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Exactly solvable models of two-dimensional statistical mechanics: the Ising model, dimers and spanning trees

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Even though the manuscript is in English, I decided to write most of this section in French.

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Foreword

This report aims at giving an overview of my research work since the PhD thesis, by presenting the papers [2, 3, 4, 5, 10, 11, 1].

The unifying theme of these articles is Statistical Mechanics, whose goal is to understand the large scale properties of a physics system, based on a model describing interactions between microscopic components. These spatially extended systems, consisting of a large number of mutually interacting components, exhibit intriguing phenomena, and require the use of powerful techniques from different fields of mathematics: *probability* by the very nature of these models and *combinatorics* since a central quantity, known as the *partition function*, plays the role of generating function for macroscopic quantities of the model. The nature of the mathematical objects representing the systems require the use of *discrete geometry* and *graph theory*. Recently, links have also appeared with *complex analysis* [Duf68, Mer01b, CS11], and *algebraic geometry* [KOS06].

Statistical mechanics started with Maxwell and Boltzmann in the 19^{th} century, and was part of physics; mathematical activity followed, and rich interactions have now emerged between the two communities. Many models belong to the field of statistical mechanics, as for example *percolation*, the *Ising model*, the *dimer model*, *spanning trees*, *the random cluster model*, *polymer models*... Within this diversity, common features appear, the most remarkable certainly being the phenomenon of *phase transition* between different states of the system. At the phase transition, also known as the *critical point*, models often exhibit a specific behavior. The understanding of the critical behavior of two-dimensional models has been going through a phase of rapid and exciting development since the introduction of SLE by Schramm [Sch00], in particular with the work of Lawler, Schramm and Werner [LSW04] and Smirnov [Smi01, Smi10].

The six papers [2, 3, 4, 5, 10, 11] concentrate on the *dimer model*, the *Ising model* and *spanning trees* in two dimensions. These three models have the common feature of being *exactly solvable*, meaning that there is an exact, explicit expression for the *partition function*. Since the partition function encodes much of the macroscopic behavior of the system, having an explicit expression opens the way to a deep understanding of the models. The title of this report is motivated by these papers.

The last paper [1] is about a *polymer model*, which is *one*-dimensional and *not* exactly solvable, thus bringing a different flavor. It is not referred to in the title, since we wanted to keep it relatively short. We now give an outline of this report.

- Chapter 1 is devoted to preliminaries. We define the Ising model, the dimer model, and spanning trees. In each case we state a few founding result, aiming at setting the basis for future chapters. We present Fisher's mapping from the Ising model to the dimer model on

a decorated graph. We also dedicate one section to defining critical versions of the models on isoradial graphs, since these play an important role all along.

- Chapter 2 presents the papers [3, 4] in collaboration with Cédric Boutillier. Our main theorems give a full description of the critical dimer model corresponding to the critical Ising model on infinite isoradial graphs, by proving an explicit expression for the free energy and for a natural Gibbs measure. In the first paper [3] we consider the $\mathbb{Z}^2$-periodic case, in the second [4] we relax this assumption and prove *local* formulas in the spirit of [Ken02].

- Chapter 3 is concerned with the papers [11, 10], both having a combinatorial flavor. In the first paper [11], we provide an explicit mapping from critical cycle rooted spanning forests - the pendent of spanning trees when working on the torus - to the double critical Ising model on the torus, via characteristic polynomials. This proves a surprising relation between two well known models of statistical mechanics. The second paper [10] answers a question raised when working on the first paper [11]. The main theorem can actually be phrased in a more general context: we prove a half-tree theorem for the Pfaffian of a skew-symmetric matrix, whose column-sum is zero. This is a Pfaffian version of the classical matrix-tree theorem of Kirchhoff [Kir47].

- Chapter 4 describes two papers [2, 5] in collaboration with Cédric Boutillier, focusing on loop configurations related to bipartite dimer models. The first paper [2] considers the loop representation of the uniform dimer model on a subgraph of the honeycomb lattice embedded in the torus. Our main theorem proves that, when the mesh of the lattice tends to zero and the aspect ratio of the torus is fixed, the winding number of the loops converge to a two-dimensional discrete Gaussian random variable. The second paper [5] relates XOR-loop configurations, constructed from two independent Ising models, to loop configurations of a dimer model on a bipartite graph. At criticality, this allows to shed a light on a conjecture of Wilson [Wil11], stating that these loops are, in the scaling limit, level lines of the Gaussian free field.

- Chapter 5 presents the paper [1] in collaboration with Erwin Bolthausen and Francesco Caravenna. Our main theorem determines the quenched critical point for the localization/delocalization phase transition of a polymer interacting with an attractive wall through a diluted, disordered potential.

- Chapter 6 mentions work in progress and perspectives.

CONTENTS

Chapter 1

Preliminaries

In this chapter, we define the models of statistical mechanics on which we have worked: the Ising model, the dimer model and spanning trees. In each case, the physics system is represented by a graph $\mathsf{G} = (\mathsf{V}, \mathsf{E})$, where V denotes the set of vertices, and E the set of edges. Throughout this report, we suppose that G is simple. When it is infinite, we suppose moreover that degrees of vertices are uniformly bounded. For each model, we state a few founding results. This is by no means a state of the art, but rather aims at setting the basis for future chapters. The last section is devoted to defining critical versions of the models on isoradial graphs, since these play an important role all along.

1.1 The two-dimensional Ising model

The *Ising model* is a model of ferromagnetism introduced by the physicist Wilhelm Lenz in 1920 as PhD subject for his student Ernst Ising. Consider a finite graph $\mathsf{G} = (\mathsf{V}, \mathsf{E})$ together with a collection of positive real numbers $(J_\mathsf{e})_{\mathsf{e}\in\mathsf{E}}$ indexed by the edges of G. The *Ising model on* G*, with coupling constants* J, is defined as follows. A *spin-configuration* σ of G is a function of the vertices of G taking values in $\{-1, 1\}$. The probability of occurrence of a spin configuration σ is given by the *Ising Boltzmann measure*, denoted $\mathbb{P}_{\text{Ising}}$:

$$\mathbb{P}_{\text{Ising}}(\sigma) = \frac{1}{Z_{\text{Ising}}(\mathsf{G}, J)} \exp\Big(\sum_{\mathsf{e}=\mathsf{uv}\in\mathsf{E}} J_\mathsf{e}\sigma_\mathsf{u}\sigma_\mathsf{v} \Big),$$

where $Z_{\text{Ising}}(\mathsf{G}, J)$ is the normalizing constant, known as the *Ising partition function*. Note that the definition of the Ising model usually involves the inverse temperature β. To simplify notations, we suppose that it is included in the coupling constants.

In the two-dimensional case, when the inverse temperature is 0, all spin configurations have the same probability of occurring and a typical configuration consists of a mixture of $+1$ and -1. When the inverse temperature is infinite, spins tend to align with their neighbors and a typical configuration consists of all $+1$ or all -1. For a specific value of the inverse temperature, the Ising model undergoes a phase transition between the disordered and ordered phases, the model is said to be *critical*.

Low and high temperature expansions

Suppose that the graph G is embedded in the torus $\mathbb{T}^2 = \{(z,w) \in \mathbb{C}^2 : |z| = |w| = 1\}$. The low and high temperature expansions of Kramers and Wannier [KW41a, KW41b] allow to rewrite the Ising partition function as a sum over polygon configurations of the dual graph G^*, and over polygon configurations of the graph G respectively.

A *polygon configuration* of the graph G is an even subgraph, that is a subset of edges having even degree at every vertex. Let $\mathcal{P}(\mathsf{G})$ denote the set of polygon configurations of the graph G, and $\mathcal{P}^{(0,0)}(\mathsf{G})$ denote polygon configurations winding around the torus an even number of times horizontally and vertically, *i.e.* having $(0,0)$ homology class in $H_1(\mathbb{T}, \mathbb{Z}/2\mathbb{Z})$. Polygon configurations of the dual graph G^* are defined in a similar way.

The *low temperature expansion* of the Ising partition function is based on the following identity:

$$\forall\, \mathsf{e} = \mathsf{uv} \in \mathsf{E}, \quad e^{J_\mathsf{e}\sigma_\mathsf{u}\sigma_\mathsf{v}} = e^{J_\mathsf{e}}(\delta_{\{\sigma_\mathsf{u}=\sigma_\mathsf{v}\}} + e^{-2J_\mathsf{e}}\delta_{\{\sigma_\mathsf{u}\neq\sigma_\mathsf{v}\}}).$$

Injecting the RHS into the Ising partition function and expanding the product, yields a sum over polygon configurations of the dual graph G^* separating clusters of $\pm$ spins, with a contribution of e^{-2J_e} for each edge e^* present in a polygon configuration P^*. Each such polygon configuration is obtained from two spin configurations, one being obtained from the other by negating all spins. We have thus sketched the proof of:

Proposition 1.1. [KW41a, KW41b][Low temperature expansion] *The Ising partition function can be rewritten as:*

$$Z_{\mathrm{Ising}}(\mathsf{G}, J) = 2\Big(\prod_{\mathsf{e}\in\mathsf{E}} e^{J_\mathsf{e}}\Big) \sum_{\mathsf{P}^*\in\mathcal{P}^{(0,0)}(\mathsf{G}^*)} \Big(\prod_{\mathsf{e}^*\in\mathsf{P}^*} e^{-2J_\mathsf{e}}\Big).$$

The *high temperature expansion* is based on the identity:

$$\forall\, \mathsf{e} = \mathsf{uv} \in \mathsf{E}, \quad e^{J_\mathsf{e}\sigma_\mathsf{u}\sigma_\mathsf{v}} = \cosh J_\mathsf{e}(1 + \sigma_\mathsf{u}\sigma_\mathsf{v}\tanh J_\mathsf{e}).$$

Using this identity, the Ising model partition function is expanded as a sum over monomials in $(\sigma_\mathsf{u})_{\mathsf{u}\in\mathsf{V}}$. In the expansion, spin variables come in pairs of neighbors $\sigma_\mathsf{u}\sigma_\mathsf{v}$ and can thus be formally identified with the edge connecting u and v, associated with a weight $\tanh J_\mathsf{e}$. Each monomial is then interpreted as a subgraph of G, the degree of σ_u being the degree of the corresponding edge configuration. Because of the symmetry $\sigma \leftrightarrow -\sigma$, when resumming over spin configurations, only terms having an even degree in each variable remain, giving a factor 2 per dual vertex, and other contributions cancel. We have thus proved:

Proposition 1.2. [KW41a, KW41b][High temperature expansion] *The Ising partition function can be rewritten as:*

$$Z_{\mathrm{Ising}}(\mathsf{G}, J) = 2^{|\mathsf{V}|}\Big(\prod_{\mathsf{e}\in\mathsf{E}} \cosh J_\mathsf{e}\Big) \sum_{\mathsf{P}\in\mathcal{P}(\mathsf{G})} \Big(\prod_{\mathsf{e}\in\mathsf{P}} \tanh J_\mathsf{e}\Big).$$

Remark 1.1.

- Propositions 1.1 and 1.2 yield a relation between the sum over polygon configurations of the primal and dual graphs with appropriate weights, this is known as Kramers and Wannier (low/high temperature) *duality relation.*

- In the paper [5], we prove an extension of the low and high temperature expansions to Ising models defined on graphs embedded in compact, orientable surfaces of genus g *with* boundary. In the literature, we did find versions of this duality for graphs embedded in surfaces of genus g [LG94], but we could not find versions taking into account boundaries. This extension involves homology theory relative to a boundary and the intersection form.

1.2 The dimer model

The *dimer model* represents the adsorption of di-atomic molecules on the surface of a crystal. It is first mentionned in a paper of Fowler and Rushbrooke [FR37]. Consider a finite graph $\mathsf{G} = (\mathsf{V}, \mathsf{E})$, and suppose that edges of G are assigned a positive weight function $\nu = (\nu_{\mathsf{e}})_{\mathsf{e}\in\mathsf{E}}$. The *dimer model on* G *with weight function* ν is defined as follows. A *dimer configuration*, also known as a *perfect matching*, is a subset of edges which covers each vertex of the graph exactly once, see Figure 1.1 for an example.

Figure 1.1: A dimer configuration of a portion of the honeycomb lattice embedded in the torus, opposite sides are identified.

Let $\mathcal{M}(\mathsf{G})$ denote the set of dimer configurations of the graph G, and assume that $\mathcal{M}(\mathsf{G})$ is non-empty. The probability of occurrence of a dimer configuration M is given by the *dimer Boltzmann measure*, denoted $\mathbb{P}_{\text{dimer}}$:

$$\mathbb{P}_{\text{dimer}}(\mathsf{M}) = \frac{\prod_{\mathsf{e}\in M} \nu_{\mathsf{e}}}{Z_{\text{dimer}}(\mathsf{G}, \nu)},$$

where $Z_{\text{dimer}}(\mathsf{G}, \nu)$ is the normalizing constant known as the *dimer partition function*.

1.2.1 Explicit formulas in the finite, planar case

Suppose that the graph G is finite and planar, and consider G as an embedded graph. Then, the dimer model has the very nice feature of having an explicit formula for the partition function, due to Kasteleyn [Kas61, Kas67] and independently to Temperley and Fisher [TF61]. We now introduce the definitions needed to state their result.

An *admissible orientation* of the edges of G is an orientation of the edges such that all cycles surrounding inner faces of the graph are *clockwise odd*, *i.e.* when traveling clockwise around such a cycle, the number of co-oriented edges is odd. Kasteleyn proves that a planar, finite graph always has an admissible orientation [Kas67].

Given such an orientation, and writing the set of vertices as $\mathsf{V} = \{\mathsf{u}_1, \cdots, \mathsf{u}_n\}$, the *Kasteleyn matrix* K is the corresponding weighted, oriented, adjacency matrix; it has size $n \times n$, it is skew-symmetric and is defined by:

$$\forall\, i, j \in \{1, \cdots, n\}, \qquad K_{\mathsf{u}_i,\mathsf{u}_j} = \begin{cases} \nu_{\mathsf{u}_i\mathsf{u}_j} & \text{if } \mathsf{u}_i \sim \mathsf{u}_j \text{ and } \mathsf{u}_i \to \mathsf{u}_j \\ -\nu_{\mathsf{u}_i\mathsf{u}_j} & \text{if } \mathsf{u}_i \sim \mathsf{u}_j \text{ and } \mathsf{u}_i \leftarrow \mathsf{u}_j \\ 0 & \text{else.} \end{cases}$$

It is sometimes useful to interpret K as an operator acting on $\mathbb{C}^{\mathsf{V}}$:

$$\forall\, f \in \mathbb{C}^{\mathsf{V}}, \quad (Kf)_{\mathsf{u}} = \sum_{\mathsf{v}\in\mathsf{V}} K_{\mathsf{u},\mathsf{v}} f_{\mathsf{v}}. \tag{1.1}$$

The *Pfaffian* of a skew-symmetric matrix K is 0 if n is odd; if n is even, it is given by:

$$\mathrm{Pf}(K) = \sum_{\pi\in\Pi_n} \mathrm{sgn}(\sigma_\pi) K_{\mathsf{u}_{\pi(1)},\mathsf{u}_{\pi(2)}} \cdots K_{\mathsf{u}_{\pi(n-1)},\mathsf{u}_{\pi(n)}}, \tag{1.2}$$

where Π_n is the set of partitions of the numbers $\{1,\cdots,n\}$ into $n/2$ unordered pairs, and σ_π is a permutation describing the partition π.

Remark 1.2. Non-zero terms in the expansion of the Pfaffian correspond to perfect matchings of the graph G, so that the Pfaffian can also be written as a sum over perfect matchings of G.

Theorem 1.1. [Kas61, Kas67, TF61] *The dimer partition function is equal to the absolute value of the Pfaffian of the Kasteleyn matrix:*

$$Z_{\mathrm{dimer}}(\mathsf{G},\nu) = |\,\mathrm{Pf}(K)|.$$

Idea of the proof. By Remark 1.2, non-zero terms in the expansion of $\mathrm{Pf}(K)$ correspond to dimer configurations of G. The only issue preventing the Pfaffian from being a generating function are the signatures of the permutations. Kasteleyn shows [Kas67] that if edges are oriented according to an admissible orientation, signs of coefficients exactly compensate signatures of permutations. □

Using the explicit formula for the dimer partition function, Kenyon [Ken97] derives an explicit formula for the dimer Boltzmann measure:

Theorem 1.2. [Ken97] *Let* $\mathsf{E}_k = \{\mathsf{e}_1 = \mathsf{u}_1\mathsf{v}_1, \cdots, \mathsf{e}_k = \mathsf{u}_k\mathsf{v}_k\}$ *be a subset of edges of the graph* G. *Then, the probability of these edges occurring in a dimer configuration of* G *chosen with respect to the dimer Boltzmann measure, is equal to:*

$$\mathbb{P}_{\mathrm{dimer}}[\mathsf{e}_1,\cdots,\mathsf{e}_k] = \Big|\Big(\prod_{i=1}^{k} K_{\mathsf{u}_i,\mathsf{v}_i}\Big)\,\mathrm{Pf}([K^{-1}_{\mathsf{E}_k}]^t)\Big|,$$

where $K^{-1}_{\mathsf{E}_k}$ *is the submatrix of the inverse Kasteleyn matrix* K^{-1} *whose lines and columns are indexed by* $\{\mathsf{u}_1,\mathsf{v}_1,\cdots,\mathsf{u}_k,\mathsf{v}_k\}$.

Idea of the proof. The key to the proof of this result is Jacobi's Pfaffian Identity, see for example [IW00], stating that:

$$\Big|\mathrm{Pf}(K_{\mathsf{E}_k^{\mathrm{c}}})\Big| = \Big|\mathrm{Pf}([K^{-1}_{\mathsf{E}_k}]^t)\cdot\mathrm{Pf}(K)\Big|, \tag{1.3}$$

where $K_{\mathsf{E}_k^{\mathrm{c}}}$ is the submatrix of the matrix K whose lines and columns are indexed by vertices of $\mathsf{V}\setminus\{\mathsf{u}_1,\mathsf{v}_1,\cdots,\mathsf{u}_k,\mathsf{v}_k\}$. □

Bipartite case

When the graph G is moreover bipartite, explicit formulas simplify and involve the determinant rather than the Pfaffian. The set of vertices V can naturally be split into two subsets $\mathsf{W}\cup\mathsf{B}$,

where W denotes white vertices, B black ones, and vertices in W are only adjacent to vertices in B. With a slight abuse of notation, let us denote by K the submatrix of the Kasteleyn matrix whose lines are indexed by white vertices and columns by black vertices. Then, in the bipartite case, the full Kasteleyn matrix takes the following simpler form:

$$\begin{pmatrix} 0 & K \\ -K^t & 0 \end{pmatrix}.$$

All information is contained in the matrix K, so that we refer to it as the *Kasteleyn matrix*.

Since the Pfaffian of a skew-symmetric matrix is the square root of the determinant, we have:

Theorem 1.3. [Kas61, Per69] *When the graph G is bipartite, the dimer partition function is equal to:*

$$Z_{\mathrm{dimer}}(\mathsf{G}, \nu) = |\det(K)|.$$

An analog of Theorem 1.2, involving the determinant instead of the Pfaffian, yields an explicit expression for the dimer Boltzmann measure.

1.2.2 Explicit formulas in the toroidal case

When the graph G is embedded in the torus, the dimer partition function and the dimer Boltzmann measure can also be computed explicitly. The result for the partition function in the case of portions of $\mathbb{Z}^2$ embedded in the torus is due to Kasteleyn [Kas67]. Extensions to graphs embedded in higher genus surfaces are due to [DZM$^+$96, GL99, Tes00, CR07, CR08].

If the graph G is embedded in the torus, the choice of an admissible orientation of the edges does not suffice to cancel signatures of permutations in the expansion of the Pfaffian. One needs a signed sum of 4 Kasteleyn matrices (2^{2g} matrices when the graph is embedded in a genus g surface), defined as follows.

Consider an admissible orientation of the edges of G and the corresponding Kasteleyn matrix K. Let γ_1 and γ_2 be two edge-cycles in the dual graph G^*, winding around two non-trivial directions of the torus. For $\theta, \tau \in \{0, 1\}$, let $K^{\theta\tau}$ be the matrix K in which the weight of edges crossing the cycle γ_1 are multiplied by $(-1)^\theta$, and the weight of edges crossing the cycle γ_2 are multiplied by $(-1)^\tau$.

Then, the dimer partition function and the Boltzmann measure are given by the following.

Theorem 1.4. [Kas67, DZM+96, GL99, Tes00, CR07, CR08] [Ken97]

- *The dimer partition function is equal to:*

$$Z_{\text{dimer}}(\mathsf{G},\nu) = \frac{1}{2}\big|-\operatorname{Pf}(K^{00}) + \operatorname{Pf}(K^{10}) + \operatorname{Pf}(K^{01}) + \operatorname{Pf}(K^{11})\big|. \tag{1.4}$$

- *Let* $\mathsf{E}_k = \{\mathsf{e}_1 = \mathsf{u}_1\mathsf{v}_1, \cdots, \mathsf{e}_k = \mathsf{u}_k\mathsf{v}_k\}$ *be a subset of edges of the graph* G*. Then, the probability of these edges occurring in a dimer configuration of* G *chosen with respect to the dimer Boltzmann measure, is equal to:*

$$\mathbb{P}_{\text{dimer}}[\mathsf{e}_1, \cdots, \mathsf{e}_k] = \frac{\Big|\big(\prod_{i=1}^k K_{\mathsf{u}_i,\mathsf{v}_i}\big)\big(-\operatorname{Pf}(K^{00}_{\mathsf{E}_k^c}) + \operatorname{Pf}(K^{10}_{\mathsf{E}_k^c}) + \operatorname{Pf}(K^{01}_{\mathsf{E}_k^c}) + \operatorname{Pf}(K^{11}_{\mathsf{E}_k^c})\big)\Big|}{2Z_{\text{dimer}}(\mathsf{G},\nu)}, \tag{1.5}$$

where $K^{\theta\tau}_{\mathsf{E}_k^c}$ *is the submatrix of the matrix* $K^{\theta\tau}$ *whose lines and columns are indexed by* $\mathsf{V}\setminus\{\mathsf{u}_1,\mathsf{v}_1,\cdots,\mathsf{u}_k,\mathsf{v}_k\}$.

When the graph G is moreover bipartite, the dimer partition function and the Boltzmann measure are computed explicitly by similar formulas, involving determinants instead of Pfaffians.

1.2.3 Dimer model on infinite periodic graphs

Assume that the graph G is infinite and $\mathbb{Z}^2$-periodic, meaning that G is embedded in the plane so that it is invariant under translations by an underlying lattice $\mathbb{Z}^2$. Suppose that edges are assigned a positive, *periodic* weight function $\nu = (\nu_{\mathsf{e}})_{\mathsf{e}\in\mathsf{E}}$. The dimer model on the infinite graph G is studied through the natural exhaustion $\{\mathsf{G}_n = \mathsf{G}/n\mathbb{Z}^2\}_{n\geq 1}$ by toroidal graphs. The smallest graph G_1 of this exhaustion plays a specific role and is known as the *fundamental domain*. For every $n \geq 1$, let us denote by $\mathbb{P}^n_{\text{dimer}}$ the Boltzmann measure of the graph G_n, and by $Z_{\text{dimer}}(\mathsf{G}_n,\nu)$ the dimer partition function.

The two key results one aims for in the first place, are an explicit expression for the *free energy* and for the *Gibbs measure(s)*, defined as follows. The *free energy* of the dimer model on G, denoted by f, is minus the exponential growth rate of the partition function, that is:

$$f = -\lim_{n\to\infty}\frac{1}{n^2}\log Z_{\text{dimer}}(\mathsf{G}_n,\nu).$$

A *Gibbs measure* on the set of dimer configurations $\mathcal{M}(\mathsf{G})$ of G, is a probability measure on $\mathcal{M}(\mathsf{G})$, satisfying the DLR (Dobrushin, Lanford, Ruelle) conditions: if a perfect matching is fixed in an annular region of G, then perfect matchings inside and outside of this annulus are independent. Moreover, the probability of any interior matching is proportional to the product of its edge weights.

Bipartite case

In 2006, Kenyon, Okounkov and Sheffield [KOS06] wrote a breakthrough paper, giving a full description of the dimer model in the case where the graph G is bipartite. They prove an explicit formula for the free energy, for a two-parameter family of ergodic, translation invariant Gibbs measures. The authors also give a full description of the phase diagram of the model, using algebraic geometry and Harnack curves. Some parts of the paper use techniques of the paper

[CKP01] by Cohn, Kenyon and Propp, where a very thorough study of the dimer model on $\mathbb{Z}^2$ is established.

Non-bipartite case

When the graph G is not bipartite, there is no general theory for the dimer model. In the paper [3] we give a full description of the dimer model on the non-bipartite graph corresponding to the critical Ising model on isoradial graphs, see Sections 1.3 and 2.1. Some parts of the proof are robust, but what is missing is a general understanding of the zeros of the dimer characteristic polynomial, defined in Section 2.1.1. This is one of the big challenges in the dimer model.

1.2.4 Loop representation

Dimer configurations can be interpreted as loop configurations in the following way. Fix a reference dimer configuration M_0 of the graph G and consider a generic dimer configuration M. Define an *alternating cycle* to be a cycle whose edges alternate between edges of M and edges of M_0, it has even length and if the cycle has length 2, it consists of a *doubled edge*, that is an edge covered by both M and M_0.

The superimposition of the dimer configuration M and the dimer configuration M_0, denoted $\mathsf{M}_0 \cup \mathsf{M}$, is a collection of disjoint alternating cycles covering all vertices of the graph G. This is because, by definition of a perfect matching, each vertex is incident to exactly one edge of the matching, so that in the superimposition of two perfect matchings M_0 and M, each vertex is incident to exactly one edge of M and one edge of M_0. We shall also refer to cycles as *loops*.

When the graph G is bipartite, orienting edges of the reference dimer configuration M_0 from the black vertex to the white one, and edges of the dimer configuration M from the white vertex to the black one, yields a natural orientation of loops, see Figure 1.2. This collection of disjoint, oriented loops is denoted by $\mathsf{M} - \mathsf{M}_0$.

Figure 1.2: Oriented superimposition $\mathsf{M} - \mathsf{M}_0$ of a portion of the honeycomb lattice embedded in the torus.

When the graph G is bipartite and planar (finite or infinite), dimer configurations can be interpreted as discrete surfaces in dimension 2+1, via a *height function*. A *height function* is a random variable h on the set of dimer configurations $\mathcal{M}(\mathsf{G})$, assigning to each dimer configuration M an $\mathbb{R}$-valued function h^{M} on the faces of G. In the mathematics community, this was first established by Thurston [Thu90] in the case of the square and honeycomb lattices. Following [KOS06], a height function can be defined using the oriented superimposition $\mathsf{M} - \mathsf{M}_0$: set the height to be 0 at some fixed face, then change the height by ± 1 each time an oriented cycle is crossed, where the sign depends on the orientation of the cycle. For a family of simply connected subgraphs of the square lattice, Kenyon proves that the height function, interpreted as a random generalized function, converges weakly in law to $1/\sqrt{\pi}$ times a Gaussian free field [Ken00, Ken01]. I have generalized this result to all critical, bipartite dimer models on isoradial graphs, when the graph G is infinite [9]. Loops of the superimpositions are conjectured to converge to a CLE(4) process.

1.3 Fisher's mapping from the Ising model to the dimer model

In the paper [Fis66], Fisher introduces a weight preserving mapping from the Ising model defined on a graph G embedded in the torus, to the dimer model defined on a decorated version G^{F} of G[1]. Since then, dimer techniques have been a powerful tool for solving pertinent questions about the Ising model, as for example the simple derivation of the free energy by Fisher [Fis66], first computed by Onsager [Ons44], and the book by Mc Coy and Wu [MW73].

Consider the high temperature expansion of the Ising model defined on a graph G embedded in the torus, see Proposition 1.2:

$$Z_{\mathrm{Ising}}(\mathsf{G}, J) = 2^{|\mathsf{V}|}\Big(\prod_{\mathsf{e}\in\mathsf{E}} \cosh J_{\mathsf{e}}\Big) \sum_{\mathsf{P}\in\mathcal{P}(\mathsf{G})} \Big(\prod_{\mathsf{e}\in\mathsf{P}} \tanh J_{\mathsf{e}}\Big).$$

The decorated graph $\mathsf{G}^{\mathrm{F}} = (\mathsf{V}^{\mathrm{F}}, \mathsf{E}^{\mathrm{F}})$ on which the dimer model lives, is constructed from G as follows. Every vertex of degree d of G is replaced by a *decoration* consisting of $3d$ vertices: a triangle is attached to every edge incident to this vertex, and these triangles are linked in a circular way, see Figure 1.3. Edges of decorations are referred to as *short edges*, and those corresponding to edges of the original graph G as *long edges*. The graph G^{F} is also embedded in the torus, and is known as the *Fisher graph* of G.

Figure 1.3: Left: a vertex of G with its incoming edges. Right: corresponding decoration of the Fisher graph G^{F}.

Here comes Fisher's mapping [Fis66]. To any polygon configuration P coming from the high temperature expansion, we associated $2^{|\mathsf{V}|}$ dimer configurations of G^{F}: edges present (respectively absent) in P are absent (respectively present) in the corresponding dimer configuration of G^{F}. Once the state of these edges is fixed, there is exactly two ways to fill each decoration so as to obtain a dimer configuration of G^{F}, see Figure 1.4 for an example.

Figure 1.4: (Left) Polygon configuration of a piece of $\mathbb{Z}^2$ embedded in the torus. Right: corresponding dimer configurations of the associated Fisher graph.

Let us assign to every edge e of G^{F} weight $\nu_{\mathsf{e}} = 1$ if it is a short edge, and weight $\nu_{\mathsf{e}} = \coth J_{\mathsf{e}}$ if it is a long edge corresponding to an edge of the original graph G, also denoted by e. Then, Fisher's mapping is measure preserving, *i.e.* the probability of a polygon configuration P of the Ising model's high temperature expansion is equal to the probability of dimer configurations of G^{F} containing edges $\mathsf{E} \setminus \mathsf{P}$.

[1] Prior to Fisher, Kasteleyn [Kas63] introduced a mapping from the Ising model to the dimer model on a *non-planar* decorated version of the graph. Since planar graphs are more convenient to study the dimer model, Fisher's mapping is most often used.

1.4 Spanning trees

The study of *spanning trees* goes back to the work of Kirchhoff [Kir47], who studied its relation to electrical networks. Random spanning trees have deep and fruitful connections to many objects as random walks, harmonic functions and dimers, see [Lyo98] for an overview and references.

Consider a finite graph G. A *spanning tree* is a connected subset of edges containing all vertices of the graph G and no cycle. Let us denote by $\mathcal{T}(\mathsf{G})$ the set of spanning trees of the graph G. Assigning positive weights $(\rho_{\mathsf{e}})_{\mathsf{e}\in\mathsf{E}}$ to edges, the probability of occurrence of a spanning tree T, chosen with respect to the *spanning tree measure* $\mathbb{P}_{\text{tree}}$, is defined by:

$$\mathbb{P}_{\text{tree}}(\mathsf{T}) = \frac{\prod_{\mathsf{e}\in\mathsf{T}} \rho_{\mathsf{e}}}{Z_{\text{tree}}(\mathsf{G},\rho)},$$

where $Z_{\text{tree}}(\mathsf{G},\rho)$ is the *spanning tree partition function.*

1.4.1 Explicit formula for the partition function

An explicit formula for the partition function is given by Kirchhoff's matrix-tree theorem [Kir47]. Consider a labeling $\mathsf{V} = \{\mathsf{u}_1, \cdots, \mathsf{u}_n\}$ of the vertices of the graph G, then the *Laplacian matrix* Δ, is the $n \times n$ symmetric matrix, defined by:

$$\forall\, i,j \in \{1,\cdots,n\}, \qquad \Delta_{\mathsf{u}_i,\mathsf{u}_j} = \begin{cases} \sum\limits_{\mathsf{u}_j\sim\mathsf{u}_i} \rho_{\mathsf{u}_i\mathsf{u}_j} & \text{if } i=j \\ -\rho_{\mathsf{u}_i\mathsf{u}_j} & \text{if } i\sim j \\ 0 & \text{else.} \end{cases}$$

The terminology *Laplacian matrix* comes from the fact that when interpreted as an operator acting on $\mathbb{C}^{\mathsf{V}}$, Δ is a discrete version of the continuous Laplacian. The matrix-tree theorem [Kir47] states that the weighted number of spanning trees is equal, up to sign, to any diagonal cofactor of the Laplacian matrix. One way of proving this result is to use Cauchy-Binet's formula.

1.4.2 Extensions

Kirchhoff's matrix-tree theorem [Kir47] has been generalized in many directions. Let us state the extension to directed graphs (di-graphs), due to Tutte [Tut48].

Consider the complete di-graph G on vertices $\mathsf{V} = \{\mathsf{u}_1, \cdots, \mathsf{u}_n\}$, fix a vertex and label it by r. An *oriented spanning tree* of G, rooted at the vertex r, is a spanning tree with edges oriented towards the vertex r, referred to as the *root vertex*. Let us denote by $\mathcal{T}^{\mathsf{r}}(\mathsf{G})$ the set of *oriented* spanning trees of the graph G, rooted at r.

To every oriented edge $(\mathsf{u}_i, \mathsf{u}_j)$, $i \neq j$, assign a variable $a_{\mathsf{u}_i,\mathsf{u}_j}$, and denote by A the $n \times n$ matrix defined by:

$$A_{\mathsf{u}_i,\mathsf{u}_j} = \begin{cases} \sum_{j\neq i} a_{\mathsf{u}_i,\mathsf{u}_j} & \text{if } i=j \\ -a_{\mathsf{u}_i,\mathsf{u}_j} & \text{if } i\neq j. \end{cases}$$

Theorem 1.5. [Tut48] *Let A^{r} be the matrix obtained from the matrix A by removing the line and the column corresponding to the vertex r, then:*

$$\det(A^{\mathsf{r}}) = \pm \sum_{\mathsf{T} \in \mathcal{T}^{\mathsf{r}}(\mathsf{G})} \prod_{\mathsf{e}=(\mathsf{u}_i,\mathsf{u}_j)\in \mathsf{T}} a_{\mathsf{u}_i,\mathsf{u}_j}.$$

Remark 1.3.

- The interest of this generalization is that the only condition needed for a matrix-tree type theorem to hold, is that the column-sum of the matrix is zero. This condition is in particular satisfied by the Laplacian matrix.
- A very nice combinatorial proof of this result and extensions are given in the paper [Cha82] by Chaiken. The phrasing of this theorem is also taken from [Cha82].

1.5 Critical models on isoradial graphs

In this section, we define the critical versions of the Ising model, the Fisher dimer model, the bipartite dimer model and spanning trees, in the case where the underlying graph G is *isoradial*. Other critical models on isoradial graphs have been studied, as for example bond percolation, see [GM11, GM12].

A graph G is *isoradial* [Duf68, Ken02], if it can be embedded in the plane in such a way that all faces are inscribable in a circle of radius 1. We suppose moreover that circumcenters are in the closure of the faces. An isoradial embedding of the dual graph G^* is obtained by taking as dual vertices the circumcenters of the corresponding faces. When we speak of an isoradial graph, we actually mean an isoradial graph with a choice of embedding. Example of isoradial graphs are the square, triangular and honeycomb lattice. A more general example is given in Figure 1.5.

Figure 1.5: Left: example of isoradial graph. Center: corresponding diamond graph. Right: rhombus half-angle associated to an edge e of the graph.

To such a graph is naturally associated the *diamond graph*, denoted $\mathsf{G}^\diamond$, defined as follows. Vertices of $\mathsf{G}^\diamond$ consist in the vertices of G and vertices of G^*. Dual vertices are then joined to all primal vertices that are on the boundary of the corresponding face, see Figure 1.5 (center). Since G is isoradial, all faces of $\mathsf{G}^\diamond$ are side-length-1 rhombi (or half-rhombi on the boundary). Each edge e of G is the diagonal of exactly one rhombus (or half-rhombus) of $\mathsf{G}^\diamond$; we let θ_{e} be the half-angle of the rhombus at the vertex it has in common with e, see Figure 1.5 (right).

Isoradial graphs have two remarkable properties. First, they yield a natural setting to define discrete complex analysis [Duf68, Mer01b, Ken02, CS11]. Second, the property of being isoradial is preserved through $Y-\Delta$ *transformations* of the graph, also known as *star-triangle transformations*, underlying the concept of Z-invariance introduced by Baxter [Bax86]: a model is said to be *Z-invariant* if the partition function only changes by a global constant when performing a $Y-\Delta$ transformation.

We now state for each model, its critical version when defined on an isoradial graph. This amounts to specifying the positive weights assigned to edges. Note that criticality is also defined away from isoradiality, so that we are handling a specific family of critical models.

Critical coupling constants for the Ising model on G:

$$\forall\, \mathsf{e} \in \mathsf{E}, \quad J_{\mathsf{e}} = \frac{1}{2} \log \left(\frac{1+\sin\theta_{\mathsf{e}}}{\cos\theta_{\mathsf{e}}} \right). \tag{1.6}$$

These coupling constants were first derived by Baxter [Bax86], by imposing that the Ising model is Z-invariant, satisfies a generalized form of self-duality (the high and low temperature expansions yield the 'same' model), and by assuming uniqueness of the critical point. When $\mathsf{G} = \mathbb{Z}^2$, one recovers the critical temperature computed by Kramers and Wannier [KW41a, KW41b]. These coupling constants have now been proved to be critical when the underlying graph is periodic [CD13, Li12].

Very precise predictions for the critical Ising model were established by physicists in the last 30-50 years, in particular by Cardy, Duplantier, Nienhuis and many others. In the mathematics community, following the introduction of SLE by Schramm [Sch00], huge progresses have occurred in the last 10 years: in 2006, Smirnov [Smi10, Smi06] proves conformal invariance of the FK-representation of the critical Ising model on $\mathbb{Z}^2$, and the convergence of an interface to SLE(16/3). This work has been generalized to the critical Ising model on isoradial graphs by Chelkak and Smirnov [CS12]. On this subject, let us also mention the paper by Mercat [Mer01b], where the author defines a new theory of discrete Riemann surfaces and proves equivalence between criticality and existence of Dirac spinors. Recently, Chelkak, Hongler and Izyurov [CHI12] computed n-points correlation functions, a result obtained independently by Dubédat in a weaker form [Dub11b].

Critical weights for the dimer model on the Fisher graph G^{F} are obtained by applying Fisher's mapping, see Section 1.3, starting from a critical Ising model on an isoradial graph G:

$$\forall\, \mathsf{e} \in \mathsf{E}^{\mathrm{F}},\ \nu_{\mathsf{e}} = \begin{cases} 1 & \text{if } \mathsf{e} \text{ is a short edge,} \\ \coth J_{\mathsf{e}} = \coth\Big(\log \sqrt{\frac{1+\sin\theta_{\mathsf{e}}}{\cos\theta_{\mathsf{e}}}}\Big) = \cot\frac{\theta_{\mathsf{e}}}{2} & \text{if } \mathsf{e} \text{ comes from an edge } \mathsf{e} \text{ of } \mathsf{G}. \end{cases}$$

Critical weights for the bipartite dimer model on G:

$$\forall \mathsf{e} \in \mathsf{E}, \quad \nu_{\mathsf{e}} = \sin\theta_{\mathsf{e}}. \tag{1.7}$$

Critical weights for spanning trees on G:

$$\forall \mathsf{e} \in \mathsf{E}, \quad \rho_{\mathsf{e}} = \tan\theta_{\mathsf{e}}.$$

The above weights for the bipartite dimer model and for spanning trees have been introduced by Kenyon [Ken02], and are referred to as *critical* weights. They are not derived explicitely, and one may wonder whether they can be obtained using Z-invariance. A straightforward application of Z-invariance yields nothing but, for spanning trees, a way out might be to use the primal and dual graphs simultaneously. Critical dimer weights are then obtained in a specific bipartite case using Temperley's bijection [Tem74] and its generalizations [BP93, KPW00].

In the breakthrough paper [Ken02], Kenyon proves an explicit *local* expression for the inverse of the Kasteleyn matrix of the critical bipartite dimer model, and for the Green's function. Locality means that the expressions only depend on the local geometry of the graph. For example, an entry $K^{-1}_{\mathsf{x},\mathsf{y}}$ of the inverse Kasteleyn matrix is expressed as a contour integral of an integrand which only depends on a path joining vertices x and y in the isoradial embedding of the graph G. This is a very surprising result since one would expect the combinatorics of the whole graph to contribute.

Chapter 2

The critical Ising model via dimers

Let G be an infinite isoradial graph and consider the critical Ising model on G as defined in Section 1.5, *i.e.* suppose that edges are assigned the critical coupling constants:

$$\forall\, \mathsf{e} \in \mathsf{E}, \quad J_{\mathsf{e}} = \frac{1}{2} \log \left(\frac{1 + \sin \theta_{\mathsf{e}}}{\cos \theta_{\mathsf{e}}} \right).$$

Consider the corresponding critical dimer model on the Fisher graph G^{F} of G, obtained through Fisher's mapping, see Sections 1.3 and 1.5: edges of G^{F} are assigned Fisher-critical dimer weights:

$$\forall\, \mathsf{e} \in \mathsf{E}^{\mathrm{F}}, \quad \nu_{\mathsf{e}} = \begin{cases} 1 & \text{if } \mathsf{e} \text{ is a short edge, } \textit{i.e.} \text{ an edge of a decoration} \\ \cot \frac{\theta_{\mathsf{e}}}{2} & \text{if } \mathsf{e} \text{ is a long edge corresponding to an edge } \mathsf{e} \text{ of } \mathsf{G}. \end{cases}$$

In this chapter, we present two papers in collaboration with Cédric Boutillier giving a full description of the critical dimer model on the Fisher graph G^{F}: we prove an explicit expression for the free energy and for a natural Gibbs measure. In the first paper [3], we consider the periodic case, and prove explicit expressions in the spirit of [KOS06]. In the second paper [4], we prove *local expressions* in the spirit of Kenyon [Ken02], and remove the periodicity assumption.

2.1 The periodic case

[3] C. Boutillier, B. de Tilière. The critical Z-invariant Ising model via dimers: the periodic case. *Probab. Theory Related Fields* **147** (2009), no. 3, 379–413.

Suppose that the isoradial graph G is infinite and $\mathbb{Z}^2$-periodic, then the corresponding Fisher graph G^{F} is also $\mathbb{Z}^2$-periodic, and the dimer model is studied through the natural exhaustion $\{\mathsf{G}^{\mathrm{F}}_n = \mathsf{G}^{\mathrm{F}}/n\mathbb{Z}^2\}_{n\geq 1}$ of G^{F} by toroidal graphs, see Section 1.2.3 for definitions.

In the paper [3], we prove an explicit expression for the free energy and for the Gibbs measure obtained as weak limit of the Boltzmann measures of the graphs $\mathsf{G}^{\mathrm{F}}_n$. These results are of the same type as those obtained by Kenyon [Ken97], Cohn, Kenyon, Propp [CKP01] in the case of the dimer model on the honeycomb and the square lattice, and by Kenyon, Okounkov, Sheffield [KOS06] for the dimer model on general bipartite graphs. The results of [KOS06] do not apply to our setting because the Fisher graph is *not* bipartite, the bipartite assumption being a key

requirement of their paper. The two main difficulties that we had to face are: understanding the zeros of the dimer characteristic polynomial, see Section 2.1.1 and proving convergence of the Boltzmann measures for *every* n, see Section 2.1.2. The second part is robust and could be used for other dimer models on non-bipartite graphs.

2.1.1 Dimer and Laplacian characteristic polynomials

Similarly to the bipartite case [CKP01, KOS06], the key object involved in the explicit expressions of the free energy and the Gibbs measure is the *dimer characteristic polynomial*. It is constructed from the Kasteleyn matrix of the fundamental domain in the following way.

Dimer characteristic polynomial

Suppose that edges of the fundamental domain $\mathsf{G}_1^{\mathrm{F}}$ are oriented according to a Kasteleyn orientation, and let K_1 be the corresponding Kasteleyn matrix. Let γ_1 and γ_2 be two oriented edge-cycles in the dual graph G_1^*, winding around the two non-trivial directions of the torus, and let $z, w \in \mathbb{C}^*$. Multiply the coefficient $(K_1)_{\mathsf{u},\mathsf{v}}$ of the edge uv of $\mathsf{G}_1^{\mathrm{F}}$ by $z^{\pm 1}$ (resp. $\mathsf{w}^{\pm 1}$) whenever it crosses γ_1 (resp. γ_2), the $+$ sign (resp. $-$ sign) is chosen if the vertex u is on the left (resp. on the right) of the path, see Figure 2.1. This defines a modified weight Kasteleyn matrix $K_1(z, w)$, which is skew-hermitian when $|z| = |w| = 1$. The *dimer characteristic polynomial*, denoted $P_{\mathrm{dimer}}(z, w)$, is the determinant of $K_1(z, w)$:

$$P_{\mathrm{dimer}}(z, w) = \det K_1(z, w).$$

Figure 2.1: Left: example of fundamental domain G_1 (opposite sides are identified). Middle: isoradial embedding of G_1, underlying rhombus graph (dotted lines) and rhombus half-angles. Right: Kasteleyn orientation of the corresponding fundamental domain $\mathsf{G}_1^{\mathrm{F}}$, and modified weights of the Kasteleyn matrix $K_1(z, w)$.

In the paper [CKP01], Cohn, Kenyon and Propp provide a general argument allowing to prove explicit expressions for the free energy and for Gibbs measure(s), see Section 2.1.2. One of the key points is to have a precise description of the zeros of the dimer characteristic polynomial on the torus $\mathbb{T}^2$.

In the case of the bipartite dimer model, Kenyon, Okounkov and Sheffield [KOS06] prove that the zero set of the dimer characteristic polynomial is a Harnack curve of genus 0, implying in particular that the polynomial has at most two conjugate zeros on $\mathbb{T}^2$. Their result is very powerful since it applies to all bipartite dimer models, but it does not extend to the non-bipartite case.

Quite surprisingly, we prove that for the non-bipartite Fisher graph G^{F}, the zero set of $P_{\mathrm{dimer}}(z, w)$ is also a Harnack curve, by relating it to the Laplacian. This is one of the main results of our paper [3], in order to precisely state it, we need to define the *Laplacian characteristic polynomial*.

Laplacian characteristic polynomial

Consider the Laplacian matrix Δ_1 of the fundamental domain G_1 of the isoradial graph G, where edges of the graph G_1 are assigned the critical spanning tree weights, see Section 1.5:

$$\forall\, \mathsf{e} \in \mathsf{E}, \quad \rho_{\mathsf{e}} = \tan \theta_{\mathsf{e}}.$$

The modified weight Laplacian matrix $\Delta_1(z,w)$ is constructed in a way similar to the modified weight Kasteleyn matrix, see Figure 2.2. The *Laplacian characteristic polynomial* $P_{\mathrm{Lap}}(z,w)$ is the determinant of $\Delta_1(z,w)$:

$$P_{\mathrm{Lap}}(z,w) = \det \Delta_1(z,w).$$

Figure 2.2: Modified weights of the Laplacian matrix $\Delta_1(z,w)$. The Laplacian characteristic polynomial is equal to $P_{\mathrm{Lap}}(z,w) = \tan(\theta_{\mathsf{e}_1})(2 - w - w^{-1}) + \tan(\theta_{\mathsf{e}_2})(2 - z - z^{-1})$.

We prove the following:

Theorem 2.1. [3] *There exists a constant $C \neq 0$, such that:*

$$\forall\, (z,w) \in \mathbb{C}^2, \quad P_{\mathrm{dimer}}(z,w) = C P_{\mathrm{Lap}}(z,w).$$

Moreover $\{(z,w) \in \mathbb{C}^2 : P_{\mathrm{dimer}}(z,w) = 0\}$ is a Harnack curve of genus 0 and its unique point on $\mathbb{T}^2$ is $(1,1)$ and it has multiplicity 2.

Remark 2.1.

- The proof of the first part consists in showing that $P_{\mathrm{Lap}}(z,w)$ divides $P_{\mathrm{dimer}}(z,w)$ and that the Newton polygon of $P_{\mathrm{dimer}}(z,w)$ is included in the Newton polygon of $P_{\mathrm{Lap}}(z,w)$. We don't give more details here since we provide a combinatorial proof of this equality in the next chapter. The remainder of the proof consists in using results of [Ken02, KPW00] to relate the polynomial $P_{\mathrm{Lap}}(z,w)$ to the characteristic polynomial of a *bipartite* dimer model, implying that its zero set is a Harnack curve [KOS06]. The fact that the zero is located precisely at $(1,1)$ is a consequence of a result of Forman [For93], see Theorem 3.1.
- Since the publication of our paper [3], two related results have been obtained. The Ising partition function can also be computed using Kac-Ward matrices [KW52]. In the paper [Cim12], Cimasoni proves a result similar to Theorem 2.1, relating the determinant of the Kac-Ward matrix to the determinant of the critical Laplacian matrix. Using a slight variation of Fisher's mapping, Dubédat [Dub11b] proves that the dimer characteristic polynomial of $\mathsf{G}_1^{\mathrm{F}}$ is equal, up to a constant, to the characteristic polynomial of yet another *bipartite* dimer model, thus also implying that its zero set is a Harnack curve [KOS06]. The result of Dubédat is not directly related to the Laplacian, but also holds away from the critical point.

2.1.2 Free energy and Gibbs measure

We now state the result of [3] proving an explicit expression for the free energy of the critical dimer model on the infinite, $\mathbb{Z}^2$-periodic Fisher graph G^{F}, and for the Gibbs measure obtained as weak limit of the Boltzmann measures of $\mathsf{G}_n^{\mathrm{F}}$. In order to state our result, we need the following notations and definitions:

- $\mathcal{F}$ is the σ-field generated by cylinders, a *cylinder* being the set of dimer configurations containing a fixed, finite subset of edges of G^{F}.

- Since the graph G^{F} is $\mathbb{Z}^2$-periodic, vertices of G^{F} can be identified with $\mathsf{V}_1^{\mathrm{F}} \times \mathbb{Z}^2$, *i.e.* (u, x, y) is the vertex u in the (x, y)-copy of the fundamental domain $\mathsf{G}_1^{\mathrm{F}}$. The *inverse Kasteleyn matrix* K^{-1} is defined to be the infinite matrix whose coefficients are given by:

$$K^{-1}_{(\mathsf{u},x,y),(\mathsf{u}',x',y')} = \frac{1}{(2\pi i)^2} \iint_{\mathbb{T}^2} \frac{\mathrm{Cof}(K_1(z,w))_{\mathsf{u}',\mathsf{u}}}{P_{\mathrm{dimer}}(z,w)} z^{x'-x} w^{y'-y} \frac{\mathrm{d}z}{z} \frac{\mathrm{d}w}{w}. \tag{2.1}$$

  In Proposition 5 of [3], we prove that K^{-1} is indeed an inverse of the infinite Kasteleyn matrix K, in the sense that $KK^{-1} = \mathrm{Id}$, and that it is the unique inverse tending to 0 as $\|(x'-x, y'-y)\| \to \infty$. It is computed using Fourier techniques, as in [CKP01].

Theorem 2.2. [3]

1. *The free energy f of the critical dimer model on the Fisher graph G^{F} is equal to:*

$$f = -\frac{1}{2(2\pi i)^2} \iint_{\mathbb{T}^2} \log P_{\mathrm{dimer}}(z,w) \frac{\mathrm{d}z}{z} \frac{\mathrm{d}w}{w}. \tag{2.2}$$

2. *There is a unique probability measure $\mathbb{P}_{\mathrm{dimer}}$ on $(\mathcal{M}(\mathsf{G}^{\mathrm{F}}), \mathcal{F})$, such that the probability of occurrence of a subset of edges $\mathsf{E}_k = \{\mathsf{e}_1 = \mathsf{u}_1\mathsf{v}_1, \cdots, \mathsf{e}_k = \mathsf{u}_k\mathsf{v}_k\}$ of G^{F} in a dimer configuration is equal to:*

$$\mathbb{P}_{\mathrm{dimer}}[\mathsf{e}_1, \cdots, \mathsf{e}_k] = \Big(\prod_{i=1}^{k} K_{\mathsf{u}_i,\mathsf{v}_i}\Big) \operatorname{Pf}\big([K^{-1}_{\mathsf{E}_k}]^t\big), \tag{2.3}$$

   where $K^{-1}_{\mathsf{E}_k}$ is the submatrix of the infinite matrix K^{-1} of (2.1), whose lines and columns are indexed by $\{\mathsf{u}_1, \mathsf{v}_1, \cdots, \mathsf{u}_k, \mathsf{v}_k\}$. Moreover, $\mathbb{P}_{\mathrm{dimer}}$ is a translation invariant Gibbs measure.

Outline of the proof. As in [KOS06], the beginning of the proof follows [CKP01].

1. Using Theorem 1.4 of Section 1.2.2, one expresses, for every toroidal graph $\mathsf{G}_n^{\mathrm{F}}$ of the exhaustion of G^{F}, the partition function $Z_{\mathrm{dimer}}(\mathsf{G}_n^{\mathrm{F}}, \nu)$ and the Boltzmann measure $\mathbb{P}^n_{\mathrm{dimer}}$. Both expressions (1.4) and (1.5) involve the four Kasteleyn matrices $(K_n^{\theta\tau})_{\theta,\tau\in\{0,1\}}$ of the graph $\mathsf{G}_n^{\mathrm{F}}$, where we have added a subscript n to emphasize the dependence in n.

2. Using translation invariance of the graph $\mathsf{G}_n^{\mathrm{F}}$ and Fourier techniques, for every $\theta, \tau \in \{0,1\}$, the matrix $K_n^{\theta\tau}$ can be block-diagonalized, with $(n^2)^2$ blocks of size $|\mathsf{V}_1^{\mathrm{F}}|$, the number of vertices of the fundamental domain. For every $j, k \in \{0, \cdots, n-1\}$, the (j,k)-th block is explicitly expressed using the modified weight Kasteleyn matrix as:

$$K_1\left(e^{i\frac{(2j+\theta)\pi}{n}}, e^{i\frac{(2k+\tau)\pi}{n}}\right). \tag{2.4}$$

The idea then is to use Points 1 and 2 above, to obtain analogs of Formulas (2.2) and (2.3) with Riemann sums instead of integrals, and to show that Riemann sums converge to the corresponding integrals as $n \to \infty$. This is where we need information on the zeros of the dimer characteristic polynomial on $\mathbb{T}^2$, provided by Theorem 2.1:

3. The dimer characteristic polynomial $P_{\mathrm{dimer}}(z,w) = \det K_1(z,w)$, has a single double zero at $(1,1)$ on $\mathbb{T}^2$. In the discretization of $\mathbb{T}^2$ obtained from the block-diagonal decomposition, see Equation (2.4), the zero $(1,1)$ is reached by $\left(e^{i\frac{(2j+\theta)\pi}{n}}, e^{i\frac{(2k+\tau)\pi}{n}}\right)$ if and only if $(j,k) = (0,0)$ and $(\theta,\tau) = (0,0)$. For all other indices, the points are at distance at least $O\big(\frac{1}{n}\big)$ from the zero $(1,1)$.

Free energy. Using the block decomposition of the matrices $K_n^{\theta\tau}$ of Point 2, one has, for every $\theta, \tau \in \{0,1\}$:

$$\det K_n^{\theta\tau} = \prod_{j=0}^{n-1}\prod_{k=0}^{n-1} P_{\mathrm{dimer}}\left(e^{i\frac{(2j+\theta)\pi}{n}}, e^{i\frac{(2k+\tau)\pi}{n}}\right). \tag{2.5}$$

By Point 3, this determinant is 0 if and only if $(\theta,\tau) = (0,0)$. When $(\theta,\tau) \neq (0,0)$, since the determinant is the square of the Pfaffian and since the points $\left(e^{i\frac{(2j+\theta)\pi}{n}}, e^{i\frac{(2k+\tau)\pi}{n}}\right)$ are at distance at least $O\big(\frac{1}{n}\big)$, one has:

$$\begin{aligned}\lim_{n\to\infty}\frac{1}{n^2}\log\mathrm{Pf}(K_n^{\theta\tau}) &= \frac{1}{2}\frac{1}{(2\pi)^2}\lim_{n\to\infty}\frac{(2\pi)^2}{n^2}\sum_{j=0}^{n-1}\sum_{k=0}^{n-1}\log P_{\mathrm{dimer}}\left(e^{i\frac{(2j+\theta)\pi}{n}}, e^{i\frac{(2k+\tau)\pi}{n}}\right),\\ &= \frac{1}{2}\frac{1}{(2\pi)^2}\int_0^{2\pi}\int_0^{2\pi}\log P_{\mathrm{dimer}}(e^{i\eta}, e^{i\xi})\,d\eta\,d\xi.\end{aligned}$$

Details on the convergence of the Riemann sums to the corresponding integral are given in the paper [CKP01]. Then, by Theorem 1.4 we have, $\max_{\theta,\tau}\{\mathrm{Pf}(K_n^{\theta\tau})\} \leq Z_{\mathrm{dimer}}(\mathsf{G}_n^{\mathrm{F}}, \nu) \leq 2\max_{\theta,\tau}\{\mathrm{Pf}(K_n^{\theta\tau})\}$, which allows to conclude the proof.

Gibbs measure. As long as $(\theta,\tau) \neq (0,0)$, as in the planar case, one uses the Jacobi identity of Equation (1.3) for each of the terms $\mathrm{Pf}((K_n^{\theta\tau})_{\mathsf{E}_k^c})$ of the Boltzmann measure (1.5). Jacobi's identity involves the Pfaffian of the full matrix, which will be compensated by the partition function

$Z_{\text{dimer}}(\mathsf{G}_n^{\mathrm{F}},\nu)$ in the denominator, and the Pfaffian of a restriction of the inverse Kasteleyn matrix. Coefficients of the inverse Kasteleyn matrix are expressed using the block diagonalization of Point 2, as Riemann sums for the integral (2.1):

$$(K_n^{\theta\tau})^{-1}_{(\mathsf{u},x,y),(\mathsf{u}',x',y')} = \frac{1}{n^2}\sum_{j=0}^{n-1}\sum_{k=0}^{n-1} e^{i\frac{(2j+\theta)\pi}{n}(x'-x)}e^{i\frac{(2k+\tau)\pi}{n}(y'-y)}\frac{\mathrm{Cof}\left(K_1\left(e^{i\frac{(2j+\theta)\pi}{n}},e^{i\frac{(2k+\tau)\pi}{n}}\right)\right)_{\mathsf{u}',\mathsf{u}}}{P_{\text{dimer}}\left(e^{i\frac{(2j+\theta)\pi}{n}},e^{i\frac{(2k+\tau)\pi}{n}}\right)},$$

Convergence to the integral as $n\to\infty$ is proved using Point 3, which tells us that we are far enough from the zero of the characteristic polynomial.

The term involving $(\theta,\tau)=(0,0)$ in the expression of the Boltzmann measure (1.5) needs to be treated separately. Indeed, since $\mathrm{Pf}(K_n^{00})=0$, Jacobi's identity does not hold in this case. Using the Cauchy-Binet formula, rather involved expansions of the determinant [Bou70], and careful estimates, we prove that:

$$\frac{\mathrm{Pf}(K_n^{00})_{\mathsf{E}^c}}{Z_{\text{dimer}}(\mathsf{G}_n^{\mathrm{F}},\nu)} = O\Big(\frac{1}{n}\Big).$$

This ends the sketch of the proof of the convergence of the Boltzmann measure $\mathbb{P}^n_{\text{dimer}}[\mathsf{e}_1,\cdots,\mathsf{e}_k]$ to the right hand side of (2.3). Existence of a unique probability measure $\mathbb{P}_{\text{dimer}}$ equal to the RHS of (2.3) on cylinder sets is then given by Kolmogorov's extension theorem. □

Remark 2.2. In the case of the bipartite dimer model, when proving convergence of the Boltzmann measures on cylinder sets, Kenyon, Okounkov and Sheffield [KOS06] actually prove convergence on a subsequence of n's where they use Jacobi's formula for each (θ,τ). They conclude using a theorem of Sheffield [She05] proving a priori existence of the limit. Sheffield's theorem strongly relies on the bipartite assumption and is thus not available for the Fisher graph. Moreover, in our case, the Pfaffian of K_n^{00} cancels for every n, so that one can never use Jacobi's' identity for $(\theta,\tau)=(0,0)$. Finding a subsequence where the general argument of [CKP01] works was not an option either. The method we provide is robust and can be used for other non-bipartite graphs. As mentionned before, what is missing for the non-bipartite dimer model is a general understanding of the zeros of the characteristic polynomial.

2.2 Non periodic case, local expressions

[4] C. Boutillier, B. de Tilière. The critical Z-invariant Ising model via dimers: locality property. *Comm. Math. Phys.* **301** (2011), no. 2, 473–516.

In the paper [4] we suppose that the graph G is infinite, isoradial, but not necessarily $\mathbb{Z}^2$-periodic. We consider the critical dimer model on the Fisher graph G^{F}. Our main result is an explicit, local formula for the inverse Kasteleyn matrix, in the spirit of [Ken02], it is stated in Section 2.2.1. Using the results of [3] and an argument of [8], this yields an explicit local formula for a natural Gibbs measure, see Section 2.2.2. Using techniques of [Ken02], we recover Baxter's formula for the free energy of the critical Ising model, and thus a new proof of it. The latter is equal, up to a constant, to $\frac{1}{2}$ of the logarithm of the normalized determinant of the Laplacian obtained in [Ken02]. This is the subject of Section 2.2.3.

2.2.1 Local expression for the inverse Kasteleyn matrix

Let us first introduce some notations for vertices of the isoradial graph G and for vertices of the corresponding Fisher graph G^{F}.

Notations. Vertices of G^{F} are written in plain symbols, those of G in boldface. Let x be a vertex of G^{F}, then x belongs to a decoration corresponding to a unique vertex of G, denoted by $\mathbf{x}$. Conversely, vertices of a decoration of G^{F} corresponding to a vertex $\mathbf{x}$ of G are labeled as follows, refer to Figure 2.3 for an example. Let $d(\mathbf{x})$ be the degree of the vertex $\mathbf{x}$ in G, then the corresponding decoration consists of $d(\mathbf{x})$ triangles, labeled from 1 to $d(\mathbf{x})$ in counterclockwise order. For the k-th triangle, let $\mathsf{v}_k(\mathbf{x})$ be the vertex incident to an edge of G, and let $\mathsf{w}_k(\mathbf{x})$, $\mathsf{z}_k(\mathbf{x})$ be the other two vertices in counterclockwise order, starting from $\mathsf{v}_k(\mathbf{x})$. Later on, when no confusion occurs, we will drop the argument $\mathbf{x}$ in the above labeling. Define a vertex x of G^{F} to be of type 'v', if $\mathsf{x} = \mathsf{v}_k(\mathbf{x})$ for some $k \in \{1, \cdots, d(\mathbf{x})\}$, and similarly for 'w' and 'z'.

Figure 2.3: Notations for vertices of G^{F}.

The isoradial embedding of G fixes an isoradial embedding of the corresponding diamond graph $\mathsf{G}^{\diamond}$. There is a natural way of assigning rhombus unit-vectors of $\mathsf{G}^{\diamond}$ to vertices of G^{F}: for every vertex $\mathbf{x}$ of G, let us assign the rhombus unit-vector $e^{i\alpha_{\mathsf{w}_k(\mathbf{x})}}$ to $\mathsf{w}_k(\mathbf{x})$, $e^{i\alpha_{\mathsf{z}_k(\mathbf{x})}}$ to $\mathsf{z}_k(\mathbf{x})$, and the two rhombus unit-vectors $e^{i\alpha_{\mathsf{w}_k(\mathbf{x})}}$, $e^{i\alpha_{\mathsf{z}_k(\mathbf{x})}}$ to $\mathsf{v}_k(\mathbf{x})$, as in Figure 2.4. Note that $e^{i\alpha_{\mathsf{w}_k(\mathbf{x})}} = e^{i\alpha_{\mathsf{z}_{k+1}(\mathbf{x})}}$.

Figure 2.4: Rhombus vectors of the diamond graph $\mathsf{G}^{\diamond}$ (dotted lines) assigned to vertices of G^{F}.

The integrand of the local formula for the inverse Kasteleyn matrix has two contributions: a complex valued function defined on vertices of G^{F}, and the discrete exponential function of [Mer01a], see also [Ken02], which only depends on the underlying isoradial graph G. We now define these two functions.

Complex valued function on vertices of G^{F}. Define $f : \mathsf{V}^{\mathrm{F}} \times \mathbb{C} \to \mathbb{C}$, by:

$$\begin{aligned} f(\mathsf{w}_k(\mathbf{x}), \lambda) &:= f_{\mathsf{w}_k(\mathbf{x})}(\lambda) = \frac{e^{i\frac{\alpha_{\mathsf{w}_k(\mathbf{x})}}{2}}}{e^{i\alpha_{\mathsf{w}_k(\mathbf{x})}} - \lambda} \\ f(\mathsf{z}_k(\mathbf{x}), \lambda) &:= f_{\mathsf{z}_k(\mathbf{x})}(\lambda) = \frac{e^{i\frac{\alpha_{\mathsf{z}_k(\mathbf{x})}}{2}}}{e^{i\alpha_{\mathsf{z}_k(\mathbf{x})}} - \lambda} \\ f(\mathsf{v}_k(\mathbf{x}), \lambda) &:= f_{\mathsf{v}_k(\mathbf{x})}(\lambda) = f_{\mathsf{w}_k(\mathbf{x})}(\lambda) + f_{\mathsf{z}_k(\mathbf{x})}(\lambda), \end{aligned} \tag{2.6}$$

for every $\mathbf{x} \in \mathsf{G}$ and every $k \in \{1, \cdots, d(\mathbf{x})\}$. For these functions to be well defined, the angles $\alpha_{\mathsf{w}_k(\mathbf{x})}$ and $\alpha_{\mathsf{z}_k(\mathbf{x})}$ need to be well defined modulo 4π, indeed half angles need to be well defined modulo 2π. We refer to Lemma 4 of the original paper [4] for this definition. It is related to spin structures on surface graphs, see [Kup98, CR07, CR08].

Discrete exponential function [Mer01a, Ken02]. Let $\mathbf{x}$ and $\mathbf{y}$ be two vertices of G, and let $\mathbf{y} = \mathbf{x}_1, \cdots, \mathbf{x}_{n+1} = \mathbf{x}$ be an edge-path of G from $\mathbf{y}$ to $\mathbf{x}$. The complex vector $\mathbf{x}_{j+1} - \mathbf{x}_j$ is the

sum of two unit complex numbers $e^{i\beta_j} + e^{i\gamma_j}$ representing edges of the rhombus of $\mathsf{G}^\diamond$ associated to the edge $\mathbf{x}_j\mathbf{x}_{j+1}$. Then, $\mathrm{Exp} : \mathsf{V} \times \mathsf{V} \times \mathbb{C} \to \mathbb{C}$ is defined by:

$$\mathrm{Exp}(\mathbf{x}, \mathbf{y}, \lambda) := \mathrm{Exp}_{\mathbf{x},\mathbf{y}}(\lambda) = \prod_{i=1}^{n} \Big(\frac{e^{i\beta_j} + \lambda}{e^{i\beta_j} - \lambda}\Big)\Big(\frac{e^{i\gamma_j} + \lambda}{e^{i\gamma_j} - \lambda}\Big).$$

The function is well defined, *i.e.* independent of the choice of edge-path of G from $\mathbf{y}$ to $\mathbf{x}$, since the product of the multipliers around a rhombus is 1.

We can now state Theorem 2.3 proving an explicit, local expression for the coefficients of an inverse K^{-1} of the Kasteleyn matrix K. The vertex x of G^{F} in the statement should be thought of as being one of $\mathsf{w}_k(\mathbf{x})$, $\mathsf{z}_k(\mathbf{x})$, $\mathsf{v}_k(\mathbf{x})$ for some $\mathbf{x}$ of G and some $k \in \{1, \cdots, d(\mathbf{x})\}$, and similarly for the vertex y of G^{F}.

Theorem 2.3. [4] *Let* x *and* y *be any two vertices of* G^{F}*. Then, the infinite matrix* K^{-1}*, whose coefficient* $K^{-1}_{\mathsf{x},\mathsf{y}}$ *is given by* (2.7) *below, is an inverse Kasteleyn matrix,*

$$K^{-1}_{\mathsf{x},\mathsf{y}} = \frac{1}{(2\pi)^2} \oint_{\mathcal{C}_{\mathsf{x},\mathsf{y}}} f_{\mathsf{x}}(\lambda) f_{\mathsf{y}}(\lambda) \,\mathrm{Exp}_{\mathbf{x},\mathbf{y}}(\lambda) \log \lambda \,\mathrm{d}\lambda + C_{\mathsf{x},\mathsf{y}}. \tag{2.7}$$

The contour of integration $\mathcal{C}_{\mathsf{x},\mathsf{y}}$ *is a simple closed curve oriented counterclockwise containing all poles of the integrand, and avoiding a half-line* $d_{\mathsf{x},\mathsf{y}}$[1] *with base point zero. The constant* $C_{\mathsf{x},\mathsf{y}}$ *is zero whenever* x *and* y *do not belong to the same decoration, and is equal to* $\pm\frac{1}{4}$*, when they do.*

Remark 2.3.

- Now that the theorem is stated, we can explain what is meant by a *local* formula, see also Section 1.5. The integrand of the explicit expression for $K^{-1}_{\mathsf{x},\mathsf{y}}$ given by Equation (2.7) only depends on the geometry of the embedding of the isoradial graph G and on a path joining vertices which are at distance at most 2 from $\mathbf{x}$ and $\mathbf{y}$. This is surprising since one would expect the combinatorics of the whole graph to contribute.

- The infinite inverse Kasteleyn matrix K^{-1} in the statement of Theorem 2.3 satisfies $KK^{-1} = \mathrm{Id}$. This raises the natural question of whether it is unique. When the graph G^{F} is assumed to be $\mathbb{Z}^2$-periodic, the answer is yes. Indeed, in Proposition 5 of [3], we prove uniqueness of the inverse such that $K^{-1}_{\mathsf{x},\mathsf{y}} \to 0$, as $\|\mathsf{x} - \mathsf{y}\| \to \infty$. In Corollary 7 of [4], we compute asymptotics of $K^{-1}_{\mathsf{x},\mathsf{y}}$ from which one easily deduces that this condition is satisfied. Uniqueness has an interesting consequence: it means that, in the $\mathbb{Z}^2$-periodic case, the expressions (2.1) and (2.7) are equal. We have not yet been able to directly prove this.

Proof (Idea). The proof consists in showing that $KK^{-1} = \mathrm{Id}$. The idea of the argument comes from [Ken02], where Kenyon computes a local explicit expression for the inverse Kasteleyn matrix of the critical dimer model on a bipartite, isoradial graph. The results of the paper [Ken02] cannot be applied to the critical dimer model on the Fisher graph G^{F}, since G^{F} is not isoradial, nor bipartite.

• The first part of the argument consists in finding a complex valued function, depending on a complex parameter, in the kernel of the Kasteleyn operator[2]. The difficulty lies in the fact

[1] Refer to the paper [4] for the precise definition of the half-line $d_{\mathsf{x},\mathsf{y}}$.

[2] The Kasteleyn operator is defined from the Kasteleyn matrix in Equation (1.1).

that there is no general method to construct such a function, nor even to determine whether it exists. Nevertheless, we were able to find one: the next proposition proves that for every $\mathsf{y} \in \mathsf{G}$, and every $\lambda \in \mathbb{C}$, the function $(f_{\mathsf{x}}(\lambda)\,\mathrm{Exp}_{\mathbf{x},\mathbf{y}}(\lambda))_{\mathsf{x}\in\mathsf{V}^{\mathrm{F}}}$, is in the kernel of the Kasteleyn operator K of the Fisher graph G^{F}.

Proposition 2.1. [4] *Let* x,y *be two vertices of* G^{F}*, and let* $\mathsf{x}_1,\mathsf{x}_2,\mathsf{x}_3$ *be the three neighbors of* x *in* G^{F}*, then for every* $\lambda \in \mathbb{C}$*:*

$$\sum_{i=1}^{3} K_{\mathsf{x},\mathsf{x}_i} f_{\mathsf{x}_i}(\lambda)\,\mathrm{Exp}_{\mathbf{x}_i,\mathbf{y}}(\lambda) = 0.$$

The proof of Proposition 2.1 consists of explicit computations, which we omit here. Rather, let us give an idea of how we identified the functions $(f_{\mathsf{x}}(\lambda)\,\mathrm{Exp}_{\mathbf{x},\mathbf{y}}(\lambda))_{\mathsf{x}\in\mathsf{V}^{\mathrm{F}}}$. By Theorem (2.1), we know that the characteristic polynomial of the critical dimer model on the Fisher graph G^{F} is equal, up to a constant, to the characteristic polynomial of the critical Laplacian on the isoradial graph G. This led us to think that the discrete exponential functions, which are the integrand of the local expression of the Green's function [Ken02], should also appear in the integrand of the inverse Kasteleyn operator. Then, since the model is defined on the Fisher graph G^{F} and not on the isoradial graph G, we believed that there should be some additional information coming from the Fisher graph, hoping that it would be local, *i.e.* only involve the decoration of the vertex in question. It turned out to be the case. Finding the functions $(f_{\mathsf{x}}(\lambda))_{\mathsf{x}\in\mathsf{V}^{\mathrm{F}}}$ involved explicitly solving the case of $\mathbb{Z}^2$, and understanding how this could be extended to the case of a general isoradial graph.

• Here is the idea of the second part of the argument. Let x, y be two vertices of G^{F}, and let $\mathcal{C}_{\mathsf{x},\mathsf{y}}$ be a a simple closed curve oriented counterclockwise, containing all poles of the integrand of (2.7), and avoiding an angular sector $s_{\mathsf{x},\mathsf{y}}$ containing the half-line $d_{\mathsf{x},\mathsf{y}}$, see Figure 2.5.

Figure 2.5: Definition of the contour of integration $\mathcal{C}_{\mathsf{x},\mathsf{y}}$ of the integral term of $K^{-1}_{\mathsf{x},\mathsf{y}}$. The poles of the integrand are on the unit circle, and are drawn as thick points.

Denote by $\mathsf{x}_1, \mathsf{x}_2, \mathsf{x}_3$ the three neighbors of x in G^{F}. The goal is to define angular sectors $s_{\mathsf{x},\mathsf{y}}$, in such a way that:

- When $\mathsf{x} \neq \mathsf{y}$, the intersection $\cap_{i=1}^{3} s_{\mathsf{x}_i,\mathsf{y}}$ is non-empty, meaning that we are working on a single branch of the logarithm. The three contours $\mathcal{C}_{\mathsf{x}_i,\mathsf{y}}$ can then be continuously deformed into a common contour $\mathcal{C}$ without meeting any pole which, using Proposition 2.1, yields:

$$\sum_{i=1}^{3} K_{\mathsf{x},\mathsf{x}_i} \oint_{\mathcal{C}_{\mathsf{x}_i,\mathsf{y}}} f_{\mathsf{x}_i}(\lambda) f_{\mathsf{y}}(-\lambda) \operatorname{Exp}_{\mathbf{x}_i,\mathbf{y}}(\lambda) \log \lambda \frac{\mathrm{d}\lambda}{(2\pi)^2} = \oint_{\mathcal{C}} \Big(\sum_{i=1}^{3} K_{\mathsf{x},\mathsf{x}_i} f_{\mathsf{x}_i}(\lambda) \operatorname{Exp}_{\mathbf{x}_i,\mathbf{y}}(\lambda) \Big) f_{\mathsf{y}}(-\lambda) \log \lambda \frac{\mathrm{d}\lambda}{(2\pi)^2} = 0. \quad (2.8)$$

- When $\mathsf{x} = \mathsf{y}$, the intersection of the three sectors is empty, meaning that we are sitting on different branches of the logarithm. Using this fact, one aims at having

$$\sum_{i=1}^{3} K_{\mathsf{x},\mathsf{x}_i} \oint_{\mathcal{C}_{\mathsf{x}_i,\mathsf{y}}} f_{\mathsf{x}_i}(\lambda) f_{\mathsf{y}}(-\lambda) \operatorname{Exp}_{\mathbf{x}_i,\mathbf{y}}(\lambda) \log \lambda \frac{\mathrm{d}\lambda}{(2\pi)^2}, \quad (2.9)$$

  equal to 1.

It turns out that it is not possible to construct angular sectors such that Equation (2.8) holds and such that (2.9) is equal to 1. Additional constants $C_{\mathsf{x},\mathsf{y}}$ need to be introduced, refer to the statement of Theorem 2.3.

In the paper [Ken02], angular sectors are not constructed explicitly since geometric considerations suffice to define an appropriate branched covering of the plane. Angular sectors appear in [BMS05] for the isoradial graph G. In our case, the construction is complicated by the presence of the decorations. The construction and the proof is rather involved (13 pages) and we do not want to enter into more details here. Let us just mention that it heavily uses the underlying rhombus graph $\mathsf{G}^{\diamond}$, and train-tracks, where a *train-track* is a path of edge-adjacent rhombi of $\mathsf{G}^{\diamond}$, which does not turn: on entering a face, it exits along the opposite edge. □

2.2.2 Local formula for a natural Gibbs measure

We let G be an infinite isoradial graph, not necessarily periodic, and let G^{F} be the corresponding Fisher graph. Using an argument of [8] and the explicit local expression for the inverse Kasteleyn matrix, we prove an explicit, local expression for a Gibbs measure of the critical dimer model on the Fisher graph G^{F}. The strength of this result lies in the fact that the graph is not assumed to be periodic.

Theorem 2.4. [4] *There is a unique probability measure $\mathbb{P}_{\mathrm{dimer}}$ on $(\mathcal{M}(\mathsf{G}^{\mathrm{F}}), \mathcal{F})$, such that the probability of occurrence of a subset of edges $\mathsf{E}_k = \{\mathsf{e}_1 = \mathsf{x}_1\mathsf{y}_1, \cdots, \mathsf{e}_k = \mathsf{x}_k\mathsf{y}_k\}$ of G^{F} in a dimer configuration is:*

$$\mathbb{P}_{\mathrm{dimer}}[\mathsf{e}_1, \cdots, \mathsf{e}_k] = \Big(\prod_{i=1}^{k} K_{\mathsf{x}_i,\mathsf{y}_i} \Big) \operatorname{Pf}\Big([K^{-1}_{\mathsf{E}_k}]^t \Big), \quad (2.10)$$

where $K^{-1}_{\mathsf{E}_k}$ is the submatrix of the infinite inverse Kasteleyn matrix K^{-1} of (2.7), whose lines and columns are indexed by $\{\mathsf{x}_1, \mathsf{y}_1, \cdots, \mathsf{x}_k, \mathsf{y}_k\}$; moreover $\mathbb{P}_{\text{dimer}}$ is a Gibbs measure. When G^{F} is $\mathbb{Z}^2$-periodic, $\mathbb{P}_{\text{dimer}}$ is the Gibbs measure of (2.3), obtained as weak limit of the Boltzmann measures $\mathbb{P}^n_{\text{dimer}}$ on the toroidal exhaustion $\{\mathsf{G}^{\mathrm{F}}_n = \mathsf{G}^{\mathrm{F}}/(n\mathbb{Z}^2)\}_{n\geq 1}$ of G^{F}.

Proof (Idea). The idea of the argument comes from the paper [8]. The goal is to construct the measure $\mathbb{P}_{\text{dimer}}$ using Kolmogorov's extension theorem. Recall that $\mathcal{F}$ denotes the σ-algebra generated by cylinder sets, where a cylinder consists of dimer configurations containing a fixed, finite subset of edges of G^{F}, that is:

$$\mathcal{F} = \sigma(A_{\mathcal{E}} : \mathcal{E} \subset \mathsf{E}^{\mathrm{F}}), \text{ where } A_{\mathcal{E}} = \{M \in \mathcal{M}(\mathsf{G}^{\mathrm{F}}) : \mathcal{E} \subset M\}.$$

To simplify notations, we have identified $A_{\mathcal{E}}$ with $\mathcal{E}$ in the statement of the theorem.

Fix a subset $\mathcal{E}$ of edges of G^{F} and denote by $\mathcal{F}_{\mathcal{E}}$ the σ-algebra generated by the cylinders $(A_{\mathcal{E}'})_{\mathcal{E}'\subset\mathcal{E}}$. Define an additive function $\mathbb{P}_{\mathcal{E}}$ on $\mathcal{F}_{\mathcal{E}}$, by setting its value on every cylinder:

$$\forall\, \mathcal{E}' = \{\mathsf{e}_{i_1}, \cdots, \mathsf{e}_{i_k}\} \subset \mathcal{E}, \quad \mathbb{P}_{\mathcal{E}}(A_{\mathcal{E}'}) = \Big(\prod_{j=1}^{k} K_{\mathsf{x}_{i_j},\mathsf{y}_{i_j}}\Big)\, \mathrm{Pf}\Big([K^{-1}_{\mathcal{E}'}]^t\Big). \tag{2.11}$$

Our goal is to prove that $\mathbb{P}_{\mathcal{E}}$ is a probability measure. This is not clear a priori, because K^{-1} is given by the local formula (2.7) and the graph G^{F} is not assumed to be periodic. Indeed, this implies that we cannot directly use uniqueness of the inverse Kasteleyn matrix in the periodic case, see Remark 2.3, to prove that $\mathbb{P}_{\mathcal{E}}$ is a weak limit of Boltzmann measures. Working directly with the expression of K^{-1} given by (2.7) turned out to be difficult, we could, for instance, not even establish positiveness of $\mathbb{P}_{\mathcal{E}}$.

Nevertheless, using the locality property of the inverse Kasteleyn matrix K^{-1}, a result of [8] and uniqueness in the $\mathbb{Z}^2$-periodic case, we can still prove that $\mathbb{P}_{\mathcal{E}}$ is a weak limit of Boltzmann measures. The argument is the following.

The first fact is that there exists a $\mathbb{Z}^2$-periodic Fisher graph $\mathsf{G}^{\mathrm{F}}_p$ which is identical to G^{F} on a finite, simply connected region containing all edges of $\mathcal{E}$. This is proved using Proposition 1 of [8] stating the following: any finite, simply connected subgraph of a rhombus tiling of the plane can be embedded into a $\mathbb{Z}^2$-periodic rhombus tiling of the plane. To extend this result to the Fisher graph G^{F}, one uses the isoradial graph G underlying the Fisher graph and the rhombus graph $\mathsf{G}^{\diamond}$ corresponding to the isoradial graph G.

Let K_p^{-1} denote the inverse Kasteleyn matrix of the critical dimer model on the periodic Fisher graph $\mathsf{G}^{\mathrm{F}}_p$, given by Theorem 2.3. Then, the second fact is that by the locality property of the inverse Kasteleyn matrix, see also Remark 2.3, $(K_p^{-1})_{\mathsf{x},\mathsf{y}}$ and $K^{-1}_{\mathsf{x},\mathsf{y}}$ are equal for all pairs of vertices of $\mathcal{E}$.

Using uniqueness of the inverse Kasteleyn matrix decreasing at infinity in the $\mathbb{Z}^2$-periodic case, see Remark 2.3, we know that K_p^{-1} is equal to the inverse Kasteleyn matrix computed using Fourier transforms given in Equation (2.1). By Theorem 2.2, we thus know that $\mathbb{P}_{\mathcal{E}}$ is a weak limit of Boltzmann measures, and as such is a probability measure.

Similar arguments are used to prove Kolmogorov's compatibility relations, thus allowing to conclude the proof.

□

Example. The critical Ising model on isoradial graphs satisfies a generalized form of self-duality, see Section 1.5. This means that the critical dimer model on the Fisher graph G^{F} can be obtained, either from the high temperature expansion of a critical Ising model on the graph G (as in Section 1.3), or from the low temperature expansion of a critical Ising model on the dual graph G^*. The advantage of the second approach is that polygon configurations of the low temperature expansion have a direct interpretation as separating clusters of ± 1 spins.

As a consequence, consider a critical Ising model on the dual graph G^*, then the probability that the spins at two neighboring vertices $\mathsf{x}^*, \mathsf{y}^*$ are equal, is equal to the probability of the dual edge $\mathsf{e} = \mathsf{xy}$ being absent in the low temperature expansion polygon configuration. Through Fisher's mapping, this is equal to the probability that the edge e is present in the dimer configuration of G^{F}. This probability can be computed explicitly using Theorem 2.4, and yields:

$$\mathbb{P}_{\mathrm{Ising}}[\sigma_{\mathsf{x}^*} = \sigma_{\mathsf{y}^*}] = \frac{1}{2} + \frac{\pi - 2\theta_{\mathsf{e}}}{2\pi \cos \theta_{\mathsf{e}}},$$

where θ_{e} is the rhombus half-angle corresponding to the edge e. Details of this computation can be found in the appendix of the original paper [4].

2.2.3 Local formula for the free energy

Let us assume that the isoradial graph G is $\mathbb{Z}^2$-periodic. Using Theorem 2.3 and techniques of [Ken02], we obtain an explicit expression for the free energy of the critical dimer model on the Fisher graph G^{F}. Using Fisher's correspondence, this yields a new proof of Baxter's formula for the free energy f_{Ising} of the critical Ising model on the isoradial graph G:

Theorem 2.5. [Bax89] *The free energy of the critical Ising model on the isoradial graph G, is equal to:*

$$f_{\mathrm{Ising}} = -|\mathsf{V}|\frac{\log 2}{2} - \sum_{\mathsf{e}\in\mathsf{E}}\Big[\frac{\theta_{\mathsf{e}}}{\pi}\tan\theta_{\mathsf{e}} + \frac{1}{\pi}\Big(L(\theta_{\mathsf{e}}) + L\Big(\frac{\pi}{2} - \theta_{\mathsf{e}}\Big)\Big)\Big],$$

where L is the Lobachevsky function, $L(x) = -\int_0^x \log|2\sin(t)|\mathrm{d}t$.

Note that, up to a constant, this is $\frac{1}{2}$ of the logarithm of the normalized determinant of the Laplacian operator obtained by Kenyon [Ken02].

Chapter 3

CRSFs and the double critical Ising model

In this chapter, we present two papers with a combinatorial flavor. The first paper [11] focuses on the critical dimer model on the Fisher graph G^{F}, corresponding to a critical Ising model on a periodic isoradial graph G. We establish a combinatorial proof of Theorem 2.1, thus explicitly relating cycle rooted spanning forests (CRSFs) - the pendent of spanning trees when working on the torus - and the double, critical Ising model on the torus. The second paper [10] answers a question raised when working on [11]. The main theorem can actually be rephrased in a more general context: we prove a half-tree theorem for the Pfaffian of a skew-symmetric matrix whose column-sum is zero. This is a Pfaffian version of the classical matrix-tree theorem of Kirchhoff [Kir47].

3.1 From CRSFs to the double critical Ising model: an explicit construction

[11] B. de Tilière. From cycle rooted spanning forests to the critical Ising model: an explicit construction. *Comm. Math. Phys.* **319** (2013), no. 1, 69–110.

Let us first give an idea of the main theorem of [11], and postpone the precise statement until Section 3.1.3. Let G be an infinite, $\mathbb{Z}^2$-periodic, isoradial graph and consider the critical dimer model on the Fisher graph G^{F}, see Section 1.5. Recall the following definitions from Section 2.1.1.

• The *dimer characteristic polynomial*, $P_{\mathrm{dimer}}(z,w)$, is the determinant of the modified weight Kasteleyn matrix $K_1(z,w)$ of the fundamental domain $\mathsf{G}_1^{\mathrm{F}}$. By Theorem 1.4, the dimer partition function of the fundamental domain $\mathsf{G}_1^{\mathrm{F}}$ is a combination of $\mathrm{Pf}\, K_1(\pm,\pm 1)$. Since the determinant of a skew-symmetric matrix is the square of the Pfaffian, one deduces that $P_{\mathrm{dimer}}(z,w)$ is a weighted sum over 'superimpositions of dimer configurations', also known as 'double dimer configurations'[1]. Using Fisher's mapping, this means that $P_{\mathrm{dimer}}(z,w)$ is a weighted sum over configurations of the 'double critical Ising' model on G_1. For larger graphs $\mathsf{G}_n^{\mathrm{F}}$, the partition function is expressed using the characteristic polynomial, see Equation (2.5), so that it is actually stronger to work with the characteristic polynomial rather than partition functions.

[1] When on the torus, there are non-trivial cycles which might have odd length but all other cycles are indeed superimpositions of dimer configurations.

• The *Laplacian characteristic polynomial*, $P_{\mathrm{Lap}}(z,w)$, is the determinant of the modified weight Laplacian matrix $\Delta_1(z,w)$ of the fundamental domain G_1, where we have assigned critical spanning tree weights to edges, see Section 1.5. By a theorem of Forman [For93], $P_{\mathrm{Lap}}(z,w)$ is a weighted sum over cycle rooted spanning forests, defined as follows.

A *cycle-rooted tree* (CRT) of the graph G_1 embedded in the torus $\mathbb{T}^2$, is a connected subgraph of G_1 with a unique non-trivial cycle, where a *non-trivial cycle* is a cycle having non-zero homology in $H_1(\mathbb{T}^2,\mathbb{Z})$. An *oriented* CRT (OCRT) is a CRT in which edges of the branches are oriented towards the non-trivial cycle, and the non-trivial cycle is oriented in one of the two possible directions. Given an OCRT $\mathbf{T}$, denote by $(h(\mathbf{T}),v(\mathbf{T}))$ the homology class of its non-trivial cycle in $H_1(\mathbb{T}^2,\mathbb{Z})\simeq\mathbb{Z}^2$.

A *cycle-rooted spanning forest* (CRSF) is a collection of disjoint cycle-rooted trees covering every vertex of G_1. An *oriented* CRSF (OCRSF) is a CRSF consisting of OCRTs, see Figure 3.1 for an example. Let us denote by $\mathcal{F}(\mathsf{G}_1)$ the set of OCRSFs of G_1. Then, Forman [For93] proves the following version of the matrix-tree theorem for toroidal graphs (which also holds for other weights than the critical ones).

Theorem 3.1. [For93] *The critical Laplacian characteristic polynomial is the following weighted sum:*

$$P_{\mathrm{Lap}}(z,w)=\sum_{\mathbf{F}\in\mathcal{F}(\mathsf{G}_1)}\Big(\prod_{\mathbf{e}\in\mathbf{F}}\tan\theta_{\mathbf{e}}\Big)\prod_{\mathbf{T}\in\mathbf{F}}(1-z^{h(\mathbf{T})}w^{v(\mathbf{T})}). \tag{3.1}$$

Figure 3.1: An example of OCRSF counted by the Laplacian characteristic polynomial, consisting of two OCRTs.

The main result of the paper [11] is a combinatorial proof of Theorem 2.1, stating that the critical dimer and Laplacian characteristic polynomials are equal. This explains, on the level of configurations, the relation between the double critical Ising model on the torus, and cycle rooted spanning forests. Working on planar domains would certainly yield an explicit relation between the double critical Ising model and spanning trees (the pendent of CRSFs when working in planar domains), thus proving an explicit relation between two well known and very classical models of statistical mechanics.

Notations. In the remainder of this section, we use the notations introduced in Section 2.2.1 for vertices of the Fisher graph G^{F}, and the definition of the complex valued functions $(f_{\mathsf{x}}(\lambda))_{\mathsf{x}\in\mathsf{V}^{\mathrm{F}}}$ of Equation (2.6).

3.1.1 Matrix-tree theorem for the dimer characteristic polynomial

We first establish a matrix-tree type theorem for the dimer characteristic polynomial of $\mathsf{G}_1^{\mathrm{F}}$. The proof is not combinatorial, that is, we do not explain on the level of configurations how the dimer characteristic polynomial can be expressed, on the one side as a sum over 'double dimer' configurations, and on the other as a sum over CRSFs of $\mathsf{G}_1^{\mathrm{F}}$. A combinatorial argument is provided in a more general framework, in the paper [10]. This is the subject of Section 3.2.

Let D be the diagonal matrix whose lines and columns are indexed by vertices of $\mathsf{G}_1^{\mathrm{F}}$, such that $D_{\mathsf{x},\mathsf{x}} = f_{\mathsf{x}}$, where f_{x} is the complex valued function $f_{\mathsf{x}}(\lambda)$ of Equation (2.6) evaluated at $\lambda = 0$. Define,

$$K_1^0(z,w) := D^* K_1(z,w) D,$$

and let $P^0_{\mathrm{dimer}}(z,w)$ denote the characteristic polynomial $\det K_1^0(z,w)$. This diagonal transformation of the Kasteleyn matrix only changes the characteristic polynomial by an overall constant. Using Proposition 2.1 at $\lambda = 0$, we prove that $K^0(1,1)$ has the nice property of having column-sum equal to 0. Using the extension of the matrix-tree theorem of Section 1.4.2, we know that any cofactor of $K^0(1,1)$ is a sum over oriented spanning trees. If instead we keep all lines and columns and add parameters z and w, we have a toroidal version of this theorem, in the spirit of the theorem of Forman [For93]. We prove it using the Cauchy-Binet formula.

Theorem 3.2. [11] *The dimer characteristic polynomial $P^0_{\mathrm{dimer}}(z,w)$ can be rewritten as:*

$$P^0_{\mathrm{dimer}}(z,w) = \sum_{\mathsf{F}\in\mathcal{F}(\mathsf{G}_1^{\mathrm{F}})} \Big(\prod_{\mathsf{e}=(\mathsf{x},\mathsf{y})\in\mathsf{F}} f_{\mathsf{x}}\overline{f_{\mathsf{y}}} K_{\mathsf{x},\mathsf{y}} \Big) \prod_{\mathsf{T}\in\mathsf{F}} (1 - z^{h(\mathsf{T})} w^{v(\mathsf{T})}).$$

Figure 3.2: An example of OCRSF counted by the dimer characteristic polynomial.

3.1.2 OCRSFs contributing to the characteristic polynomial

It turns out that the contributions of some OCRSFs to $P^0_{\mathrm{dimer}}(z,w)$ cancel. The next step consists in characterizing the OCRSFs that actually contribute. Denote by F a generic OCRSF of $\mathsf{G}_1^{\mathrm{F}}$. We first look at the restriction of F to long edges, that is to edges of the graph G_1, and establish the following necessary condition:

(1) The restriction $\mathbf{L}(\mathsf{F})$ of the OCRSF F to edges of G_1 must be an oriented edge configuration of G_1 such that there is at least one outgoing edge from every vertex of G_1.

At each decoration, a vertex of type 'v' with an outgoing edge of $\mathbf{L}(\mathsf{F})$ is referred to as a *root vertex*, and a vertex of type 'v' with either an incoming edge of $\mathbf{L}(\mathsf{F})$, or no edge incident is referred to as *non-root* vertex, see Figure 3.3.

Figure 3.3: Example of a restriction to long edges (edges of G_1), root vertices and non-root vertices.

Next, we characterize the restriction to a decoration $\mathbf{x}$ of OCRSFs contributing to $P^0_{\mathrm{dimer}}(z,w)$. There are two symmetric cases *cw* and *cclw*, we only describe the *cw* case here.

(2) If the OCRSF F contributes to $P^0_{\mathrm{dimer}}(z,w)$, then the restriction to every decoration consists of, see also Figure 3.4.

– all edges joining the triangles in a circular way, oriented clockwise,

– one of the three following 2-edge configurations at the triangle of every non-root vertex v_i:

$$\{(\mathsf{w}_i,\mathsf{v}_i),(\mathsf{v}_i,\mathsf{z}_i)\},\{(\mathsf{w}_i,\mathsf{z}_i),(\mathsf{v}_i,\mathsf{z}_i)\},\{(\mathsf{v}_i,\mathsf{w}_i),(\mathsf{w}_i,\mathsf{z}_i)\},$$

– one of the following 1-edge configuration at the triangle of every root vertex v_i:

$$\{(\mathsf{w}_i,\mathsf{v}_i)\},\{(\mathsf{w}_i,\mathsf{z}_i)\},$$

with the additional constraint that the triangle of at least one root vertex contains the configuration $(\mathsf{w}_i,\mathsf{v}_i)$.

Figure 3.4: Restriction to a decoration of an OCRSF that contributes to $P^0_{\mathrm{dimer}}(z,w)$.

A priori, the restriction to a decoration of an OCRSF of $\mathsf{G}_1^{\mathrm{F}}$ can take many other forms. The proof consists in showing that other contributions cancel.

Define an *essential OCRSF* of $\mathsf{G}_1^{\mathrm{F}}$ to be an oriented edge configuration of F of $\mathsf{G}_1^{\mathrm{F}}$ such that the restriction of F to edges of G_1 satisfies Condition (1), the restriction of F to every decoration satisfies Condition (2), and such that F contains no trivial cycle consisting of long edges and short edges. Denote by $\mathcal{F}^0(\mathsf{G}_1^{\mathrm{F}})$ this set of OCRSFs. Then, as a consequence of the above we have:

Corollary 3.3. [11] *The dimer characteristic polynomial can be written as:*

$$P^0_{\mathrm{dimer}}(z,w)=\sum_{\mathsf{F}\in\mathcal{F}^0(\mathsf{G}_1^{\mathrm{F}})}\Big(\prod_{\mathsf{e}=(\mathsf{x},\mathsf{y})\in\mathsf{F}} f_{\mathsf{x}}\overline{f_{\mathsf{y}}}K_{\mathsf{x},\mathsf{y}}\Big)\prod_{\mathsf{T}\in\mathsf{F}}(1-z^{h(\mathsf{T})}w^{v(\mathsf{T})}). \tag{3.2}$$

3.1.3 Main result

We can now give a precise statement of the main theorem of the paper [11].

Theorem 3.4. [11] *Consider a critical Ising model defined on an infinite, $\mathbb{Z}^2$-periodic isoradial graph G. Then, one can explicitly construct essential OCRSFs of $\mathsf{G}_1^{\mathrm{F}}$ counted by the critical dimer characteristic polynomial $P^0_{\mathrm{dimer}}(z,w)$, from OCRSFs of G_1 counted by the critical Laplacian characteristic polynomial $P_{\mathrm{Lap}}(z,w)$.*

Sketch of proof. Both P^0_{Lap} and P^0_{dimer} are weighted sums over OCRSFs of a graph embedded in the torus, see Equations (3.1) and (3.2). We first need a rewriting of these polynomials as sums over pairs of primal and dual OCRSFs.

Rewriting of the characteristic polynomials. It is a general fact that if F is a CRSF, then the complementary configuration consisting exactly of the dual edges of the edges absent in F, is a CRSF of the dual graph, with non-trivial cycles parallel to those of F, such that primal and dual non-trivial cycles alternate along the torus, see Figure 3.5. It is referred to as the *dual CRSF of* F.

Figure 3.5: Primal and dual CRSFs of a graph embedded in the torus.

OCRSFs are said to be dual to each other if their unoriented versions are. Define the *homology class* of an OCRSF to be the sum of the homology classes of its OCRT components. Then, we prove a rewriting of polynomials of the form (3.1) and (3.2), using pairs of primal and dual OCRSFs. Denote by $\mathcal{F}(\mathsf{G}_1, \mathsf{G}_1^*)$ the set of pairs of dual OCRSFs of G_1 and G_1^*. Similarly, denote by $\mathcal{F}^0(\mathsf{G}_1^{\mathrm{F}}, \mathsf{G}_1^{\mathrm{F}^*})$ the set of pairs of dual OCRSFs, such that the primal is an essential OCRSF of $\mathsf{G}_1^{\mathrm{F}}$.

Lemma 3.1. *The critical Laplacian and dimer characteristic polynomials can be rewritten as:*

$$P_{\mathrm{Lap}}(z,w) = \sum_{(\mathbf{F},\mathbf{F}^*)\in\mathcal{F}(\mathsf{G}_1,\mathsf{G}_1^*)} \left(\prod_{\mathbf{e}=\mathbf{xy}\in\mathbf{F}} \tan\theta_{\mathbf{xy}}\right) (-z^{\frac{h(\mathbf{F})+h(\mathbf{F}^*)}{2}} w^{\frac{v(\mathbf{F})+v(\mathbf{F}^*)}{2}})$$

$$P^0_{\mathrm{dimer}}(z,w) = \sum_{(\mathsf{F},\mathsf{F}^*)\in\mathcal{F}^0(\mathsf{G}_1^{\mathrm{F}},\mathsf{G}_1^{\mathrm{F}^*})} \left(\prod_{\mathsf{e}=(\mathsf{x},\mathsf{y})\in\mathsf{F}} f_{\mathsf{x}}\overline{f_{\mathsf{y}}} K_{\mathsf{x},\mathsf{y}}\right) (-z^{\frac{h(\mathsf{F})+h(\mathsf{F}^*)}{2}} w^{\frac{v(\mathsf{F})+v(\mathsf{F}^*)}{2}}).$$

Let us now describe the mapping, constructing pairs of OCRSFs of $\mathcal{F}^0(\mathsf{G}_1^{\mathrm{F}}, \mathsf{G}_1^{\mathrm{F}^*})$ from pairs of OCRSFs of $\mathcal{F}(\mathsf{G}_1, \mathsf{G}_1^*)$.

Idea of the mapping. The mapping consists in assigning to every pair $(\mathbf{F}, \mathbf{F}^*)$ of dual OCRSFs of G_1 and G_1^* a family $\mathcal{S}(\mathbf{F}, \mathbf{F}^*)$ of dual essential OCRSFs of $\mathsf{G}_1^{\mathrm{F}}$ and $\mathsf{G}_1^{\mathrm{F}^*}$, such that:

1. $\bigcup_{(\mathbf{F},\mathbf{F}^*)\in\mathcal{F}(\mathsf{G}_1,\mathsf{G}_1^*)} \mathcal{S}(\mathbf{F},\mathbf{F}^*) = \mathcal{F}^0(\mathsf{G}_1^{\mathrm{F}}, \mathsf{G}_1^{\mathrm{F}^*})$.
2. When $(\mathbf{F}_1, \mathbf{F}_1^*) \neq (\mathbf{F}_2, \mathbf{F}_2^*)$, then $\mathcal{S}(\mathbf{F}_1, \mathbf{F}_1^*) \cap \mathcal{S}(\mathbf{F}_2, \mathbf{F}_2^*) = \emptyset$.
3. For every $(\mathsf{F}, \mathsf{F}^*) \in \mathcal{S}(\mathbf{F}, \mathbf{F}^*)$, we have:

$$\Big(\frac{h(\mathbf{F})+h(\mathbf{F}^*)}{2}, \frac{v(\mathbf{F})+v(\mathbf{F}^*)}{2}\Big) = \Big(\frac{h(\mathsf{F})+h(\mathsf{F}^*)}{2}, \frac{v(\mathsf{F})+v(\mathsf{F}^*)}{2}\Big)$$

Construction of $\mathcal{S}(\mathbf{F}, \mathbf{F}^*)$. The family $\mathcal{S}(\mathbf{F}, \mathbf{F}^*)$ will be very large, indeed the decorated graph $\mathsf{G}_1^{\mathrm{F}}$ contains many more OCRSFs than the graph G_1. The idea of the construction is to use a inductive procedure, allowing to guarantee that, at each step, we are indeed constructing pairs of OCRSFs and that these satisfy Point 3. The induction is on the number of long edges present. More precisely, let $\mathbf{e}_1, \cdots, \mathbf{e}_m$ be a labeling of the unoriented edges of $\mathsf{E}_1 \setminus \mathbf{F}$. For $k \in \{0, \cdots, m\}$, let $J_k = \{(i_1, \cdots, i_k) \in \{1, \cdots, m\}^k \,|\, 1 \leq i_1 < \cdots < i_k \leq m\}$, with the convention that $J_k = \emptyset$, when $k = 0$. Then,

$$\mathcal{S}(\mathbf{F}, \mathbf{F}^*) = \bigcup_{k=0}^{m} \bigcup_{(i_1,\cdots,i_k)\in J_k} \mathcal{F}^{(\mathbf{F},\mathbf{F}^*),\{\mathbf{e}_{i_1},\cdots,\mathbf{e}_{i_k}\}}(\mathsf{G}_1^{\mathrm{F}}, \mathsf{G}_1^{\mathrm{F}^*}),$$

where $\mathcal{F}^{(\mathbf{F},\mathbf{F}^*),\{\mathbf{e}_{i_1},\cdots,\mathbf{e}_{i_k}\}}(\mathsf{G}_1^{\mathrm{F}}, \mathsf{G}_1^{\mathrm{F}^*})$ is constructed by induction on k. It consists of pairs of dual essential OCRSFs of $\mathsf{G}_1^{\mathrm{F}}$ and $\mathsf{G}_1^{\mathrm{F}^*}$ such that the primal contains exactly the oriented long edges $\mathbf{F}$ and the long edges $\mathbf{e}_{i_1}, \cdots, \mathbf{e}_{i_k}$ in one of the two possible directions.

- Initial step of the induction. Primal OCRSFs of $\mathcal{F}^{(\mathbf{F},\mathbf{F}^*),\emptyset}(\mathsf{G}_1^{\mathrm{F}},\mathsf{G}_1^{\mathrm{F}^*})$ are required to contain exactly the oriented long edges $\mathbf{F}$, and to have the restriction to every decoration satisfy Condition (2) of Section 3.1.2. It is not so difficult to prove that the pairs of oriented edge configurations obtained in this way are indeed OCRSFs and satisfy Point 3.
- Induction step. The idea is to construct pairs of dual OCRSFs of $\mathcal{F}^{(\mathbf{F},\mathbf{F}^*),\{\mathbf{e}_{i_1},\cdots,\mathbf{e}_{i_j};\cdot\}}(\mathsf{G}_1^{\mathrm{F}},\mathsf{G}_1^{\mathrm{F}^*})$ from pairs of dual OCRSFs of $\mathcal{F}^{(\mathbf{F},\mathbf{F}^*),\{\mathbf{e}_{i_1},\cdots,\mathbf{e}_{i_{j-1}};\cdot\}}(\mathsf{G}_1^{\mathrm{F}},\mathsf{G}_1^{\mathrm{F}^*})$, by adding the long edge $\mathbf{e}_{i_j}$ to the primal OCRSFs. Simply adding an edge to an OCRSF does *not* yield an OCRSF, one also needs to remove an edge. There are quite a few subtleties in doing this procedure, and we do not want to enter into too many details here. The point is that we define *licit primal/dual edge moves* that have to be performed on the primal and the dual OCRSFs *simultaneously*. These moves guarantee that a long edge is added, and that the resulting pair of oriented edge configuration is indeed a pair of OCRSFs, satisfying Point 3. Note that performing such moves on the primal OCRSF only does not guarantee Point 3, so that writing characteristic polynomials as a sum over pairs of dual OCRSFs as in Lemma 3.1 is really a key requirement.

Conclusion of the proof. Once the construction of $\mathcal{S}(\mathbf{F},\mathbf{F}^*)$ is done, we consider

$$\bigcup_{(\mathbf{F},\mathbf{F}^*)\in\mathcal{F}(\mathsf{G}_1,\mathsf{G}_1^*)} \mathcal{S}(\mathbf{F},\mathbf{F}^*),$$

and need to prove Points 1 and 2, *i.e.* we need to show that we have constructed all pairs of OCRSFs of $\mathcal{F}^0(\mathsf{G}_1^{\mathrm{F}},\mathsf{G}_1^{\mathrm{F}^*})$, and that we have constructed each pair only once. This still requires a lot of work. Given a pair $(\mathsf{F},\mathsf{F}^*)$ of dual OCRSFs of $\mathcal{F}^0(\mathsf{G}_1^{\mathrm{F}},\mathsf{G}_1^{\mathrm{F}^*})$, we need to define a reverse procedure, allowing to recover from which pair $(\mathbf{F},\mathbf{F}^*)$ they arise. We refer to the original paper [11] for the full proof. □

3.1.4 Weighted mapping

The mapping presented in Section 3.1.3 is between pairs of OCRSFs of G_1 and G_1^* and pairs of OCRSFs of $\mathsf{G}_1^{\mathrm{F}}$ and $\mathsf{G}_1^{\mathrm{F}^*}$. Keeping track of the weights along the different steps yields the following.

Theorem 3.5. [11] *Let $(\mathbf{F},\mathbf{F}^*)$ be a pair of dual OCRSFs of G_1 and G_1^*. Then,*

$$\sum_{(\mathsf{F},\mathsf{F}^*)\in\mathcal{S}(\mathbf{F},\mathbf{F}^*)} \Big(\prod_{(\mathsf{x},\mathsf{y})\in\mathsf{F}} f_{\mathsf{x}}\overline{f_{\mathsf{y}}}K_{\mathsf{x},\mathsf{y}}\Big)(-z^{\frac{h(\mathbf{F})+h(\mathbf{F}^*)}{2}}w^{\frac{v(\mathbf{F})+v(\mathbf{F}^*)}{2}}) =$$

$$= C\left(\prod_{(\mathbf{x},\mathbf{y})\in\mathbf{F}}\tan\theta_{\mathbf{xy}}\right)(-z^{\frac{h(\mathsf{F})+h(\mathsf{F}^*)}{2}}w^{\frac{v(\mathsf{F})+v(\mathsf{F}^*)}{2}}),$$

where $C = 2^{4|\mathsf{E}_1|+|\mathsf{V}_1|}\prod_{\mathbf{xy}\in\mathsf{E}_1}\sin^2\big(\frac{\theta_{\mathbf{xy}}}{2}\big)\cos\theta_{\mathbf{xy}}$.

As a consequence we recover, by an explicit computation, Theorem 2.1. Note that the constant could not be explicited in Theorem 2.1, but could only be recovered a posteriori.

Corollary 3.6. [3, 4]

$$P_{\mathrm{dimer}}(z,w) = \Big(2^{|\mathsf{V}_1|}\prod_{\mathbf{xy}\in\mathsf{E}_1}[\cot^2\Big(\frac{\theta_{\mathbf{xy}}}{2}\Big)-1]\Big)P_{\mathrm{Lap}}(z,w).$$

3.2 Pfaffian half-tree theorem

[10] B. de Tilière. Principal Minors Pfaffian Half-Tree Theorem. *Arxiv: 1207.2759*

In the paper [10], we prove a half-tree theorem for the Pfaffian principal minors of a skew-symmetric matrix whose column-sum is zero. This is a Pfaffian version of the classical matrix-tree theorem of Kirchoff [Kir47].

3.2.1 Setting

The idea of proving such a theorem comes from a question asked by David Wilson when talking about the matrix-tree theorem for the Kasteleyn matrix, Theorem 3.2. On the one side, the dimer characteristic polynomial $\det K_1(z,w)$ is a sum over weighted 'superimpositions of dimer configurations', consisting of disjoint unions of alternating cycles, see Section 1.2.4. On the other side, the matrix-tree theorem for the Kasteleyn matrix states that the dimer characteristic polynomial $\det K_1(z,w)$ is a sum over weighted OCRSFs, which are the pendent of spanning trees when working on the torus. The question is: how are these configurations constructed from one another ? From the work of Temperley [Tem74], and Kenyon, Propp, Wilson [KPW00] it is known that spanning trees are related to dimer configurations of a related graph, but how are double-dimer configurations and spanning trees of the same graph related ?

We looked at the many available proofs of the matrix-tree theorem, in particular the combinatorial one by Chaiken [Cha82], but none explained how to construct spanning trees from double-dimer configurations. The paper [10] provides an answer. It turns out that:

- The matrix does not have to be a Kasteleyn matrix, *i.e.* a weighted oriented adjacency matrix of a planar graph or a graph embedded on the torus. It suffices that it is a skew-symmetric matrix.
- A key requirement is that the column-sum is equal to 0, as in the statement of Theorem 1.5. This is indeed the case for the matrix $K_1(0,0)$, see Section 3.1.1, a fact related to the model being critical.
- Not only can spanning trees be constructed from weighted double perfect matchings counted by the determinant, but actually half-spanning trees can be constructed from weighted perfect matchings counted by the Pfaffian. This is why we have called the main theorem of the paper [10] the Pfaffian half-tree theorem.

3.2.2 Statement

The main theorem of [10] involves principal minors and half spanning-forests. We only state the half-tree version here. Let $\mathsf{V} = \{1, \cdots, n, \mathsf{r}\}$, where n is even, and r denotes a specific vertex referred to as the *root*. Let $A = (a_{i,j})$ be a skew-symmetric matrix of size $(n+1) \times (n+1)$, whose column-sum is zero. Consider the graph $\mathsf{G} = (\mathsf{V}, \mathsf{E})$, with edges corresponding to non-zero coefficients of the matrix A, and such that every oriented edge (i,j) is assigned the weight $a_{i,j}$. Oriented edges are thus assigned a skew-symmetric weight function. Consider also the graph $\mathsf{G}^{\mathsf{r}} = (\mathsf{V}^{\mathsf{r}}, \mathsf{E}^{\mathsf{r}})$ obtained from G by removing the vertex r and edges connected to it.

For the remainder of this section, a spanning tree will actually mean an oriented spanning tree, oriented towards r. We now define spanning trees compatible with perfect matchings, and half-spanning trees: fix a perfect matching M_0 of G^r, then a spanning tree T of G is said to be *compatible* with M_0 if it consists of the $n/2$ edges of M_0 and $n/2$ edges of $\mathsf{E} \setminus \mathsf{M}_0$. The oriented edge configuration $\mathsf{T} \setminus \mathsf{M}_0$ is referred to as a *half-spanning tree*.

Example. Let $\mathsf{V} = \{1, 2, 3, 4, \mathsf{r}\}$. Consider the graph G of Figure 3.6 below, and the graph G^r obtained by removing the vertex r and edges connected to it. A choice of perfect matching M_0 of G^r is pictured in white. $\mathsf{T}_1, \mathsf{T}_2, \mathsf{T}_3$ are examples of spanning trees compatible with M_0. Black edges are half-spanning trees.

Figure 3.6: Spanning trees compatible with M_0 and half-spanning trees (black edges).

Theorem 3.7. [10][Pfaffian half-tree theorem] *Let A be a skew-symmetric matrix of size $(n+1) \times (n+1)$, whose column-sum is zero, such that n is even; and let A^r be the matrix obtained from A by removing the last line and column. Let G and G^r be the graphs naturally constructed from the matrices A and A^r respectively. Then, for every perfect matching M_0 of G^r, the Pfaffian of A^r is equal to:*

$$\mathrm{Pf}(A^\mathsf{r}) = \sum_{\mathsf{T} \in \mathcal{T}^\mathsf{r}_{\mathsf{M}_0}(\mathsf{G})} \mathrm{sgn}(\sigma_{\mathsf{M}_0(\mathsf{T} \setminus \mathsf{M}_0)}) \prod_{\mathsf{e} \in \mathsf{T} \setminus \mathsf{M}_0} a_\mathsf{e},$$

where a_e is the coefficient of the matrix A corresponding to the oriented edge e; $\mathrm{sgn}(\sigma_{\mathsf{M}_0(\mathsf{T} \setminus \mathsf{M}_0)})$ is the signature of a permutation naturally constructed from T and M_0[2]; $\mathcal{T}^\mathsf{r}_{\mathsf{M}_0}(\mathsf{G})$ is the set of spanning trees of G compatible with M_0, satisfying Condition (C) *of Definition 3.1 below.*

Here is the algorithm used to characterize half-spanning trees contributing to $\mathrm{Pf}(A^\mathsf{r})$.

Trimming algorithm

Input: a spanning tree T of G compatible with M_0.

Initialization: $\mathsf{T}_1 = \mathsf{T}$.

Step i, $i \geq 1$

Let ℓ_1^i be the largest leaf of T_i and consider the connected component containing ℓ_1^i. Start from ℓ_1^i along the unique path joining ℓ_1^i to the root, until the first time one of the following vertices is reached:

- the root vertex r,
- a fork, that is a vertex with more than one incoming edge,
- a vertex which is smaller that the leaf ℓ_1^i.

This yields a loopless path $\lambda_{\ell_1^i}$ starting from ℓ_1^i, of length ≥ 1. Let $\mathsf{T}_{i+1} = \mathsf{T}_i \setminus \lambda_{\ell_1^i}$. If T_{i+1} is empty, stop; else go to Step $i+1$.

End: since edges are removed at every step, and since T contains a finite number of edges, the trimming algorithm ends in finite time N.

[2]Refer to the paper [10] for the precise definition

Definition 3.1. A spanning tree T compatible with M_0 is said to satisfy *Condition* (C) if each of the paths $\lambda_{\ell_1^1}, \cdots, \lambda_{\ell_1^N}$ obtained from the trimming algorithm, starts from an edge of M_0 and has even length. We let $\mathcal{T}^{\mathsf{r}}_{\mathsf{M}_0}(\mathsf{G})$ denote the set of spanning trees compatible with M_0, satisfying Condition (C).

Example. Applying the trimming algorithm to each of the spanning trees $\mathsf{T}_1, \mathsf{T}_2, \mathsf{T}_3$ of Figure 3.6 yields:

$$\begin{aligned}
\mathsf{T}_1: &\quad \text{Step 1: } \ell_1^1 = 2, \lambda_2 = 2,3,1. \quad \text{Step 2: } \ell_1^2 = 1, \lambda_1 = 1,4,\mathsf{r}.\\
\mathsf{T}_2: &\quad \text{Step 1: } \ell_1^1 = 2, \lambda_2 = 2,3,\mathsf{r}. \quad \text{Step 2: } \ell_1^2 = 1, \lambda_1 = 1,4,\mathsf{r}.\\
\mathsf{T}_3: &\quad \text{Step 1: } \ell_1^1 = 4, \lambda_4 = 4,1. \quad\quad \text{Step 2: } \ell_1^2 = 1, \lambda_1 = 1,2,3,\mathsf{r}.
\end{aligned}$$

The spanning trees T_1 and T_2 satisfy Condition (C) but not T_3.

Remark 3.1. In the paper [MV02], Masbaum and Vaintrob assign to a 3-uniform hypergraph a specific skew-symmetric matrix whose column sum is zero, and prove that the Pfaffian of any principal minor of this matrix enumerates signed spanning trees of the 3-uniform graph. A combinatorial proof of their result is then given by Hirschman and Reiner [HR04], and yet another proof using Grassman variables is provided by Abdesselam [Abd04]. The matrix considered by Masbaum and Vaintrob satisfies the assumptions of Theorem 3.7, and it would interesting to investigate whether there is an explicit relation between half-spanning trees of Theorem 3.7 and spanning trees of the 3-uniform graph.

Using the fact that the determinant of a skew-symmetric matrix is the square of the Pfaffian, we obtain the following corollary.

Corollary 3.8. [10] *Let A be a skew-symmetric matrix of size $(n+1)\times(n+1)$, whose column sum is zero, such that n is even; and let A^{r} be the matrix obtained from A by removing the last line and column. Let G and G^{r} be the graphs naturally constructed from the matrices A and A^{r} respectively. Then, the determinant of the matrix A^{r} is equal to:*

$$\det(A^{\mathsf{r}}) = \sum_{\mathsf{M}_0 \in \mathcal{M}(\mathsf{G}^{\mathsf{r}})} \sum_{\mathsf{T} \in \mathcal{T}^{\mathsf{r}}_{\mathsf{M}_0}(\mathsf{G}^{\mathsf{r}})} \prod_{\mathsf{e} \in \mathsf{T}} a_{\mathsf{e}}, \tag{3.3}$$

where a_{e} is the coefficient of the matrix A corresponding to the oriented edge e, and $\mathcal{T}^{\mathsf{r}}_{\mathsf{M}_0}(\mathsf{G})$ is the set of spanning forests compatible with M_0, satisfying Condition (C).

Remark 3.2.

- From Theorem 1.5, which holds for all matrices whose column-sum is zero, we know that:

$$\det(A^{\mathsf{r}}) = \sum_{\mathsf{T} \in \mathcal{T}^{\mathsf{r}}(\mathsf{G})} \prod_{\mathsf{e} \in \mathsf{T}} a_{\mathsf{e}}.$$

  When the matrix A is moreover skew-symmetric, we also have the equality given by Equation (3.3). This is a refined version, because we prove in [10] that the following union is disjoint, and the following inclusion is strict,

$$\bigcup_{\mathsf{M}_0 \in \mathcal{M}(\mathsf{G}^{\mathsf{r}})} \mathcal{T}^{\mathsf{r}}_{\mathsf{M}_0}(\mathsf{G}) \subset \mathcal{T}^{\mathsf{r}}(\mathsf{G}).$$

We also give an intrinsic definition of spanning trees of the union, without using reference perfect matchings. This means that when the matrix A is skew-symmetric, specific cancellations occur between spanning trees, a fact hard to characterize without using Corollary 3.8.

- We also prove a line bundle version of this result, in the spirit of [For93] and [Ken11]. In particular this allows to have a version of the theorem for the modified weight case $\det K_1(z, w)$.

3.2.3 Sketch of the proof of the Pfaffian half-tree theorem

The proof uses an explicit algorithm. In order to describe it, we need a few facts and definitions. First, recall from Remark 1.2, that the Pfaffian of the matrix A^{r} can be written as a sum over perfect matchings of the graph G^{r}:

$$\mathrm{Pf}(A^{\mathsf{r}}) = \sum_{\mathsf{M}\in\mathcal{M}(\mathsf{G}^{\mathsf{r}})} \mathrm{sgn}(\sigma_{\mathsf{M}}) a_{\sigma_{\mathsf{M}}(1)\sigma_{\mathsf{M}}(2)} \cdots a_{\sigma_{\mathsf{M}}(n-1)\sigma_{\mathsf{M}}(n)},$$

where σ_{M} is a permutation describing the perfect matching. Next, recall from Section 1.2.4 that the superimposition $\mathsf{M}_0 \cup \mathsf{M}$ of a fixed reference matching M_0 and of a generic perfect matching M of G^{r}, is a collection of disjoint alternating cycles covering all vertices of the graph G^{r}. As a consequence, the sum in the Pfaffian can graphically be interpreted as a sum over such superimpositions, see Figure 3.7 for an example.

Figure 3.7: The sum over perfect matchings of G^{r} can graphically be interpreted as a sum over collections of disjoint alternating cycles of the graph G^{r}.

The tool of the algorithm is to use the reference configuration M_0 as a skeleton for 'opening' doubled edges of the superimposition $\mathsf{M}_0 \cup \mathsf{M}$, as in Figure 3.8. Indeed, because of the condition $\sum_{j=1}^{n+1} a_{i,j} = 0$, configurations of Figure 3.8 have opposite weights, so that up to a global sign change, this 'opening' procedure is weight preserving. In this way, one obtains a new set of edge configurations of the graph G.

Figure 3.8: 'Opening' of doubled edges procedure: $a_{i',i} = -\sum_{j\sim i,\, j\neq i'} a_{i',j}$.

Our goal is to do the 'opening' procedure in such a way that part of the new edge configurations are spanning trees, and all the others have contributions that cancel. We face two difficulties: the first is that, a priori, there is no natural way of deciding whether to 'open' the doubled edge at the vertex i or at the vertex i'. The second is that we need to keep track of edge configurations constructed, and to be able to characterize them. It turns out that the 'opening' procedure depends strongly on the labeling of vertices.

Edge configurations that we shall construct from the 'opening' procedure are *half-RC-rooted spanning forests*, defined as follows. An *RC-rooted spanning forest*, referred to as an RC-RSF, is

an oriented edge configuration of G, spanning vertices of G^{r}, such that each connected component is, either a tree rooted on the root-vertex r, or a tree rooted on a cycle of length ≥ 3 of G^{r}. Edges of each of the components are oriented towards its root, and edges of the cycles are oriented in one of the two possible directions. An RC-RSF F is said to be compatible with M_0, it is consists of the $n/2$ edges of M_0 and $n/2$ edges of $\mathsf{E} \setminus \mathsf{M}_0$. The oriented edge $\mathsf{F} \setminus \mathsf{M}_0$ is referred to as a *half-RC-RSF*.

We now describe the algorithm. It is performed separately for each perfect matching M of the graph G^{r}. Since it is notationally cumbersome to rigorously write it, we only state the main ideas, and show its application to the explicit example M_1 of Figure 3.7.

Algorithm

Let M be a perfect matching of G^{r}. Consider the superimposition $\mathsf{M}_0 \cup \mathsf{M}$ and do the following steps.

Step 1

- If the superimposition $\mathsf{M}_0 \cup \mathsf{M}$ consists of cycles of length ≥ 4 only: stop.

  Example. This is not the case for $\mathsf{M}_0 \cup \mathsf{M}_1$, so we go to the next step.

- If the superimposition $\mathsf{M}_0 \cup \mathsf{M}$ contains doubled edges, consider the one with the smallest index. Do the 'opening' procedure, starting from the vertex with the smallest index, yielding a new set of configurations.

  Example. The superimposition $\mathsf{M}_0 \cup \mathsf{M}_1$ contains doubled edges. The one with the smallest index is the doubled edge 14*. Performing the 'opening' procedure yields three new configurations $\mathsf{M}_{1,2}$, $\mathsf{M}_{1,3}$, $\mathsf{M}_{1,\mathsf{r}}$.*

Figure 3.9: *Black edges consist of the new configurations $\mathsf{M}_{1,2}$, $\mathsf{M}_{1,3}$, $\mathsf{M}_{1,\mathsf{r}}$ obtained from the 'opening' procedure.*

- For each new configuration, consider the terminal vertex of the 'opening' procedure:
  - It it does not belong to a doubled edge: stop. Note that the algorithm stops when either the root is reached or a cycle is created.

    Example. Step 1 of the algorithm stops for $\mathsf{M}_{1,\mathsf{r}}$, since the root is reached.
  - If it belongs to a doubled edge, iterate the 'opening' procedure,

    Example. The terminal vertex of the configurations $\mathsf{M}_{1,2}$ and $\mathsf{M}_{1,3}$ belong to a doubled edge. For each of them, we iterate the 'opening' procedure, yielding new configurations $\mathsf{M}_{1,2,1}$, $\mathsf{M}_{1,2,4}$, $\mathsf{M}_{1,2,\mathsf{r}}$ and $\mathsf{M}_{1,3,1}$, $\mathsf{M}_{1,3,4}$ respectively, see Figure 3.10.

Figure 3.10: First line, from left to right, black edges consist of the configurations $\mathsf{M}_{1,2,1}$, $\mathsf{M}_{1,2,4}$, $\mathsf{M}_{1,2,\mathsf{r}}$. Second line, from left to right, black edges consist of the configurations $\mathsf{M}_{1,3,1}$, $\mathsf{M}_{1,3,4}$.

  - Iterate this step of the algorithm with the new configurations.

    Example. For each new configuration $\mathsf{M}_{1,2,1}$, $\mathsf{M}_{1,2,4}$, $\mathsf{M}_{1,2,\mathsf{r}}$, $\mathsf{M}_{1,3,1}$, $\mathsf{M}_{1,3,4}$, *the terminal vertex does not belong to a doubled edge, so the algorithm stops.*

- The output of Step 1 is the set of configurations obtained each time the algorithm stopped.

  Example. The output of Step 1 is $\mathsf{M}_{1,\mathsf{r}}, \mathsf{M}_{1,2,1}$, $\mathsf{M}_{1,2,4}$, $\mathsf{M}_{1,2,\mathsf{r}}$, $\mathsf{M}_{1,3,1}$, $\mathsf{M}_{1,3,4}$.

Following steps

Step 1 is iterated until no doubled edge remains.

Example. From the output of Step 1, only the configuration $\mathsf{M}_{1,\mathsf{r}}$ *contains a doubled edge. We look for the doubled edge with the smallest index. Since there is only one doubled edge* 23*, we perform the 'opening' procedure starting from the vertex with the smallest index, that is the vertex 2. This yields a new set of configurations* $\mathsf{M}_{1,\mathsf{r};2,1}, \mathsf{M}_{1,\mathsf{r};2,4}, \mathsf{M}_{1,\mathsf{r};2,\mathsf{r}}$. *All the terminal vertices of the 'opening' procedure do not belong to a doubled edge, so the algorithm stops.*

Figure 3.11: Output of Step 2 of the algorithm.

The output of the full algorithm consists in the output of the different steps, applied to all perfect matchings of the graph G^{r}. Note that when we apply the algorithm to the superimposition $\mathsf{M}_0 \cup \mathsf{M}_2$ and $\mathsf{M}_0 \cup \mathsf{M}_3$, it stops right from the beginning because these configurations contain no doubled edges.

Example. Figure 3.12 below summarizes the output of the algorithm. Observe that half-RC-RSFs are obtained, and that half-RC-RSFs which are not half-spanning trees come into pairs.

Figure 3.12: Output of the full algorithm

Conclusion. In the example, half RC-RSFs which are not half- trees come into pairs. One shows that this is always the case, and that their contributions to $\mathrm{Pf}(A^{\mathsf{r}})$ have opposite signs, thus cancel. What then remains is the contributions of half-spanning trees. The fact that pairs of half-RC-RSFs have opposite signs is established by keeping track of the signs when the 'opening' procedure is performed and of the signatures in the Pfaffian. In order to show that half-RC-RSFs come into pairs, we need to carefully analyze the algorithm and characterize the configurations obtained. Although it is quite clear that half-RC-RSFs are obtained, they are special in that they have specific conditions on the labeling of their vertices. The characterization is done by introducing a reverse algorithm. Specified to the case of half-spanning trees, the reverse algorithm is exactly the *trimming algorithm* used to characterize half-spanning trees involved in the expansion of the Pfaffian of Theorem 3.7.

Chapter 4

Loops and bipartite dimer models

In this chapter, we present two papers in collaboration with Cédric Boutillier focusing on loop configurations related to bipartite dimer models. The first paper [2] considers the loop representation of the uniform dimer model on a subgraph of the honeycomb lattice embedded in the torus. We prove that, when the mesh of the graph tends to 0 and the aspect ratio is fixed, the winding number of the loops converge to a two-dimensional discrete Gaussian random variable. The second paper [5] relates XOR-loop configurations, constructed from two independent Ising models living on the same graph, to loop configurations of a dimer model defined on a decorated, bipartite version of the graph. Using results of [9], this allows to shed a light on Wilson's conjecture [Wil11] stating that, at criticality and in the scaling limit, these loops are level lines of the Gaussian free field.

4.1 Winding of loops in the toroidal honeycomb dimer model

[2] C. Boutillier, B. de Tilière. Loops statistics in the honeycomb dimer model. *Ann. Probab.* **37** (2009), no. 5, 1747–1777.

Let G denote the honeycomb lattice embedded in the plane so that all faces are regular, side-length 1 hexagons. The lattice and its bipartite coloring are invariant under translations by the vectors e_1, e_2 of Figure 4.1. We let $\mathbb{L}_{m,n}$ be the lattice generated by $m\mathsf{e}_1$ and $n\mathsf{e}_1$, and $\mathsf{G}_{m,n}$ be the toroidal graph obtained by quotienting the honeycomb lattice by $\mathbb{L}_{m,n}$.

Figure 4.1: Left: the honeycomb lattice G and its bipartite coloring are invariant under translations by e_1 and e_2. Right: the graph $\mathsf{G}_{2,3}$ (opposite sides are identified).

The setting of the paper [2] is the loop representation of the uniform dimer model on the toroidal graph $\mathsf{G}_{m,n}$, see Section 1.2.4. Let M_0 be the fixed periodic reference dimer configuration of Figure 4.2. For a generic dimer configuration M of $\mathsf{G}_{m,n}$, consider the oriented superimposition $\mathsf{M}-\mathsf{M}_0$, consisting of a collection of disjoint, oriented alternating loops of $\mathsf{G}_{m,n}$. An example for $\mathsf{G}_{2,3}$ is provided in Figure 4.2, an example for $\mathsf{G}_{6,6}$ is given in Figure 1.2.

Figure 4.2: Oriented superimposition $\mathsf{M}-\mathsf{M}_0$ of $\mathsf{G}_{2,3}$. The winding number of M is, $\mathrm{wind}_{2,3}(\mathsf{M}) = (0,-1)$.

The *winding number* of the dimer configuration M counts the winding of this collection in the two non-trivial directions of the torus. More precisely, it is defined as follows. Consider the torus $\mathbb{T}^2_{m,n} = \mathbb{C}/\mathbb{L}_{m,n}$, and let $H_1(\mathbb{T}^2_{m,n},\mathbb{Z}) \cong \mathbb{Z}^2$ denote its first homology group in $\mathbb{Z}$. The graph $\mathsf{G}_{m,n}$ being embedded in the torus $\mathbb{T}^2_{m,n}$, we choose a representative of a basis of $H_1(\mathbb{T}^2_{m,n},\mathbb{Z})$ using this embedding: we take as first (respectively second) basis vector $m\mathsf{e}_1$ (respectively $n\mathsf{e}_2$), embedded in the torus. Then, the *winding number of* M, denoted by $\mathrm{wind}_{m,n}(\mathsf{M})$ is the homology class of the oriented superimposition $\mathsf{M}-\mathsf{M}_0$ expressed in this basis. For example, the winding number of the dimer configuration of Figure 4.2 is $\mathrm{wind}_{2,3}(\mathsf{M}) = (0,-1)$. The *winding number* $\mathrm{wind}_{m,n}(\cdot)$ is a $\mathbb{Z}^2$-valued random variable defined on dimer configurations of $\mathsf{G}_{m,n}$. Note that changing the reference dimer configuration changes the winding number by an overall constant, so that it is not restrictive to fix M_0 and to remove this dependence from the notations.

Prior to citing our main result, we need one more definition. The *modulus of the torus* $\mathbb{T}^2_{m,n}$ is the ratio of the two vectors $n\mathsf{e}_2$ and $m\mathsf{e}_1$ (seen as complex numbers) generating the lattice $\mathbb{L}_{m,n}$, *i.e.* the modulus of $\mathbb{T}^2_{m,n}$ is equal to $\frac{n\mathsf{e}_2}{m\mathsf{e}_1} = \frac{i\sqrt{3}n}{3m} = i\frac{n}{\sqrt{3}m}$.

Theorem 4.1. [2] *Suppose that in the joint limit $m,n\to\infty$, the modulus $i\frac{n}{\sqrt{3}m}$ of the torus $\mathbb{T}_{m,n}$ converges to $i\rho$ for some $\rho>0$. Then, the sequence of random variables* $(\mathrm{wind}_{m,n})$ *converges in law to the two-dimensional discrete Gaussian random variable* wind_ρ*, whose law is given by:*

$$\forall\,(k,\ell)\in\mathbb{Z}^2,\quad \mathbb{P}[\mathrm{wind}_\rho=(k,\ell)] = \frac{1}{Z_\rho}e^{-\frac{\pi}{2}\left(\frac{k^2}{\rho}+\ell^2\rho\right)}, \tag{4.1}$$

where $Z_\rho = \sum_{(k,\ell)\in\mathbb{Z}^2} e^{-\frac{\pi}{2}\left(\frac{k^2}{\rho}+\ell^2\rho\right)}$.

Remark 4.1.

- A similar result is obtained by Kenyon and Wilson [KW] in the case of the square lattice embedded in the *cylinder*. Working on the torus makes computations much more difficult, since it means dealing with the toroidal partition function in the proof, which is a combination of four terms instead of one in the cylinder case. Moreover, we have to extract information about the two components of the winding number, instead of one in the cylinder case. Note also that in proving Theorem 4.1, we give a *full asymptotic expansion* of a perturbation of the uniform partition function, see Theorem 4.2 below.
- A certain asymptotic expansion of the uniform partition function is obtained in the physics literature by Ferdinand [Fer67], in the case of the square lattice. Nevertheless, the expansion of [Fer67] is not pertubative, it is not done at the same level of mathematical rigor, and no information about the distribution of the winding is inferred.
- In Section 1.2.4, we outline the definition of the height function for the dimer model on planar graphs. When the dimer model is defined on a toroidal graph, as is the case here, the height function is additively multivalued, and it splits into a scalar and an instanton

component. After the publication of our paper, Dubédat [Dub11a] proved the functional convergence of the height function of the dimer model on the torus to the compactified free field. Theorem 4.1 corresponds to proving the convergence of the instanton marginal. It is in agreement with what is predicted by the compactified free field.

In the next two sections, we describe an outline of the proof.

4.1.1 Moment generating function and dimer partition function

Convergence in law of the random variables $(\text{wind}_{m,n})$ to the two-dimensional Gaussian distribution wind_ρ of (4.1) is equivalent to pointwise convergence of the moment generating functions $(F_{m,n})$ of $(\text{wind}_{m,n})$ to the moment generating function of wind_ρ. By definition, for all $(\alpha, \beta) \in \mathbb{R}^2$,

$$F_{m,n}(\alpha,\beta) = \mathbb{E}[e^{-\pi \,\text{wind}_{m,n}\cdot(\alpha,\beta)}] = \sum_{(k,\ell)\in\mathbb{Z}^2} \mathbb{P}[\text{wind}_{m,n} = (k,\ell)] e^{-\pi(\alpha k+\beta\ell)}.$$

The first step of the proof consists in expressing the moment generating function $F_{m,n}$ in terms of a perturbed uniform dimer partition function. Define edges to be of type I, II, III as in Figure 4.3.

Figure 4.3: Labeling of the edges of the honeycomb lattice.

Let us assign weight $e^{-\frac{\alpha\pi}{2m}}$ to edges of type I, weight $e^{-\frac{\beta\pi}{2n}}$ to edges of type II, and weight $e^{\frac{\beta\pi}{2n}}$ to those of type III. Note that the new edge-weights tend to 1 in the limit m, n tend to infinity, and thus yield a perturbation of the uniform partition function. Denote by $Z_{m,n}(\alpha,\beta)$ the partition function of the dimer model on the graph $\mathsf{G}_{m,n}$ with these perturbed weights. Then,

Lemma 4.1. [2] *The moment generating function of the winding number and the perturbed partition function are related as follows:*

$$\forall\,(\alpha,\beta)\in\mathbb{R}^2, \quad F_{m,n}(\alpha,\beta) = e^{\frac{\pi\alpha n}{3}} \frac{Z_{m,n}(\alpha,\beta)}{Z_{m,n}(0,0)}.$$

Proof. Let $N_i(\mathsf{M})$ denote the number of edges of type i (i =I, II, III) in a dimer configuration M. Then by definition, the partition function $Z_{m,n}(\alpha,\beta)$ is equal to:

$$\begin{aligned} Z_{m,n}(\alpha,\beta) &= \sum_{\mathsf{M}\in\mathcal{M}(\mathsf{G}_{m,n})} \left(e^{-\frac{\alpha\pi}{2m}}\right)^{N_{\mathrm{I}}(\mathsf{M})} \left(e^{-\frac{\beta\pi}{2n}}\right)^{N_{\mathrm{II}}(\mathsf{M})} \left(e^{\frac{\beta\pi}{2n}}\right)^{N_{\mathrm{III}}(\mathsf{M})} \\ &= \sum_{\mathsf{M}\in\mathcal{M}(\mathsf{G}_{m,n})} (e^{-\pi\alpha})^{\frac{N_{\mathrm{I}}(\mathsf{M})}{2m}} (e^{-\pi\beta})^{\frac{N_{\mathrm{II}}(\mathsf{M})-N_{\mathrm{III}}(\mathsf{M})}{2n}}. \end{aligned}$$

The next step consists in computing the two components $(\text{wind}^1_{m,n}, \text{wind}^2_{m,n})$ of $\text{wind}_{m,n}$ as a function of N_{I}, N_{II}, N_{III}. We only show the computation of the second component. Consider the $2n$ left-to-right horizontal paths of the dual graph $\mathsf{G}^*_{m,n}$ as in Figure 4.4, fix one of these paths and denote it by γ.

Figure 4.4: Choice of $2n$ left-to-right horizontal paths to compute $\mathrm{wind}^2_{m,n}$.

Then, for any dimer configuration M, $\mathrm{wind}^2_{m,n}(\mathsf{M})$ is equal to the number of positive crossings (right-to-left) minus the number of negative crossings (left-to-right) of γ by $\mathsf{M} - \mathsf{M}_0$. Recalling that edges of M are oriented from the white vertex to the black one, and edges of M_0 are oriented from the black vertex to the white one, one deduces that each edge of type II of M contributes a $+1$, each edge of type III contributes a -1, and the opposite for edges of M_0. Since this holds for any such horizontal path, summing over all $2n$ paths yields:

$$2n\,\mathrm{wind}^2_{m,n}(\mathsf{M}) = N_{\mathrm{II}}(\mathsf{M}) - N_{\mathrm{III}}(\mathsf{M}) - N_{\mathrm{II}}(\mathsf{M}_0) + N_{\mathrm{III}}(\mathsf{M}_0).$$

Observing that $N_{\mathrm{II}}(\mathsf{M}_0) = N_{\mathrm{III}}(\mathsf{M}_0)$ with our choice of reference matching, we obtain $\mathrm{wind}^2_{m,n}(\mathsf{M}) = \frac{N_{\mathrm{II}}(\mathsf{M}) - N_{\mathrm{III}}(\mathsf{M})}{2n}$.

Similarly, introducing $2m$ vertical paths, one shows that $\mathrm{wind}^1_{m,n}(\mathsf{M}) = -\frac{n}{3} + \frac{N_{\mathrm{I}}(\mathsf{M})}{2m}$. As a consequence:

$$\begin{aligned} Z_{m,n}(\alpha,\beta) &= e^{-\frac{\pi\alpha n}{3}} \sum_{\mathsf{M}\in\mathcal{M}(\mathsf{G}_{m,n})} (e^{-\pi\alpha})^{\mathrm{wind}^1_{m,n}(\mathsf{M})} (e^{-\pi\beta})^{\mathrm{wind}^2_{m,n}(\mathsf{M})} \\ &= e^{-\frac{\pi\alpha n}{3}} \sum_{(k,\ell)\in\mathbb{Z}^2} \mathsf{N}_{k,\ell} e^{-\pi(\alpha k + \beta\ell)}, \end{aligned}$$

where $\mathsf{N}_{k,\ell}$ is the number of dimer configurations whose winding number is (k,ℓ). The proof is ended by recalling that dimer configurations of $\mathsf{G}_{m,n}$ are chosen with respect to the uniform measure. □

4.1.2 Asymptotic expansion of the perturbed weight partition function

The second part of the proof is rather technical. It consists in establishing a precise asymptotic expansion of $Z_{m,n}(\alpha,\beta)$ in the limit of Theorem 4.1. As one would expect, the exponential growth rate mn of the partition function is governed by the free energy f of the uniform dimer model on G. By the results of Kenyon, Okounkov and Sheffield [KOS06], we have:

$$f = \frac{1}{(2\pi)^2} \int_{\mathbb{T}^2} P_{\mathrm{dimer}}(z,w) \frac{\mathrm{d}z}{z} \frac{\mathrm{d}w}{w},$$

where $P_{\mathrm{dimer}}(z,w)$ is the corresponding dimer characteristic polynomial. It is defined similarly to the non-bipartite case, see Section 2.1.1, the only difference being that the matrix $K_1(z,w)$ has lines indexed by white vertices and columns indexed by black ones. Using the fundamental domain $\mathsf{G}_{1,1}$, the computation is straightforward and yields:

$$P_{\mathrm{dimer}}(z,w) = w + w^{-1} + 2 - z.$$

We then prove the following precise asymptotic expansion:

Theorem 4.2. [2] *In the joint limit $m,n \to \infty$, $\frac{n}{\sqrt{3}m} \to \rho$, $\rho > 0$, we have the following asymptotic expansion of the perturbed partition function $Z_{m,n}(\alpha,\beta)$. For all $(\alpha,\beta) \in \mathbb{R}^2$:*

$$Z_{m,n}(\alpha,\beta) = (-1)^{mn} e^{-\frac{\pi\alpha n}{3}} e^{-mn\,f} \frac{e^{\frac{\pi}{6}\rho}}{\sqrt{2\rho} P(e^{-\pi\rho})^2} \sum_{(k,\ell)\in\mathbb{Z}^2} e^{\pi(\alpha k+\beta\ell)} e^{-\frac{\pi}{2}\left(\frac{k^2}{\rho}+\rho\ell^2\right)} (1+o(1)),$$

where $P(q) = \prod_{k=1}^{\infty}(1-q^{2k})$, and f is the free energy of the uniform dimer model on the graph G.

Before sketching the proof of Theorem 4.2, let us mention that the proof of Theorem 4.1 is concluded by combining Lemma 4.1 and Theorem 4.2.

Sketch of proof. By Kasteleyn theory on the torus, see Section 1.2.2, the partition function $Z_{m,n}(\alpha,\beta)$ is equal to[1]:

$$Z_{m,n}(\alpha,\beta) = \frac{1}{2}\Big((-1)^n\big(-\det K^{00}_{m,n}(\alpha,\beta) + \det K^{01}_{m,n}(\alpha,\beta)\big) + \det K^{10}_{m,n}(\alpha,\beta) + \det K^{11}_{m,n}(\alpha,\beta)\Big), \tag{4.2}$$

where $(K^{\theta\tau}_{m,n}(\alpha,\beta))_{\theta,\tau\in\{0,1\}}$ are the four Kasteleyn matrices of the toroidal graph $\mathsf{G}_{m,n}$ with perturbed weights. Then, we perform an asymptotic expansion for each of the four terms. Let us only state the result for $\det K^{11}_{m,n}(\alpha,\beta)$, and give an idea of the proof. It involves the *third Jacobi theta function* ϑ_3, defined by:

$$\forall\,\zeta,\, q\in\mathbb{C},\, |q|<1, \quad \vartheta_3(\zeta,q) = \sum_{k=-\infty}^{\infty} e^{2ki\zeta} q^{k^2} = P(q)\prod_{\ell=0}^{\infty}(1+2q^{2\ell+1}\cos(2\zeta)+q^{4\ell+2}),$$

where $P(q) = \prod_{k=1}^{\infty}(1-q^{2k})$.

Proposition 4.1. [2] *In the joint limit $m,n \to \infty$, $\frac{n}{\sqrt{3}m} \to \rho$, $\rho > 0$, we have the following asymptotic expansion for $\det K^{11}_{m,n}(\alpha,\beta)$:*

$$\det K^{11}_{m,n}(\alpha,\beta) = (-1)^{mn} e^{-\frac{\pi\alpha n}{3}}\, e^{-mn\,f}\, \frac{e^{\frac{\pi}{2}\alpha^2\rho} e^{\frac{\pi}{6}\rho}}{P(q)^2} \cdot \vartheta_3(\zeta,q)\vartheta_3(\bar{\zeta},q) + o(1).$$

where $\zeta = \frac{\pi}{2}(\rho\alpha + i\beta)$, $q = e^{-\rho\pi}$, f is the free energy, and ϑ_3 is the third Jacobi theta function.

Idea of the proof. The beginning of the computation is inspired from [KW]. Using the arguments of [CKP01], detailed in Section 2.1.2, we have:

$$\det K^{11}_{m,n}(\alpha,\beta) = \prod_{z^m=-1}\prod_{w^n=-1} P^{(\alpha,\beta)}_{\text{dimer}}(z,w),$$

where $P^{(\alpha,\beta)}_{\text{dimer}}(z,w)$ is the dimer characteristic polynomial with the perturbed weights. An explicit computation using the fundamental domain $\mathsf{G}_{1,1}$ yields:

$$\begin{aligned}\det K^{11}_{m,n}(\alpha,\beta) &= \prod_{z^m=-1}\prod_{w^n=-1}\left(\frac{w}{\left(e^{\frac{\beta\pi}{2n}}\right)^2} + \frac{\left(e^{\frac{\beta\pi}{2n}}\right)^2}{w} + (2-\left(e^{-\frac{\alpha\pi}{2m}}\right)^2 z)\right),\\ &= \prod_{z^m=-1}\prod_{w^n=-1}\frac{1}{\left(e^{\frac{\beta\pi}{2n}}\right)^2 w}\left(w-\left(e^{\frac{\beta\pi}{2n}}\right)^2 r_1(z)\right)\left(w-\left(e^{\frac{\beta\pi}{2n}}\right)^2 r_2(z)\right),\end{aligned} \tag{4.3}$$

[1] When n is odd, the linear combination is not the same as the one of Theorem 1.4, this is because we fix a Kasteleyn orientation of the fundamental domain and extend it periodically.

where

$$r_1(z), r_2(z) = -1 + \frac{\left(e^{-\frac{\alpha\pi}{2m}}\right)^2 z}{2} \pm \sqrt{\left(1 - \frac{\left(e^{-\frac{\alpha\pi}{2m}}\right)^2 z}{2}\right)^2 - 1}.$$

Since w's are n-th roots of -1 we have, $\forall\, \lambda \in \mathbb{C}$, $\prod_{w^n=-1}(\lambda - w) = \lambda^n + 1$. Performing the product over w's in (4.3) yields:

$$\det K^{11}_{m,n}(\alpha,\beta) = (-1)^{mn}\left(e^{-\beta\pi m}\right) \prod_{z^m=-1} (1 + e^{\beta\pi} r_1(z)^n)(1 + e^{\beta\pi} r_2(z)^n). \tag{4.4}$$

The remainder of the proof is quite technical and involves rather heavy and detailed computations. Here are the main ideas. The first issue addressed is to clarify the determination of the square root. Then, one observes that there is a change of behavior in the product (4.4) when $|r_j|$ is smaller, greater or close to 1. In Lemma 11 of [2], we characterize the three regimes. Then we separate the product in three parts accordingly. In a sequence of lemmas and propositions, we then prove that the part of the product with terms:

- $|r_j(z)| > 1$, grows exponentially and the growth rate is governed by the free energy f,
- $|r_j(z)| < 1$, does not contribute,
- $|r_j(z)|$ close to one, contributes the part involving the third Jacobi theta function.

□

The asymptotic expansions of $\det K^{00}_{m,n}(\alpha,\beta)$, $\det K^{01}_{m,n}(\alpha,\beta)$, $\det K^{10}_{m,n}(\alpha,\beta)$ are similar, the only difference being that they involve the first ϑ_1, second ϑ_2 and fourth ϑ_4 Jacobi theta function respectively, instead of the third one (and eventually a global minus sign).

The next proposition explicitly computes the recombination of theta functions $\sum_{i=1}^4 \vartheta_i(\zeta,q)\vartheta_i(\bar{\zeta},q)$ occurring when taking the signed sum of partition functions $\det K^{\theta\tau}_{m,n}(\alpha,\beta)$ of Equation (4.2).

Proposition 4.2. [2] *When $\zeta = \frac{\pi}{2}(\rho\alpha + i\beta)$, and $q = e^{-\pi\rho}$,*

$$\sum_{i=1}^4 \vartheta_i(\zeta,q)\vartheta_i(\bar{\zeta},q) = \sqrt{\frac{2}{\rho}} e^{-\frac{\pi}{2}\alpha^2\rho} \sum_{(k,\ell)\in\mathbb{Z}^2} e^{\pi(\alpha k + \beta\ell)} e^{-\frac{\pi}{2}\left(\frac{k^2}{\rho} + \rho\ell^2\right)}.$$

The proof of Proposition 4.2 is an explicit computation with the four Jacobi theta functions. The proof of Theorem 4.2 is concluded by combining Proposition 4.1, its pendent for $\det K^{00}_{m,n}(\alpha,\beta)$, $\det K^{01}_{m,n}(\alpha,\beta)$, $\det K^{10}_{m,n}(\alpha,\beta)$, and Proposition 4.2. □

4.2 XOR-Ising loops via bipartite dimers

[5] C. Boutillier, B. de Tilière. Height representation of XOR-Ising loops via dimers. *Arxiv: 1211.4825.*

The *XOR-Ising model* is constructed from two independent Ising models living on the same graph by taking at every vertex the product of the spins of the two Ising models. More precisely,

let G be a graph embedded in a compact, orientable surface Σ of genus g, and let G^* denote its dual graph. Consider two independent Ising models defined on vertices of the dual graph G^*. To every pair of spin configurations (σ, σ'), one associates the corresponding *XOR-spin configuration* $\xi \in \{-1,1\}^{\mathsf{V}^*}$, obtained by taking the product of the spins:

$$\forall\, \mathsf{v}^* \in \mathsf{V}^*, \quad \xi_{\mathsf{v}^*} = \sigma_{\mathsf{v}^*}\sigma'_{\mathsf{v}^*}.$$

The terminology XOR-Ising comes from the paper [Wil11]; in the physics community this model is known as the *polarization* of the double-Ising model [KB79]. The double-Ising model is related [KW71, Wu71, Fan72, Weg72] to other models of statistical mechanics as the 8-vertex model [Sut70, FW70] and the Ashkin-Teller model [AT43].

Interfaces separating ± 1 clusters of XOR-spin configurations are polygon configurations of the graph G, referred to as *XOR-loop configurations*. In the paper [5], we explicitly relate XOR-loop configurations to loop configurations in a dimer model defined on a decorated, bipartite version of the graph. This is done in two steps: the first step consists in a mixed contour expansion of the double-Ising partition function, allowing to keep track of the XOR-loop configurations, it is the subject of Section 4.2.1; the second step consists in relating loop configurations obtained from the mixed contour expansion to loop configurations of a dimer model on a decorated, bipartite version of the graph, using the 6-vertex model; this is the subject of Section 4.2.2. Then, in Section 4.2.3 we prove that XOR-loop configurations have the same law as level lines of the height function of the dimer model. Using results of [9], this allows to shed a light on Wilson's conjecture [Wil11], stating that in the scaling limit these loops are level lines of the Gaussian free field.

Results of our paper are proved for the XOR-Ising model defined on a graph embedded in a compact, orientable surface Σ of genus g. There are quite a few difficulties related to taking a general genus, involving homology theory and homology theory relative to a boundary. Since our goal here is to give main ideas of arguments, we choose not to address issues related to the genus. We shall nevertheless state results in full generality, adding footnotes whenever necessary. All details can be found in the original paper [5].

4.2.1 Mixed contour expansion

Consider the low temperature expansion of the two Ising models, see Section 1.1, *i.e.* consider pairs of polygon configurations separating ± 1 clusters of the two models, and assign them different colors, say black and white. Since the two models are independent, we consider the probability measure $\mathbb{P}_{\text{d-Ising}}$ on pairs of polygon configurations $\mathfrak{P}$ [2] of the graph G, defined by:

$$\forall\, (\mathsf{P_b}, \mathsf{P_w}) \in \mathfrak{P}, \quad \mathbb{P}_{\text{d-Ising}}(\mathsf{P_b}, \mathsf{P_w}) = \frac{C^2\big(\prod_{\mathsf{e}\in\mathsf{P_b}} e^{-2J_{\mathsf{e}^*}}\big)\big(\prod_{\mathsf{e}\in\mathsf{P_w}} e^{-2J_{\mathsf{e}^*}}\big)}{Z_{\text{d-Ising}}(\mathsf{G}^*, J)}, \tag{4.5}$$

where the constant C and the choice of weights come from the low temperature expansion, see Proposition 1.1. The normalizing constant $Z_{\text{d-Ising}}(\mathsf{G}^*, J)$ is the *double-Ising partition function*:

$$Z_{\text{d-Ising}}(\mathsf{G}^*, J) = \sum_{(\mathsf{P_b},\mathsf{P_w})\in\mathfrak{P}} C^2\Big(\prod_{\mathsf{e}\in\mathsf{P_b}} e^{-2J_{\mathsf{e}^*}}\Big)\Big(\prod_{\mathsf{e}\in\mathsf{P_w}} e^{-2J_{\mathsf{e}^*}}\Big). \tag{4.6}$$

[2] Polygon configurations of the low temperature of the two Ising models are restricted to having the same homology class in $H_1(\Sigma, \mathbb{Z}/2\mathbb{Z})$, we then take the union over all homology classes of $H_1(\Sigma, \mathbb{Z}/2\mathbb{Z})$; the double-Ising partition function is thus the sum of 2^{2g} restricted partition functions. In the spin representation, this amounts to considering Ising models with *defects*, both Ising models should then have the same defect condition.

Consider the superimposition of a pair of polygon configurations $(\mathsf{P_b}, \mathsf{P_w})$ of $\mathfrak{P}$, see Figure 4.5. One then defines two new edge configurations:

- $\mathrm{Mono}(\mathsf{P_b}, \mathsf{P_w})$, consisting of *monochromatic edges*, that is edges covered by exactly one of $\mathsf{P_b}$ or $\mathsf{P_w}$,
- $\mathrm{Bi}(\mathsf{P_b}, \mathsf{P_w})$, consisting of *bichromatic edges*, that is edges covered by both polygon configurations.

Figure 4.5: Left: superimposition of a pair of polygon configurations $(\mathsf{P_b}, \mathsf{P_w})$ of the low temperature expansion of two Ising models defined on a portion of $\mathbb{Z}^2$ embedded in the torus. Right: corresponding monochromatic edge configuration.

Moreover, denote by $\mathrm{XOR}(\mathsf{P_b}, \mathsf{P_w})$ the XOR-loop configuration corresponding to the pair $(\mathsf{P_b}, \mathsf{P_w})$. Observing that monochromatic edges are present exactly when spins of one of the two Ising models have opposite signs, one deduces that monochromatic edges are present when XOR-spins have opposite signs and absent when XOR-spins have the same sign. We have thus sketched the proof of the following:

Lemma 4.2. [5] *For every pair of polygon configuration* $(\mathsf{P_b}, \mathsf{P_w}) \in \mathfrak{P}$*, the monochromatic edge configuration* $\mathrm{Mono}(\mathsf{P_b}, \mathsf{P_w})$ *is exactly the XOR-loop configuration* $\mathrm{XOR}(\mathsf{P_b}, \mathsf{P_w})$*. In particular, it is a polygon configuration of* $\mathcal{P}^0(\mathsf{G})$[3].

Understanding XOR-loop configurations thus amounts to understanding monochromatic edge configurations. Each such configuration separates the surface Σ into connected components $\Sigma_1, \cdots, \Sigma_N$. Inside each connected component, bichromatic edges are the low temperature expansion of an Ising model, with coupling constants that are doubled, thus allowing for a rewriting of the double-Ising partition function using XOR-loop configurations and bichromatic edge configurations. Refer to Proposition 4.3. of the original paper [5] for an explicit form of this rewriting.

The next part of the argument follows an idea of Nienhuis [Nie84]. It consists in fixing a monochromatic loop configuration, and applying Kramers and Wannier's low/high temperature duality, see Remark 1.1, to the single Ising model corresponding to bichromatic edges in each of the connected components $\Sigma_1, \cdots, \Sigma_N$. This is referred to as the *mixed contour expansion*. As a result, we obtain a rewriting of the double-Ising partition function, as a sum over pairs of non-intersecting polygon configurations of the primal and dual graph, where primal polygon configurations exactly correspond to monochromatic loop configurations.

Proposition 4.3. [5] *The double-Ising partition function on the graph* G^* *can be rewritten as:*

$$Z_{\text{d-Ising}}(\mathsf{G}^*, J) = C_{\mathrm{I}} \sum_{\substack{\{(\mathsf{P},\mathsf{P}^*)\in \mathcal{P}^0(\mathsf{G})\times\mathcal{P}^0(\mathsf{G}^*):\\ \mathsf{P}\cap\mathsf{P}^*=\emptyset\}}} \left(\prod_{\mathsf{e}\in\mathsf{P}} \frac{2e^{-2J_{\mathsf{e}^*}}}{1+e^{-4J_{\mathsf{e}^*}}}\right)\left(\prod_{\mathsf{e}^*\in\mathsf{P}^*} \frac{1-e^{-4J_{\mathsf{e}^*}}}{1+e^{-4J_{\mathsf{e}^*}}}\right),$$

where primal polygon configurations of $\mathcal{P}^0(\mathsf{G})$ *are the monochromatic/XOR loop configurations, and* $C_{\mathrm{I}} = 2^{|\mathsf{V}^*|+2g+1}\left(\prod_{\mathsf{e}\in\mathsf{E}}\cosh(2J_{\mathsf{e}^*})\right)$.

[3] $\mathcal{P}^0(\mathsf{G})$ denotes the set of polygon configurations of the graph G having 0 homology class in $H_1(\Sigma, \mathbb{Z}/2\mathbb{Z})$.

Remark 4.2.

- In the paper [Nie84], Nienhuis performs a similar expansion for a double-Ising model defined on a finite, simply connected, planar graph. There are quite a few difficulties in generalizing this result to graphs embedded in surfaces of genus g. First, we need a version of Kramers and Wannier's low and high temperature expansions for Ising models defined on graphs embedded in compact surfaces of genus g *with* boundary. This is the content of Propositions 6 and 7 of [5]. The formulas obtained are rather natural, but we do not cite them here in order to avoid introducing homology theory relative to a boundary and related notations. Second, the double-Ising partition function $Z_{\text{d-Ising}}(\mathsf{G}^*, J)$ is a sum over pairs of polygon configurations in $\mathfrak{P}$, which is a sum of the restricted partition functions over the 2^{2g} homology classes of $H_1(\Sigma, \mathbb{Z}/2\mathbb{Z})$, see Equation (4.6) and Footnote 1. There is quite some work involved in proving that the restricted partition functions recombine nicely as in the statement, and that only the 0 homology class remains.
- The weight of edges of monochromatic loop configurations have changed from $e^{-2J_{\mathsf{e}^*}}$ in Equation (4.5) to $\frac{2e^{-2J_{\mathsf{e}^*}}}{1+e^{-4J_{\mathsf{e}^*}}}$ in the statement of Proposition 4.3. This comes from the constants which occur when performing Kramers and Wannier's low/high temperature duality in each of the connected components $\Sigma_1, \cdots, \Sigma_N$. The weights of edges of dual polygon configurations are equal to $\tanh(2J_{\mathsf{e}^*})$, these are exactly the weights of the high temperature expansion of an Ising model with doubled coupling constants $(2J_{\mathsf{e}^*})$, see Proposition 1.2.

4.2.2 Polygon configurations in a related dimer model

In this section, we explicitly construct pairs of non-intersecting polygon configurations of the primal and dual graph G and G^* of Proposition 4.3, from a dimer model defined on a bipartite, decorated version G^{Q} of the graph G. This construction is done in two steps: the first step uses a mapping of Nienhuis [Nie84], which constructs pairs of non-intersecting polygon configurations from the 6-vertex model on the medial graph G^{M}; the second step consists in using Wu-Lin/Dubédat's mapping [WL75, Dub11b] from the 6-vertex model on the medial graph G^{M} to the bipartite dimer model on the decorated graph G^{Q}.

The *medial graph* G^{M} of the graph G is defined as follows. Vertices of G^{M} correspond to edges of G. Two vertices of the medial graph are joined by an edge if the corresponding edges in G are incident. Observe that G^{M} is also the medial graph of the dual graph G^*, and that vertices of the medial graph all have degree four. Figure 4.6 represents the medial graph of a piece of $\mathbb{Z}^2$.

Figure 4.6: The medial graph of a piece of $\mathbb{Z}^2$: plain lines represent $\mathbb{Z}^2$, dotted lines represent the dual graph $\mathbb{Z}^2$, and thick plain lines represent the medial graph $(\mathbb{Z}^2)^{\mathrm{M}}$. Grey (resp. white) faces of the medial graph correspond to primal (resp. dual) vertices of the initial graph.

A *6V-configuration*, also called *ice-type configuration*, is an orientation of edges of G^{M}, such that every vertex has exactly two incoming edges [Lie67]. An equivalent way of defining 6-vertex configurations uses edge configurations instead of orientations. This approach is more useful in

our context, so that we define a 6-*vertex configuration* to be an edge configuration, such that around every vertex of G^{M}, there is an even number of consecutive present edges, see Figure 4.7.

Figure 4.7: The six possible local configurations of the 6-vertex model around a vertex of the medial graph G^{M}.

Weights ω_{12} (respectively ω_{34}, ω_{56}) are assigned to configurations 1 and 2 of Figure 4.7 (respectively 3 and 4, 5 and 6):

$$\omega_{12} = \frac{2e^{-2J_{\mathsf{e}^*}}}{1+e^{-4J_{\mathsf{e}^*}}}, \quad \omega_{34} = \frac{1-e^{-4J_{\mathsf{e}^*}}}{1+e^{-4J_{\mathsf{e}^*}}}, \quad \omega_{56} = 1.$$

Mapping I [Nie84]. Consider the following combinatorial mapping from 6-vertex configurations to edge configurations of the primal and dual graph: whenever a vertex of G^{M} has two neighboring edges in the 6-vertex configuration, put the edge of G or G^* separating the present and the absent edges, see Figure 4.8.

Figure 4.8: Mapping I on the local level (top), and on the global level (bottom).

Mapping I associates to a 6-vertex configuration a pair of non-intersecting polygon configurations $(\mathsf{P},\mathsf{P}^*)$[4]. Given such a pair of polygon configurations $(\mathsf{P},\mathsf{P}^*)$, there are exactly two 6-vertex configurations which are mapped to $(\mathsf{P},\mathsf{P}^*)$, see Figure 4.8. As a consequence the 6-vertex partition function $Z_{\text{6-vertex}}(\mathsf{G}^{\mathrm{M}},J)$ can be rewritten as:

$$Z_{\text{6-vertex}}(\mathsf{G}^{\mathrm{M}},J) = 2 \sum_{\{(\mathsf{P},\mathsf{P}^*)\in\mathcal{P}^0(\mathsf{G}\cup\mathsf{G}^*):\,\mathsf{P}\cap\mathsf{P}^*=\emptyset\}} \left(\prod_{\mathsf{e}\in\mathsf{P}} \frac{2e^{-2J_{\mathsf{e}^*}}}{1+e^{-4J_{\mathsf{e}^*}}}\right)\left(\prod_{\mathsf{e}^*\in\mathsf{P}^*} \frac{1-e^{-4J_{\mathsf{e}^*}}}{1+e^{-4J_{\mathsf{e}^*}}}\right).$$

Let us construct yet another graph from the graph G. The *quadri-tiling graph* G^{Q} of G is the decorated graph obtained from G^{M} by replacing every vertex by a decoration which is a quadrangle. The graph G^{Q} is bipartite and can be drawn on the same surface as G. Edges shared by G^{Q} and G^{M} are referred to as *external* edges, and those inside the decorations as *internal*. Weights ω_{12}, ω_{34}, ω_{56} are assigned to edges as in Figure 4.9.

Figure 4.9: Left: a vertex of the medial graph G^{M}. Right: corresponding decoration of the quadri-tiling graph G^{Q}.

Mapping II [WL75, Dub11b]. Requiring exterior edges to match yields a mapping from dimer configurations of G^{Q} to 6-vertex configurations of G^{M}, see Figure 4.10. This mapping between local configurations is one-to-one except in the empty edge case where this mapping is two-to-one.

[4]The pair $(\mathsf{P},\mathsf{P}^*)$ is such that the superimposition $\mathsf{P}\cup\mathsf{P}^*$ has 0 homology class in $H_1(\Sigma,\mathbb{Z}/2\mathbb{Z})$, *i.e.* belongs to $\mathcal{P}^0(\mathsf{G}\cup\mathsf{G}^*)$.

Figure 4.10: Mapping II from the dimer model on G^{Q} to the 6-vertex model on G^{M}.

Observing that the weights ω_{12}, ω_{34}, ω_{56} satisfy the *free-fermionic* condition:

$$(\omega_{12})^2 + (\omega_{34})^2 = (\omega_{56})^2 \quad \Leftrightarrow \quad \frac{2e^{-2J_{\mathsf{e}^*}}}{1+e^{-4J_{\mathsf{e}^*}}}^2 + \frac{1-e^{-4J_{\mathsf{e}^*}}}{1+e^{-4J_{\mathsf{e}^*}}}^2 = 1,$$

we deduce that Mapping II is weight preserving, so that the dimer model partition function $Z_{\mathrm{dimer}}(\mathsf{G}^{\mathrm{Q}}, J)$ is equal to the 6-vertex partition function $Z_{\text{6-vertex}}(\mathsf{G}^{\mathrm{M}}, J)$.

Consider a dimer configuration M of the graph G^{Q}, then Mapping II assigns to M a 6-vertex configuration, and Mapping I assigns to this 6-vertex configuration a pair of non-intersecting polygon configuration of $\mathcal{P}^0(\mathsf{G} \cup \mathsf{G}^*)$. Let us denote this pair by $\mathrm{Poly}(\mathsf{M}) = (\mathrm{Poly}_1(\mathsf{M}), \mathrm{Poly}_2(\mathsf{M}))$. Combining this with Proposition 4.3 yields:

Theorem 4.3. [5]

- *The double-Ising partition function and the dimer partition function are equal up to an explicit constant:*

$$Z_{\text{d-Ising}}(\mathsf{G}^*, J) = 2^{|\mathsf{V}^*|+2g}\Big(\prod_{\mathsf{e}\in\mathsf{E}} \cosh(2J_{\mathsf{e}^*})\Big) Z^0_{\mathrm{dimer}}(\mathsf{G}^{\mathrm{Q}}, J).$$

- *XOR-loop configurations of the double-Ising model on G^* have the same law as Poly_1 configurations of the corresponding dimer model on the bipartite graph G^{Q}:*

$$\forall \mathsf{P} \in \mathcal{P}^0(\mathsf{G}), \quad \mathbb{P}_{\text{d-Ising}}[\mathrm{XOR} = \mathsf{P}] = \mathbb{P}^0_{\mathrm{dimer}}[\mathrm{Poly}_1 = \mathsf{P}].$$

Remark 4.3.

- In the paper [Dub11b], Dubédat relates a version of the double Ising model and the bipartite dimer model on the graph G^{Q}, with one Ising model living on the primal graph G and the second on the dual graph G^*. Using Kramers and Wannier's duality relation, the first part of Theorem 4.3 can be derived from his results in the case of genus 0 and 1. Nevertheless, our goal was to keep track of XOR-loop configurations, an information which is lost in the mappings used in [Dub11b]. Rather, Dubédat keeps track of the order-disorder variables in the vein of [KC71]. Using results of the paper [Dub11a], this allows him to derive critical correlators in the plane. For simply connected regions, this result has been independently obtained by Chelkak, Hongler and Izyurov [CHI12].

- The superscript 0 in the dimer partition function and the dimer Boltzmann measure indicates that each component of the pairs of non-intersecting polygon configurations, is restricted to having 0 homology class.

4.2.3 XOR-Ising loops as level lines of the height function

In this section we consider the infinite volume, critical case: we suppose that the graph G is infinite, isoradial and that the two independent Ising models are assigned the *critical* Ising coupling constants of Equation (1.6). Using Mappings I and II, one finds that weights of the corresponding dimer model on the decorated, bipartite graph $\mathsf{G}^{\diamond}$ are the *critical* dimer weights of Equation (1.7).

From our papers [8] and [4], we have the existence of infinite volume Gibbs measures $\mathbb{P}^{\infty}_{\text{dimer}}$ for the critical dimer model on $\mathsf{G}^{\diamond}$, and $\mathbb{P}^{\infty}_{\text{d-Ising}}$ for the critical double-Ising model. Using the locality of the Gibbs measures, the fact that they are obtained as weak limit of Boltzmann measures on the torus, and Theorem 4.3, we deduce that the second part of Theorem 4.3 still holds in the critical, infinite volume case:

Theorem 4.4. [5] *For any finite subset of edges* $\mathsf{E}_k = \{\mathsf{e}_1, \cdots, \mathsf{e}_k\}$ *of* G*:*

$$\mathbb{P}^{\infty}_{\text{d-Ising}}[\mathsf{E}_k \subset \text{XOR}] = \mathbb{P}^{\infty}_{\text{dimer}}[\mathsf{E}_k \subset \text{Poly}_1].$$

Dimer configurations of the bipartite graph $\mathsf{G}^{\diamond}$ can, like all bipartite dimer models, be interpreted as discrete random surfaces, via a height function, see Section 1.2.4. Using the definition of height function of the paper [8], we then prove in Lemma 22 of [5], that Poly_1 configurations of the dimer model on $\mathsf{G}^{\diamond}$ are the level lines of a restriction of the height function h on dimer configurations of $\mathsf{G}^{\diamond}$. Using Theorem 4.4, we deduce:

Theorem 4.5. [5] *At criticality, XOR-loop configurations have the same law as level lines of a restriction of the height function of dimer configurations of* $\mathsf{G}^{\diamond}$*.*

In [Wil11], Wilson presents extensive numerical simulations on loops of the critical XOR-Ising model on the honeycomb lattice, on the base of which he conjectures the following:

Conjecture 1 (Wilson [Wil11])**.** *The scaling limit of the XOR-loop configurations are the level lines of the Gaussian free field corresponding to levels that are odd multiples of* $\frac{\sqrt{\pi}}{2}$*.*

Note that similar conjectures, involving SLE rather than the Gaussian free field, are obtained through conformal field theory [IR11, PS11].

The results of our paper bring some elements in the direction of the proof of the conjecture of Wilson. Indeed in [9], I have proved that the height function of critical dimer models, interpreted as a random generalized function, converges weakly in law to $\frac{1}{\sqrt{\pi}}$ times a Gaussian free field. This also holds for the restriction of the height function. Wrapping up, we know that loops of the XOR-Ising model have the same law as level lines of a restricted height function on the discrete level, we know that this height function converges to a Gaussian free field. Unfortunately, when establishing this convergence we consider the height function as a random generalized function, thus loosing information on the convergence of the level lines to those of the Gaussian free field.

Chapter 5

A polymer model

[1] E. Bolthausen, F. Caravenna, B. de Tilière. The quenched critical point of a diluted disordered polymer model. *Stochastic Process. Appl.* **119** (2009), 1479–1504.

The issue addressed in this paper is the determination of the quenched critical point for the localization/delocalization phase transition of a polymer interacting with an attractive wall through a diluted disordered potential. The model we consider was first introduced by Bodineau and Giacomin in [BG04], as a *reduced* model for the so-called *copolymer near a selective interface model* [BdH97], with the hope that it could have the same behavior as the full copolymer model, in the limit of weak coupling constants. The main result of this paper shows that this is not the case.

5.1 Setting: a diluted disordered polymer model

Let $S = (S_n)_{n\geq 0}$ be the simple symmetric random walk on $\mathbb{Z}$, and denote by $\mathbb{P}$ its law. For $N \in \mathbb{N}$, denote by $\mathbb{P}_N^+(\,\cdot\,)$ the law of the random walk conditioned to stay non-negative up to time N. The trajectories $((n, S_n))_{0\leq n\leq N}$ under $\mathbb{P}_N^+$ model the configurations of a polymer chain of length N above an impenetrable wall.

The interaction of the polymer with the wall is tuned by two parameters $\beta \geq 0$ and $p \in [0, 1]$. For fixed β and p, we introduce a sequence $\omega^{\beta,p} = (\omega_n^{\beta,p})_{n\geq 1}$ of i.i.d. random variables, taking values in $\{0, \beta\}$ and with law P given by:

$$\mathsf{P}(\omega_1^{\beta,p} = \beta) = p, \quad \mathsf{P}(\omega_1^{\beta,p} = 0) = 1 - p. \tag{5.1}$$

We are ready to define the model: for a fixed realization $\omega^{\beta,p}$ and $N \in \mathbb{N}$, we introduce the probability measure $\mathbb{P}_{N,\omega^{\beta,p}}$ defined by

$$\frac{\mathrm{d}\mathbb{P}_{N,\omega^{\beta,p}}}{\mathrm{d}\mathbb{P}_N^+}(S) = \frac{1}{Z_{N,\omega^{\beta,p}}} \exp\left(\sum_{n=1}^{N} \omega_n^{\beta,p}\, \mathbb{I}_{\{S_n=0\}}\right), \tag{5.2}$$

where $Z_{N,\omega^{\beta,p}}$ is the normalizing constant known as the the *partition function*.

We focus on the regime of large β and small p, *i.e.* $\omega^{\beta,p}$ represents a random sequence of charges sitting on the wall, represented by the x-axis, which are rare, but of strong intensity, and which attract the polymer, see Figure 5.1.

Figure 5.1: A typical path of the polymer measure $\mathbb{P}^{\beta,p}_{N,\omega}$.

We are interested in the behavior of the polymer measure $\mathbb{P}_{N,\omega^{\beta,p}}$ in the limit of large N: in particular, we want to understand if the attractive effect of the environment $\omega^{\beta,p}$ is strong enough to pin the polymer at the wall (*localization*), or if it is still more convenient for the polymer to wander away from it (*delocalization*), as it happens when there are no charges. We are facing a competition between energy and entropy.

The classical way of detecting the transition between the two regimes just sketched is to study the *free energy* of the model, defined as:

$$f(\omega^{\beta,p}) = \lim_{N\to\infty} \frac{1}{N} \log Z_{N,\omega^{\beta,p}}. \tag{5.3}$$

The existence of this limit, both $\mathsf{P}(\mathrm{d}\omega^{\beta,p})$–a.s. and in $L^1(\mathsf{P})$, and the fact that $f(\omega^{\beta,p})$ is non-random are proven in [Gia07] via super-additivity arguments. Analyzing the behavior of the free energy yields that it is non-negative, and allows for the following partition of the phase space:

- the *localized* region $\mathcal{L} := \{(\beta,p) : f(\omega^{\beta,p}) > 0\}$
- the *delocalized* region $\mathcal{D} := \{(\beta,p) : f(\omega^{\beta,p}) = 0\}$.

By a standard coupling on the environment, for every $\beta \geq 0$ there exists a critical value $p_c(\beta) \in [0,1]$ such that the model is localized for $p > p_c(\beta)$ and delocalized for $p \leq p_c(\beta)$. The main goal of the paper [1] is to study the asymptotic behavior of $p_c(\beta)$ as $\beta \to \infty$.

5.2 Main result

Some bounds on $p_c(\beta)$ can be obtained quite easily, as is shown in [BG04]. These are stated in the following lemma.

Lemma 5.1. [BG04] *The following relations holds:*

$$\begin{aligned} -\liminf_{\beta\to\infty} \frac{1}{\beta} \log p_c(\beta) &\leq 1 \\ -\limsup_{\beta\to\infty} \frac{1}{\beta} \log p_c(\beta) &\geq \frac{2}{3}. \end{aligned}$$

The upper bound is that of the *annealed* model, obtained by averaging over the environment; it is proved using Jensen's inequality. The lower bound uses a lower bound on the partition function, obtained by summing over all trajectories that touch the wall whenever there is a non-zero charge.

We can summarize Lemma 5.1 by stating that for β large we have

$$p_c(\beta) \asymp e^{-c_{red}\,\beta} \qquad \text{where} \qquad \frac{2}{3} \le c_{red} \le 1,$$

where the subscript *red* stands for *reduced model*, see the discussion below. The main result of this paper is that in fact $c_{red} = \frac{2}{3}$. More precisely:

Theorem 5.1. [1] *For every $c > \frac{2}{3}$ there exists $\beta_0 = \beta_0(c)$ such that*

$$\forall\, \beta \ge \beta_0, \quad f(\omega^{\beta, e^{-c\,\beta}}) \;=\; 0$$

i.e. forall $\beta \ge \beta_0$, $(\beta, e^{-c\,\beta}) \in \mathcal{D}$. Therefore

$$-\lim_{\beta\to\infty} \frac{1}{\beta} \log p_c(\beta) \;=\; \frac{2}{3}\,.$$

Remark 5.1.

- Recall that the model $\mathbb{P}_{N,\omega^{\beta,p}}$ was first introduced in [BG04] as a simplified version, or *reduced model*, of the so-called *copolymer near a selective interface model* [BdH97] (see also [Gia07] for a more recent overview). It is known that the copolymer model undergoes a localization/delocalization phase transition. An interesting object is the *critical line* separating the two phases, in particular in the limit of weak coupling constants, where it becomes a straight line with positive slope c_{cop}.

  A lot of effort has been put in finding the exact value of c_{cop}. This is motivated by the fact that c_{cop} appears to be a *universal* quantity: it is independent of the law of the environment [GT05] and it determines the phase transition of a continuous copolymer model, arising as the scaling limit of the discrete one [BdH97]. At the time where our paper was written, it was only known that $\frac{2}{3} \le c_{cop} \le 1$. Notice that $\frac{2}{3}$ and 1 are exactly the same bounds that were previously known for c_{red}, and this is not a case: the definition of the model $\mathbb{P}_{N,\omega^{\beta,p}}$ has been inspired by the strategy behind the proof of $c_{cop} \ge \frac{2}{3}$ [BG04].

  The reason for introducing the reduced model was to have a more tractable model, which could possibly have the same behavior as the full copolymer model in the limit of weak coupling constants, *i.e.* for which possibly $c_{red} = c_{cop}$. However, the numerical results obtained in [CGG06] provide strong indications for the fact that $c_{cop} > \frac{2}{3}$. Since the publication of our paper, this has indeed been proved to be the case by Bolthausen, den Hollander and Opoku [BHO11]. Our result thus shows that the reduced model does not catch the full complexity of the copolymer model, *i.e.* the 'missing free energy' comes from a different strategy than the one which is at the basis of the lower bound $c_{cop} \ge \frac{2}{3}$.

- Theorem 5.1 also provides a non-trivial example of a linear chain pinning model where, for large β, the quenched critical point is different from the annealed one. The proof we present relies on quenched arguments, based on a rigorous renormalization procedure (somewhat in the spirit of [Mon00]). The idea is to remove from the environment $(\omega_n^{\beta,p})_n$ the positive charges that are well-spaced (and therefore give no sensible contribution to the partition function) and to cluster together the positive charges that are very close. This procedure produces a new environment sequence $(T(\omega)_n)_{n\ge1}$, which has fewer charges but of stronger intensity. The key point is that replacing $\omega^{\beta,p}$ by $T(\omega)$ in the partition function

yields an upper bound on the free energy. Then, by iterating the map T several times, we obtain environment sequences for which the free energy can be estimated and shown to be arbitrarily small. A more detailed description of this approach, is given in Section 5.3.

- The same result has been obtained independently and at the same time by Toninelli [Ton08] with a simpler (though more indirect) argument, avoiding the renormalization technique. We however believe that our more direct approach gives a better understanding of the behavior of the system.

5.3 Strategy of the proof: a renormalization procedure

In this section, we explain the strategy behind the proof of Theorem 5.1. To this purpose, we introduce some notations, and a preliminary transformation of the partition function $Z_{N,\omega^{\beta,p}}$.

5.3.1 Notations

Suppose that $\omega = (\omega_n)_{n\geq 1}$ is an i.i.d. sequence of non-negative random variables with marginal law μ:

$$\mathsf{P}(\omega_1 \in \mathrm{d}x) = \mu(\mathrm{d}x),$$

representing an environment of non-negative charges, and consider a fixed realization ω of the environment. Then, define the sequence $(t_n)_{n\geq 0} = (t_n(\omega))_{n\geq 0}$ as representing the location of the positive charges:

$$t_0(\omega) \;:=\; 0 \qquad\qquad t_n(\omega) \;:=\; \min\left\{k > t_{n-1}(\omega) : \omega_k > 0\right\}. \tag{5.4}$$

We also introduce the sequence $(\eta_n)_{n\geq 1} = (\eta_n(\omega))_{n\geq 1}$ giving the intensities of the positive charges, that is:

$$\eta_n(\omega) \;:=\; \omega_{t_n(\omega)}, \qquad n \geq 1. \tag{5.5}$$

For $C > 0$, and 'reasonable'[1] sequences ω, define the following 'partition function':

$$\mathsf{Z}_{n,\omega}(C) := \sum_{k=1}^{n} \sum_{\substack{j_1,\ldots,j_{k-1}\in\mathbb{N} \\ 0=:j_0<j_1<\ldots<j_{k-1}<j_k:=N}} \prod_{\ell=1}^{k} e^{\eta_{j_\ell}} \cdot \frac{C}{(t_{j_\ell}(\omega) - t_{j_{\ell-1}}(\omega))^{3/2}},$$

and the corresponding free energy:

$$\mathsf{f}(\omega, C) := \lim_{n\to\infty} \frac{1}{t_n(\omega)} \log \mathsf{Z}_{n,\omega}(C).$$

[1] All sequences ω that we shall consider are such that the following partition function and free energy are well defined.

5.3.2 Preliminary transformation

Consider $\omega = \omega^{\beta,p}$ to be a fixed realization of our initial environment. Its marginal law can be written as $\mu^{\beta,p} := (1-p)\delta_{\{0\}} + p\delta_{\{\beta\}}$, and in this case the intensity of the positive charges is identically equal to β.

Using estimates from renewal theory, see [Gia07], one shows that for some $C > 0$:

$$f(\omega^{\beta,p}) \leq \mathsf{f}(\omega^{\beta,p}, C),$$

so that to prove Theorem 5.1, it suffices to prove that for every $C > 0$, and every $c > \frac{2}{3}$, there exists $\beta_0 = \beta_0(C, c)$ such that:

$$\forall\, \beta \geq \beta_0, \quad \mathsf{f}(\omega^{\beta,e^{-c\beta}}, C) = 0.$$

To simplify notation, let us denote $\omega^{\beta,e^{-c\beta}}$ as ω^{β}.

5.3.3 The renormalization procedure

The proof of Theorem 5.1 is achieved through an inductive argument. The steps of the induction are labeled by $\{\beta, \beta+1, \beta+2, \ldots\}$, and we call them *level* β,*level* $\beta+1$,... .

Each induction step consists of a renormalization procedure that acts both on the environment ω, and on the partition function $\mathsf{Z}_n(\omega, C)$, and produces an upper bound on the free energy $\mathsf{f}(\omega, C)$. Let us be more precise.

Renormalizing the environment. At the starting point (level β) the environment ω^{β} is i.i.d. with marginal law:

$$\mu^{\beta} = (1 - e^{-c\beta})\delta_{\{0\}} + e^{-c\beta}\delta_{\{\beta\}},$$

supported on $\{0\} \cup \{\beta\}$. More generally, at level $b \geq \beta$ the environment ω^b will be i.i.d. with marginal law μ^b supported on $\{0\} \cup \{b, b+1, \ldots\}$. If we are at level b, we define a renormalization map T_b acting on ω^b that produces a renormalized environment $\omega^{b+1} := T_b(\omega^b)$ as follows.

We first need to define *isolated charges*, *good charges* and *bad blocks* at level b. To this purpose, we fix the threshold $L_b := \lfloor e^{\frac{2}{3}(b+K_b)} \rfloor$, where K_b is defined explicitly. A positive charge is said to be an *isolated charge* if both its neighboring positive charges are at distance greater than L_b. Among the isolated charges, we call *good charges* those that have intensity exactly equal to b, *i.e.* the least possible intensity. Finally, a group of adjacent positive charges is said to be a *bad block* if all the distances between neighboring charges inside the group are smaller than L_b. Note that a charge is either isolated, or it belongs to a bad block (see Figure 5.2 for a graphical illustration).

Figure 5.2: Good charges, isolated charges and bad blocks at level b.

Then the renormalized environment ω^{b+1} is obtained from ω^b in the following way: each *bad block* is clustered into one single larger charge, each *good charge* is erased, the *isolated charges* that are not good are left unchanged and finally the distances between charges are suitably shortened. We prove that the new environment ω^{b+1} constructed in this way is still i.i.d. and we

obtain an explicit expression for the marginal law of ω_1^{b+1}, denoted by $\mu^{b+1} := T_b(\mu^b)$. Observe that by construction μ^{b+1} is supported on $\{0\} \cup \{b+1, b+2, \dots\}$.

Renormalizing the partition function. The idea behind the definition of good charges and bad blocks is the following:

- if a charge is good, it is not worth for the polymer to visit it, because this would entail a substantial entropy loss;
- on the other hand, if a charge belongs to a bad block and the polymer visits it, it is extremely convenient for the polymer to visit all the charges in the block.

The work then consists in making these rough considerations precise. If we replace the environment ω^b with the renormalized one $\omega^{b+1} = T_b(\omega^b)$, we get an upper bound on the partition function. More precisely, if we also denote by T_b the transformation acting on $C > 0$, $T_b(C) := C \cdot (1 + B\, e^{-K_b}\, C)$ (where B is some absolute constant), then we show that the partition function satisfies, for every $N \in \mathbb{N}$,

$$\mathsf{Z}_n(\omega^b, C) \;\le\; (const.)\, \mathsf{Z}_N\big(\omega^{b+1}, T_b(C)\big)\,, \tag{5.6}$$

for a suitable $n = n(\omega, N)$ such that $n \ge N$ and $t_n(\omega^b) \ge t_N(\omega^{b+1})$. Taking '$\frac{1}{t_n}\log$' on both sides of (5.6), and letting $n \to \infty$, we then obtain for every $b \ge \beta$,

$$\mathsf{f}\big(\omega^b, C\big) \;\le\; \mathsf{f}\big(\omega^{b+1}, T_b(C)\big).$$

By iteration we have for $b \ge \beta$,

$$\mathsf{f}\big(\omega^\beta, C\big) \;\le\; \mathsf{f}\big(\omega^b, C_b\big), \qquad \text{where} \qquad C_b \;:=\; \big(T_{b-1} \cdot T_{b-2} \cdots T_\beta\big)(C)\,. \tag{5.7}$$

Completing the proof. The last step is to get a control on the law μ^b and on the constant C_b, in order to extract explicit bounds from (5.7). By easy estimates, we show that $C_b \le 2C$ for every b, so that this yields no problem. The crucial point is rather in estimating the law μ^b: we prove (when β is large but fixed) an explicit stochastic domination of μ^b, which allows to show that

$$\lim_{b \to \infty} \mathsf{f}\big(\omega^b, C_b\big) \;=\; 0\,.$$

By (5.7) this implies that $\mathsf{f}\big(\omega^\beta, C\big) = 0$, and Theorem 5.1 is proved.

Chapter 6

Perspectives

As a conclusion, I would like to mention work in progress and a few perspectives.

6.1 Double Ising model in dimension two

In the paper [11], described in Section 3.1, I constructed an explicit mapping from cycle-rooted spanning forests to the double critical Ising model on the torus, via Fisher's mapping and characteristic polynomials. This provides an unexpected relation, on the level of configurations, between two well known models of statistical mechanics. The drawback is that the construction is rather complicated, and reaching other observables than the partition function seems difficult.

Since then, thanks to the work of Wu and Lin [WL75], Dubédat [Dub11b], and the paper written in collaboration with C. Boutillier [5], presented in Section 4.2, we know that the double Ising model on the graph G is also related to the dimer model defined on the decorated, bipartite graph G^{Q}.

In the remainder of this section, we consider this bipartite dimer model on the graph G^{Q}, when the Ising graph G, is finite, planar and isoradial. Note that the setting of finite, planar graphs is on the one side simpler than toroidal ones, because the topology of the plane is easier than that of the torus; on the other hand it is not, because it requires to handle boundary conditions.

Double-Ising model, spanning forests (Work in progress).

- At criticality. I am trying to explicitly construct the dimer model on the graph G^{Q} from spanning trees on the graph G, with appropriate boundary conditions. I think that I am close to a solution. As a consequence the partition function of the double Ising model would be equal to the partition function of the corresponding spanning trees, thus proving a pendent for planar regions of the result for toroidal graphs [3, 11]. I believe that using this method, other observables can be reached, in particular I am working on expressing the square of Ising spin-correlations using spanning trees and related objects. Quite remarkably, the latter have recently been computed in [CHI12] for finite, simply connected regions, and in [Dub11b] for the whole plane. In the first paper, the authors use a variant of Smirnov's observable [Smi10] in a double cover of the plane. In the second paper, the author proves that spin correlations are equal to monomer correlations in the dimer model, which he had computed in the paper [Dub11a]. My goal is not to compute them yet another

time. I would like to shed a different light, by proving how square of Ising spin correlations are related, on the level of configurations, to objects naturally arising from spanning trees. Indeed spanning trees are one of the models of statistical mechanics where random walks and discrete holomorphic functions arise the most naturally. Our method might provide a natural way of understanding, from configurations, how the winding numbers arise in Smirnov's observable [Smi10].

- Away from criticality. I am working on tackling the same questions. Things seem to be more complicated on the combinatorial level, but maybe tractable. In the subcritical regime, spanning trees seem to be replaced by spanning forests, which are naturally related to massive random walk. This would be in agreement with the work of [BDC12], where the authors relate the correlation length of the Ising model to the large deviation behavior of a massive random walk, by using the random-cluster representation of the Ising model, and a massive version of Smirnov's observable. The connection with massive random walk was first observed by Messikh [Mes06].

If things work out as I hope, then combining the two would relate in an explicit way, the critical Ising model to random walk, and the sub-critical Ising model to massive random walk, thus explaining the phase transition in the Ising model by the recurrent or transient behavior of the random walks.

Extension to random-cluster model

The two-dimensional Ising model has a random-cluster model representation for $q = 2$, see [FK72] and also [Gri06]. Since it is possible to construct the double critical Ising model from spanning trees, it should be possible to construct its random cluster representation from spanning trees as well. I would like to understand how this could be done, in the hope of being able to generalize it other values of q. I think it would be interesting to identify the pendent of spanning trees when $q \neq 2$. What kind of statistical mechanics model would it be ? Has it been studied in another context ?

Conformal invariance of the height function

In collaboration with Cédric Boutillier

Consider the bipartite dimer model on the graph G^{Q}, approximating a simply connected domain. We would like to prove conformal invariance of the corresponding height function, in the scaling limit. Results of this type, for simply connected domains, have been obtained by Kenyon [Ken00, Ken01, Ken08]. Nevertheless, boundary conditions arising from the double Ising model do not enter the framework of previously known results, and seem to be non-trivial to handle.

6.2 Statistical mechanics on isoradial graphs

We have seen that the critical bipartite dimer model, the critical Fisher dimer model and spanning trees have very special features when defined on isoradial graphs: the inverse Kasteleyn operator of the first two models [Ken02, 4] and the Green's function of the third [Ken02] have *local expressions*. Recall that one would expect the combinatorics of the whole graph to contribute.

Non-critical models on isoradial graphs

In collaboration with Cédric Boutillier, maybe also Christian Mercat

We believe that the locality property is, in part due to the geometry of the isoradial graphs, and in part to the $Y-\Delta$ transformation used to determine the weights (the use of the $Y-\Delta$ transformation is clear for the Ising model [Bax86], although not so much for the other ones). Recall that the weights are Z-invariant if the partition function only changes by a global constant when a $Y-\Delta$ transformation of the graph is performed. In particular, this implies that probabilities do not get affected if a local transformation of the graph is performed. This is, we believe, one of the deep reasons underlying the *locality* property.

If what we think is correct, *locality* should remain true when the weights are Z-invariant and *non-critical*. Such weights exist for Ising and for the corresponding bipartite dimer model [Bax86] on the graph G^{Q}. We aim at proving local expressions for the inverse Kasteleyn matrix of this bipartite dimer model.

The main difficulties that we have to face are the following. A crucial role is played by Mercat's discrete exponential functions [Mer04, Ken02]. Following the work of Kenyon and Okounkov [KO06], it seems clear that these functions are specific to the critical case, since they are related to Harnack curves of genus 0. In order to generalize the discrete exponential functions, one would need to find a way of parametrizing Harnack curves of higher genus. This brings us close to algebraic geometry, where we reach our limits. But we do hope, that by explicitly solving specific cases, we might be able to generalize.

Dimer model on non-bipartite isoradial graphs

Consider the dimer model on an isoradial graph G. When the underlying graph is not bipartite, the Kasteleyn operator cannot be naturally related to the Dirac operator. This was one of the key requirement of the paper [Ken02]. We would like to understand if the bipartite dimer critical weights are also critical in some sense in the non-bipartite case. We would also like to compute the inverse Kasteleyn matrix. This seems like a difficult problem, because there are very few tools available for non-bipartite dimer models.

6.3 One last thought

Recently, when at a conference on the Gaussian free field in Marseille, Nathalie Eisenbaum mentioned non-Gaussian processes constructed from non-symmetric covariance matrices (if I recall correctly). If these processes are well defined, is it possible to find models of statistical mechanics where such processes arise? One candidate would be oriented spanning trees with different weights on the two possible orientation of the edges. Considering the corresponding dimer model given by [Tem74], [KPW00], is there some regime where the height function converges to such a non-Gaussian process? I think this would be fun.

We might ask this question to the PhD student we are taking with Cédric Boutillier.

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