Duality, flows and improved Sobolev inequalities

Jean Dolbeault

http://www.ceremade.dauphine.fr/∼dolbeaul

Ceremade, Université Paris-Dauphine

September 16, 2015

Workshop on *Nonlocal Nonlinear Partial Differential Equations and Applications*, Anacapri, 14-18 September, 2015

Improvements of Sobolev's inequality

A brief (and incomplete) review of improved Sobolev inequalities
involving the fractional Laplacian

The fractional Sobolev inequality

$$\|u\|^2_{\mathring{\mathrm{H}}^{\frac{s}{2}}} \geq \mathsf{S} \left(\int_{\mathbb{R}^d} |u|^q \, dx \right)^{\frac{2}{q}} \quad \forall\, u \in \mathring{\mathrm{H}}^{\frac{s}{2}}(\mathbb{R}^d)$$

where $0 < s < d$, $q = \frac{2d}{d-s}$

$\mathring{\mathrm{H}}^{\frac{s}{2}}(\mathbb{R}^d)$ is the space of all tempered distributions u such that

$$\hat{u} \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^d) \quad \text{and} \quad \|u\|^2_{\mathring{\mathrm{H}}^{\frac{s}{2}}} := \int_{\mathbb{R}^d} |\xi|^s |\hat{u}|^2 dx < \infty$$

Here $\hat{u}$ denotes the Fourier transform of u

$$\mathsf{S} = \mathsf{S}_{d,s} = 2^s \pi^{\frac{s}{2}} \frac{\Gamma(\frac{d+s}{2})}{\Gamma(\frac{d-s}{2})} \left(\frac{\Gamma(\frac{d}{2})}{\Gamma(d)} \right)^{s/d}$$

▷ Non-fractional: [Bliss], [Rosen], [Talenti], [Aubin] (+link with Yamabe flow)

▷ Fractional: dual form on the sphere [Lieb, 1983]; the case $s = 1$: [Escobar, 1988]; [Swanson, 1992], [Chang, Gonzalez, 2011]; moving planes method: [Chen, Li, Ou, 2006]

The dual Hardy-Littlewood-Sobolev inequality

$$\left|\iint_{\mathbb{R}^d\times\mathbb{R}^d}\frac{f(x)\,g(y)}{|x-y|^\lambda}\,dx\,dy\right|\le\pi^{\lambda/2}\frac{\Gamma(\frac{d-\lambda}{2})}{\Gamma(d-\frac{\lambda}{2})}\left(\frac{\Gamma(d)}{\Gamma(d/2)}\right)^{1-\frac{\lambda}{d}}\|f\|_{\mathrm{L}^p(\mathbb{R}^d)}\,\|g\|_{\mathrm{L}^p(\mathbb{R}^d)}$$

for all $f,g\in L^p(\mathbb{R}^d)$, where $0<\lambda<d$ and $p=\frac{2d}{2d-\lambda}$

The equivalence with the Sobolev inequality follows by a duality argument: for every $f\in L^{\frac{q}{q-1}}(\mathbb{R}^d)$ there exists a unique solution $(-\Delta)^{-s/2}f\in\mathring{\mathrm{H}}^{\frac{s}{2}}(\mathbb{R}^d)$ of $(-\Delta)^{s/2}u=f$ given by

$$(-\Delta)^{-s/2}f\,(x)=2^{-s}\pi^{-\frac{d}{2}}\frac{\Gamma(\frac{d-s}{2})}{\Gamma(s/2)}\int_{\mathbb{R}^d}\frac{1}{|x-y|^{d-s}}f(y)\,dy$$

[Lieb, 83]: identification of the extremal functions
(on the sphere; then use the stereographic projection)

Up to translations, dilations and multiplication by a nonzero constant, the optimal function is

$$U(x)=(1+|x|^2)^{-\frac{d-s}{2}}\,,\quad x\in\mathbb{R}^d$$

Biachi-Egnell type improvements

Theorem

There exists a positive constant $\alpha = \alpha(s,d)$ with $s \in (0,d)$ such that

$$\int_{\mathbb{R}^d} u\,(-\Delta)^{s/2} u\,dx - \mathsf{S}\left(\int_{\mathbb{R}^d} |u|^q\,dx\right)^{\frac{2}{q}} \geq \alpha\, d^2(u,\mathcal{M})$$

where $d(u,\mathcal{M}) = \min\{\|u-\varphi\|^2_{\dot{\mathrm{H}}^{\frac{s}{2}}} \;:\; \varphi \in \mathcal{M}\}$
and $\mathcal{M}$ is the set of optimal functions [Chen, Frank, Weth, 2013]

▷ Existence of a weak $\mathrm{L}^{2^*/2}$-remainder term in bounded domains in the case $s=2$: [Brezis, Lieb, 1985]
[Gazzola, Grunau, 2001] when $s \in \mathbb{N}$ is even, positive, and $s < d$
▷ [Bianchi, Egnell, 1991] for $s=2$, [Bartsch, Weth, Willem, 2003] and [Lu, Wei, 2000] when $s \in \mathbb{N}$ is even, positive, and $s < d$ (ODE)
▷ Inverse stereographic projection (eigenvalues): [Ding, 1986], [Beckner, 1993], [Morpurgo, 2002], [Bartsch, Schneider, Weth, 2004]
▷ Symmetrization [Cianchi, Fusco, Maggi, Pratelli, 2009] and [Figalli, Maggi, Pratelli, 2010] + many others: ask for experts in Naples!

Nonlinear flows as a tool for getting sharp/improved functional inequalities

- Prove inequalities: Gagliardo-Nireberg inequalities in sharp form [del Pino, JD, 2002]

$$\|w\|_{L^{2p}(\mathbb{R}^d)} \le \mathcal{C}^{\rm GN}_{p,d}\,\|\nabla w\|^{\theta}_{L^2(\mathbb{R}^d)}\,\|w\|^{1-\theta}_{L^{p+1}(\mathbb{R}^d)}$$

equivalent to entropy – entropy production inequalities: [Carrillo, Toscani, 2000], [Carrillo, Vázquez, 2003]; also see [Arnold, Markowich, Toscani, Unterreiter], [Arnold, Carrillo, Desvillettes, JD, Jüngel, Lederman, Markowich, Toscani, Villani, 2004], [Carrillo, Jüngel, Markowich, Toscani, Unterreiter]... and many other papers

- Establish sharp symmetry breaking conditions in Caffarelli-Kohn-Nirenberg inequalities [JD, Esteban, Loss, 2015]

$$\left(\int_{\mathbb{R}^d}\frac{|v|^p}{|x|^{b\,p}}\,dx\right)^{2/p} \le \mathsf{C}_{a,b}\int_{\mathbb{R}^d}\frac{|\nabla v|^2}{|x|^{2\,a}}\,dx\,,\quad p=\frac{2\,d}{d-2+2\,(b-a)}$$

with the conditions $a\le b\le a+1$ if $d\ge 3$, $a<b\le a+1$ if $d=2$, $a+1/2<b\le a+1$ if $d=1$, and $a<a_c:=(d-2)/2$

Linear flow: improved Bakry-Emery method

[Arnold, JD, 2005], [Arnold, Bartier, JD, 2007], [JD, Esteban, Kowalczyk, Loss, 2014]

- Consider the heat flow / Ornstein-Uhlenbeck equation written for $u = w^p$: with $\kappa = p - 2$, we have

$$w_t = \mathcal{L}\, w + \kappa \, \frac{|w'|^2}{w} \, \nu$$

If $p > 1$ and either $p < 2$ (flat, Euclidean case) or $p < \frac{2\,d^2+1}{(d-1)^2}$ (case of the sphere), there exists a positive constant γ such that

$$\frac{d}{dt} (\mathsf{i} - \, d\, \mathsf{e}) \le -\gamma \int_{-1}^{1} \frac{|w'|^4}{w^2} \, d\nu_d \le -\gamma \, \frac{|\mathsf{e}'|^2}{1 - \,(p-2)\,\mathsf{e}}$$

Recalling that $\mathsf{e}' = -\,\mathsf{i}$, we get a differential inequality

$$\mathsf{e}'' + \, d\, \mathsf{e}' \ge \gamma \, \frac{|\mathsf{e}'|^2}{1 - \,(p-2)\,\mathsf{e}}$$

After integration: $d\,\Phi(\mathsf{e}(0)) \le \mathsf{i}(0)$

Improvements - From linear to nonlinear flows

What does "improvement" mean ? (Case of the sphere $\mathbb{S}^d$)

An *improved* inequality is

$$d\,\Phi\,(\mathsf{e}) \le \mathsf{i} \quad \forall\, u \in \mathrm{H}^1(\mathbb{S}^d) \quad \text{s.t.} \quad \|u\|^2_{\mathrm{L}^2(\mathbb{S}^d)} = 1$$

for some function Φ such that $\Phi(0) = 0$, $\Phi'(0) = 1$, $\Phi' > 0$ and $\Phi(s) > s$ for any s. With $\Psi(s) := s - \Phi^{-1}(s)$

$$\mathsf{i} - d\,\mathsf{e} \ge d\,(\Psi \circ \Phi)(\mathsf{e}) \quad \forall\, u \in \mathrm{H}^1(\mathbb{S}^d) \quad \text{s.t.} \quad \|u\|^2_{\mathrm{L}^2(\mathbb{S}^d)} = 1$$

▷ When such an improvement is available, the best constant is achieved by linearizing

- Fast diffusion equation: [Blanchet, Bonforte, JD, Grillo, Vázquez, 2010], [JD, Toscani, 2011]
- With $\mathsf{i}[u] = \|\nabla u\|^2_{\mathrm{L}^2(\mathbb{S}^d)}$ and $\mathsf{e}[u] = \frac{1}{p-2}\left(\|u\|^2_{\mathrm{L}^2(\mathbb{S}^d)} - \|u\|^2_{\mathrm{L}^p(\mathbb{S}^d)}\right)$

$$\inf_{u \in \mathrm{H}^1(\mathbb{S}^d)} \frac{\mathsf{i}[u]}{\mathsf{e}[u]} = d$$

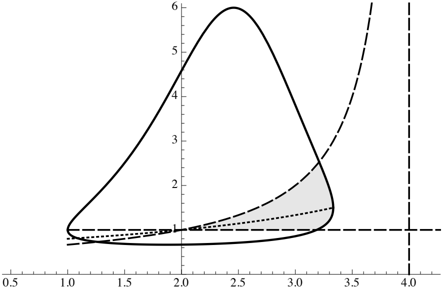
Improvements based on nonlinear flows

Manifolds: [Bidaut-Véron, Véron, 1991], [Beckner, 1993], [Bakry, Ledoux, 1996], [Demange, 2008]
[Demange, PhD thesis], [JD, Esteban, Kowalczyk, Loss, 2014]... the sphere

$$w_t = w^{2-2\beta}\left(\mathcal{L}\, w + \kappa\,\frac{|w'|^2}{w}\right)$$

with $p \in [1, 2^*]$ and $\kappa = \beta\,(p-2)+1$

▷ Admissible (p, β) for $d = 5$



Other type of improvements based on nonlinear flows flows

- Hardy-Littlewood-Sobolev: a proof based on Gagliardo-Nirenberg inequalities [E. Carlen, J.A. Carrillo and M. Loss]

The fast diffusion equation

$$\frac{\partial v}{\partial t} = \Delta v^{\frac{d}{d+2}} \quad t > 0 \,, \quad x \in \mathbb{R}^d$$

- Hardy-Littlewood-Sobolev: a proof based on the Yamabe flow

$$\frac{\partial v}{\partial t} = \Delta v^{\frac{d-2}{d+2}} \quad t > 0 \,, \quad x \in \mathbb{R}^d$$

[JD, 2011], [JD, Jankowiak, 2014]

- The limit case $d = 2$ of the logarithmic Hardy-Littlewood-Sobolev is covered. The dual inequality is the Onofri inequality, which can be established directly by the fast diffusion flow [JD, Esteban, Jankowiak]

Sobolev and Hardy-Littlewood-Sobolev inequalities: duality, flows

Joint work with G. Jankowiak

Preliminary observations

Legendre duality: Onofri and log HLS

Legendre's duality: $F^*[v] := \sup \left(\int_{\mathbb{R}^2} u\, v\, dx - F[u] \right)$

$$F_1[u] := \log\left(\int_{\mathbb{R}^2} e^u\, d\mu \right),\; F_2[u] := \frac{1}{16\,\pi} \int_{\mathbb{R}^2} |\nabla u|^2\, dx + \int_{\mathbb{R}^2} u\, \mu\, dx$$

Onofri's inequality ($d = 2$) amounts to $F_1[u] \le F_2[u]$ with $d\mu(x) := \mu(x)\, dx$, $\mu(x) := \frac{1}{\pi\,(1+|x|^2)^2}$

Proposition

For any $v \in \mathrm{L}^1_+(\mathbb{R}^2)$ with $\int_0^\infty v\, r\, dr = 1$, such that $v \log v$ and $(1 + \log|x|^2)\, v \in \mathrm{L}^1(\mathbb{R}^2)$, we have

$F_1^*[v] - F_2^*[v] =$

$\int_0^\infty v\, \log\left(\frac{v}{\mu}\right) r\, dr - 4\,\pi \int_0^\infty (v - \mu)\,(-\Delta)^{-1}(v - \mu)\, r\, dr \ge 0$

[E. Carlen, M. Loss] [W. Beckner] [V. Calvez, L. Corrias]

A puzzling result of E. Carlen, J.A. Carrillo and M. Loss

[E. Carlen, J.A. Carrillo and M. Loss] The fast diffusion equation

$$\frac{\partial v}{\partial t} = \Delta v^m \quad t > 0\,, \quad x \in \mathbb{R}^d$$

with exponent $m = d/(d+2)$, when $d \geq 3$, is such that

$$\mathsf{H}_d[v] := \int_{\mathbb{R}^d} v\,(-\Delta)^{-1} v\; dx - \mathsf{S}_d\,\|v\|^2_{\mathrm{L}^{\frac{2\,d}{d+2}}(\mathbb{R}^d)}$$

obeys to

$$\frac{1}{2}\,\frac{d}{dt}\mathsf{H}_d[v(t,\cdot)] = \frac{1}{2}\,\frac{d}{dt}\left[\int_{\mathbb{R}^d} v\,(-\Delta)^{-1} v\; dx - \mathsf{S}_d\,\|v\|^2_{\mathrm{L}^{\frac{2\,d}{d+2}}(\mathbb{R}^d)}\right]$$
$$= \frac{d\,(d-2)}{(d-1)^2}\,\mathsf{S}_d\,\|u\|^{4/(d-1)}_{\mathrm{L}^{q+1}(\mathbb{R}^d)}\,\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{R}^d)} - \|u\|^{2q}_{\mathrm{L}^{2q}(\mathbb{R}^d)}$$

with $u = v^{(d-1)/(d+2)}$ and $q = \frac{d+1}{d-1}$. The r.h.s. is nonnegative. Optimality is achieved simultaneously in both functionals (Barenblatt regime): the Hardy-Littlewood-Sobolev inequalities can be improved by an integral remainder term

... and the two-dimensional case

Recall that $(-\Delta)^{-1} v = G_d * v$ with

- $G_d(x) = \frac{1}{d-2} \, |\mathbb{S}^{d-1}|^{-1} \, |x|^{2-d}$ if $d \ge 3$
- $G_2(x) = \frac{1}{2\,\pi} \, \log|x|$ if $d = 2$

Same computation in dimension $d = 2$ with $m = 1/2$ gives

$$\frac{\|v\|_{\mathrm{L}^1(\mathbb{R}^2)}}{8} \frac{d}{dt} \left[\frac{4\,\pi}{\|v\|_{\mathrm{L}^1(\mathbb{R}^2)}} \int_0^\infty v\,(-\Delta)^{-1} v \; r\,dr - \int_0^\infty v \, \log v \; r\,dr \right]$$
$$= \|u\|^4_{\mathrm{L}^4(\mathbb{R}^2)} \, \|\nabla u\|^2_{\mathrm{L}^2(\mathbb{R}^2)} - \pi \, \|u\|^6_{\mathrm{L}^6(\mathbb{R}^2)}$$

The r.h.s. is one of the Gagliardo-Nirenberg inequalities ($d = 2$, $q = 3$)
The l.h.s. is bounded from below by the logarithmic HLS inequality and achieves its minimum if $v = \mu$ with

$$\mu(x) := \frac{1}{\pi\,(1 + |x|^2)^2} \quad \forall\, x \in \mathbb{R}^2$$

Sobolev and HLS

As it has been noticed by E. Lieb, Sobolev's inequality in $\mathbb{R}^d$, $d \geq 3$,

$$\|u\|^2_{\mathrm{L}^{2^*}(\mathbb{R}^d)} \leq \mathsf{S}_d \, \|\nabla u\|^2_{\mathrm{L}^2(\mathbb{R}^d)} \quad \forall\, u \in \mathcal{D}^{1,2}(\mathbb{R}^d) \tag{1}$$

and the Hardy-Littlewood-Sobolev inequality

$$\mathsf{S}_d \, \|v\|^2_{\mathrm{L}^{\frac{2d}{d+2}}(\mathbb{R}^d)} \geq \int_{\mathbb{R}^d} v \, (-\Delta)^{-1} v \, dx \quad \forall\, v \in \mathrm{L}^{\frac{2d}{d+2}}(\mathbb{R}^d) \tag{2}$$

are dual of each other. Here S_d is the Aubin-Talenti constant and $2^* = \frac{2d}{d-2}$. Can we recover this using a nonlinear flow approach ? Can we improve it ?

Using the Yamabe / Ricci flow

Using a nonlinear flow to relate Sobolev and HLS

Consider the *fast diffusion* equation

$$\frac{\partial v}{\partial t} = \Delta v^m \quad t > 0\,, \quad x \in \mathbb{R}^d \tag{3}$$

If we define $\mathsf{H}(t) := \mathsf{H}_d[v(t,\cdot)]$, with

$$\mathsf{H}_d[v] := \int_{\mathbb{R}^d} v\,(-\Delta)^{-1} v\;dx - \mathsf{S}_d\,\|v\|^2_{\mathrm{L}^{\frac{2d}{d+2}}(\mathbb{R}^d)}$$

then we observe that

$$\frac{1}{2}\,\mathsf{H}' = -\int_{\mathbb{R}^d} v^{m+1}\;dx + \mathsf{S}_d \left(\int_{\mathbb{R}^d} v^{\frac{2d}{d+2}}\;dx\right)^{\frac{2}{d}} \int_{\mathbb{R}^d} \nabla v^m \cdot \nabla v^{\frac{d-2}{d+2}}\;dx$$

where $v = v(t,\cdot)$ is a solution of (3). With the choice $m = \frac{d-2}{d+2}$, we find that $m+1 = \frac{2d}{d+2}$

A first statement

Proposition

[JD] *Assume that* $d \geq 3$ *and* $m = \frac{d-2}{d+2}$. *If* v *is a solution of* (3) *with nonnegative initial datum in* $\mathrm{L}^{2d/(d+2)}(\mathbb{R}^d)$, *then*

$$\frac{1}{2}\frac{d}{dt}\left[\int_{\mathbb{R}^d} v\,(-\Delta)^{-1} v\,dx - \mathsf{S}_d\,\|v\|^2_{\mathrm{L}^{\frac{2d}{d+2}}(\mathbb{R}^d)}\right]$$
$$= \left(\int_{\mathbb{R}^d} v^{m+1}\,dx\right)^{\frac{2}{d}}\left[\mathsf{S}_d\,\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{R}^d)} - \|u\|^2_{\mathrm{L}^{2^*}(\mathbb{R}^d)}\right] \geq 0$$

The HLS inequality amounts to $\mathsf{H} \leq 0$ and appears as a consequence of Sobolev, that is $\mathsf{H}' \geq 0$ if we show that $\limsup_{t>0} \mathsf{H}(t) = 0$
Notice that $u = v^m$ is an optimal function for (1) if v is optimal for (2)

Improved Sobolev inequality

By integrating along the flow defined by (3), we can actually obtain optimal integral remainder terms which improve on the usual Sobolev inequality (1), but only when $d \geq 5$ for integrability reasons

Theorem

[JD] *Assume that $d \geq 5$ and let $q = \frac{d+2}{d-2}$. There exists a positive constant $\mathcal{C} \leq \left(1+\frac{2}{d}\right)\left(1-e^{-d/2}\right) \mathsf{S}_d$ such that*

$$\mathsf{S}_d \, \|w^q\|^2_{\mathrm{L}^{\frac{2d}{d+2}}(\mathbb{R}^d)} - \int_{\mathbb{R}^d} w^q \, (-\Delta)^{-1} w^q \, dx \\ \leq \mathcal{C} \, \|w\|^{\frac{8}{d-2}}_{\mathrm{L}^{2^*}(\mathbb{R}^d)} \left[\|\nabla w\|^2_{\mathrm{L}^2(\mathbb{R}^d)} - \mathsf{S}_d \, \|w\|^2_{\mathrm{L}^{2^*}(\mathbb{R}^d)} \right]$$

for any $w \in \mathcal{D}^{1,2}(\mathbb{R}^d)$

Solutions with *separation of variables*

Consider the solution of $\frac{\partial v}{\partial t} = \Delta v^m$ vanishing at $t = T$:

$$\overline{v}_T(t,x) = c\,(T-t)^\alpha\,(F(x))^{\frac{d+2}{d-2}}$$

where F is the Aubin-Talenti solution of

$$-\Delta F = d\,(d-2)\,F^{(d+2)/(d-2)}$$

Let $\|v\|_* := \sup_{x\in\mathbb{R}^d}(1+|x|^2)^{d+2}\,|v(x)|$

Lemma

[M. del Pino, M. Saez], [J. L. Vázquez, J. R. Esteban, A. Rodriguez]
For any solution v with initial datum $v_0 \in \mathrm{L}^{2d/(d+2)}(\mathbb{R}^d)$, $v_0 > 0$, there exists $T > 0$, $\lambda > 0$ and $x_0 \in \mathbb{R}^d$ such that

$$\lim_{t\to T_-}(T-t)^{-\frac{1}{1-m}}\,\|v(t,\cdot)/\overline{v}(t,\cdot)-1\|_* = 0$$

with $\overline{v}(t,x) = \lambda^{(d+2)/2}\,\overline{v}_T(t,(x-x_0)/\lambda)$

Improved inequality: proof (1/2)

The function $\mathsf{J}(t) := \int_{\mathbb{R}^d} v(t,x)^{m+1}\,dx$ satisfies

$$\mathsf{J}' = -(m+1)\,\|\nabla v^m\|^2_{\mathrm{L}^2(\mathbb{R}^d)} \le -\frac{m+1}{\mathsf{S}_d}\,\mathsf{J}^{1-\frac{2}{d}}$$

If $d \ge 5$, then we also have

$$\mathsf{J}'' = 2\,m\,(m+1)\int_{\mathbb{R}^d} v^{m-1}\,(\Delta v^m)^2\,dx \ge 0$$

Notice that

$$\frac{\mathsf{J}'}{\mathsf{J}} \le -\frac{m+1}{\mathsf{S}_d}\,\mathsf{J}^{-\frac{2}{d}} \le -\kappa \quad \text{with} \quad \kappa\,T = \frac{2\,d}{d+2}\,\frac{T}{\mathsf{S}_d}\left(\int_{\mathbb{R}^d} v_0^{m+1}\,dx\right)^{-\frac{2}{d}} \le \frac{d}{2}$$

Improved inequality: proof (2/2)

By the Cauchy-Schwarz inequality, we have

$$\frac{\mathsf{J}'^2}{(m+1)^2} = \|\nabla v^m\|^4_{\mathrm{L}^2(\mathbb{R}^d)} = \left(\int_{\mathbb{R}^d} v^{(m-1)/2}\,\Delta v^m \cdot v^{(m+1)/2}\,dx\right)^2$$
$$\le \int_{\mathbb{R}^d} v^{m-1}\,(\Delta v^m)^2\,dx \int_{\mathbb{R}^d} v^{m+1}\,dx = Cst\;\mathsf{J}''\,\mathsf{J}$$

so that $\mathsf{Q}(t) := \|\nabla v^m(t,\cdot)\|^2_{\mathrm{L}^2(\mathbb{R}^d)} \left(\int_{\mathbb{R}^d} v^{m+1}(t,x)\,dx\right)^{-(d-2)/d}$ is monotone decreasing, and

$$\mathsf{H}' = 2\,\mathsf{J}\,(\mathsf{S}_d\,\mathsf{Q} - 1)\,,\quad \mathsf{H}'' = \frac{\mathsf{J}'}{\mathsf{J}}\,\mathsf{H}' + 2\,\mathsf{J}\,\mathsf{S}_d\,\mathsf{Q}' \le \frac{\mathsf{J}'}{\mathsf{J}}\,\mathsf{H}' \le 0$$

$$\mathsf{H}'' \le -\kappa\,\mathsf{H}' \quad \text{with} \quad \kappa = \frac{2\,d}{d+2}\,\frac{1}{\mathsf{S}_d}\left(\int_{\mathbb{R}^d} v_0^{m+1}\,dx\right)^{-2/d}$$

By writing that $-\mathsf{H}(0) = \mathsf{H}(T) - \mathsf{H}(0) \le \mathsf{H}'(0)\,(1 - e^{-\kappa\,T})/\kappa$ and using the estimate $\kappa\,T \le d/2$, the proof is completed □

$d=2$: Onofri's and log HLS inequalities

$$\mathsf{H}_2[v] := \int_0^\infty (v-\mu)\,(-\Delta)^{-1}(v-\mu)\;r\,dr - \frac{1}{4\,\pi}\int_0^\infty v\,\log\left(\frac{v}{\mu}\right)\,r\,dr$$

With $\mu(x) := \frac{1}{\pi}\,(1+|x|^2)^{-2}$. Assume that v is a positive solution of

$$\frac{\partial v}{\partial t} = \Delta\log\left(v/\mu\right)\quad t>0\,,\quad x\in\mathbb{R}^2$$

Proposition

If $v=\mu\,e^{u/2}$ is a solution with nonnegative initial datum v_0 in $\mathrm{L}^1(\mathbb{R}^2)$ such that $\int_0^\infty v_0\;r\,dr=1$, $v_0\,\log v_0\in\mathrm{L}^1(\mathbb{R}^2)$ and $v_0\,\log\mu\in\mathrm{L}^1(\mathbb{R}^2)$, then

$$\frac{d}{dt}\mathsf{H}_2[v(t,\cdot)] = \frac{1}{16\,\pi}\int_0^\infty |\nabla u|^2\;r\,dr - \int_{\mathbb{R}^2}\left(e^{\frac{u}{2}}-1\right)u\,d\mu$$
$$\geq \frac{1}{16\,\pi}\int_0^\infty |\nabla u|^2\;r\,dr + \int_{\mathbb{R}^2} u\,d\mu - \log\left(\int_{\mathbb{R}^2} e^u\,d\mu\right)\geq 0$$

Improvements

Improved Sobolev inequality by duality

Theorem

[JD, G. Jankowiak] *Assume that $d \geq 3$ and let $q = \frac{d+2}{d-2}$. There exists a positive constant $\mathcal{C} \leq 1$ such that*

$$\mathsf{S}_d \, \|w^q\|^2_{\mathrm{L}^{\frac{2d}{d+2}}(\mathbb{R}^d)} - \int_{\mathbb{R}^d} w^q \, (-\Delta)^{-1} w^q \, dx \\ \leq \mathcal{C} \, \mathsf{S}_d \, \|w\|^{\frac{8}{d-2}}_{\mathrm{L}^{2^*}(\mathbb{R}^d)} \left[\|\nabla w\|^2_{\mathrm{L}^2(\mathbb{R}^d)} - \mathsf{S}_d \, \|w\|^2_{\mathrm{L}^{2^*}(\mathbb{R}^d)} \right]$$

for any $w \in \mathcal{D}^{1,2}(\mathbb{R}^d)$

Proof: the completion of a square

Integrations by parts show that

$$\int_{\mathbb{R}^d} |\nabla(-\Delta)^{-1} v|^2 \, dx = \int_{\mathbb{R}^d} v \, (-\Delta)^{-1} v \, dx$$

and, if $v = u^q$ with $q = \frac{d+2}{d-2}$,

$$\int_{\mathbb{R}^d} \nabla u \cdot \nabla(-\Delta)^{-1} v \, dx = \int_{\mathbb{R}^d} u \, v \, dx = \int_{\mathbb{R}^d} u^{2^*} \, dx$$

Hence the expansion of the square

$$0 \le \int_{\mathbb{R}^d} \left| \mathsf{S}_d \, \|u\|_{\mathrm{L}^{2^*}(\mathbb{R}^d)}^{\frac{4}{d-2}} \nabla u - \nabla(-\Delta)^{-1} v \right|^2 dx$$

shows that

$$0 \le \mathsf{S}_d \, \|u\|_{\mathrm{L}^{2^*}(\mathbb{R}^d)}^{\frac{8}{d-2}} \left[\mathsf{S}_d \, \|\nabla u\|_{\mathrm{L}^2(\mathbb{R}^d)}^2 - \|u\|_{\mathrm{L}^{2^*}(\mathbb{R}^d)}^2 \right] - \left[\mathsf{S}_d \, \|u^q\|_{\mathrm{L}^{\frac{2d}{d+2}}(\mathbb{R}^d)}^2 - \int_{\mathbb{R}^d} u^q \, (-\Delta)^{-1} u^q \, dx \right]$$

The equality case

Equality is achieved if and only if

$$\mathsf{S}_d \, \|u\|_{\mathrm{L}^{2^*}(\mathbb{R}^d)}^{\frac{4}{d-2}} \, u = (-\Delta)^{-1} \, v = (-\Delta)^{-1} \, u^q$$

that is, if and only if u solves

$$-\Delta u = \frac{1}{\mathsf{S}_d} \, \|u\|_{\mathrm{L}^{2^*}(\mathbb{R}^d)}^{-\frac{4}{d-2}} \, u^q$$

which means that u is an Aubin-Talenti extremal function

$$u_\star(x) := (1 + |x|^2)^{-\frac{d-2}{2}} \quad \forall \, x \in \mathbb{R}^d$$

An identity

$$0 = \mathsf{S}_d \, \|u\|_{\mathrm{L}^{2^*}(\mathbb{R}^d)}^{\frac{8}{d-2}} \left[\mathsf{S}_d \, \|\nabla u\|_{\mathrm{L}^2(\mathbb{R}^d)}^2 - \|u\|_{\mathrm{L}^{2^*}(\mathbb{R}^d)}^2 \right] - \left[\mathsf{S}_d \, \|u^q\|_{\mathrm{L}^{\frac{2d}{d+2}}(\mathbb{R}^d)}^2 - \int_{\mathbb{R}^d} u^q \, (-\Delta)^{-1} \, u^q \; dx \right] - \int_{\mathbb{R}^d} \left| \mathsf{S}_d \, \|u\|_{\mathrm{L}^{2^*}(\mathbb{R}^d)}^{\frac{4}{d-2}} \nabla u - \nabla (-\Delta)^{-1} \, u^q \right|^2 dx$$

Another improvement

$$\mathsf{J}_d[v] := \int_{\mathbb{R}^d} v^{\frac{2d}{d+2}}\,dx \quad\text{and}\quad \mathsf{H}_d[v] := \int_{\mathbb{R}^d} v\,(-\Delta)^{-1} v\,dx - \mathsf{S}_d\,\|v\|^2_{\mathrm{L}^{\frac{2d}{d+2}}(\mathbb{R}^d)}$$

Theorem (J.D., G. Jankowiak)

Assume that $d \ge 3$. Then we have

$$0 \le \mathsf{H}_d[v] + \mathsf{S}_d\,\mathsf{J}_d[v]^{1+\frac{2}{d}}\,\varphi\left(\mathsf{J}_d[v]^{\frac{2}{d}-1}\left[\mathsf{S}_d\,\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{R}^d)} - \|u\|^2_{\mathrm{L}^{2^*}(\mathbb{R}^d)}\right]\right)$$
$$\forall\, u \in \mathcal{D}^{1,2}(\mathbb{R}^d)\,,\quad v = u^{\frac{d+2}{d-2}}$$

where $\varphi(x) := \sqrt{\mathcal{C}^2 + 2\,\mathcal{C}\,x} - \mathcal{C}$ for any $x \ge 0$

Proof: $\mathsf{H}(t) = -\,\mathsf{Y}(\mathsf{J}(t))$ $\forall\, t \in [0,T)$, $\kappa_0 := \frac{\mathsf{H}_0'}{\mathsf{J}_0}$ and consider the differential inequality

$$\mathsf{Y}'\left(\mathcal{C}\,\mathsf{S}_d\,s^{1+\frac{2}{d}} + \mathsf{Y}\right) \le \frac{d+2}{2\,d}\,\mathcal{C}\,\kappa_0\,\mathsf{S}_d^2\,s^{1+\frac{4}{d}}\,,\quad \mathsf{Y}(0) = 0\,,\quad \mathsf{Y}(\mathsf{J}_0) = -\,\mathsf{H}_0$$

... but $\mathcal{C} = 1$ is not optimal

Theorem (J.D., G. Jankowiak)

[JD, G. Jankowiak] *In the inequality*

$$\mathsf{S}_d \, \|w^q\|^2_{\mathrm{L}^{\frac{2d}{d+2}}(\mathbb{R}^d)} - \int_{\mathbb{R}^d} w^q \, (-\Delta)^{-1} w^q \, dx \le \mathcal{C} \, \mathsf{S}_d \, \|w\|^{\frac{8}{d-2}}_{\mathrm{L}^{2^*}(\mathbb{R}^d)} \left[\|\nabla w\|^2_{\mathrm{L}^2(\mathbb{R}^d)} - \mathsf{S}_d \, \|w\|^2_{\mathrm{L}^{2^*}(\mathbb{R}^d)} \right]$$

we have

$$\frac{d}{d+4} \le \mathsf{C}_d < 1$$

based on a (painful) linearization like the one used by Bianchi and Egnell

- Extensions: magnetic Laplacian [JD, Esteban, Laptev] or fractional Laplacian operator [Jankowiak, Nguyen]

Improved Onofri inequality

Theorem

Assume that $d = 2$. The inequality

$$\int_{\mathbb{R}^2} g \, \log\left(\frac{g}{M}\right) dx - \frac{4\,\pi}{M} \int_{\mathbb{R}^2} g\,(-\Delta)^{-1} g \, dx + M\,(1 + \log \pi) \\ \le M \left[\frac{1}{16\,\pi} \, \|\nabla f\|^2_{\mathrm{L}^2(\mathbb{R}^d)} + \int_{\mathbb{R}^2} f \, d\mu - \log M\right]$$

holds for any function $f \in \mathcal{D}(\mathbb{R}^2)$ such that $M = \int_{\mathbb{R}^2} e^{\,f} \, d\mu$ and $g = \pi \, e^f \, \mu$

Recall that

$$\mu(x) := \frac{1}{\pi\,(1 + |x|^2)^2} \quad \forall\, x \in \mathbb{R}^2$$

An improvement of the fractional Sobolev inequality

$\triangleright$ Some results of Gaspard Jankowiak and Van Hoang Nguyen

[Dolbeault, 2011], [Dolbeault, Jankowiak, 2014]
[Jin, Xiong, 2011], [Jankowiak, Nguyen, 2014]

Theorem (Jankowiak, Nguyen)

Let $d \geq 2$, $0 < s < \frac{d}{2}$, *and* $r = \frac{d+2s}{d-2s}$

(i) *There exists a positive constant* $\mathsf{C}_{d,s}$ *such that*

$$\mathsf{S}_{d,2s} \|u^r\|^2_{L^{\frac{2d}{d+2s}}(\mathbb{R}^d)} - \int_{\mathbb{R}^d} u^r (-\Delta)^{-s} u^r \, dx \\ \leq \mathsf{C}_{d,s} \|u\|^{\frac{8s}{d-2s}}_{L^{\frac{2d}{d-2s}}(\mathbb{R}^d)} \left(\mathsf{S}_{d,2s} \|u\|_s^2 - \|u\|^2_{L^{\frac{2d}{d-2s}}(\mathbb{R}^d)} \right)$$

holds for any positive $u \in \mathring{\mathrm{H}}^s(\mathbb{R}^d)$

(ii) *The best constant is such that*

$$\frac{d-2s+2}{d+2s+2} \mathsf{S}_{d,2s} \leq \mathsf{C}_{d,s} \leq \mathsf{S}_{d,2s}$$

If $0 < s < 1$, *then* $\mathsf{C}_{d,s} < \mathsf{S}_{d,2s}$

$$d\mu(x) = \mu(x)\,dx\,,\quad \mu(x) = \frac{1}{\pi\,(1+|x|^2)^2}\,,\quad x\in\mathbb{R}^2$$

Corollary

There exists a positive (optimal) constant C_2 such that

$$C_2\left(\int_{\mathbb{R}^2} e^f\,d\mu\right)^2\left[\frac{1}{16\pi}\,\|\nabla f\|^2_{L^2(\mathbb{R}^2} + \int_{\mathbb{R}^2} f\,d\mu - \log\left(\int_{\mathbb{R}^2} e^f\,d\mu\right)\right]$$
$$\geq \left(\int_{\mathbb{R}^2} e^f\,d\mu\right)^2\left(1+\log\pi + \int_{\mathbb{R}^2}\frac{e^f\,\mu}{\int_{\mathbb{R}^2} e^f\,d\mu}\log\left(\frac{e^f\,\mu}{\int_{\mathbb{R}^2} e^f\,d\mu}\right)dx\right.$$
$$\left. - 4\pi\int_{\mathbb{R}^2} e^f\,\mu\,(-\Delta)^{-1}(e^f\,\mu)\,dx\right)$$

and

$$\frac{1}{3}\leq C_2\leq 1$$

Proof and some open questions

- The proof based on the Yamabe flow does not require integration by parts: positivity arises from a simple Cauchy-Schwarz inequality and the result follows from the analysis of $\mathsf{J}=\int_{\mathbb{R}^d} v^{\frac{2d}{d+2s}}\,dx$ and

$$\mathcal{C}\left(-\frac{\kappa_0}{p}\,\mathsf{S}_{d,2s}^2\,\frac{\mathsf{J}^{1+\frac{4s}{d}}}{\mathsf{Y}'}+\mathsf{S}_{d,2s}\,\mathsf{J}^{1+\frac{2s}{n}}\right)+\mathsf{Y}\le 0$$

To justify the computations, it is simpler to analyze the extinction profile on the sphere (inverse stereographic projection) and analyze the spectrum of the linearized problem in this setting
- In the four other flows, monotonicity along the flow is based on a property of positivity obtained by integration by parts: can one give an other proof ?
- Because of the improvements in the inequalities, best constants are obtained by inearization. Is it the same for fractional operators ?

These slides can be found at

http://www.ceremade.dauphine.fr/~dolbeaul/Conferences/
▷ Lectures

Thank you for your attention !