}_n)} (\tanh\beta)^{|P|}}.$$

The set $\mathcal{P}(\bar{\Lambda}_n)$ has two elements, one of length 0 and the other of length $2(n+1)$, while the set $\mathcal{P}_0(\bar{\Lambda}_n)$ has two elements of size $n+1$, see Figure 4.5.

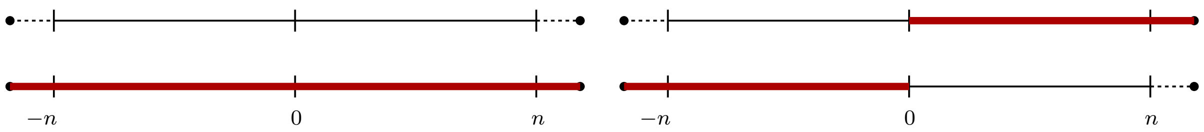


Figure 4.5: The two polygon configurations of $\mathcal{P}(\bar{\Lambda}_n)$ (left), resp. $\mathcal{P}_0(\bar{\Lambda}_n)$ (right).

As a consequence, for all $\beta \geq 0$, since $0 \leq \tanh\beta < 1$,

$$\langle \sigma_0 \rangle^+_{\Lambda_n,\beta} = \frac{2(\tanh\beta)^{n+1}}{1+(\tanh\beta)^{2(n+1)}} \xrightarrow{n\to\infty} 0,$$

thus concluding the proof.

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