Multiplicity results for the assigned Gauss curvature problem in $\mathbb{R}^2$

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(A JOINT WORK WITH M. ESTEBAN AND G. TARANTELLO)

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Outline

- From assigned Gauss curvature problems with conical singularities to weighted Liouville-type equations in $\mathbb{R}^2$
- Known results on weighted Liouville-type equations in $\mathbb{R}^2$
- Multiplicity results for radially symmetric solutions
- More numerical observations and conjectures

1. From assigned Gauss curvature problems with conical singularities to weighted Liouville-type equations in $\mathbb{R}^2$

The assigned Gauss curvature problem

$$\Delta u + K(x)\, e^{2u} = 0 \ \text{ in } \ \mathbb{R}^2$$

K a given function on $\mathbb{R}^2$

If u is a solution, then the metric $g = e^{2u}\,|dx|^2$ is conformal to the flat metric $|dx|^2$ and such that K is the Gaussian curvature of the new metric g

- Analysis of gravitating systems
- statistical mechanics description of the vorticity in fluid mechanics
- self-dual gauge field vortices

K is negative: uniqueness$K$is positive: either uniqueness or multiplicity of solutions holds

The assigned Gauss curvature with conical singularities

For a given Riemann surface (M, g), we aim at determining the range of the parameters λ, $\rho \in \mathbb{R}$ one can solve

$$\Delta_g u + \lambda \left(\frac{e^{2u}}{\int_M e^{2u}\, d\sigma_g} - \frac{1}{|M|} \right) - 2\pi\rho \left(\delta_P - \frac{1}{|M|} \right) = f$$

- Δ_g is the Laplace-Beltrami operator
- $d\sigma_g$ is the volume element corresponding to the metric g
- $f \in C(M)$ with $\int_M f\, d\sigma_g = 0$
- δ_P is the Dirac measure with singularity at $P \in M$

In case M has a non-empty boundary, both Dirichlet or Neumann boundary conditions on ∂M are of interest

Two basic examples:

- 2-sphere $M = \mathbb{S}^2$
- the flat 2-torus $M = \mathbb{C}/(\xi_1\mathbb{Z} + \xi_2\mathbb{Z})$, with periodic cell domain generated by ξ_1 and ξ_2 (vortex-like configurations in periodic settings)

A brief review of the Onsager vortex problem ($\rho = 0$)

Closed surfaces ($\partial M = \emptyset$), no singularity ($\rho = 0$): solvability depends of the topological properties of M

- [Y. Y. Li], [C.-C. Chen, C.-S. Lin]: if $\rho = 0$, the solutions of are uniformly bounded for any $\lambda \in \mathbb{R} \setminus 4\pi\mathbb{N}$
- for any $\lambda \in (4\pi(m-1), 4\pi m)$, $m \in \mathbb{N}^*$, Leray-Schauder degree of the corresponding Fredholm operator is

$$d_\lambda = 1 \text{ if } m = 1\,, \quad d_\lambda = \frac{(-\chi(M)+1) \ \cdots \ (-\chi(M)+m-1)}{(m-1)!} \text{ if } m \geq 2$$

  where $\chi(M)$ is the Euler characteristics of M
- for the flat 2-torus, $d_\lambda = 1$ also when $\lambda \in 4\pi\mathbb{N}$: there is a solution for every $\lambda \in \mathbb{R}$
- For the standard 2-sphere, $d_\lambda = 0$ for all $\lambda > 8\pi$, but still, there is a solution for any $\lambda \in \mathbb{R} \setminus 4\pi\mathbb{N}$ (multiplicity results)

Case with a Dirac measure ($\rho \neq 0$)

The Leray-Schauder degree, given by $d_\lambda = -\chi(M) + 2$, is available only when $\rho \geq 1$ and $\lambda \in (4\pi, 8\pi)$

- On $\mathbb{S}^2$: the solvability is very delicate [Troyanov]
- On the flat 2-torus with $\lambda = 4\pi$ and $\rho = 2$, consider

$$\Delta_g u + 4\pi \left(\frac{e^{2u}}{\int_M e^{2u} \, d\sigma_g} - \delta_0 \right) = 0 \quad \text{in } M = \mathbb{C}/(\xi_1 \mathbb{Z} + \xi_2 \mathbb{Z})$$

  [C.-S. Lin , C. L. Wang]: no solution in the case of a rectangular lattice (*i.e.* $\xi_1 = a$, $\xi_2 = i\,b$, a, $b > 0$), but there is a solution for a rhombus lattice (*i.e.* $\xi_1 = a$, $\xi_2 = a\,e^{i\pi/3}$, $a > 0$)

As usual for such difficulties... lack of compactness for the solution set

Weighted Liouville-type equations in $\mathbb{R}^2$: first derivation

Consider the square 2-torus $\approx \Omega = (-1,1)^2 \subset \mathbb{R}^2$ and suppose that $P = 0 \in \Omega$. Let u_ε be the solution of (existence is ok)

$$\begin{cases} \Delta u_\varepsilon + \lambda \left(\dfrac{e^{2u_\varepsilon}}{\int_\Omega e^{2u_\varepsilon}\, dx} - \dfrac{1}{|\Omega|} \right) = 2\pi\rho \left(\dfrac{\varepsilon^2}{\pi\,(\varepsilon^2 + |x|^2)^2} - \dfrac{c_\varepsilon}{|\Omega|} \right) & \text{in } \Omega \\ u_\varepsilon \quad \text{doubly periodic on } \partial\Omega \end{cases}$$

where $c_\varepsilon := \int_\Omega \frac{\varepsilon^2}{\pi\,(\varepsilon^2+|x|^2)^2}\, dx \to 1$ as $\varepsilon \to 0$. Let $u_\varepsilon = v_\varepsilon + u_{\varepsilon,0}$

$$\begin{cases} \Delta u = 2\pi\rho \left(\dfrac{\varepsilon^2}{\pi\,(\varepsilon^2 + |x|^2)^2} - \dfrac{c_\varepsilon}{|\Omega|} \right) & \text{in } \Omega \\ u \quad \text{doubly periodic on } \partial\Omega\,, \quad \int_\Omega u\, dx = 0 \end{cases}$$

which takes the form $u_{\varepsilon,0}(x) = \frac{\rho}{2} \log(\varepsilon^2 + |x|^2) + \psi_\varepsilon(x)$, for some suitable function ψ_ε (uniformly bounded in $C^{2,\alpha}$-norm, with respect to $\varepsilon > 0$)

Then $v_\varepsilon = u_\varepsilon - u_{\varepsilon,0}$ satisfies

$$\Delta v_\varepsilon + \lambda \left(\frac{e^{2(u_{\varepsilon,0}+v_\varepsilon)}}{\int_\Omega e^{2(u_{\varepsilon,0}+v_\varepsilon)}\,dx} - \frac{1}{|\Omega|} \right) = 0 \quad \text{in } \Omega$$

The function $e^{u_{\varepsilon,0}}$ is bounded from above and from below away from zero in $C^0_{\text{loc}}(\Omega \setminus \{0\})$... u_ε is bounded uniformly in $C^{2,\alpha}_{\text{loc}}(\Omega \setminus \{0\})$ for $\lambda \notin 4\pi\mathbb{N}$

Close to $0 \in \Omega$? $w_\varepsilon := v_\varepsilon + \frac{1}{2}\,\log\lambda - \frac{1}{2}\,\log\left(\int_\Omega e^{2u_\varepsilon}\,dx\right)$ and $W_\varepsilon := e^{2\psi_\varepsilon}$

$$\begin{cases} -\Delta w_\varepsilon = (\varepsilon^2 + |x|^2)^\rho\, W_\varepsilon(x)\, e^{2w_\varepsilon} - \frac{\lambda}{|\Omega|} \quad \text{in } B_r(0) \\ \displaystyle\int_{B_r(0)} (\varepsilon^2 + |x|^2)^\rho\, W_\varepsilon(x)\, e^{2w_\varepsilon}\,dx = \lambda\, \frac{\int_{B_r(0)} e^{2u_\varepsilon}\,dx}{\int_\Omega e^{2u_\varepsilon}\,dx} \le \lambda \end{cases}$$

By contradiction: $\lim_{n\to\infty}\varepsilon_n = 0$, $\lim_{n\to\infty} x_n = 0$,
$w_{\varepsilon_n}(x_n) = \max_{\bar B_r(0)} w_{\varepsilon_n} \to +\infty$

Let $s_n := \max\left\{\varepsilon_n, |x_n|, \exp(-\frac{w_{\varepsilon_n}(x_n)}{2(1+\rho)})\right\} \to 0$, $R_n := W_{\varepsilon_n}(x_n + s_n\, x)$, $B_n := B_{r/s_n}(0)$, $U_n(x) := w_{\varepsilon_n}(x_n + s_n\, x) + 2\,(1+\rho)\,\log s_n$

$$\begin{cases} -\Delta U_n = \left(\left|\frac{\varepsilon_n}{s_n}\right|^2 + \left|\frac{x_n}{s_n} + x\right|^2\right)^\rho R_n\, e^{2U_n} + o(1) \quad \text{in } B_n \\ U_n(0) = w_{\varepsilon_n}(x_n) + 2\,(1+\rho)\,\log s_n \\ \displaystyle\int_{B_n} \left(\left|\frac{\varepsilon_n}{s_n}\right|^2 + \left|\frac{x_n}{s_n} + x\right|^2\right)^\rho R_n\, e^{2U_n}\, dx \le \lambda \end{cases}$$

By definition of s_n, we know that $\limsup_{n\to\infty} s_n\, \exp(\frac{w_{\varepsilon_n}(x_n)}{2(1+\rho)}) \ge 1$. We do not know whether this limit is finite or not. If it is finite (conjectured), by Harnack's estimates

$$U_n \;\to\; U_\infty \;\text{ in }\; C^{2,\alpha}_{\text{loc}}(\mathbb{R}^2)\,, \quad \frac{\varepsilon_n}{s_n} \to \varepsilon_\infty \in [0,1]\,, \quad \frac{x_n}{s_n} \to x_\infty \in B(0,1) \subset \mathbb{R}^2$$

$$\begin{cases} -\Delta U_\infty = (\varepsilon_\infty^2 + |x_\infty + x|^2)^\rho \, W_\infty \, e^{2U_\infty} \quad \text{in } \mathbb{R}^2 \\ U_\infty(0) = \max_{\mathbb{R}^2} U_\infty \geq 0 \\ \int_{\mathbb{R}^2} (\varepsilon_\infty^2 + |x_\infty + x|^2)^\rho \, W_\infty \, e^{2U_\infty} \, dx \leq \lambda \end{cases}$$

If $\varepsilon_\infty = 0$, classification results give

$$W_\infty \int_{\mathbb{R}^2} |x_\infty + x|^{2\rho} \, e^{2U_\infty} \, dx = 4\pi \, (1 + \rho)$$

... impossible if $\lambda < 4\pi \, (1 + \rho)$. If $\varepsilon_\infty > 0$, then (a much harder question)

$$\begin{cases} -\Delta U = (1 + |x|^2)^\rho \, e^{2U} \quad \text{in } \mathbb{R}^2 \\ \int_{\mathbb{R}^2} (1 + |x|^2)^\rho \, e^{2U} \, dx \leq \lambda \end{cases}$$

Weighted Liouville-type equations in $\mathbb{R}^2$: second derivation

Consider on $\mathbb{R}^2$ the solutions of

$$\begin{cases} -\Delta u = (1+|x|^2)^N\, e^{2u} \quad \text{in } \mathbb{R}^2 \\ \displaystyle\int_{\mathbb{R}^2} (1+|x|^2)^N\, e^{2u}\, dx = \lambda \end{cases}$$

Let $\Sigma : \mathbb{S}^2 \to \mathbb{R}^2$ be the stereographic projection with respect to the north pole, N $:= (0,0,1)$. With $x = \Sigma(y)$

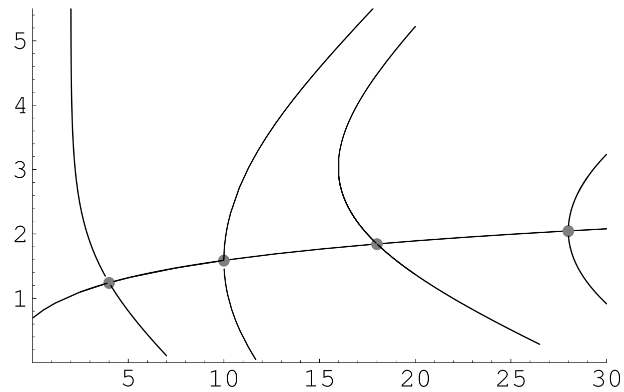
$$v(y) := u(x) - \frac{1}{2} \log\left(\frac{\lambda}{(1+|x|^2)^{N+2}}\right) - \log 2$$

$$\Delta_g v + \lambda \frac{e^{2v}}{\int_{\mathbb{S}^2} e^{2v}\, d\sigma_g} = \frac{2\pi\,(N+2)}{|\mathbb{S}^2|} + \Big(\lambda - 2\pi\,(N+2)\Big)\, \delta_{\mathsf{N}} \quad \text{on } \mathbb{S}^2$$

If $\lambda = 2\pi\,(N+2)$, we find that v is a bounded solution on $\mathbb{S}^2$ with $\rho = 0$

Main result

Theorem 1. *For all $k \geq 2$ and $N > k(k+1) - 2$, there are at least $2(k-2)+2$ distinct radial solutions of the weighted Liouville-type equation in $\mathbb{R}^2$ with $\lambda = 2\pi\,(N+2)$, one of them being the function $u_N^*(r) := \frac{1}{2}\,\log\left(\frac{2(N+2)}{(1+r^2)^{N+2}}\right)$ with $r = |x|$.*



Bifurcation diagram for the weighted Liouville-type equation in $\mathbb{R}^2$ with $\lambda = 2\pi(N+2)$ (right). Non trivial branches bifurcate from $N_k = 4, 10, 18, 28, \ldots$

2. Known results on weighted Liouville-type equations in $\mathbb{R}^2$

[Mostly C.-S. Lin *et al.*]

Radially symmetric solutions

[C.-S. Lin *et al.*]: the solution set can be parametrized by a parameter $a \in \mathbb{R}$

$$\begin{cases} u_a'' + \frac{u_a'}{r} + (1+r^2)^N e^{2u_a} = 0 \quad \text{in } (0,+\infty) \\ u_a(0) = a\,, \quad u_a'(0) = 0\,, \quad \int_0^\infty (1+r^2)^N e^{2u_a}\, r\, dr < +\infty \end{cases}$$

For every $a \in \mathbb{R}$, there exists a unique solution u_a such that

$$\lim_{r\to\infty} \left(u_a(r) + \alpha(a)\, \log r\right) = \beta(a)\,,$$

$$\alpha(a) = \int_0^\infty (1+r^2)^N e^{2u_a}\, r\, dr \quad \text{and} \quad \beta(a) = \int_0^\infty (1+r^2)^N e^{2u_a}\, r\, \log r\, dr$$

For any $N > 0$

$$\lim_{a\to-\infty} \alpha(a) = 2(N+1) \quad \text{and} \quad \lim_{a\to+\infty} \alpha(a) = 2\, \min\{1,\, N\}$$

Uniqueness

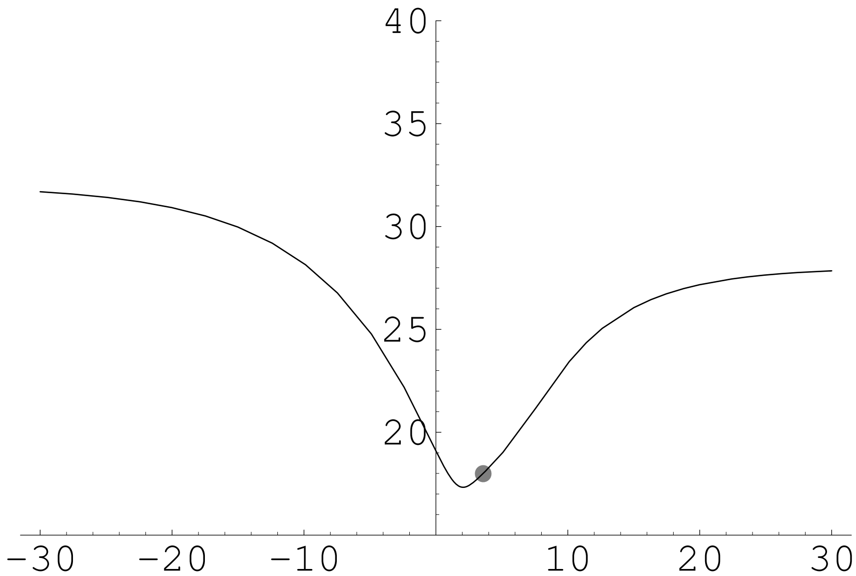
Pohozaev's identity shows that $\alpha(a) \in (2, 2(N+1))$. For integrability reasons, we also know that $\alpha(a) > N+1$, and so

$$\max\{2\,, N+1\} < \alpha(a) < 2(N+1) \quad \forall\, a \in \mathbb{R}$$

[C.-S. Lin]: there is a unique a such that $\alpha(a) = \alpha$ if $\alpha \in (2N, 2(N+1))$, $N > 1$ and for all $\alpha \in (2, 2(N+1))$ if $N \leq 1$. On the other hand, it is easy to verify that for all N, the function

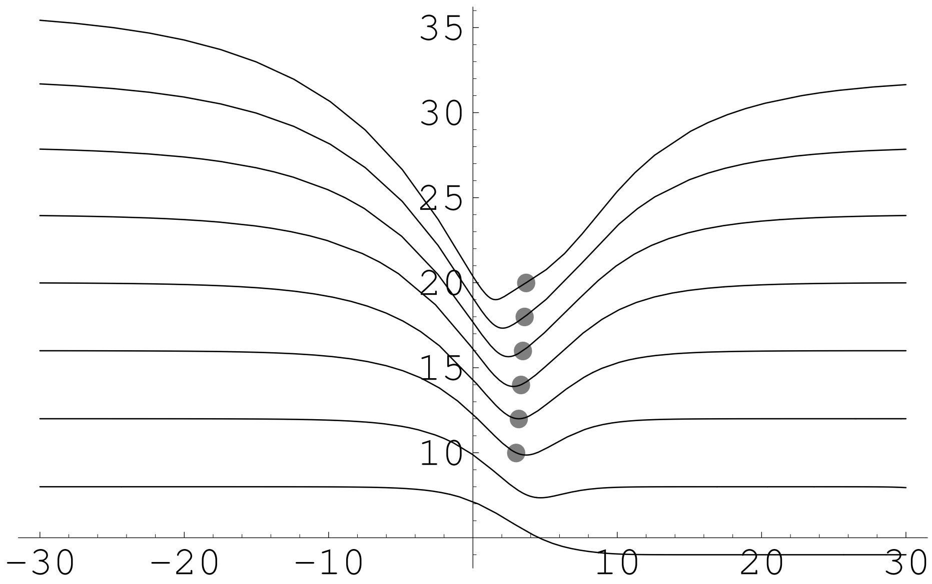
$$u_N^*(r) := \frac{1}{2}\log\left(\frac{2(N+2)}{(1+r^2)^{N+2}}\right) \quad \text{for} \quad \begin{cases} a = a_N^* := \frac{1}{2}\log\left(2(N+2)\right) \\ \alpha(a_N^*) = N+2 \end{cases}$$

Since $N+2 < 2N < 2(N+1)$ for all $N > 2$, by continuity of $a \mapsto \alpha(a)$, it appears that there exists at least two different values of a such that $\alpha(a) = \alpha$, for any $\alpha \in (\min_{a\in\mathbb{R}} \alpha(a), 2N)$



Curve $a \mapsto \alpha(a)$ for $N = 7$. Recall that

$$\alpha(a) = \int_0^\infty (1 + r^2)^N e^{2u_a} \, r \, dr$$



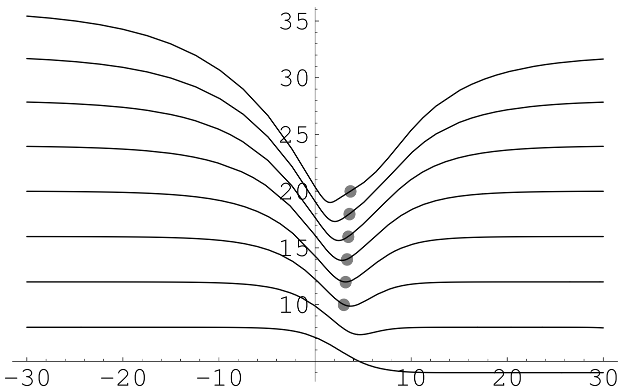
Curve $a \mapsto \alpha(a)$ for $N = 1$, 2,... 8

Theorem 2. [K.-S. Cheng and C.-S. Lin], [C.-S. Lin] *Let N be any positive real number*

(i) *If $N \leq 1$, then the curve $a \mapsto \alpha(a)$ is monotone decreasing. Moreover, there exists a radially symmetric solution u if and only if $\alpha \in (2, 2(N+1))$, and such a solution is unique*

(ii) *If $N > 1$, then for all $\alpha \in (2N, 2(N+1))$, there exists a unique $a \in \mathbb{R}$ such that $\alpha(a) = \alpha$. In other words, for such α, there is a unique radial solution*

(iii) *If $N \geq 2$, then $\min_{a\in\mathbb{R}} \alpha(a) < 2N$, and for all $\alpha \in (\min_{a\in\mathbb{R}} \alpha(a),\ 2N)$, there exists at least two radial solutions*



Non radially symmetric solutions

- To any solution u_a, we can associate a function v_a on $\mathbb{S}^2$, such that $\int_{\mathbb{S}^2} e^{2v}\, d\sigma_g = 1$ for $\lambda = 2\pi\, \alpha(a)$
- At level $\alpha = N + 2$, v_a is a bounded solution on $\mathbb{S}^2$ (with $f = 0$, $\rho = 0$) of

$$\Delta_g u + \lambda \left(\frac{e^{2u}}{\int_M e^{2u}\, d\sigma_g} - \frac{1}{|M|} \right) - 2\pi\rho \left(\delta_P - \frac{1}{|M|} \right) = f$$

  which is axially symmetric with respect to the unit vector $(0, 0, 1)$ pointing towards the north pole N of $\mathbb{S}^2$
- Since $v_N^* = v_{a_N^*}$ is the unique constant solution, if we know the existence of more than one solution at level $\alpha = N + 2$, then there is an axially symmetric solution on $\mathbb{S}^2$ which is not constant, and that can be thus rotated in order to be axially symmetric with respect to any vector $\mathsf{e} \in \mathbb{S}^2 \setminus \{\mathsf{N}, \mathsf{S}\}$. Let us denote by v_{e} such a solution
- Applying the stereographic projection to v_{e}, we find a solution u_{e} which is not radially symmetric

Linearization

$$\begin{cases} \varphi_a'' + \frac{\varphi_a'}{r} + 2\,(1+r^2)^N e^{2u_a}\,\varphi_a = 0\,, & r \in (0,+\infty) \\ \varphi_a(0) = 1\,, \quad \varphi_a'(0) = 0 \end{cases}$$

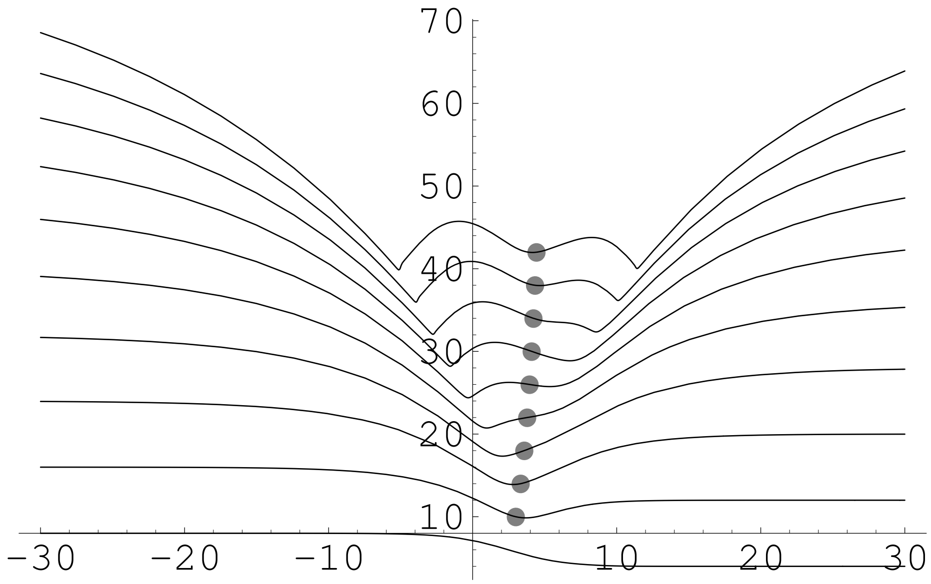
The number of critical points of $a \mapsto \alpha(a)$ is also connected with the number of zeroes of φ_a in the range $(\min_{a\in\mathbb{R}} \alpha(a),\, 2N)$

$$\varphi_a(r) \sim -\alpha'(a)\log r + b'(a) + o(1)$$

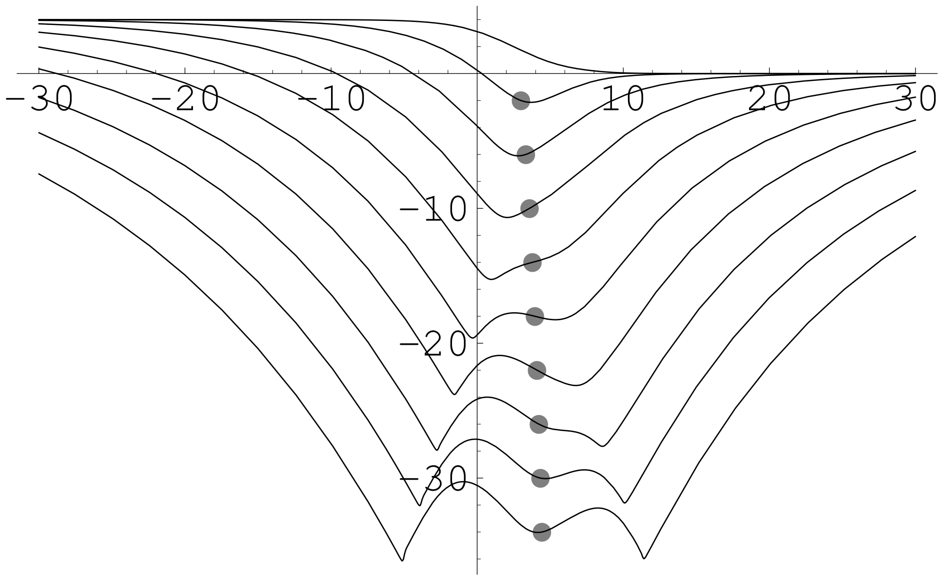
φ_a is a bounded function if and only if $a \in \mathbb{R}$ is a critical point of the function α

As a special case, for all N, if $\min_{a\in\mathbb{R}} \alpha(a)$ is achieved for some finite $\underline{a}$, then $\varphi_{\underline{a}}$ is bounded

3. Multiplicity results for radially symmetric solutions



Curve $a \mapsto \alpha(a)$ for $N = 1$, 3,... 19



Curve $a \mapsto \alpha(a) - 4N$ for $N = 1, 3, \ldots 19$

- For given N, critical levels of the curve $a \mapsto \alpha(a)$ determine the multiplicity of the radial solutions at a given level
- The number of zeroes of the solutions of the linearized problem can change only at critical points of α
- In the special case $\alpha = N + 2$, a bifurcation argument provides us with a very precise multiplicity result

Study of the linearized problem

Emden-Fowler transformation in the linearized equation

$$t = \log r\,, \quad w_a(t) := \varphi_a(r)$$

The equation is then transformed into

$$w_a''(t) + 2\,e^{2t}(1+e^{2t})^N e^{2u_a(e^t)}\,w_a(t) = 0\,, \quad t \in (-\infty, +\infty)$$

When $a = a_N^*$, the equation for $w_N^* := w_{a_N^*}$ reads

$${w_N^*}''(t) + \frac{(N+2)}{2\,(\cosh t)^2}\,w_N^*(t) = 0\,, \quad t \in (-\infty, +\infty)$$

With one more change of variables, $w(t) = \psi(s)$, $s = \tanh t$, we find Legendre's equation

$$\frac{d}{ds}\left((1-s^2)\,\frac{d\psi}{ds}\right) + \frac{N+2}{2}\,\psi = 0$$

Legendre polynomials

Legendre polynomial of order $k \in \mathbb{N}^*$ if $N + 2 = k(k+1)$

Legendre's equation has bounded solutions if and only if there is a positive integer k such that $1 + 2k = \sqrt{1 + 4(N+2)}$, that is, if and only if

$$\mathfrak{N}(N) := \frac{-1 + \sqrt{1 + 4(N+2)}}{2}$$

is a positive integer: $w^*_{N_k}(t) \equiv P_k(s)$ for all integer $k \geq 2$ with $s = \tanh t$

Lemma 3. [Landau-Lifschitz] *Take $N \geq 1$. Then, there are bounded solutions if and only if $\mathfrak{N}(N)$ is a positive integer: $N = N_k := k(k+1) - 2$, $k \in \mathbb{N}^*$. In such a case, $\varphi_{a^*_N}$ has exactly $\mathfrak{N}(N)$ zeroes in the interval $(-\infty, +\infty)$*

Lemma 4. *For any* $N > 0$, $a_0 > 0$ *and* $R > 0$, *if* φ_{a_0} *has* k *zeroes in* $(0, R)$ *and* $\varphi_{a_0}(R) \neq 0$, *then there exists an* $\varepsilon > 0$ *such that* φ_a *also has exactly* k *zeroes in* $(0, R)$ *for any* $a \in (a_0 - \varepsilon, a_0 + \varepsilon)$

Corollary 5. *For any* $N \in [N_k, N_{k+1})$, $k \in \mathbb{N}$, $k \geq 1$, *solutions* w_N^* *have exactly* $k + 1$ *zeroes in the interval* $(-\infty, +\infty)$

Lemma 6. *Take* $N \geq 1$ *and consider* $a_1, a_2 \in \mathbb{R}$ *such that* $\alpha'(a_1) = \alpha'(a_1) = 0$ *and* $\alpha'(a) \neq 0$ *if* $a \in (a_1, a_2)$. *Then, for all* $a \in (a_1, a_2)$, *the functions* φ_a *have the same number of zeroes.*

$J_N(a) := \int_0^{+\infty} (1 + r^2)^N e^{2u_a} \varphi_a^3 \, r \, dr$ governs the dynamics of the zeroes of φ_a at infinity as follows. Let $r(a) := \max\{r > 0 \, : \, \varphi_a(r) = 0\}$

Lemma 7. *Let* $\bar{a} > 0$ *be such that, for* $\zeta = \pm 1$, $\lim_{a \to \bar{a}, \, \zeta(a - \bar{a}) > 0} r(a) = \infty$. *Then there exists* $\varepsilon > 0$ *such that, on* $(\bar{a} - \varepsilon, \bar{a})$ *if* $\zeta = -1$, *on* $(\bar{a}, \bar{a} + \varepsilon)$ *if* $\zeta = +1$,

$$\frac{dr}{da}(a) = -\frac{4}{r(a) \, |\varphi_a'(r(a))|^2} \int_0^{r(a)} (1 + r^2)^N \, e^{2u_a} \, \varphi_a^3 \, r \, dr$$

$$\frac{dr}{da} \, J_N(a) < 0 \text{ if } J_N(\bar{a}) \neq 0$$

Corollary 8. *Let $\tilde{a}$ be a critical point of α. There exists $\varepsilon > 0$, small enough, such that the following properties hold.*

(i) *If $J_N(\tilde{a}) > 0$ and if, for any $a \in (\tilde{a} - \varepsilon, \tilde{a})$, all functions φ_a are unbounded and have k zeroes in $(0, +\infty)$, then $\varphi_{\tilde{a}}$ is bounded and has k zeroes, and for any $a \in (\tilde{a}, \tilde{a} + \varepsilon)$, φ_a is unbounded and has either k or $k+1$ zeroes in $(0, +\infty)$.*

(ii) *If $J_N(\tilde{a}) < 0$ and if, for any $a \in (\tilde{a} - \varepsilon, \tilde{a})$, all functions φ_a are unbounded and have k zeroes in $(0, +\infty)$, then $\varphi_{\tilde{a}}$ is bounded and has either k or $k-1$ zeroes, and for any $a \in (\tilde{a}, \tilde{a} + \varepsilon)$, φ_a has the same number of zeroes as $\varphi_{\tilde{a}}$.*

Proposition 9. *Let us define $j(k) := J_N(a^*_{N_k})$ for any integer $k \geq 2$. Then, $j(k) = 0$ if k is odd, and $j(k) > 0$ if k even.*

Proof.

$$j(k) = \frac{1}{2}k(k+1)\int_{-1}^{1} P_k(s)^3 \, ds$$

Now, when k is odd, P_k is also odd and so, $j(k) = 0$. On the contrary, Gaunt's formula shows that $j(k) > 0$ if k is even □

A multiplicity result at level $\alpha = N + 2$

Let u be a radial solution of the weighted Liouville-type equation in $\mathbb{R}^2$ with $\alpha = N + 2$. We may reformulate this problem in terms of $f := u - u_N^* \in \mathcal{D}^{1,2}(\mathbb{R}^2)$ as a solution to

$$\Delta f + \frac{\mu}{(1+|x|^2)^2}\left(e^{2f} - 1\right) = 0 \;\text{ in }\; \mathbb{R}^2\,, \quad \int_{\mathbb{R}^2} \frac{e^{2f}}{(1+|x|^2)^2}\,dx = \pi$$

with $\mu = 2(N+2)$

Lemma 10 (Kelvin's transformation). *The function $x \mapsto f\left(\frac{x}{|x|^2}\right)$ is also a solution*

Theorem 11. *For any $k \geq 2$, there are two continuous half-branches, $\mathcal{C}_k^+$ and $\mathcal{C}_k^-$, of solutions (μ, f) bifurcating from the branch of trivial solutions at, and only at, $(\mu_k = 2k(k+1), 0)$, and solutions in $\mathcal{C}_k^\pm$ are such that $\pm f(0) > 0$ Branches are disjoint, unbounded and characterized by the number of zeroes. In $\mathcal{C}_k^\pm$, the solutions have exactly k zeroes. If k is odd, the branch $\mathcal{C}_k^\mp$ is the image of $\mathcal{C}_k^\pm$ by the Kelvin transform. If k is even, the half-branches $\mathcal{C}_k^\pm$ are invariant under the Kelvin transform. Finally, $\mathcal{C}_k^\pm$ for $k \geq 3$ and $\mathcal{C}_2^-$ are locally bounded in μ*

Sketch of the proof

Step 1: branches of solutions

- There is local bifurcation from the trivial line $\{(\mu, 0)\}$ at the points $(\mu_k := 2(N_k + 2), 0)$, and there is no other bifurcation point in this trivial branch
- On $\mathcal{C}_k^{\pm}$, the number of zeroes of the solutions is constant, namely equal to k. Non-trivial branches with different k cannot intersect or join two different points of bifurcation in the trivial branch
- For any $k \geq 2$, the half-branches $\mathcal{C}_k^{\pm}$ are therefore unbounded and we can distinguish them as follows: if $(\mu, f) \in \mathcal{C}_k^{+}$, resp. $(\mu, f) \in \mathcal{C}_k^{-}$, then $f(0) > 0$, resp. $f(0) < 0$

Step 2: Symmetry under Kelvin transform

- If k is odd, the solutions in the branches $\mathcal{C}_k^{\pm}$ have an odd number of zeroes in $(0, +\infty)$ and so, they cannot be invariant under the Kelvin transform, because they take values of different sign at 0 and near $+\infty$. Since $\mu_k = 2(N_k + 2)$ is a simple bifurcation point, the only possibility is that the branches $\mathcal{C}_k^{\pm}$ transform into each other through the Kelvin transform. Otherwise, there would be at least four half-branches bifurcating from $(\mu_k, 0)$, which is impossible.
- If k is even, the solutions of $\mathcal{C}_k^{\pm}$ have an even number of zeroes. So, they take values of the same sign at 0 and near $+\infty$. If they were not invariant under the Kelvin transform, we would find two new branches, $\tilde{\mathcal{C}}_k^{\pm}$, bifurcating from $(\mu_k, 0)$, which is again impossible.

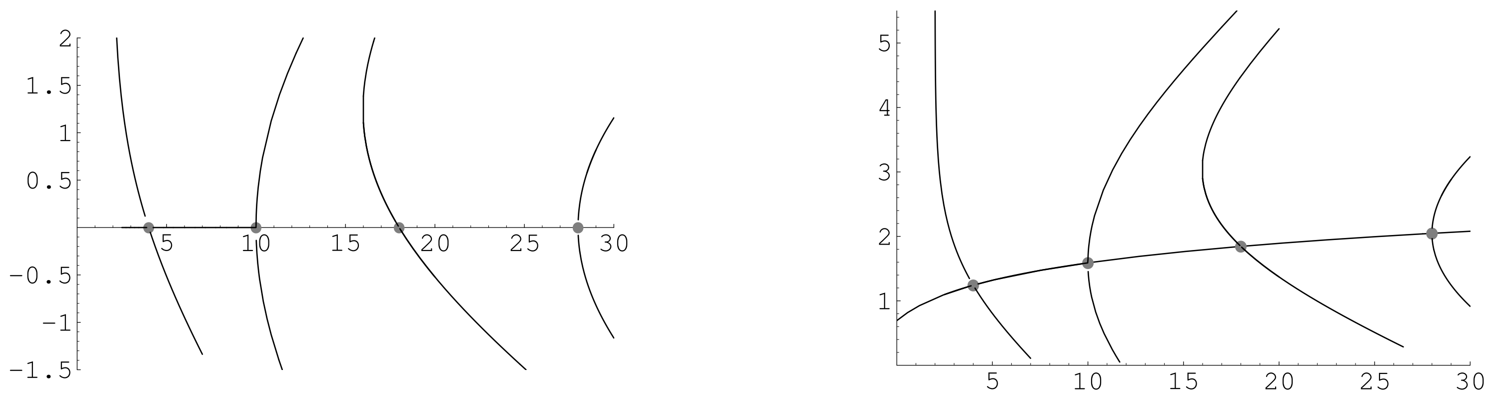
Step 3: Asymptotic behaviour of the branches

- Non trivial branches of radial solutions are contained in the region $\{\mu > 8\}$, that is $N > 2$, and there exists a unique $a(N) \in \mathbb{R}$ such that $\alpha(a) > 2N$ for all $a < a(N)$. For $N > 2$, since $N + 2 < 2N < 2(N+1)$, if $\alpha(a) = N + 2$, then $a > a(N)$. Hence $f(0) = u(0) - u_N^*(0) > a(N) - u_N^*(0)$ with $u = u_a$
- As a consequence, the branches $\mathcal{C}_k^-$ are locally bounded for $\mu \in [8, +\infty)$ for any $k \geq 2$. By Step 2, $\mathcal{C}_3^+$ is also locally bounded for $\mu \in [8, +\infty)$. Since non trivial branches do not intersect, $\mathcal{C}_k^\pm$, $k \geq 3$, are all locally bounded for $\mu \in [8, +\infty)$

Corollary 12. *For all $k \geq 2$, for all $\mu > \mu_k = 2k(k+1)$, there at least $2(k-2)+2$ distinct radial solutions, one of them being the zero solution.*

Bifurcation diagrams

The bifurcation diagram obtained for f (left) is easily transformed into a bifurcation diagram for the solutions of the weighted Liouville-type equation in $\mathbb{R}^2$ (right) with $\lambda = 2\pi(N+2)$ through the transformation $u = f + u_N^*$. In the case of equation the weighted Liouville-type equation in $\mathbb{R}^2$, branches bifurcate from the set of trivial solutions $\mathcal{C} := \{(N, \frac{1}{2}\log(2(N+2))\}$, in the representation $(N, a = u(0))$.



Bifurcation diagram in the representation $(N, f(0))$ for f (left) and (N, a) for equation the weighted Liouville-type equation in $\mathbb{R}^2$ with $\lambda = 2\pi(N+2)$ (right). Non trivial branches bifurcate from $N_k = 4, 10, 18, 28, \ldots$

The asymptote of $\mathcal{C}^+_{k=2}$

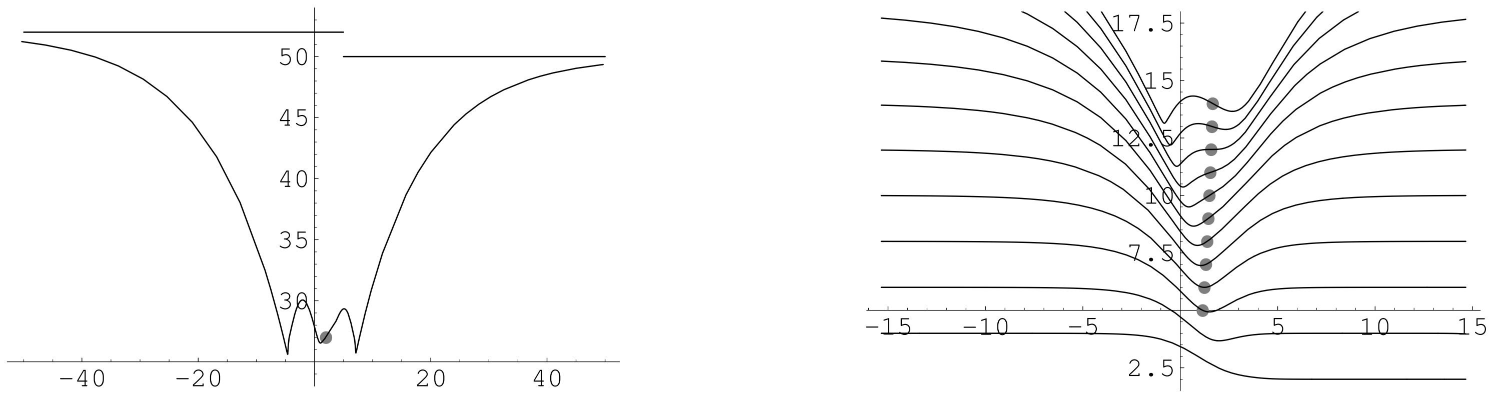
Based on numerical evidence, it is reasonable to conjecture that, in contrast, the branch $\mathcal{C}^+_{k=2}$ admits a vertical asymptote in the sense that as $s \longrightarrow +\infty$, then N converges to 2, which is the only admissible value. So for $(\mu(s), f_s) \in \mathcal{C}^+_2$, f_s should develop a concentration phenomenon at the origin, and as $s \longrightarrow +\infty$, we should have: $a \longrightarrow +\infty$, $N \longrightarrow 2_+$ and $\frac{\mu(s)}{(1+|x|^2)^2}\, e^{2f_s} \rightharpoonup 8\pi\, \delta_{z=0}$, weakly in the sense of measures

Non radially symmetric solutions

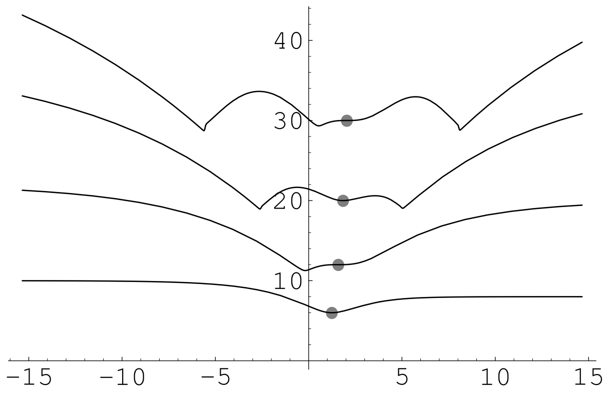
To any solution $u \neq u_N^*$ such that $\int_{\mathbb{R}^2} (1+|x|^2)^N \, e^{2u} \, dx = 2\pi \, (N+2)$, we can associate a punctured sphere of non radially symmetric solutions, u_{e} with $\mathsf{e} \in \mathbb{S}^2 \setminus \{\mathsf{N}, \mathsf{S}\}$, satisfying also $\int_{\mathbb{R}^2} (1+|x|^2)^N \, e^{2u_{\mathsf{e}}} \, dx = 2\pi \, (N+2)$ for all $\mathsf{e} \in \mathbb{S}^2 \setminus \{\mathsf{N}, \mathsf{S}\}$. And so, for $N > N_k$, there are at least $2(k-2)+1$ punctured spheres of non radially symmetric solutions at level $\lambda = 2\pi(N+2)$

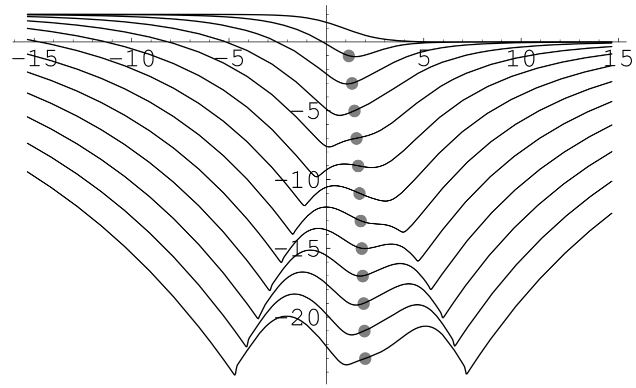
4. More numerical observations and conjectures

Curves $a \mapsto \alpha(a)$



Curves $a \mapsto \alpha(a)$ for various values of N: $N = 25$ (left) and $N = 1, 2, 3, \ldots 12$ (right). The point $(a_N^*, N+2)$ corresponding to the explicit solution is represented by a gray dot



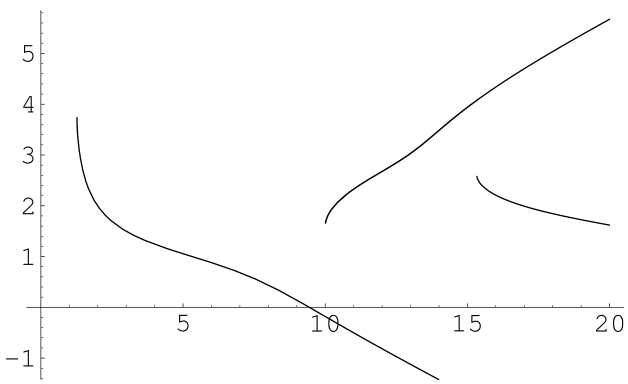


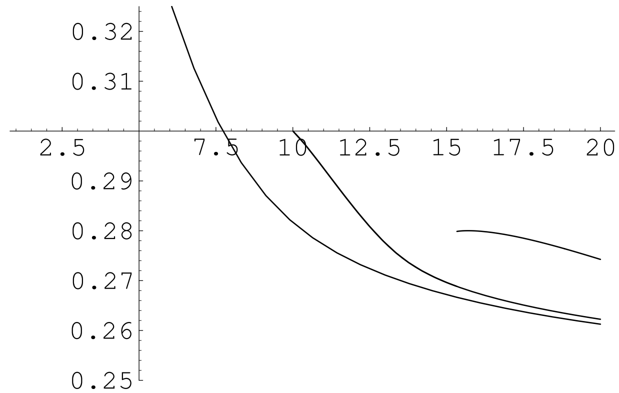
Curves $a \mapsto \alpha(a)$ for $N = N_k$, $k = 2, 3, 4, 5$ (left) and $a \mapsto \alpha(a) - 2N$ for $N = 1$, 3, 5,… 19 (right). The function $N \mapsto \alpha_N - 2N$ is monotone decreasing

As a function of $N > 0$, we observe that $\alpha_N = \inf_{a\in\mathbb{R}} \alpha(a) < 2N$ if and only if $N > N_0$, where N_0 is numerically found of the order of 1.27 ± 0.02

Proposition 13. *There exists $N_0 \in (1, 2)$ such that, for all $N \in (N_0, \infty)$, $\alpha_N = \min_{a\in\mathbb{R}} \alpha(a) < 2N$, and for all $\alpha \in (\alpha_N,\, 2N)$, there exists at least two solutions*

Local minima as a function of N

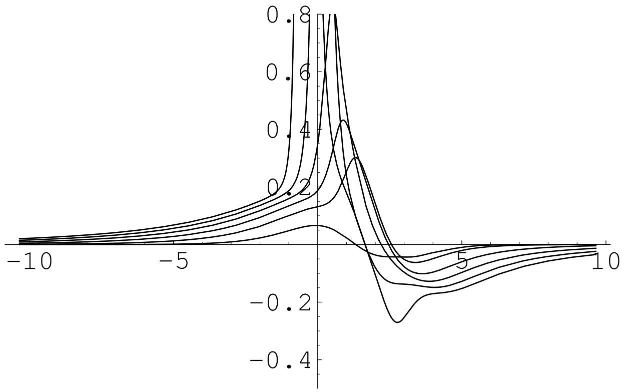


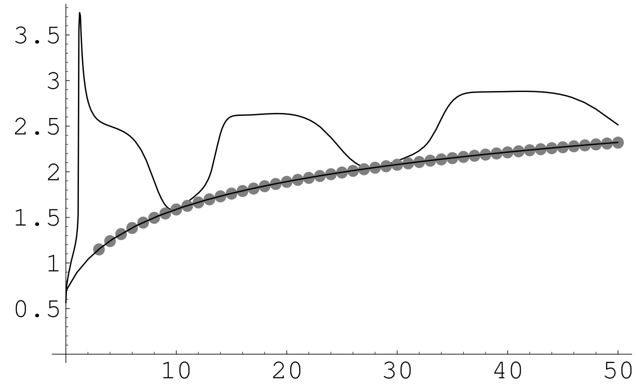


Points of local minimum (left) and corresponding critical values (divided by $4N$, right) of α, as a function of N.

On the basis of our numerical results, we may also conjecture that for $N_0 < N < 10$ and $\alpha \in (\alpha_N, 2N)$, there exist exactly two radially symmetric solutions of the weighted Liouville-type equation in $\mathbb{R}^2$. This conjecture is supported by the bifurcation analysis concerning the specific value $\alpha = N + 2 \in (\alpha_N, 2N)$ for $N > 2$ and $N \neq 4$. Note that for $N = 4$, $\alpha_N = N + 2$ should hold. As N increases, the curves $a \mapsto \alpha(a)$ appear to have more and more critical points:the number of solutions increases

The function $a \mapsto J_N(a)$



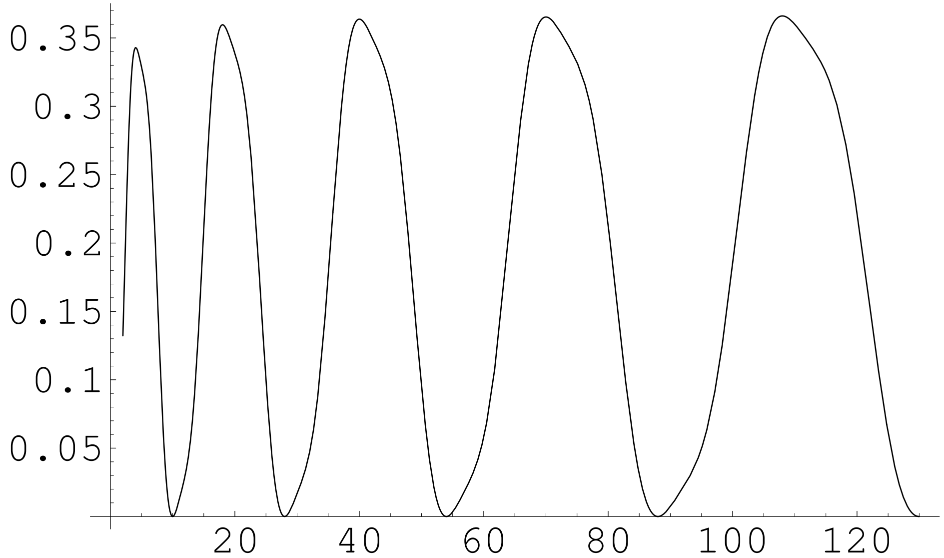


The function $a \mapsto J_N(a)$ for $N = 1, 3, 5, \ldots 11$ (left) and the curve $N \mapsto c(N)$, where, at N fixed, $c(N)$ is the first positive zero of $a \mapsto J_N(a)$; the dotted line corresponds to $N \mapsto \frac{1}{2}\log(2(N+2))$. These two curves are tangent at $N = 10 = N_3$ and $N = 28 = N_5$ (right)

Conjectures

- *There exists a function $N \mapsto c(N)$ on $(0, +\infty)$ such that $J_N(a) = 0$ if and only if $a = c(N)$ and $J_N(a) > 0$ if and only if $a < c(N)$.*
- *For any $N > 2$, $a_N^* \leq c(N)$, with equality if and only if $N = N_{2l+1}$, $l \geq 1$*

These conjectures are observed numerically with a very high accuracy for $k = 3, 5, 7, 9, 11$



The curve $N \mapsto J_N(a_N^*)$ is nonnegative and achieves its minimum value, 0, (resp. local maxima) for $N = N_{2l+1}, l \geq 1$ (resp. $N = N_{2l}$)

$j(k) := J_N(a_{N_k}^*) = 0$ if k is odd and $j(k) > 0$ if k is even

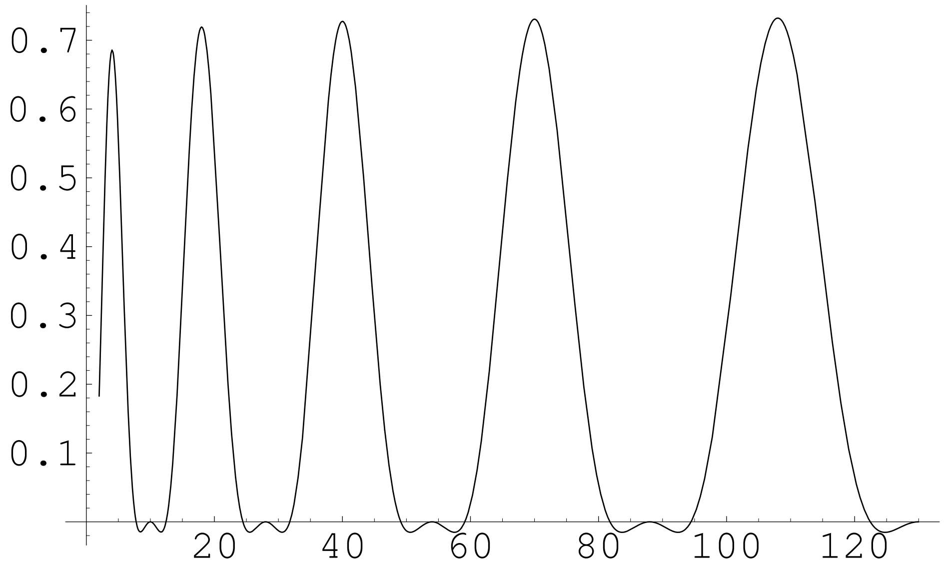
To investigate whether a critical point of α is a local minimum, we may look at the functional

$$K_N(a) := \int_0^{+\infty} (1+r^2)^N e^{2u_a} (\psi_a + 2\varphi_a^2)\, r\, dr$$

where ψ_a solves the ordinary differential equation

$$\begin{cases} \psi_a'' + \frac{\psi_a'}{r} + 2\,(1+r^2)^N e^{2u_a}\,(\psi_a + 2\,\varphi_a^2) = 0\,, \quad r \in (0, +\infty) \\ \psi_a(0) = 0\,, \quad \psi_a'(0) = 0 \end{cases}$$

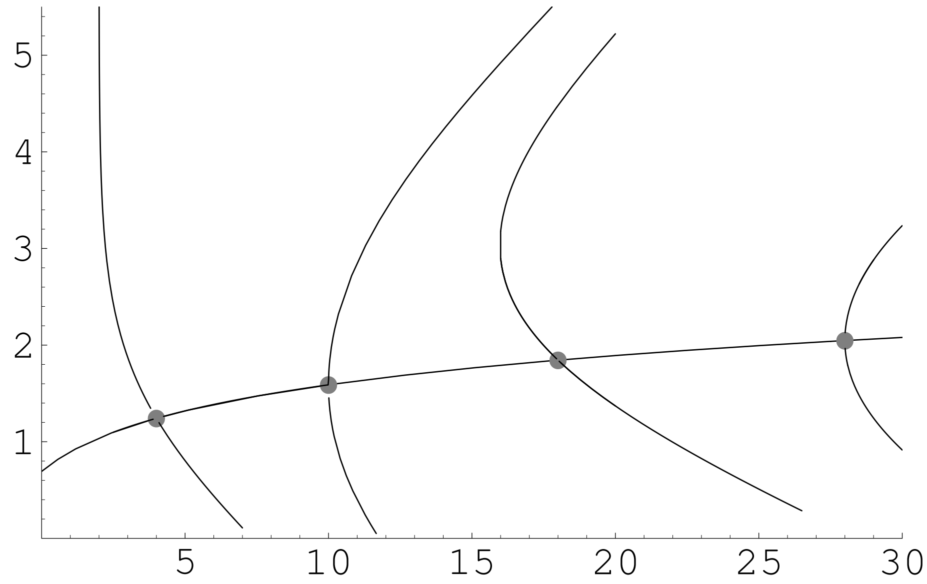
We have indeed $\alpha''(a) = 2\,K_N(a)$. No simple criterion for the positivity of $K_N(a)$ is known, but our numerical results at level $\alpha = N+2$ combine very well with our main results



The curve $N \mapsto K_N(a_N^*)$ *changes sign, but is always nonnegative when* $\alpha'(N) = 0$*.*
When $N = N_{2l}, l \geq 1$, $K_N(a_N^*)$ *is positive*

5. Conclusion

Concluding remarks



- A set of solutions with a rich structure, by far richer than for $N = 0$
- Even at level $\lambda = 2\pi(N+2)$, bifurcation diagram is not completely understood
- For $\lambda \neq 2\pi(N+2)$, multiplicity is essentially an open question
- There are plenty of non radially symmetric solutions: classification ?