Symmetry of the ground state of the nonlinear Schrödinger equation in presence of a magnetic field

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September 18, 2019

Nonlinear days in Alghero
September 16-20, 2019

Outline

- *Without magnetic fields: symmetry and symmetry breaking in interpolation inequalities*

▷ Gagliardo-Nirenberg-Sobolev inequalities on the sphere
▷ Keller-Lieb-Thirring inequalities on the sphere
▷ Caffarelli-Kohn-Nirenberg inequalities

- *With magnetic fields in dimensions* 2 *and* 3

▷ Interpolation inequalities and spectral estimates
▷ Estimates, numerics; an open question on constant magnetic fields

- *Magnetic rings: the case of* $\mathbb{S}^1$

▷ A one-dimensional magnetic interpolation inequality
▷ Consequences: Keller-Lieb-Thirring estimates, Aharonov-Bohm magnetic fields and a new Hardy inequality in $\mathbb{R}^2$

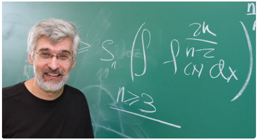
- *Aharonov-Bohm magnetic fields in* $\mathbb{R}^2$

▷ Aharonov-Bohm effect
▷ Interpolation and Keller-Lieb-Thirring inequalities in $\mathbb{R}^2$
▷ Aharonov-Symmetry and symmetry breaking

A joint research program (mostly) with...

M.J. Esteban, Ceremade, Université Paris-Dauphine
▷ *symmetry, interpolation, Keller-Lieb-Thirring, magnetic fields*

M. Loss, Georgia Institute of Technology (Atlanta)
▷ *symmetry, interpolation, Keller-Lieb-Thirring, magnetic fields*



A. Laptev, Imperial College London
▷ *Keller-Lieb-Thirring, magnetic fields*

D. Bonheure, Université Libre de Bruxelles
▷ *Aharonov-Bohm magnetic fields*

Definitions in presence of a magnetic field

The *magnetic covariant derivative / magnetic gradient*

$$\nabla_{\mathbf{A}} := \nabla + i\,\mathbf{A}$$

The *magnetic Dirichlet energy*: if $\psi = u\,e^{iS}$, then

$$\int_{\mathbb{R}^d} |\nabla_{\mathbf{A}}\,\psi|^2\,dx = \int_{\mathbb{R}^d} |\nabla\psi|^2\,dx + \int_{\mathbb{R}^d} |\nabla S + \mathbf{A}|^2\,|\psi|^2\,dx$$

The *magnetic Sobolev space*

$$\mathrm{H}^1_{\mathbf{A}}(\mathbb{R}^d) := \left\{\psi \in \mathrm{L}^2(\mathbb{R}^d)\,:\,\nabla_{\mathbf{A}}\,\psi \in \mathrm{L}^2(\mathbb{R}^d)\right\}$$

The *magnetic Laplacian*

$$-\Delta_{\mathbf{A}}\,\psi = -\Delta\,\psi - 2\,i\,\mathbf{A}\cdot\nabla\psi + |\mathbf{A}|^2\psi - i\,(\operatorname{div}\mathbf{A})\,\psi$$

Magnetic interpolation, ground state of the nonlinear Schrödinger equation and symmetry

For any $p \in (2, 2^*)$, the magnetic interpolation inequality

$$\|\nabla_{\mathbf{A}}\psi\|^2_{\mathrm{L}^2(\mathbb{R}^d)} + \alpha\,\|\psi\|^2_{\mathrm{L}^2(\mathbb{R}^d)} \geq \mu(\alpha)\,\|\psi\|^2_{\mathrm{L}^p(\mathbb{R}^d)} \quad \forall\,\psi \in \mathrm{H}^1_{\mathbf{A}}(\mathbb{R}^d)$$

is a consequence of the Gagliardo-Nirenberg inequality

$$\|\nabla\psi\|^2_{\mathrm{L}^2(\mathbb{R}^d)} + \alpha\,\|\psi\|^2_{\mathrm{L}^2(\mathbb{R}^d)} \geq \mu_{\mathrm{GN}}(\alpha)\,\|\psi\|^2_{\mathrm{L}^p(\mathbb{R}^d)} \quad \forall\,\psi \in \mathrm{H}^1(\mathbb{R}^d)$$

and of the diamagnetic inequality

$$\|\nabla_{\mathbf{A}}\,\psi\|^2_{\mathrm{L}^2(\mathbb{R}^d)} \geq \|\nabla u\|^2_{\mathrm{L}^2(\mathbb{R}^d)}\,, \quad u = |\psi|$$

An optimal function solves

$$-\Delta_{\mathbf{A}}\,\psi + \alpha\,\psi = |\psi|^{p-2}\,\psi$$

If $\mathbf{A}$ is invariant under rotation, is there a (complex valued) ground state (minimum of the energy) which is depending only on $|x|$?

Symmetry and symmetry breaking in interpolation inequalities *without magnetic field*

- Gagliardo-Nirenberg-Sobolev inequalities on the sphere
- Keller-Lieb-Thirring inequalities on the sphere
- Caffarelli-Kohn-Nirenberg inequalities on $\mathbb{R}^2$

A result of uniqueness on a classical example

On the sphere $\mathbb{S}^d$, let us consider the positive solutions of

$$-\Delta u + \lambda\, u = u^{p-1}$$

$p \in [1,2) \cup (2,2^*]$ if $d \geq 3$, $2^* = \frac{2\,d}{d-2}$

$p \in [1,2) \cup (2,+\infty)$ if $d = 1,\, 2$

Theorem

If $\lambda \leq d$, $u \equiv \lambda^{1/(p-2)}$ is the unique solution

[Gidas & Spruck, 1981], [Bidaut-Véron & Véron, 1991]

Bifurcation point of view and symmetry breaking

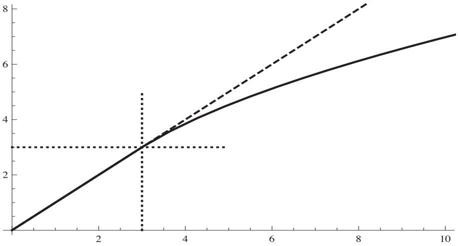


Figure: $(p-2)\,\lambda \mapsto (p-2)\,\mu(\lambda)$ *with* $d=3$

$$\boxed{\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{S}^d)} + \lambda\,\|u\|^2_{\mathrm{L}^2(\mathbb{S}^d)} \ge \mu(\lambda)\,\|u\|^2_{\mathrm{L}^p(\mathbb{S}^d)}}$$

Taylor expansion of $u = 1 + \varepsilon\,\varphi_1$ as $\varepsilon \to 0$ with $-\Delta\varphi_1 = d\,\varphi_1$

$$\mu(\lambda) < \lambda \quad \text{if and only if} \quad \lambda > \frac{d}{p-2}$$

▷ The inequality holds with $\mu(\lambda) = \lambda = \frac{d}{p-2}$ [Bakry & Emery, 1985] [Beckner, 1993], [Bidaut-Véron & Véron, 1991, Corollary 6.1]

The Bakry-Emery method on the sphere

Entropy functional

$$\mathcal{E}_p[\rho] := \frac{1}{p-2}\left[\int_{\mathbb{S}^d} \rho^{\frac{2}{p}}\,d\mu - \left(\int_{\mathbb{S}^d} \rho\,d\mu\right)^{\frac{2}{p}}\right] \quad \text{if} \quad p \neq 2$$

$$\mathcal{E}_2[\rho] := \int_{\mathbb{S}^d} \rho \log\left(\frac{\rho}{\|\rho\|_{\mathrm{L}^1(\mathbb{S}^d)}}\right) d\mu$$

Fisher information functional

$$\mathcal{I}_p[\rho] := \int_{\mathbb{S}^d} |\nabla \rho^{\frac{1}{p}}|^2\,d\mu$$

[Bakry & Emery, 1985] *carré du champ* method: use the heat flow

$$\frac{\partial \rho}{\partial t} = \Delta \rho$$

and observe that $\frac{d}{dt}\mathcal{E}_p[\rho] = -\,\mathcal{I}_p[\rho]$

$$\frac{d}{dt}\Big(\mathcal{I}_p[\rho] - d\,\mathcal{E}_p[\rho]\Big) \le 0 \quad \Longrightarrow \quad \mathcal{I}_p[\rho] \ge d\,\mathcal{E}_p[\rho]$$

with $\rho = |u|^p$, if $p \le 2^\# := \frac{2\,d^2+1}{(d-1)^2}$

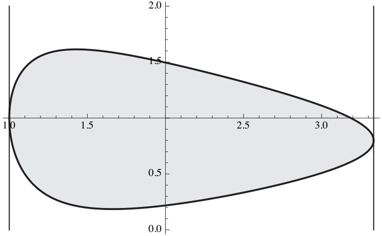
The evolution under the fast diffusion flow

To overcome the limitation $p \leq 2^{\#}$, one can consider a nonlinear diffusion of fast diffusion / porous medium type

$$\frac{\partial \rho}{\partial t} = \Delta \rho^m$$

[Demange], [JD, Esteban, Kowalczyk, Loss]: for any $p \in [1, 2^*]$

$$\mathcal{K}_p[\rho] := \frac{d}{dt}\Big(\mathcal{I}_p[\rho] - d\,\mathcal{E}_p[\rho]\Big) \leq 0$$



(p, m) *admissible region*, $d = 5$

References

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- JD, M.J. Esteban, and M. Loss. Interpolation inequalities on the sphere: linear vs. nonlinear flows. Annales de la faculté des sciences de Toulouse Sér. 6, 26 (2): 351-379, 2017
- JD, M.J. Esteban. Improved interpolation inequalities and stability. Preprint, 2019 arXiv:1908.08235

Optimal inequalities

With $\mu(\lambda)=\lambda=\frac{d}{p-2}$: [Bakry & Emery, 1985] [Beckner, 1993], [Bidaut-Véron & Véron, 1991, Corollary 6.1]

$$\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{S}^d)}\ge\frac{d}{p-2}\left(\|u\|^2_{\mathrm{L}^p(\mathbb{S}^d)}-\|u\|^2_{\mathrm{L}^2(\mathbb{S}^d)}\right)\quad\forall\,u\in\mathrm{H}^1(\mathbb{S}^d)$$

- $d\ge 3$, $p\in[1,2)$ or $p\in(2,\frac{2\,d}{d-2})$
- $d=1$ or $d=2$, $p\in[1,2)$ or $p\in(2,\infty)$
- $p=-2=2\,d/(d-2)=-2$ with $d=1$ [Exner, Harrell, Loss, 1998]

$$\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{S}^d)}+\frac{1}{4}\left(\int_{\mathbb{S}^d}\frac{1}{u^2}\,d\mu\right)^{-1}\ge\frac{1}{4}\,\|u\|^2_{\mathrm{L}^2(\mathbb{S}^d)}\quad\forall\,u\in\mathrm{H}^1_+(\mathbb{S}^d)$$

Keller-Lieb-Thirring inequalities on the sphere

- The Keller-Lieb-Tirring inequality is equivalent to an interpolation inequality of Gagliardo-Nirenberg-Sobolev type
- We measure a quantitative deviation with respect to the semi-classical regime due to finite size effects

Joint work with M.J. Esteban and A. Laptev

An introduction to (Keller)-Lieb-Thirring inequalities in $\mathbb{R}^d$

$(\lambda_k)_{k\geq 1}$: eigenvalues of the Schrödinger operator $H = -\Delta - V$ on $\mathbb{R}^d$

- Euclidean case [Keller, 1961]

$$|\lambda_1|^\gamma \leq \mathrm{L}^1_{\gamma,d} \int_{\mathbb{R}^d} V_+^{\gamma+\frac{d}{2}}$$

[Lieb-Thirring, 1976]

$$\sum_{k\geq 1} |\lambda_k|^\gamma \leq \mathrm{L}_{\gamma,d} \int_{\mathbb{R}^d} V_+^{\gamma+\frac{d}{2}}$$

$\gamma \geq 1/2$ if $d = 1$, $\gamma > 0$ if $d = 2$ and $\gamma \geq 0$ if $d \geq 3$ [Weidl], [Cwikel], [Rosenbljum], [Aizenman], [Laptev-Weidl], [Helffer], [Robert], [JD-Felmer-Loss-Paturel], [JD-Laptev-Loss]...[Frank, Hundertmark, Jex, Nam]

- Compact manifolds: log Sobolev case: [Federbusch], [Rothaus]; case $\gamma = 0$ (Rozenbljum-Lieb-Cwikel inequality): [Levin-Solomyak]; [Lieb], [Levin], [Ouabaz-Poupaud]... [Ilyin]

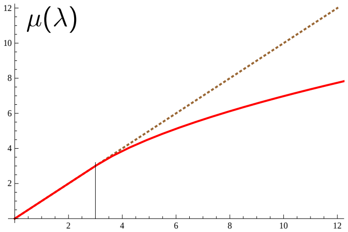
$\triangleright$ *How does one take into account the finite size effects on $\mathbb{S}^d$?*

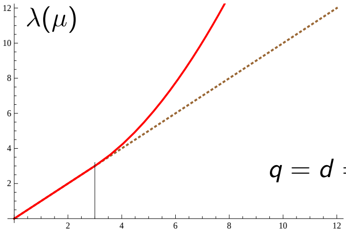
Hölder duality and link with interpolation inequalities

Let $p = \frac{q}{q-2}$. Consider the Schrödinger energy

$$\int_{\mathbb{S}^d} |\nabla u|^2 - \int_{\mathbb{S}^d} V\,|u|^2 \geq \int_{\mathbb{S}^d} |\nabla u|^2 - \mu\,\|u\|^2_{\mathrm{L}^q(\mathbb{S}^d)}$$
$$\geq -\lambda(\mu)\,\|u\|^2_{\mathrm{L}^2(\mathbb{S}^d)} \quad \text{if } \mu = \|V_+\|_{\mathrm{L}^p(\mathbb{S}^d)}$$

- *We deduce from* $\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{S}^d)} + \lambda\,\|u\|^2_{\mathrm{L}^2(\mathbb{S}^d)} \geq \mu(\lambda)\,\|u\|^2_{\mathrm{L}^q(\mathbb{S}^d)}$ *that*
$$\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{S}^d)} - \mu(\lambda)\,\|u\|^2_{\mathrm{L}^q(\mathbb{S}^d)} \geq -\lambda\,\|u\|^2_{\mathrm{L}^2(\mathbb{S}^d)}$$





$q = d = 3$

A Keller-Lieb-Thirring inequality on the sphere

Let $d \geq 1$, $p \in \big[\max\{1, d/2\}, +\infty\big)$ and $\mu_* := \frac{d}{2}\,(p-1)$

Theorem (JD-Esteban-Laptev)

There exists a convex increasing function λ s.t. $\lambda(\mu) = \mu$ if $\mu \in \big[0, \mu_*\big]$ and $\lambda(\mu) > \mu$ if $\mu \in \big(\mu_*, +\infty\big)$ and, for any $p < d/2$,

$$\boxed{|\lambda_1(-\Delta - V)| \leq \lambda\big(\|V\|_{\mathrm{L}^p(\mathbb{S}^d)}\big) \quad \forall\, V \in \mathrm{L}^p(\mathbb{S}^d)}$$

This estimate is optimal

For large values of μ, we have

$$\lambda(\mu)^{p-\frac{d}{2}} = \mathrm{L}^1_{p-\frac{d}{2},d}\,(\kappa_{q,d}\,\mu)^p\,(1+o(1))$$

If $p = d/2$ and $d \geq 3$, the inequality holds with $\lambda(\mu) = \mu$ iff $\mu \in [0, \mu_*]$

A Keller-Lieb-Thirring inequality: second formulation

Let $d \geq 1$, $\gamma = p - d/2$

Corollary (JD-Esteban-Laptev)

$$|\lambda_1(-\Delta - V)|^\gamma \lesssim \mathrm{L}^1_{\gamma,d} \int_{\mathbb{S}^d} V^{\gamma+\frac{d}{2}} \quad \textit{as} \quad \mu = \|V\|_{\mathrm{L}^{\gamma+\frac{d}{2}}(\mathbb{S}^d)} \to \infty$$

if either $\gamma > \max\{0, 1 - d/2\}$ *or* $\gamma = 1/2$ *and* $d = 1$

However, if $\mu = \|V\|_{\mathrm{L}^{\gamma+\frac{d}{2}}(\mathbb{S}^d)} \leq \mu_*$, *then we have*

$$|\lambda_1(-\Delta - V)|^{\gamma+\frac{d}{2}} \leq \int_{\mathbb{S}^d} V^{\gamma+\frac{d}{2}}$$

for any $\gamma \geq \max\{0, 1 - d/2\}$ *and this estimate is optimal*

$\mathrm{L}^1_{\gamma,d}$ is the optimal constant in the Euclidean one bound state ineq.

$$|\lambda_1(-\Delta - \phi)|^\gamma \leq \mathrm{L}^1_{\gamma,d} \int_{\mathbb{R}^d} \phi_+^{\gamma+\frac{d}{2}}\, dx$$

References

- JD, M. J. Esteban, and A. Laptev. Spectral estimates on the sphere. Analysis & PDE, 7 (2): 435-460, 2014
- JD, M.J. Esteban, A. Laptev, M. Loss. Spectral properties of Schrödinger operators on compact manifolds: Rigidity, flows, interpolation and spectral estimates. Comptes Rendus Mathématique, 351 (11-12): 437-440, 2013

Caffarelli-Kohn-Nirenberg, symmetry and symmetry breaking results, and weighted nonlinear flows

Joint work with M.J. Esteban and M. Loss

Critical Caffarelli-Kohn-Nirenberg inequality

Let $\mathcal{D}_{a,b} := \left\{ v \in \mathrm{L}^p\left(\mathbb{R}^d, |x|^{-b}\,dx\right) \,:\, |x|^{-a}\,|\nabla v| \in \mathrm{L}^2\left(\mathbb{R}^d, dx\right) \right\}$

$$\left(\int_{\mathbb{R}^d} \frac{|v|^p}{|x|^{b\,p}}\,dx\right)^{2/p} \le \mathsf{C}_{a,b} \int_{\mathbb{R}^d} \frac{|\nabla v|^2}{|x|^{2\,a}}\,dx \quad \forall\, v \in \mathcal{D}_{a,b}$$

holds under conditions on a and b

$$p = \frac{2\,d}{d-2+2\,(b-a)} \qquad \text{(critical case)}$$

▷ *An optimal function among radial functions:*

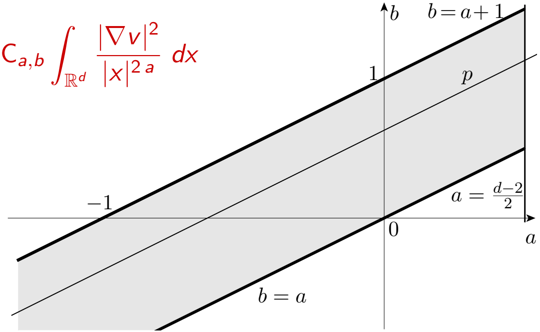
$$v_\star(x) = \left(1 + |x|^{(p-2)\,(a_c-a)}\right)^{-\frac{2}{p-2}} \quad \textit{and} \quad \mathsf{C}^\star_{a,b} = \frac{\|\,|x|^{-b}\,v_\star\,\|_p^2}{\|\,|x|^{-a}\,\nabla v_\star\,\|_2^2}$$

Question: $\mathsf{C}_{a,b} = \mathsf{C}^\star_{a,b}$ *(symmetry) or* $\mathsf{C}_{a,b} > \mathsf{C}^\star_{a,b}$ *(symmetry breaking) ?*

Critical CKN: range of the parameters

Figure: $d = 3$

$$\left(\int_{\mathbb{R}^d} \frac{|v|^p}{|x|^{b\,p}}\,dx\right)^{2/p} \le \mathsf{C}_{a,b} \int_{\mathbb{R}^d} \frac{|\nabla v|^2}{|x|^{2\,a}}\,dx$$



$$p = \frac{2\,d}{d-2+2\,(b-a)}$$

$a < a_c := (d-2)/2$

$a \le b \le a+1$ if $d \ge 3$,

$a + 1/2 < b \le a+1$ if $d = 1$

and $a < b \le a+1 < 1$,

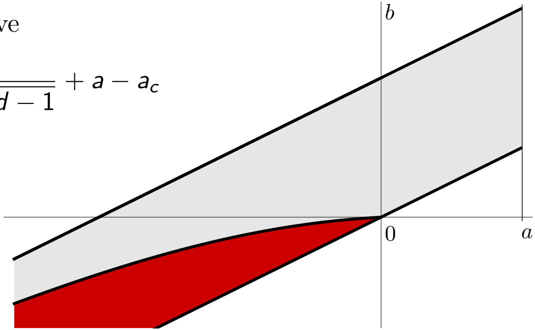
$p = 2/(b-a)$ if $d = 2$

[Il'in (1961)]
[Glaser, Martin, Grosse, Thirring (1976)]
[Caffarelli, Kohn, Nirenberg (1984)]
[F. Catrina, Z.-Q. Wang (2001)]

Linear instability of radial minimizers: the Felli-Schneider curve

The Felli & Schneider curve

$$b_{\mathrm{FS}}(a) := \frac{d\,(a_c - a)}{2\sqrt{(a_c-a)^2+d-1}} + a - a_c$$



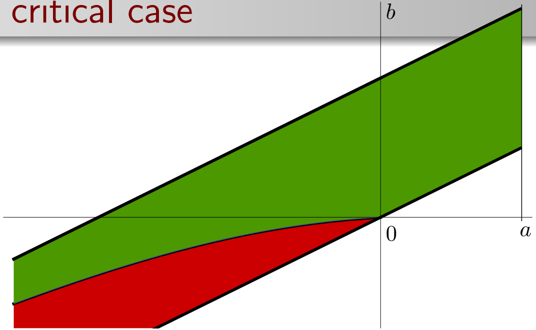
[Smets], [Smets, Willem], [Catrina, Wang], [Felli, Schneider]

$$v \mapsto \mathsf{C}^\star_{a,b}\int_{\mathbb{R}^d}\frac{|\nabla v|^2}{|x|^{2\,a}}\,dx - \left(\int_{\mathbb{R}^d}\frac{|v|^p}{|x|^{b\,p}}\,dx\right)^{2/p}$$

is linearly instable at $v = v_\star$

Symmetry *versus* symmetry breaking: the sharp result in the critical case

[JD, Esteban, Loss (2016)]



Theorem

Let $d \geq 2$ and $p < 2^*$. If either $a \in [0, a_c)$ and $b > 0$, or $a < 0$ and $b \geq b_{\mathrm{FS}}(a)$, then the optimal functions for the critical Caffarelli-Kohn-Nirenberg inequalities are radially symmetric

The symmetry proof in one slide

- A change of variables: $v(|x|^{\alpha-1}\,x)=w(x)$, $\mathsf{D}_\alpha v=\left(\alpha\,\frac{\partial v}{\partial s},\frac{1}{s}\,\nabla_\omega v\right)$

$$\|v\|_{\mathrm{L}^{2p,d-n}(\mathbb{R}^d)}\le\mathsf{K}_{\alpha,n,p}\,\|\mathsf{D}_\alpha v\|_{\mathrm{L}^{2,d-n}(\mathbb{R}^d)}^{\vartheta}\,\|v\|_{\mathrm{L}^{p+1,d-n}(\mathbb{R}^d)}^{1-\vartheta}\quad\forall\,v\in\mathrm{H}^p_{d-n,d-n}(\mathbb{R}^d)$$

- Concavity of the Rényi entropy power: with $\mathcal{L}_\alpha=-\,\mathsf{D}_\alpha^*\,\mathsf{D}_\alpha=\alpha^2\left(u''+\frac{n-1}{s}\,u'\right)+\frac{1}{s^2}\,\Delta_\omega\,u$ and $\frac{\partial u}{\partial t}=\mathcal{L}_\alpha u^m$

$$\begin{aligned}
&-\frac{d}{dt}\,\mathcal{G}[u(t,\cdot)]\left(\int_{\mathbb{R}^d}u^m\,d\mu\right)^{1-\sigma}\\
&\ge(1-m)\,(\sigma-1)\int_{\mathbb{R}^d}u^m\left|\mathcal{L}_\alpha\mathsf{P}-\frac{\int_{\mathbb{R}^d}u\,|\mathsf{D}_\alpha\mathsf{P}|^2\,d\mu}{\int_{\mathbb{R}^d}u^m\,d\mu}\right|^2d\mu\\
&+2\int_{\mathbb{R}^d}\left(\alpha^4\left(1-\frac{1}{n}\right)\left|\mathsf{P}''-\frac{\mathsf{P}'}{s}-\frac{\Delta_\omega\,\mathsf{P}}{\alpha^2\,(n-1)\,s^2}\right|^2+\frac{2\,\alpha^2}{s^2}\left|\nabla_\omega\mathsf{P}'-\frac{\nabla_\omega\mathsf{P}}{s}\right|^2\right)u^m\,d\mu\\
&+2\int_{\mathbb{R}^d}\left((n-2)\left(\alpha_{\mathrm{FS}}^2-\alpha^2\right)|\nabla_\omega\mathsf{P}|^2+c(n,m,d)\,\frac{|\nabla_\omega\mathsf{P}|^4}{\mathsf{P}^2}\right)u^m\,d\mu
\end{aligned}$$

- Elliptic regularity and the Emden-Fowler transformation: justifying the integrations by parts

The variational problem on the cylinder

▷ *With the Emden-Fowler transformation*

$$v(r,\omega)=r^{a-a_c}\,\varphi(s,\omega)\quad\text{with}\quad r=|x|\,,\quad s=-\log r\quad\text{and}\quad \omega=\frac{x}{r}$$

the variational problem becomes

$$\Lambda\mapsto\mu(\Lambda):=\min_{\varphi\in\mathrm{H}^1(\mathcal{C})}\frac{\|\partial_s\varphi\|^2_{\mathrm{L}^2(\mathcal{C})}+\|\nabla_\omega\varphi\|^2_{\mathrm{L}^2(\mathcal{C})}+\Lambda\,\|\varphi\|^2_{\mathrm{L}^2(\mathcal{C})}}{\|\varphi\|^2_{\mathrm{L}^p(\mathcal{C})}}$$

is a concave increasing function

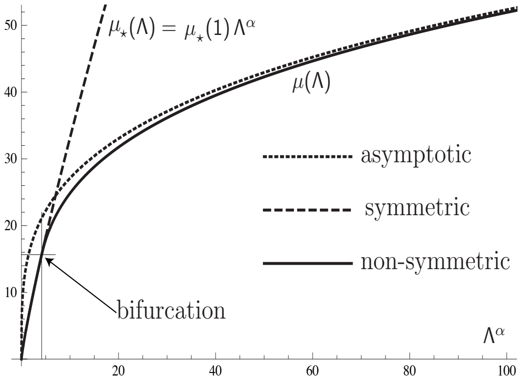
Restricted to symmetric functions, the variational problem becomes

$$\mu_\star(\Lambda):=\min_{\varphi\in\mathrm{H}^1(\mathbb{R})}\frac{\|\partial_s\varphi\|^2_{\mathrm{L}^2(\mathbb{R}^d)}+\Lambda\,\|\varphi\|^2_{\mathrm{L}^2(\mathbb{R}^d)}}{\|\varphi\|^2_{\mathrm{L}^p(\mathbb{R}^d)}}=\mu_\star(1)\,\Lambda^\alpha$$

Symmetry means $\mu(\Lambda)=\mu_\star(\Lambda)$

Symmetry breaking means $\mu(\Lambda)<\mu_\star(\Lambda)$

Numerical results



Parametric plot of the branch of optimal functions for $p = 2.8$*,* $d = 5$*. Non-symmetric solutions bifurcate from symmetric ones at a bifurcation point* Λ_1 *computed by V. Felli and M. Schneider. The branch behaves for large values of* Λ *as shown by F. Catrina and Z.-Q. Wang*

Three references

- Lecture notes on *Symmetry and nonlinear diffusion flows...* a course on entropy methods (see webpage)

- [JD, Maria J. Esteban, and Michael Loss] *Symmetry and symmetry breaking: rigidity and flows in elliptic PDEs* ... the elliptic point of view: Proc. Int. Cong. of Math., Rio de Janeiro, 3: 2279-2304, 2018.

- [JD, Maria J. Esteban, and Michael Loss] *Interpolation inequalities, nonlinear flows, boundary terms, optimality and linearization...* the parabolic point of view
Journal of elliptic and parabolic equations, 2: 267-295, 2016.

Magnetic interpolation inequalities in the Euclidean space

- ▷ Three interpolation inequalities and their dual forms
- ▷ Estimates in dimension $d = 2$ for constant magnetic fields
  - Lower estimates
  - Upper estimates and numerical results
  - A linear stability result (numerical) and an open question

- Warning: assumptions are not repeated
- Estimates are given only in the case $p > 2$ but similar estimates hold in the other cases

Joint work with M.J. Esteban, A. Laptev and M. Loss

Magnetic Laplacian and spectral gap

In dimensions $d = 2$ and $d = 3$: the *magnetic Laplacian* is

$$-\Delta_{\mathbf{A}}\,\psi = -\Delta\,\psi - 2\,i\,\mathbf{A}\cdot\nabla\psi + |\mathbf{A}|^2\psi - i\,(\operatorname{div}\mathbf{A})\,\psi$$

where the magnetic potential (resp. field) is $\mathbf{A}$ (resp. $\mathbf{B} = \operatorname{curl}\mathbf{A}$) and

$$\mathrm{H}^1_{\mathbf{A}}(\mathbb{R}^d) := \left\{\psi \in \mathrm{L}^2(\mathbb{R}^d)\,:\,\nabla_{\mathbf{A}}\psi \in \mathrm{L}^2(\mathbb{R}^d)\right\}\,, \quad \nabla_{\mathbf{A}} := \nabla + i\,\mathbf{A}$$

Spectral gap inequality

$$\|\nabla_{\mathbf{A}}\psi\|^2_{\mathrm{L}^2(\mathbb{R}^d)} \geq \Lambda[\mathbf{B}]\,\|\psi\|^2_{\mathrm{L}^2(\mathbb{R}^d)} \quad \forall\,\psi \in \mathrm{H}^1_{\mathbf{A}}(\mathbb{R}^d)$$

- Λ depends only on $\mathbf{B} = \operatorname{curl}\mathbf{A}$
- Assumption: *equality holds for some* $\psi \in \mathrm{H}^1_{\mathbf{A}}(\mathbb{R}^d)$
- If $\mathbf{B}$ is a constant magnetic field, $\Lambda[\mathbf{B}] = |\mathbf{B}|$
- If $d = 2$, $\operatorname{spec}(-\Delta_{\mathbf{A}}) = \{(2j+1)\,|\mathbf{B}|\,:\,j \in \mathbb{N}\}$ is generated by the *Landau levels*. The *Lowest Landau Level* corresponds to $j = 0$

Magnetic interpolation inequalities

$$\|\nabla_{\mathbf{A}}\psi\|^2_{\mathrm{L}^2(\mathbb{R}^d)} + \alpha\,\|\psi\|^2_{\mathrm{L}^2(\mathbb{R}^d)} \ge \mu_{\mathbf{B}}(\alpha)\,\|\psi\|^2_{\mathrm{L}^p(\mathbb{R}^d)} \quad \forall\,\psi \in \mathrm{H}^1_{\mathbf{A}}(\mathbb{R}^d)$$

for any $\alpha \in (-\Lambda[\mathbf{B}], +\infty)$ and any $p \in (2, 2^*)$,

$$\|\nabla_{\mathbf{A}}\psi\|^2_{\mathrm{L}^2(\mathbb{R}^d)} + \beta\,\|\psi\|^2_{\mathrm{L}^p(\mathbb{R}^d)} \ge \nu_{\mathbf{B}}(\beta)\,\|\psi\|^2_{\mathrm{L}^2(\mathbb{R}^d)} \quad \forall\,\psi \in \mathrm{H}^1_{\mathbf{A}}(\mathbb{R}^d)$$

for any $\beta \in (0, +\infty)$ and any $p \in (1, 2)$

$$\|\nabla_{\mathbf{A}}\psi\|^2_{\mathrm{L}^2(\mathbb{R}^d)} \ge \gamma \int_{\mathbb{R}^d} |\psi|^2 \log\left(\frac{|\psi|^2}{\|\psi\|^2_{\mathrm{L}^2(\mathbb{R}^d)}}\right) dx + \xi_{\mathbf{B}}(\gamma)\,\|\psi\|^2_{\mathrm{L}^2(\mathbb{R}^d)}$$

(limit case corresponding to $p = 2$) for any $\gamma \in (0, +\infty)$

$$\mathsf{C}_p := \begin{cases} \min_{u \in \mathrm{H}^1(\mathbb{R}^d)\setminus\{0\}} \dfrac{\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{R}^d)} + \|u\|^2_{\mathrm{L}^2(\mathbb{R}^d)}}{\|u\|^2_{\mathrm{L}^p(\mathbb{R}^d)}} & \text{if} \quad p \in (2, 2^*) \\ \min_{u \in \mathrm{H}^1(\mathbb{R}^d)\setminus\{0\}} \dfrac{\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{R}^d)} + \|u\|^2_{\mathrm{L}^p(\mathbb{R}^d)}}{\|u\|^2_{\mathrm{L}^2(\mathbb{R}^d)}} & \text{if} \quad p \in (1, 2) \end{cases}$$

$\mu_{\mathbf{0}}(1) = \mathsf{C}_p$ if $p \in (2, 2^*)$, $\nu_{\mathbf{0}}(1) = \mathsf{C}_p$ if $p \in (1, 2)$
$\xi_{\mathbf{0}}(\gamma) = \gamma \log\left(\pi\, e^2/\gamma\right)$ if $p = 2$

Technical assumptions

$\mathbf{A} \in \mathrm{L}^{\alpha}_{\mathrm{loc}}(\mathbb{R}^d)$, $\alpha > 2$ if $d = 2$ or $\alpha = 3$ if $d = 3$ and

$$\lim_{\sigma\to+\infty} \sigma^{d-2} \int_{\mathbb{R}^d} |\mathbf{A}(x)|^2\, e^{-\sigma\,|x|}\, dx = 0 \quad \text{if} \quad p \in (2, 2^*)$$

$$\lim_{\sigma\to+\infty} \frac{\sigma^{\frac{d}{2}-1}}{\log\sigma} \int_{\mathbb{R}^d} |\mathbf{A}(x)|^2\, e^{-\sigma\,|x|^2}\, dx = 0 \quad \text{if} \quad p = 2$$

$$\lim_{\sigma\to+\infty} \sigma^{d-2} \int_{|x|<1/\sigma} |\mathbf{A}(x)|^2\, dx \quad \text{if} \quad p \in (1, 2)$$

These estimates can be found in [Esteban, Lions, 1989]

A statement

Theorem

$p\in(2,2^*)$: $\mu_{\mathbf B}$ is monotone increasing on $(-\Lambda[\mathbf B],+\infty)$, concave and

$$\lim_{\alpha\to(-\Lambda[\mathbf B])_+}\mu_{\mathbf B}(\alpha)=0\quad\text{and}\quad\lim_{\alpha\to+\infty}\mu_{\mathbf B}(\alpha)\,\alpha^{\frac{d-2}{2}-\frac{d}{p}}=\mathsf C_p$$

$p\in(1,2)$: $\nu_{\mathbf B}$ is monotone increasing on $(0,+\infty)$, concave and

$$\lim_{\beta\to0_+}\nu_{\mathbf B}(\beta)=\Lambda[\mathbf B]\quad\text{and}\quad\lim_{\beta\to+\infty}\nu_{\mathbf B}(\beta)\,\beta^{-\frac{2\,p}{2\,p+d\,(2-p)}}=\mathsf C_p$$

$\xi_{\mathbf B}$ is continuous on $(0,+\infty)$, concave, $\xi_{\mathbf B}(0)=\Lambda[\mathbf B]$ and

$$\xi_{\mathbf B}(\gamma)=\tfrac{d}{2}\,\gamma\,\log\left(\tfrac{\pi\,e^2}{\gamma}\right)(1+o(1))\quad\text{as}\quad\gamma\to+\infty$$

Constant magnetic fields: equality is achieved
Nonconstant magnetic fields: only partial answers are known

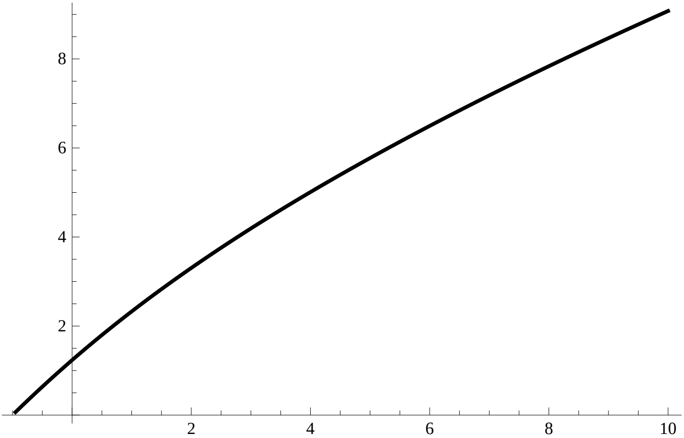


Figure: Case $d=2$, $p=3$, $B=1$: plot of $\alpha \mapsto (2\,\pi)^{\frac{2}{p}-1}\,\mu_{\mathbf{B}}(\alpha)$

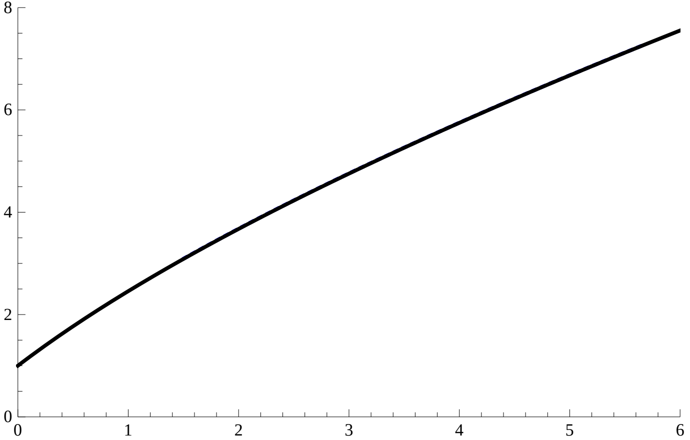


Figure: Case $d = 2$, $p = 1.4$, $B = 1$: plot of $\beta \mapsto \nu_{\mathbf{B}}(\beta)$
The horizontal axis is measured in units of $(2\,\pi)^{1-\frac{2}{p}}\,\beta$

Magnetic Keller-Lieb-Thirring inequalities

$\lambda_{\mathbf{A},V}$ is the principal eigenvalue of $-\Delta_{\mathbf{A}}+V$
$\alpha_{\mathbf{B}}:(0,+\infty)\to(-\Lambda,+\infty)$ is the inverse function of $\alpha\mapsto\mu_{\mathbf{B}}(\alpha)$

Corollary

(i) For any $q=p/(p-2)\in(d/2,+\infty)$ *and any potential* $V\in\mathrm{L}^q_+(\mathbb{R}^d)$

$$\lambda_{\mathbf{A},V}\ge-\,\alpha_{\mathbf{B}}\big(\|V\|_{\mathrm{L}^q(\mathbb{R}^d)}\big)$$

$\lim_{\mu\to0_+}\alpha_{\mathbf{B}}(\mu)=\Lambda$ *and* $\lim_{\mu\to+\infty}\alpha_{\mathbf{B}}(\mu)\,\mu^{\frac{2\,(q+1)}{d-2-2\,q}}=-\,\mathsf{C}_p^{\frac{2\,(q+1)}{d-2-2\,q}}$

(ii) For any $q=p/(2-p)\in(1,+\infty)$ *and any* $0<W^{-1}\in\mathrm{L}^q(\mathbb{R}^d)$

$$\lambda_{\mathbf{A},W}\ge\nu_{\mathbf{B}}\left(\|W^{-1}\|_{\mathrm{L}^q(\mathbb{R}^d)}^{-1}\right)$$

(iii) For any $\gamma>0$ *and any* $W\ge0$ *s.t.* $e^{-W/\gamma}\in\mathrm{L}^1(\mathbb{R}^d)$

$$\lambda_{\mathbf{A},W}\ge\xi_{\mathbf{B}}\,(\gamma)-\gamma\,\log\left(\textstyle\int_{\mathbb{R}^d}e^{-W/\gamma}\,dx\right)$$

Proofs

Interpolation without magnetic field...

Assume that $p>2$ and let C_p denote the best constant in

$$\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{R}^d)}+\|u\|^2_{\mathrm{L}^2(\mathbb{R}^d)}\ge \mathsf{C}_p\,\|u\|^2_{\mathrm{L}^p(\mathbb{R}^d)}\quad\forall\,u\in\mathrm{H}^1(\mathbb{R}^d)$$

By scaling, if we test the inequality by $u(\,\cdot\,/\lambda)$, we find that

$$\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{R}^d)}+\lambda^2\,\|u\|^2_{\mathrm{L}^2(\mathbb{R}^d)}\ge \mathsf{C}_p\,\lambda^{2-d\,(1-\frac{2}{p})}\,\|u\|^2_{\mathrm{L}^p(\mathbb{R}^d)}\quad\forall\,u\in\mathrm{H}^1(\mathbb{R}^d)\quad\forall\,\lambda>0$$

An optimization on $\lambda>0$ shows that the best constant in the scale-invariant inequality

$$\|\nabla u\|^{d\,(1-\frac{2}{p})}_{\mathrm{L}^2(\mathbb{R}^d)}\,\|u\|^{2-d\,(1-\frac{2}{p})}_{\mathrm{L}^2(\mathbb{R}^d)}\ge \mathsf{S}_p\,\|u\|^2_{\mathrm{L}^p(\mathbb{R}^d)}\quad\forall\,u\in\mathrm{H}^1(\mathbb{R}^d)$$

is given by

$$\mathsf{S}_p=\frac{1}{2\,p}\,\big(2\,p-d\,(p-2)\big)^{1-d\,\frac{p-2}{2\,p}}\,\big(d\,(p-2)\big)^{\frac{d\,(p-2)}{2\,p}}\,\mathsf{C}_p$$

... and with magnetic field

Proposition

Let $d=2$ or 3. For any $p\in(2,+\infty)$, any $\alpha>-\Lambda=-\Lambda[\mathbf{B}]<0$

$$\mu_{\mathbf{B}}(\alpha)\ge\mu_{\rm interp}(\alpha):=\begin{cases}\mathsf{S}_p\,(\alpha+\Lambda)\,\Lambda^{-d\,\frac{p-2}{2\,p}} & \text{if}\,\alpha\in\left[-\Lambda,\frac{\Lambda\,(2\,p-d\,(p-2))}{d\,(p-2)}\right]\\ \mathsf{C}_p\,\alpha^{1-d\,\frac{p-2}{2\,p}} & \text{if}\,\alpha\ge\frac{\Lambda\,(2\,p-d\,(p-2))}{d\,(p-2)}\end{cases}$$

Diamagnetic inequality: $\|\nabla|\psi|\|_{\mathrm{L}^2(\mathbb{R}^d)}\le\|\nabla_{\mathbf{A}}\psi\|_{\mathrm{L}^2(\mathbb{R}^d)}$
Non-magnetic inequality with $\lambda=\frac{\alpha+\Lambda\,t}{1-t}$, $t\in[0,1]$

$$\begin{aligned}\|\nabla_{\mathbf{A}}\psi\|^2_{\mathrm{L}^2(\mathbb{R}^d)}+\alpha\,\|\psi\|^2_{\mathrm{L}^2(\mathbb{R}^d)}&\ge t\,\left(\|\nabla_{\mathbf{A}}\psi\|^2_{\mathrm{L}^2(\mathbb{R}^d)}-\Lambda\,\|\psi\|^2_{\mathrm{L}^2(\mathbb{R}^d)}\right)\\&\quad+(1-t)\,\left(\|\nabla|\psi|\|_{\mathrm{L}^2(\mathbb{R}^d)}+\frac{\alpha+\Lambda\,t}{1-t}\,\|\psi\|^2_{\mathrm{L}^2(\mathbb{R}^d)}\right)\\&\ge\mathsf{C}_p\,(1-t)^{\frac{d\,(p-2)}{2\,p}}\,(\alpha+t\,\Lambda)^{1-d\,\frac{p-2}{2\,p}}\,\|\psi\|^2_{\mathrm{L}^p(\mathbb{R}^d)}\end{aligned}$$

and optimize on $t\in[\max\{0,-\alpha/\Lambda\},1]$

The special case of constant magnetic field in dimension $d = 2$

Constant magnetic field, $d = 2$...

Assume that $\mathbf{B} = (0, B)$ is constant, $d = 2$ and choose

$$\mathbf{A}_1 = \frac{B}{2} x_2\,, \quad \mathbf{A}_2 = -\frac{B}{2} x_1 \quad \forall\, x = (x_1, x_2) \in \mathbb{R}^2$$

Proposition

[Loss, Thaller, 1997] Consider a constant magnetic field with field strength B in two dimensions. For every $c \in [0,1]$, we have

$$\int_{\mathbb{R}^2} |\nabla_{\mathbf{A}} \psi|^2\, dx \geq \left(1 - c^2\right) \int_{\mathbb{R}^2} |\nabla \psi|^2\, dx + c\, B \int_{\mathbb{R}^2} \psi^2\, dx$$

and equality holds with $\psi = u\, e^{iS}$ and $u > 0$ if and only if

$$\left(-\,\partial_2 u^2,\, \partial_1 u^2\right) = \frac{2\, u^2}{c}\, (\mathbf{A} + \nabla S)$$

... a computation ($d=2$, constant magnetic field)

$$\begin{aligned}\int_{\mathbb{R}^2} |\nabla_{\mathbf{A}}\psi|^2\,dx &= \int_{\mathbb{R}^2} |\nabla u|^2\,dx + \int_{\mathbb{R}^2} |\mathbf{A}+\nabla S|^2\,u^2\,dx\\ &= \left(1-c^2\right)\int_{\mathbb{R}^2} |\nabla u|^2\,dx + \underbrace{\int_{\mathbb{R}^2}\left(c^2\,|\nabla u|^2 + |\mathbf{A}+\nabla S|^2\,u^2\right)dx}_{\ge \int_{\mathbb{R}^2} 2\,c\,|\nabla u|\,|\mathbf{A}+\nabla S|\,u\,dx}\end{aligned}$$

with equality only if $c\,|\nabla u| = |\mathbf{A}+\nabla S|\,u$

$$2\,|\nabla u|\,|\mathbf{A}+\nabla S|\,u = |\nabla u^2|\,|\mathbf{A}+\nabla S| \ge \left(\nabla u^2\right)^\perp \cdot (\mathbf{A}+\nabla S)$$

where $\left(\nabla u^2\right)^\perp := \left(-\,\partial_2 u^2,\,\partial_1 u^2\right)$

Equality case: $\left(-\,\partial_2 u^2,\,\partial_1 u^2\right) = \gamma\,(\mathbf{A}+\nabla S)$ for $\gamma = 2\,u^2/c$
Integration by parts yields

$$\int_{\mathbb{R}^2}\left(c^2\,|\nabla u|^2 + |\mathbf{A}+\nabla S|^2\,u^2\right)dx \ge B\,c\int_{\mathbb{R}^2} u^2\,dx$$

... a lower estimate ($d=2$, constant magnetic field)

Proposition

Consider a constant magnetic field with field strength B in two dimensions. Given any $p\in(2,+\infty)$, and any $\alpha>-B$, we have

$$\mu_{\mathbf{B}}(\alpha)\ge \mathsf{C}_p\left(1-c^2\right)^{1-\frac{2}{p}}\left(\alpha+c\,B\right)^{\frac{2}{p}}=:\mu_{\mathrm{LT}}(\alpha)$$

with

$$c=c(p,\eta)=\frac{\sqrt{\eta^2+p-1}-\eta}{p-1}=\frac{1}{\eta+\sqrt{\eta^2+p-1}}\in(0,1)$$

and $\eta=\alpha\,(p-2)/(2\,B)$

Upper estimate (1): $d = 2$, constant magnetic field

For every integer $k \in \mathbb{N}$ we introduce the special symmetry class

$$\psi(x) = \left(\frac{x_2 + i\,x_1}{|x|}\right)^k v(|x|) \quad \forall\, x = (x_1, x_2) \in \mathbb{R}^2 \qquad (\mathcal{C}_k)$$

[Esteban, Lions, 1989]: if $\psi \in \mathcal{C}_k$, then

$$\frac{1}{2\,\pi} \int_{\mathbb{R}^2} |\nabla_{\mathbf{A}} \psi|^2\, dx = \int_0^{+\infty} |v'|^2\, r\, dr + \int_0^{+\infty} \left(\tfrac{k}{r} - \tfrac{B\, r}{2}\right)^2 |v|^2\, r\, dr$$

and optimality is achieved in $\mathcal{C}_k$

Test function $v_\sigma(r) = e^{-\,r^2/(2\,\sigma)}$: an optimization on $\sigma > 0$ provides an explicit expression of $\mu_{\mathrm{Gauss}}(\alpha)$ such that

Proposition

If $p > 2$, then

$$\mu_{\mathbf{B}}(\alpha) \le \mu_{\mathrm{Gauss}}(\alpha) \quad \forall\, \alpha > -\Lambda[\mathbf{B}]$$

This estimate is not optimal because v_σ does not solve the Euler-Lagrange equations

Upper estimate (2): $d = 2$, constant magnetic field

A more numerical point of view. The Euler-Lagrange equation in $\mathcal{C}_0$ is

$$-v'' - \frac{v'}{r} + \left(\frac{B^2}{4}\,r^2 + \alpha\right) v = \mu_{\mathrm{EL}}(\alpha) \left(\int_0^{+\infty} |v|^p\,r\,dr\right)^{\frac{2}{p}-1} |v|^{p-2}\,v$$

We can restrict the problem to positive solutions such that

$$\mu_{\mathrm{EL}}(\alpha) = \left(\int_0^{+\infty} |v|^p\,r\,dr\right)^{1-\frac{2}{p}}$$

and then we have to solve the reduced problem

$$-v'' - \frac{v'}{r} + \left(\frac{B^2}{4}\,r^2 + \alpha\right) v = |v|^{p-2}\,v$$

Numerical results and the symmetry issue

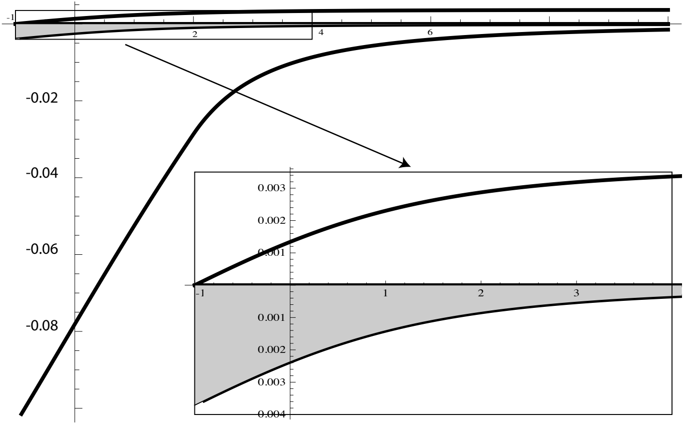


Figure: Case $d=2$, $p=3$, $B=1$

Upper estimates: $\alpha \mapsto \mu_{\rm Gauss}(\alpha)$, $\mu_{\rm EL}(\alpha)$

Lower estimates: $\alpha \mapsto \mu_{\rm interp}(\alpha)$, $\mu_{\rm LT}(\alpha)$

The exact value associated with $\mu_{\mathbf{B}}$ lies in the grey area.

Plots represent the curves $\log_{10}(\mu/\mu_{\rm EL})$

Asymptotics (1): *Lowest Landau Level*

Proposition

Let $d=2$ and consider a constant magnetic field with field strength B. If ψ_α is a minimizer for $\mu_{\mathbf{B}}(\alpha)$ such that $\|\psi_\alpha\|_{\mathrm{L}^p(\mathbb{R}^d)}=1$, then there exists a non trivial $\varphi_\alpha\in\mathrm{LLL}$ such that

$$\lim_{\alpha\to(-B)_+}\|\psi_\alpha-\varphi_\alpha\|_{\mathrm{H}^1_{\mathbf{A}}(\mathbb{R}^2)}=0$$

Let $\psi_\alpha\in\mathrm{H}^1_{\mathbf{A}}(\mathbb{R}^2)$ be an optimal function such that $\|\psi_\alpha\|_{\mathrm{L}^p(\mathbb{R}^d)}=1$ and let us decompose it as $\psi_\alpha=\varphi_\alpha+\chi_\alpha$, where $\varphi_\alpha\in\mathrm{LLL}$ and χ_α is in the orthogonal of LLL

$$\mu_{\mathbf{B}}(\alpha)\geq(\alpha+B)\|\varphi_\alpha\|^2_{\mathrm{L}^2(\mathbb{R}^d)}+(\alpha+3B)\|\chi_\alpha\|^2_{\mathrm{L}^2(\mathbb{R}^d)}\geq(\alpha+3B)\|\chi_\alpha\|^2_{\mathrm{L}^2(\mathbb{R}^d)}\sim 2B\|\chi_\alpha\|^2_{\mathrm{L}^2}$$

as $\alpha\to(-B)_+$ because $\|\nabla\chi_\alpha\|^2_{\mathrm{L}^2(\mathbb{R}^d)}\geq 3B\|\chi_\alpha\|^2_{\mathrm{L}^2(\mathbb{R}^d)}$
Since $\lim_{\alpha\to(-B)_+}\mu_{\mathbf{B}}(\alpha)=0$, $\lim_{\alpha\to(-B)_+}\|\chi_\alpha\|_{\mathrm{L}^2(\mathbb{R}^d)}=0$ and

$$\mu_{\mathbf{B}}(\alpha)=(\alpha+B)\|\varphi_\alpha\|^2_{\mathrm{L}^2(\mathbb{R}^d)}+\|\nabla_{\mathbf{A}}\chi_\alpha\|^2_{\mathrm{L}^2(\mathbb{R}^d)}+\alpha\|\chi_\alpha\|^2_{\mathrm{L}^2(\mathbb{R}^d)}\geq\frac{2}{3}\|\nabla_{\mathbf{A}}\chi_\alpha\|^2_{\mathrm{L}^2(\mathbb{R}^d)}$$

concludes the proof

Asymptotics (2): *semi-classical regime*

Let us consider the small magnetic field regime. We assume that the magnetic potential is given by

$$\mathbf{A}_1=\frac{B}{2}x_2\,,\quad \mathbf{A}_2=-\frac{B}{2}x_1\quad \forall\,x=(x_1,x_2)\in\mathbb{R}^2$$

if $d=2$. In dimension $d=3$, we choose $\mathbf{A}=\frac{B}{2}(-x_2,x_1,0)$ and observe that the constant magnetic field is $\mathbf{B}=(0,0,B)$, while the spectral gap is $\Lambda[\mathbf{B}]=B$.

Proposition

Let $d=2$ or 3 and consider a constant magnetic field $\mathbf{B}$ of intensity B with magnetic potential $\mathbf{A}$
For any $p\in(2,2^*)$ and any fixed α and $\mu>0$, we have

$$\lim_{\varepsilon\to0_+}\mu_{\varepsilon\,\mathbf{B}}(\alpha)=\mathsf{C}_p\,\alpha^{\frac{d}{p}-\frac{d-2}{2}}$$

Consider any function $\psi\in\mathrm{H}^1_{\mathbf{A}}(\mathbb{R}^d)$ and let $\psi(x)=\chi(\sqrt{\varepsilon}\,x)$, $\sqrt{\varepsilon}\,\mathbf{A}(x/\sqrt{\varepsilon})=\mathbf{A}(x)$ with our conventions on $\mathbf{A}$

Numerical stability of radial optimal functions

Let us denote by ψ_0 an optimal function in $(\mathcal{C}_0)$ such that

$$-\psi_0'' - \frac{\psi_0'}{r} + \left(\frac{B^2}{4}\,r^2 + \alpha\right)\psi_0 = |\psi_0|^{p-2}\,\psi_0$$

and consider the test function

$$\psi_\varepsilon = \psi_0 + \varepsilon\, e^{i\,\theta}\, v$$

where $v = v(r)$ and $e^{i\,\theta} = (x_1 + i\,x_2)/r$
As $\varepsilon \to 0_+$, the leading order term is

$$2\,\pi\left[\int_{\mathbb{R}^2} |v'|^2\,dx + \int_{\mathbb{R}^2}\left(\left(\tfrac{1}{r} - \tfrac{B\,r}{2}\right)^2 + \alpha\right)|v|^2\,dx - \tfrac{p}{2}\int_0^{+\infty} |\psi_0|^{p-2}\,v^2\,r\,dr\right]\varepsilon^2$$

and we have to solve the eigenvalue problem

$$-v'' - \frac{v'}{r} + \left(\left(\tfrac{1}{r} - \tfrac{B\,r}{2}\right)^2 + \alpha\right)v - \tfrac{p}{2}\,|\psi_0|^{p-2}\,v = \mu\,v$$

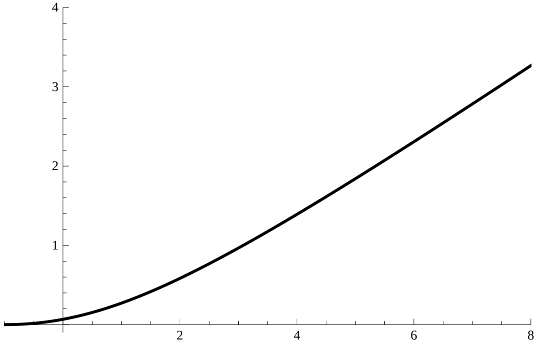


Figure: Case $p = 3$ and $B = 1$: plot of the eigenvalue μ as a function of α A careful investigation shows that μ is always positive, including in the limiting case as $\alpha \to (-B)_+$, thus proving the numerical stability of the optimal function in $\mathcal{C}_0$ with respect to perturbations in $\mathcal{C}_1$

An open question of symmetry

- [Bonheure, Nys, Van Schaftingen, 2016] for a fixed $\alpha > 0$ and for $\mathbf{B}$ small enough, the optimal functions are radially symmetric functions, *i.e.*, belong to $\mathcal{C}_0$
This regime is equivalent to the regime as $\alpha \to +\infty$ for a given $\mathbf{B}$, at least if the magnetic field is constant
- Numerically our upper and lower bounds are (in dimension $d = 2$, for a constant magnetic field) numerically extremely close
- The optimal function in $\mathcal{C}_0$ is linearly stable with respect to perturbations in $\mathcal{C}_1$

▷ *Prove that the optimality case is achieved among radial function if $d = 2$ and $\mathbf{B}$ is a constant magnetic field*

Reference

- JD, M.J. Esteban, A. Laptev, M. Loss. Interpolation inequalities and spectral estimates for magnetic operators. Annales Henri Poincaré, 19 (5): 1439-1463, May 2018

Magnetic rings

▷ A magnetic interpolation inequality on $\mathbb{S}^1$: with $p > 2$

$$\|\psi' + i\,a\,\psi\|^2_{\mathrm{L}^2(\mathbb{S}^1)} + \alpha\,\|\psi\|^2_{\mathrm{L}^2(\mathbb{S}^1)} \ge \mu_{a,p}(\alpha)\,\|\psi\|^2_{\mathrm{L}^p(\mathbb{S}^1)}$$

▷ Consequences

- A Keller-Lieb-Thirring inequality
- A new Hardy inequality for Aharonov-Bohm magnetic fields in $\mathbb{R}^2$

Joint work with M.J. Esteban, A. Laptev and M. Loss

Magnetic flux, a reduction

Assume that $a:\mathbb{R}\to\mathbb{R}$ is a 2π-periodic function such that its restriction to $(-\pi,\pi]\approx\mathbb{S}^1$ is in $\mathrm{L}^1(\mathbb{S}^1)$ and define the space

$$X_a:=\left\{\psi\in C_{\rm per}(\mathbb{R})\,:\,\psi'+i\,a\,\psi\in\mathrm{L}^2(\mathbb{S}^1)\right\}$$

- A standard change of gauge (see *e.g.* [Ilyin, Laptev, Loss, Zelik, 2016])

$$\psi(s)\mapsto e^{i\int_{-\pi}^{s}(a(s)-\bar a)\,\mathrm{d}\sigma}\,\psi(s)$$

where $\bar a:=\int_{-\pi}^{\pi}a(s)\,\mathrm{d}\sigma$ is the *magnetic flux*, reduces the problem to

$$a \textit{ is a constant function}$$

- For any $k\in\mathbb{Z}$, ψ by $s\mapsto e^{iks}\,\psi(s)$ shows that $\mu_{a,p}(\alpha)=\mu_{k+a,p}(\alpha)$

$$a\in[0,1]$$

- $\mu_{a,p}(\alpha)=\mu_{1-a,p}(\alpha)$ because
$|\psi'+i\,a\,\psi|^2=|\chi'+i\,(1-a)\,\chi|^2=\left|\overline{\psi}'-i\,a\,\overline{\psi}\right|^2$ if $\chi(s)=e^{-is}\,\overline{\psi(s)}$

$$a\in[0,1/2]$$

Optimal interpolation

We want to characterize the *optimal constant* in the inequality

$$\|\psi' + i\,a\,\psi\|^2_{\mathrm{L}^2(\mathbb{S}^1)} + \alpha\,\|\psi\|^2_{\mathrm{L}^2(\mathbb{S}^1)} \geq \mu_{a,p}(\alpha)\,\|\psi\|^2_{\mathrm{L}^p(\mathbb{S}^1)}$$

written for any $p > 2$, $a \in (0, 1/2]$, $\alpha \in (-a^2, +\infty)$, $\psi \in X_a$

$$\mu_{a,p}(\alpha) := \inf_{\psi \in X_a \setminus \{0\}} \frac{\int_{-\pi}^{\pi} \left(|\psi' + i\,a\,\psi|^2 + \alpha\,|\psi|^2\right) d\sigma}{\|\psi\|^2_{\mathrm{L}^p(\mathbb{S}^1)}}$$

$p = -2 = 2\,d/(d-2)$ with $d = 1$ [Exner, Harrell, Loss, 1998]
$p = +\infty$ [Galunov, Olienik, 1995] [Ilyin, Laptev, Loss, Zelik, 2016]
$\lim_{\alpha \to -a^2} \mu_{a,p}(\alpha) = 0$ [JD, Esteban, Laptev, Loss, 2016]

Using a Fourier series $\psi(s) = \sum_{k\in\mathbb{Z}} \psi_k\, e^{iks}$, we obtain that

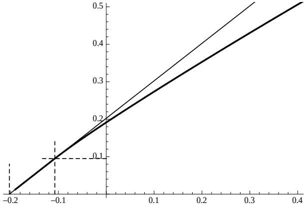
$$\|\psi' + i\,a\,\psi\|^2_{\mathrm{L}^2(\mathbb{S}^1)} = \sum_{k\in\mathbb{Z}} (a+k)^2\,|\psi_k|^2 \geq a^2\,\|\psi\|^2_{\mathrm{L}^2(\mathbb{S}^1)}$$

$\psi \mapsto \|\psi' + i\,a\,\psi\|^2_{\mathrm{L}^2(\mathbb{S}^1)} + \alpha\,\|\psi\|^2_{\mathrm{L}^2(\mathbb{S}^1)}$ is coercive for any $\alpha > -a^2$

An interpolation result for the magnetic ring

Theorem

For any $p > 2$, $a \in \mathbb{R}$, *and* $\alpha > -a^2$, $\mu_{a,p}(\alpha)$ *is achieved and*
(i) if $a \in [0, 1/2]$ *and* $a^2\,(p+2) + \alpha\,(p-2) \leq 1$, *then* $\mu_{a,p}(\alpha) = a^2 + \alpha$ *and equality is achieved only by the constant functions*
(ii) if $a \in [0, 1/2]$ *and* $a^2\,(p+2) + \alpha\,(p-2) > 1$, *then* $\mu_{a,p}(\alpha) < a^2 + \alpha$ *and equality is not achieved by the constant functions*
If $\alpha > -a^2$, $a \mapsto \mu_{a,p}(\alpha)$ *is monotone increasing on* $(0, 1/2)$



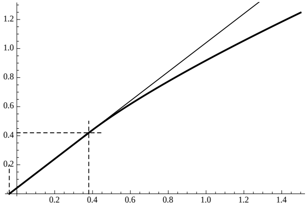


Figure: $\alpha \mapsto \mu_{a,p}(\alpha)$ with $p = 4$ and (left) $a = 0.45$ or (right) $a = 0.2$

The proof: how to eliminate the phase

Reformulations of the interpolation problem (1/3)

Any minimizer $\psi \in X_a$ of $\mu_{a,p}(\alpha)$ satisfies the Euler-Lagrange equation

$$(H_a + \alpha)\,\psi = |\psi|^{p-2}\,\psi\,, \quad H_a\psi = -\left(\frac{d}{ds} + i\,a\right)^2 \psi \qquad (*)$$

up to a multiplication by a constant and $v(s) = \psi(s)\,e^{ias}$ satisfies the condition

$$v(s + 2\pi) = e^{2i\pi a}\,v(s) \quad \forall\, s \in \mathbb{R}$$

Hence

$$\mu_{a,p}(\alpha) = \min_{v \in Y_a \setminus \{0\}} \mathsf{Q}_{p,\alpha}[v]$$

where $Y_a := \left\{v \in C(\mathbb{R}) \,:\, v' \in \mathrm{L}^2(\mathbb{S}^1)\,,\ (*) \text{ holds}\right\}$ and

$$\mathsf{Q}_{p,\alpha}[v] := \frac{\|v'\|^2_{\mathrm{L}^2(\mathbb{S}^1)} + \alpha\,\|v\|^2_{\mathrm{L}^2(\mathbb{S}^1)}}{\|v\|^2_{\mathrm{L}^p(\mathbb{S}^1)}}$$

Reformulations of the interpolation problem (2/3)

With $v = u\,e^{i\phi}$ the boundary condition becomes

$$u(\pi) = u(-\pi)\,, \quad \phi(\pi) = 2\pi\,(a+k) + \phi(-\pi) \qquad (**)$$

for some $k \in \mathbb{Z}$, and $\|v'\|^2_{\mathrm{L}^2(\mathbb{S}^1)} = \|u'\|^2_{\mathrm{L}^2(\mathbb{S}^1)} + \|u\,\phi'\|^2_{\mathrm{L}^2(\mathbb{S}^1)}$
Hence

$$\mu_{a,p}(\alpha) = \min_{(u,\phi)\in Z_a\setminus\{0\}} \frac{\|u'\|^2_{\mathrm{L}^2(\mathbb{S}^1)} + \|u\,\phi'\|^2_{\mathrm{L}^2(\mathbb{S}^1)} + \alpha\,\|u\|^2_{\mathrm{L}^2(\mathbb{S}^1)}}{\|u\|^2_{\mathrm{L}^p(\mathbb{S}^1)}}$$

where $Z_a := \left\{(u,\phi) \in C(\mathbb{R})^2 \,:\, u',\, u\,\phi' \in \mathrm{L}^2(\mathbb{S}^1)\,,\ (**)\text{ holds}\right\}$

Reformulations of the interpolation problem (3/3)

We use the Euler-Lagrange equations

$$-u'' + |\phi'|^2\,u + \alpha\,u = |u|^{p-2}\,u \quad \text{and} \quad (\phi'\,u^2)' = 0$$

Integrating the second equation, and *assuming that u never vanishes*, we find a constant L such that $\phi' = L/u^2$. Taking (*) into account, we deduce from

$$L\int_{-\pi}^{\pi}\frac{\mathrm{d}s}{u^2} = \int_{-\pi}^{\pi}\phi'\,\mathrm{d}s = 2\pi\,(a+k)$$

that

$$\|u\,\phi'\|^2_{\mathrm{L}^2(\mathbb{S}^1)} = L^2\int_{-\pi}^{\pi}\frac{\mathrm{d}\sigma}{u^2} = \frac{(a+k)^2}{\|u^{-1}\|^2_{\mathrm{L}^2(\mathbb{S}^1)}}$$

Hence

$$\phi(s) - \phi(0) = \frac{a+k}{\|u^{-1}\|^2_{\mathrm{L}^2(\mathbb{S}^1)}}\int_{-\pi}^{s}\frac{\mathrm{d}s}{u^2}$$

Let us define

$$\mathcal{Q}_{a,p,\alpha}[u] := \frac{\|u'\|^2_{\mathrm{L}^2(\mathbb{S}^1)} + a^2\,\|u^{-1}\|^{-2}_{\mathrm{L}^2(\mathbb{S}^1)} + \alpha\,\|u\|^2_{\mathrm{L}^2(\mathbb{S}^1)}}{\|u\|^2_{\mathrm{L}^p(\mathbb{S}^1)}}$$

Lemma

For any $a \in (0,1/2)$, $p > 2$, $\alpha > -a^2$,

$$\mu_{a,p}(\alpha) = \min_{u \in \mathrm{H}^1(\mathbb{S}^1)\setminus\{0\}} \mathcal{Q}_{a,p,\alpha}[u]$$

is achieved by a function $u > 0$

Proofs

- The existence proof is done on the original formulation of the problem using the diamagnetic inequality
- To prove that $|\psi| \neq 0$, we use $\psi(s)\, e^{ias} = v_1(s) + i\, v_2(s)$, solves

$$-\, v_j'' + \alpha\, v_j = (v_1^2 + v_2^2)^{\frac{p}{2}-1}\, v_j\,, \quad j = 1\,,\ 2$$

and the Wronskian $w = (v_1\, v_2' - \, v_1'\, v_2)$ is constant so that $\psi(s) = 0$ is incompatible with the twisted boundary condition

- if $a^2\,(p+2) + \alpha\,(p-2) \le 1$, then $\mu_{a,p}(\alpha) = a^2 + \alpha$ because

$$\|u'\|^2_{\mathrm{L}^2(\mathbb{S}^1)} + a^2\,\|u^{-1}\|^{-2}_{\mathrm{L}^2(\mathbb{S}^1)} + \alpha\,\|u\|^2_{\mathrm{L}^2(\mathbb{S}^1)} = (1-4\,a^2)\,\|u'\|^2_{\mathrm{L}^2(\mathbb{S}^1)} + \alpha\,\|u\|^2_{\mathrm{L}^2(\mathbb{S}^1)} + 4\,a^2\left(\|u'\|^2_{\mathrm{L}^2(\mathbb{S}^1)} + \tfrac{1}{4}\,\|u^{-1}\|^2_{\mathrm{L}^2(\mathbb{S}^1)}\right)$$

- if $a^2\,(p+2) + \alpha\,(p-2) > 1$, the test function $u_\varepsilon := 1 + \varepsilon\, w_1$

$$\mathcal{Q}_{a,p,\alpha}[u_\varepsilon] = a^2 + \alpha + \big(1 - a^2\,(p+2) - \,\alpha\,(p-2)\big)\,\varepsilon^2 + o(\varepsilon^2)$$

proves the linear instability of the constants and $\mu_{a,p}(\alpha) < a^2 + \alpha$

$$\mathcal{Q}_{a,p,\alpha}[u] := \frac{\|u'\|^2_{\mathrm{L}^2(\mathbb{S}^1)} + a^2\,\|u^{-1}\|^{-2}_{\mathrm{L}^2(\mathbb{S}^1)} + \alpha\,\|u\|^2_{\mathrm{L}^2(\mathbb{S}^1)}}{\|u\|^2_{\mathrm{L}^p(\mathbb{S}^1)}}\,,$$

$$\mu_{a,p}(\alpha) = \min_{u\in\mathrm{H}^1(\mathbb{S}^1)\setminus\{0\}} \mathcal{Q}_{a,p,\alpha}[u]$$

$$\mathsf{Q}_{p,\alpha}[u] = \mathcal{Q}_{a=0,p,\alpha}[u]\,,\quad \nu_p(\alpha) := \inf_{v\in\mathrm{H}^1_0(\mathbb{S}^1)\setminus\{0\}} \mathsf{Q}_{p,\alpha}[v]$$

Proposition

$\forall\,p>2$, $\alpha>-\,a^2$, *we have* $\mu_{a,p}(\alpha)<\mu_{1/2,p}(\alpha)\le\nu_p(\alpha)=\mu_{1/2,p}(\alpha)$

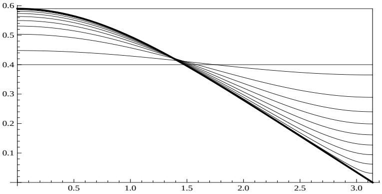


Figure: $p=4,\ \alpha=0,\ a=0.40,\ 0.41,\dots 0.49;\ u''+u^{p-1}=0$

Consequences: Keller-Lieb-Thirring inequalities and Hardy inequalities for Aharonov-Bohm magnetic fields

A Keller-Lieb-Thirring inequality

Magnetic Schrödinger operator $H_a - \varphi = -\left(\frac{d}{ds} + i\,a\right)^2 \psi - \varphi$

- The function $\alpha \mapsto \mu_{a,p}(\alpha)$ is monotone increasing, concave, and therefore has an inverse, denoted by $\alpha_{a,p} : \mathbb{R}^+ \to (-a^2, +\infty)$, which is monotone increasing, and convex

Corollary

Let $p > 2$, $a \in [0, 1/2]$, $q = p/(p-2)$ and assume that φ is a non-negative function in $\mathrm{L}^q(\mathbb{S}^1)$. Then

$$\lambda_1(H_a - \varphi) \geq -\,\alpha_{a,p}\left(\|\varphi\|_{\mathrm{L}^q(\mathbb{S}^1)}\right)$$

and $\alpha_{a,p}(\mu) = \mu - a^2$ iff $4\,a^2 + \mu\,(p-2) \leq 1$ (optimal φ is constant)

Equality is achieved

Aharonov-Bohm magnetic fields

On the two-dimensional Euclidean space $\mathbb{R}^2$, let us introduce the polar coordinates $(r,\vartheta)\in[0,+\infty)\times\mathbb{S}^1$ of $\mathbf{x}\in\mathbb{R}^2$ and consider a magnetic potential $\mathbf{a}$ in a transversal (Poincaré) gauge, or Poincaré gauge

$$(\mathbf{a},\mathbf{e}_r)=0 \quad\text{and}\quad (\mathbf{a},\mathbf{e}_\vartheta)=a_\vartheta(r,\vartheta)$$

Magnetic Schrödinger energy

$$\int_{\mathbb{R}^2}|(i\,\nabla+\mathbf{a})\,\Psi|^2\,d\mathbf{x}=\int_0^{+\infty}\int_{-\pi}^{\pi}\left(|\partial_r\Psi|^2+\frac{1}{r^2}\,|\,\partial_\vartheta\Psi+i\,r\,a_\vartheta\,\Psi|^2\right)r\,d\vartheta\,dr$$

Aharonov-Bohm magnetic fields: $a_\vartheta(r,\vartheta)=a/r$ for some constant $a\in\mathbb{R}$ (a is the *magnetic flux*), with magnetic field $b=\operatorname{curl}\mathbf{a}$

$$\int_{\mathbb{R}^2}|(i\,\nabla+\mathbf{a})\,\Psi|^2\,d\mathbf{x}\ge\tau\int_{\mathbb{R}^2}\frac{\varphi(\mathbf{x}/|\mathbf{x}|)}{|\mathbf{x}|^2}\,|\Psi|^2\,\mathrm{d}\mathbf{x}\quad\forall\,\varphi\in\mathrm{L}^q(\mathbb{S}^1)\,,\quad q\in(1,+\infty)$$

$$\Longrightarrow\tau=\tau\left(a,\|\varphi\|_{\mathrm{L}^q(\mathbb{S}^1)}\right)\ ?$$

Hardy inequalities

[Hoffmann-Ostenhof, Laptev, 2015] proved Hardy's inequality

$$\int_{\mathbb{R}^d} |\nabla \Psi|^2 \, \mathrm{d}\mathbf{x} \geq \tau \int_{\mathbb{R}^d} \frac{\varphi(\mathbf{x}/|\mathbf{x}|)}{|\mathbf{x}|^2} \, |\Psi|^2 \, \mathrm{d}\mathbf{x}$$

where the constant τ depends on the value of $\|\varphi\|_{\mathrm{L}^q(\mathbb{S}^{d-1})}$ and $d \geq 3$

Aharonov-Bohm vector potential in dimension $d = 2$

$$\mathbf{a}(\mathbf{x}) = a \left(\frac{x_2}{|\mathbf{x}|^2}, \frac{-x_1}{|\mathbf{x}|^2} \right), \quad \mathbf{x} = (x_1, x_2) \in \mathbb{R}^2, \quad a \in \mathbb{R}$$

and recall the inequality [Laptev, Weidl, 1999]

$$\int_{\mathbb{R}^2} |(i \nabla + \mathbf{a}) \Psi|^2 \, \mathrm{d}\mathbf{x} \geq \min_{k \in \mathbb{Z}} (a - k)^2 \int_{\mathbb{R}^2} \frac{|\Psi|}{|\mathbf{x}|^2} \, \mathrm{d}\mathbf{x}$$

A new Hardy inequality

$$\int_{\mathbb{R}^2} |(i\,\nabla+\mathbf{a})\,\Psi|^2\,dx \ge \tau \int_{\mathbb{R}^2} \frac{\varphi(\mathbf{x}/|\mathbf{x}|)}{|\mathbf{x}|^2}\,|\Psi|^2\,\mathrm{d}\mathbf{x} \quad \forall\,\varphi\in \mathrm{L}^q(\mathbb{S}^1)\,, \quad q\in(1,+\infty)$$

Corollary

Let $p>2$, $a\in[0,1/2]$, $q=p/(p-2)$ and assume that φ is a non-negative function in $\mathrm{L}^q(\mathbb{S}^1)$. Then the inequality holds with $\tau>0$ given by

$$\alpha_{a,p}\left(\tau\,\|\varphi\|_{\mathrm{L}^q(\mathbb{S}^1)}\right)=0$$

Moreover, $\tau=a^2/\|\varphi\|_{\mathrm{L}^q(\mathbb{S}^1)}$ if $4\,a^2+\|\varphi\|_{\mathrm{L}^q(\mathbb{S}^1)}\,(p-2)\le 1$

For any $a\in(0,1/2)$, by taking φ constant, small enough in order that $4\,a^2+\|\varphi\|_{\mathrm{L}^q(\mathbb{S}^1)}\,(p-2)\le 1$, we recover the inequality

$$\int_{\mathbb{R}^2} |(i\,\nabla+\mathbf{a})\,\Psi|^2\,\mathrm{d}\mathbf{x} \ge a^2 \int_{\mathbb{R}^2} \frac{|\Psi|^2}{|\mathbf{x}|^2}\,\mathrm{d}\mathbf{x}$$

Proofs (Keller-Lieb-Thirring inequality)

Hölder's inequality

$$\|\psi' + i\, a\, \psi\|^2_{\mathrm{L}^2(\mathbb{S}^d)} - \int_{-\pi}^{\pi} \varphi\, |\psi|^2\, d\sigma \geq \|\psi' + i\, a\, \psi\|^2_{\mathrm{L}^2(\mathbb{S}^d)} - \mu\, \|\psi\|^2_{\mathrm{L}^p(\mathbb{S}^d)}$$

where $\mu = \|\varphi\|_{\mathrm{L}^q(\mathbb{S}^d)}$ and $\frac{1}{q} + \frac{2}{p} = 1$: choose $\mu_{a,p}(\alpha) = \mu$

$$\|\psi' + i\, a\, \psi\|^2_{\mathrm{L}^2(\mathbb{S}^d)} - \mu\, \|\psi\|^2_{\mathrm{L}^p(\mathbb{S}^d)} \geq -\, \alpha\, \|\psi\|^2_{\mathrm{L}^2(\mathbb{S}^d)}$$

Proofs (Hardy inequality)

Let $\tau \geq 0$, $\mathbf{x} = (r, \vartheta) \in \mathbb{R}^2$ be polar coordinates in $\mathbb{R}^2$

$$\int_{\mathbb{R}^2} \left(|(i\,\nabla + \mathbf{a})\,\Psi|^2 - \tau\,\frac{\varphi}{|x|^2}\,|\Psi|^2 \right) \mathrm{d}\mathbf{x}$$

$$= \int_0^\infty \int_{\mathbb{S}^1} \Big(\underbrace{r\,|\partial_r \Psi|^2}_{\geq 0} + \frac{1}{r}\,|\partial_\vartheta \Psi + i\,a\,\Psi|^2 - \tau\,\frac{\varphi}{r}\,|\Psi|^2 \Big) \mathrm{d}\vartheta\,\mathrm{d}r$$

$$\geq \lambda_1\,(H_a - \tau\,\varphi) \int_0^\infty \int_{\mathbb{S}^1} \frac{1}{r}\,|\Psi|^2\,\mathrm{d}\vartheta\,\mathrm{d}r$$

$$\geq -\,\alpha_{a,p}(\tau\,\|\varphi\|_{\mathrm{L}^q(\mathbb{S}^d)}) \int_0^\infty \int_{\mathbb{S}^1} \frac{1}{r}\,|\Psi|^2\,\mathrm{d}\vartheta$$

- If $\tau = 0$, then $\alpha_{a,p}(\tau\,\|\varphi\|_{\mathrm{L}^q(\mathbb{S}^d)}) = \alpha_{a,p}(0) = -\,a^2$
- $\alpha_{a,p}(\tau\,\|\varphi\|_{\mathrm{L}^q(\mathbb{S}^d)}) > 0$ for τ large

$\Longrightarrow\ \exists\,!\ \tau > 0$ such that $\alpha_{a,p}(\tau\,\|\varphi\|_{\mathrm{L}^q(\mathbb{S}^d)}) = 0$

Comments

$\triangleright$ The region $a^2\,(p+2)+\alpha\,(p-2)<1$ is exactly the set where the constant functions are linearly stable critical points

$\triangleright$ The proof of the *rigidity result* is based
- neither on the *carré du champ* method, at least directly
- nor on a Fourier representation of the operator as it was the case in earlier proofs ($p=+\infty$, or $p>2$ and $\alpha=0$)

$\triangleright$ Magnetic rings: see [Bonnaillie-Noël, Hérau, Raymond, 2017]

$\triangleright$ Deducing *Hardy's inequality* applied with *Aharonov-Bohm* magnetic fields from a *Keller-Lieb-Thirring inequality* is an extension of [Hoffmann-Ostenhof, Laptev, 2015] to the magnetic case

$\triangleright$ Our results are not limited to the semi-classical regime

Symmetry in Aharonov-Bohm magnetic fields

- Aharonov-Bohm effect
- Interpolation and Keller-Lieb-Thirring inequalities in $\mathbb{R}^2$
  ▷ Statements
  ▷ Constants and numerics
- Symmetry and symmetry breaking

Joint work with D. Bonheure, M.J. Esteban, A. Laptev, & M. Loss

Aharonov-Bohm effect

A major difference between classical mechanics and quantum mechanics is that particles are described by a non-local object, the wave function. In 1959 Y. Aharonov and D. Bohm proposed a series of experiments intended to put in evidence such phenomena which are nowadays called *Aharonov-Bohm effects*

One of the proposed experiments relies on a long, thin solenoid which produces a magnetic field such that the region in which the magnetic field is non-zero can be approximated by a line in dimension $d = 3$ and by a point in dimension $d = 2$

▷ [Physics today, 2009] *"The notion, introduced 50 years ago, that electrons could be affected by electromagnetic potentials without coming in contact with actual force fields was received with a skepticism that has spawned a flourishing of experimental tests and expansions of the original idea."* Problem solved by considering appropriate weak solutions !
▷ Is the wave function a physical object or is its modulus ? Decisive experiments have been done only 20 years ago

The interpolation inequality

Let us consider an Aharonov-Bohm vector potential

$$\mathbf{A}(x) = \frac{a}{|x|^2}\,(x_2, -x_1)\;, \quad x = (x_1, x_2) \in \mathbb{R}^2 \setminus \{0\}\;, \quad a \in \mathbb{R}$$

Magnetic Hardy inequality [Laptev, Weidl, 1999]

$$\int_{\mathbb{R}^2} |\nabla_{\mathbf{A}}\,\psi|^2\,dx \ge \min_{k\in\mathbb{Z}}\,(a-k)^2 \int_{\mathbb{R}^2} \frac{|\psi|^2}{|x|^2}\,dx$$

where $\nabla_{\mathbf{A}}\,\psi := \nabla\psi + i\,\mathbf{A}\,\psi$, so that, with $\psi = |\psi|\,e^{iS}$

$$\int_{\mathbb{R}^2} |\nabla_{\mathbf{A}}\,\psi|^2\,dx = \int_{\mathbb{R}^2} \left[(\partial_r\,|\psi|)^2 + (\partial_r S)^2\,|\psi|^2 + \frac{1}{r^2}\,(\partial_\theta S + A)^2\,|\psi|^2\right] dx$$

Magnetic interpolation inequality

$$\int_{\mathbb{R}^2} |\nabla_{\mathbf{A}}\,\psi|^2\,dx + \lambda \int_{\mathbb{R}^2} \frac{|\psi|^2}{|x|^2}\,dx \ge \mu(\lambda) \left(\int_{\mathbb{R}^2} \frac{|\psi|^p}{|x|^2}\,dx\right)^{2/p}$$

▷ Symmetrization: [Erdös, 1996], [Boulenger, Lenzmann], [Lenzmann, Sok]

A magnetic Hardy-Sobolev inequality

Theorem

Let $a \in [0,1/2]$ and $p > 2$. For any $\lambda > -a^2$, there is an optimal, monotone increasing, concave function $\lambda \mapsto \mu(\lambda)$ which is such that

$$\boxed{\int_{\mathbb{R}^2} |\nabla_{\mathbf{A}} \psi|^2 \, dx + \lambda \int_{\mathbb{R}^2} \frac{|\psi|^2}{|x|^2} \, dx \geq \mu(\lambda) \left(\int_{\mathbb{R}^2} \frac{|\psi|^p}{|x|^2} \, dx \right)^{2/p}}$$

If $\lambda \leq \lambda_\star = 4 \, \frac{1-4\,a^2}{p^2-4} - a^2$ equality is achieved by

$$\psi(x) = \left(|x|^\alpha + |x|^{-\alpha}\right)^{-\frac{2}{p-2}} \quad \forall\, x \in \mathbb{R}^2\,, \quad \textit{with} \quad \alpha = \frac{p-2}{2} \sqrt{\lambda + a^2}$$

If $\lambda > \lambda_\bullet$ with

$$\lambda_\bullet := \frac{8\left(\sqrt{p^4 - a^2\,(p-2)^2\,(p+2)\,(3\,p-2)} + 2\right) - 4\,p\,(p+4)}{(p-2)^3\,(p+2)} - a^2$$

there is symmetry breaking: optimal functions are not radially symmetric

A magnetic Keller-Lieb-Thirring estimate

Let $q \in (1,+\infty)$ and denote by $\mathrm{L}^q_\star(\mathbb{R}^2)$ the space defined using the weighted norm $|||\phi|||_q := \left(\int_{\mathbb{R}^2} |\phi|^q\,|x|^{2\,(q-1)}\,dx\right)^{1/q}$

Theorem

Let $a \in (0,1/2)$, $q \in (1,\infty)$ and $\phi \in \mathrm{L}^q_\star(\mathbb{R}^2)$: $\mu \mapsto \lambda(\mu)$ is a convex monotone increasing function such that $\lim_{\mu\to 0^+} \lambda(\mu) = -\min_{k\in\mathbb{Z}}(a-k)^2$ and

$$\lambda_1(-\Delta_{\mathbf{A}} - \phi) \geq -\lambda\left(|||\phi|||_q\right)$$

There is an explicit $\mu_\star > 0$ such that the equality case is achieved for any $\mu \leq \mu_\star$ by

$$\phi(x) = \left(|x|^\alpha + |x|^{-\alpha}\right)^{-2} \quad \forall\, x \in \mathbb{R}^2\,, \quad \textit{with} \quad \alpha = \frac{p-2}{2}\sqrt{\lambda(\mu)+a^2}$$

There is an explicit $\mu_\bullet > \mu_\star$ such that the equality case is achieved only by non-radial functions if $\mu > \mu_\bullet$

Constants are explicit...

- For $a=1/2$, we shall see that $\mu_\bullet=\mu_\star=-1/4$
- The function $\lambda\mapsto\mu(\lambda)$ is the inverse of $\mu\mapsto\lambda(\mu)$ and

$$\mu_\star=h(\lambda_\star)\quad\text{and}\quad\mu_\bullet=h(\lambda_\bullet)$$

with

$$h(\lambda):=\frac{p}{2}\,(2\,\pi)^{1-\frac{2}{p}}\,\big(\lambda+a^2\big)^{1+\frac{2}{p}}\left(\frac{2\,\sqrt{\pi}\,\Gamma\big(\frac{p}{p-2}\big)}{(p-2)\,\Gamma\big(\frac{p}{p-2}+\frac{1}{2}\big)}\right)^{1-\frac{2}{p}}$$

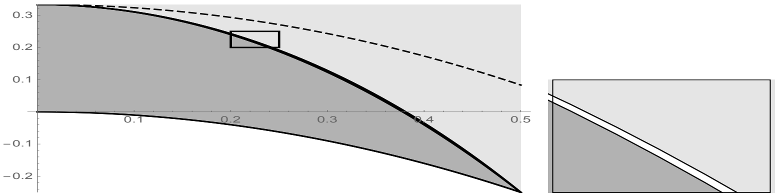


Figure: Case $p = 4$
Symmetry breaking region: $\lambda > \lambda_\bullet(a)$
Symmetry breaking region: $\lambda < \lambda_\star$

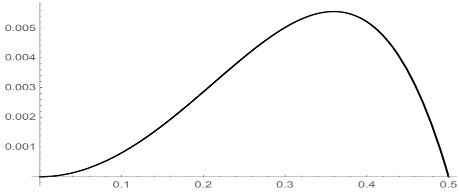


Figure: The curve $a \mapsto \lambda_\bullet(a) - \lambda_\star(a)$

Lemma

Let $a \in [0,1/2]$ *and* $\psi = u\,e^{iS} \in C^1 \cap \mathrm{H}^1_{\mathbf{A}}$ *such that* $|\psi| > 0$

$$\int_{\mathbb{R}^2} |\nabla_{\mathbf{A}}\,\psi|^2\,dx \ge \int_{\mathbb{R}^2} \left(|\partial_r u|^2 + \frac{1}{r^2}\,|\partial_\theta u|^2 + \frac{1}{r^2}\,\frac{a^2}{\int_{\mathbb{S}^2} u^{-2}\,d\sigma}\right) dx$$

Equality holds if and only if $\partial_r S \equiv 0$ *and*

$$\partial_\theta S = a - \frac{a}{u^2}\,\frac{1}{\int_{\mathbb{S}^2} u^{-2}\,d\sigma}$$

When u *does not depend on* θ, *equality is achieved iff* S *is constant*

Lemma

For all $a \in [0,1/2]$ *and* $\psi \in \mathrm{H}^1(\mathbb{S}^1)$ *with* $u = |\psi|$, *we have*

$$\int_{\mathbb{S}^2} |\partial_\theta \psi - \, i\,a\,\psi|^2\,d\sigma \ge \left(1 - 4\,a^2\right) \int_{\mathbb{S}^2} |\partial_\theta u|^2\,d\sigma + a^2 \int_{\mathbb{S}^2} u^2\,d\sigma$$

Proof (1/3): the inequality with a non-optimal constant

Diamagnetic inequality: $\|\nabla_{\mathbf{A}}\,\psi\|_{\mathrm{L}^2(\mathbb{R}^2)} \ge \|\nabla u\|_{\mathrm{L}^2(\mathbb{R}^2)}$, $u = |\psi|$

$$\begin{aligned}\int_{\mathbb{R}^2} |\nabla_{\mathbf{A}}\,\psi|^2\,dx + \lambda \int_{\mathbb{R}^2} \frac{|\psi|^2}{|x|^2}\,dx \\ \ge t\left(\|\nabla_{\mathbf{A}}\,\psi\|^2_{\mathrm{L}^2(\mathbb{R}^2)} - a^2 \int_{\mathbb{R}^2} \frac{u^2}{|x|^2}\,dx\right) \\ + (1-t)\left(\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{R}^2)} + \frac{\lambda + a^2\,t}{1-t}\int_{\mathbb{R}^2} \frac{u^2}{|x|^2}\,dx\right)\end{aligned}$$

With $\mathsf{a}^2 = \frac{\lambda + a^2\,t}{1-t}$, $t \in (0,1)$ such that $\lambda + a^2\,t > 0$: existence of a positive constant $\mu(\lambda)$

Proof (2/3): optimal estimate in the symmetry range

With $a\in[0,1/2]$, $\psi\in\mathrm H^1(\mathbb R^2)$ and $u=|\psi|$

$$\int_{\mathbb R^2}|\nabla_{\mathbf A}\,\psi|^2\,dx\ge\int_{\mathbb R^2}|\partial_r u|^2\,dx+\left(1-4\,a^2\right)\int_{\mathbb R^2}\frac{1}{r^2}\,|\partial_\theta u|^2\,dx+a^2\int_{\mathbb R^2}u^2\,dx$$

The relaxed inequality

$$\int_{\mathbb R^2}\left(|\partial_r u|^2+\tfrac{1-4\,a^2}{r^2}\,|\partial_\theta u|^2\right)dx+\left(\lambda+a^2\right)\int_{\mathbb R^2}\frac{|u|^2}{|x|^2}\,dx\ge\mu_{\rm rel}(\lambda)\left(\int_{\mathbb R^2}\frac{|u|^p}{|x|^2}\,dx\right)^{\frac2p}$$

is rewritten on the cylinder $\mathcal C:=\mathbb R\times\mathbb S^1$ using the Emden-Fowler transformation as

$$\int_{\mathcal C}\left(|\partial_s w|^2+\left(1-4\,a^2\right)\,|\partial_\theta w|^2\right)dy+\left(\lambda+a^2\right)\int_{\mathcal C}|w|^2\,dy$$
$$\ge(2\,\pi)^{\frac2p-1}\,\mu_{\rm rel}(\lambda)\left(\int_{\mathcal C}|w|^p\,dy\right)^{\frac2p}$$

If $\left(\lambda+a^2\right)\left(p^2-4\right)\le4\left(1-4\,a^2\right)\iff\lambda\le\lambda_\star$, the minimizer is symmetric

Proof (3/3): symmetry breaking range

$$\mathcal{E}_{a,\lambda}[\psi] := \int_{\mathbb{R}^2} |\nabla_{\mathbf{A}}\,\psi|^2\,dx + \lambda \int_{\mathbb{R}^2} \frac{|\psi|^2}{|x|^2}\,dx - \mu \left(\int_{\mathbb{R}^2} \frac{|\psi|^p}{|x|^2}\,dx\right)^{2/p}$$

$\mu = \left(2\pi \int_{\mathcal{C}} |w_\star|^p\,dy\right)^{1-2/p}$, $w_\star(s) = \zeta_\star\,\big(\cosh(\omega\,s)\big)^{-\frac{2}{p-2}}$

$s = -\log r$ and $\psi_\varepsilon(r,\theta) := \big(w_\star(s) + \varepsilon\,\varphi(s,\theta)\big)\,e^{i\,\varepsilon\,\chi(s,\theta)}$

$\mathcal{E}_{a,\lambda}[\psi_\varepsilon] = \varepsilon^2\,\mathcal{Q}[\varphi,\chi] + o(\varepsilon^2)$

$$\begin{aligned}\mathcal{Q}[\varphi,\chi] \;=\; & \int_{\mathcal{C}} w_\star^2\,\left(|\partial_s\chi|^2 + |\partial_\theta\chi - a|^2 - a^2\right)dy - 4\,a \int_{\mathcal{C}} w_\star\,\varphi\,\partial_\theta\chi\,dy \\ & + \int_{\mathcal{C}} \left(|\partial_s\varphi|^2 + |\partial_\theta\varphi|^2 + (\lambda + a^2)\,\varphi^2\right)dy \\ & - (p-1)\int_{\mathcal{C}} |w_\star|^{p-2}\,|\varphi|^2\,dy\end{aligned}$$

$\varphi(s,\theta) = \dfrac{\cos\theta}{\cosh(\omega\,s)^{\frac{p}{p-2}}}$, $\chi(s,\theta) = \dfrac{\zeta}{\zeta_\star}\,\dfrac{\sin\theta}{\cosh(\omega\,s)}$: $\quad \mathcal{Q}[\varphi,\chi] < 0 \Longrightarrow \lambda > \lambda_\bullet$

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These slides can be found at

http://www.ceremade.dauphine.fr/∼dolbeaul/Conferences/
▷ Lectures

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Thank you for your attention !