Hypocoercivity in kinetic equations

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Outline

- **Introduction to hypocoercivity**
  - ▷ Decay and convergence rates based on φ-entropies
  - ▷ H^1 Hypocoercivity, entropy methods and *carré du champ*
- L^2 **Hypocoercivity**
  - ▷ The diffusion limit
  - ▷ Mode-by-mode analysis in Fourier variables
- **Functional inequalities and applications**
  - ▷ Towards a systematic classification
  - ▷ Some examples and extensions

Introduction to hypocoercivity

We are interested in ***kinetic equations*** acting on a distribution function $f(t,x,v) \geq 0$ with time t, position x and velocity v

A typical example: *Vlasov-Fokker-Planck equation* with external potential ψ

$$\partial_t f + \underbrace{v \cdot \nabla_x f - \nabla_x \psi \cdot \nabla_v f}_{\mathsf{T} f} = \underbrace{\Delta_v f + \nabla_v (v\, f)}_{\mathsf{L} f}$$

▷ ***Homogeneous case:*** no dependence in x: Fokker-Planck equation, exponential rate of convergence to a stationary solution

$$\mathscr{M}(v) = \frac{e^{-\frac{1}{2}|v|^2}}{(2\pi)^{d/2}}$$

▷ ***Inhomogeneous case:*** rate of convergence to a stationary solution ? rate of decay if there is no stationary solution ?

Fokker-Planck equation and φ-entropies

The Fokker-Planck equation

$$\partial_t f = \Delta_v f + \nabla_v (v\, f)$$

acts on a probability distribution function $f(t,v) \geq 0$
which depends here on time t and velocity v (but not on x)

$$\mathscr{M}(v) = \frac{e^{-\frac{1}{2}|v|^2}}{(2\pi)^{d/2}}$$

Use φ-***entropies***... cf. Anton's lecture. With $p \in [1,2]$, for any $t \geq 0$,

$$\|f(t,v) - \mathscr{M}(v)\|^2_{\mathrm{L}^p(\mathbb{R}^d,\mathscr{M}^{-1}dv)} \leq \mathscr{C}_0\, e^{-2t}$$

- *Beckner's inequalities* with Gaussian measure $d\mu = \mathscr{M}(v)\, dv$

$$\|h\|^2_{\mathrm{L}^2(\mathbb{R}^d,d\mu)} - \|h\|^2_{\mathrm{L}^q(\mathbb{R}^d,d\mu)} \leq (2-q)\ \|\nabla h\|^2_{\mathrm{L}^2(\mathbb{R}^d,d\mu)}$$

applied to $h = (f/\mathscr{M})^p$, $q = 2/p \in (1,2]$
- $q = 2$: *Gaussian logarithmic Sobolev inequality* (Gross, 1975)

$$\int_{\mathbb{R}^d} h^2\, \log\left(h^2/\|h\|^2_{\mathrm{L}^q(\mathbb{R}^d,d\mu)}\right) d\mu \leq \|\nabla h\|^2_{\mathrm{L}^2(\mathbb{R}^d,d\mu)}$$

Vlasov-Fokker-Planck equation: methods

Vlasov-Fokker-Planck equation with external potential ψ

$$\partial_t f + v \cdot \nabla_x f - \nabla_x \psi \cdot \nabla_v f = \Delta_v f + \nabla_v (v\, f)$$

▷ Harmonic potential case: $\psi(x) = \frac{\kappa}{2}\,|x|^2$

- Decomposition on Hermite functions and spectral results
- Green's function as in Kolmogorov's computation (Kolmogorov, 1934)

$$G(t,x,v) = \frac{\exp\left(-\frac{\gamma(t)\,|x|^2 + \alpha(t)\,|v|^2 + \beta(t)\,x\cdot v}{4\,\alpha(t)\,\gamma(t) - \beta^2(t)}\right)}{(2\,\pi)^d\,\big(4\,\alpha(t)\,\gamma(t) - \beta^2(t)\big)^{d/2}}$$

- Hypoelliptic methods (Hörmander, 1965)
- H^1 ***hypocoercivity*** (Villani, 2001 & 2005)
- L^2 ***hypocoercivity*** (Mouhot, Neumann, 2006), (Hérau, 2006), (JD, Mouhot, Schmeiser 2009 & 2015)
- H^{-1} hypocoercivity (Armstrong, Mourrat, 2019), (Brigati, 2021), (Cao, Lu, Wang, 2020), (Albritton-Armstrong-Mourrat-Novack, 2021)

A toy model

$$\frac{du}{dt} = (\mathsf{L}-\mathsf{T})\,u\,,\quad \mathsf{L} = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix},\quad \mathsf{T} = \begin{pmatrix} 0 & -k \\ k & 0 \end{pmatrix},\quad k \neq 0$$

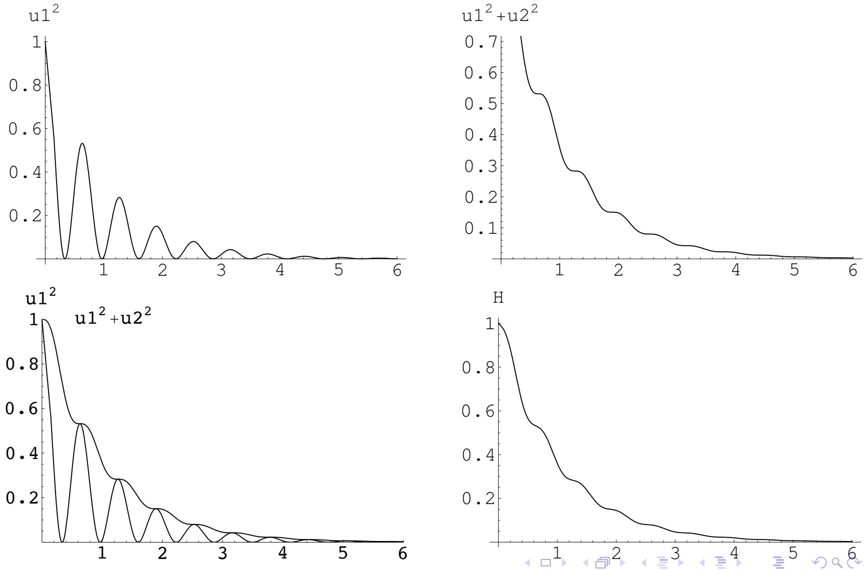
$$u = (u_1, u_2) \text{ and } |u|^2 = u_1^2 + u_2^2$$

Non-monotone decay, a well known picture:
see for instance (Filbet, Mouhot, Pareschi, 2006)

- H-theorem: $\frac{d}{dt}|u|^2 = -\,2\,u_2^2$
- macroscopic limit: $\frac{du_1}{dt} = -\,k^2\,u_1$
- generalized entropy: $\mathsf{H}(u) = |u|^2 - \frac{\delta\,k}{1+k^2}\,u_1\,u_2$

$$\begin{aligned}\frac{d\mathsf{H}}{dt} &= -\left(2 - \frac{\delta\,k^2}{1+k^2}\right)u_2^2 - \frac{\delta\,k^2}{1+k^2}\,u_1^2 + \frac{\delta\,k}{1+k^2}\,u_1\,u_2 \\ &\le -(2-\delta)\,u_2^2 - \frac{\delta\Lambda}{1+\Lambda}\,u_1^2 + \frac{\delta}{2}\,u_1 u_2\end{aligned}$$

Plots for the toy problem



φ-entropies and hypocoercivity (H^1 framework)

▷ Adapt φ-entropies to kinetic equations

▷ Villani's strategy: derive H^1 estimates (using a twisted Fisher information) and then use standard interpolation inequalities to establish decay rates for the entropy

The *kinetic Fokker-Planck equation*, or *Vlasov-Fokker-Planck equation*

$$\frac{\partial f}{\partial t} + v \cdot \nabla_x f - \nabla_x \psi \cdot \nabla_v f = \Delta_v f + \nabla_v \cdot (v\, f)$$

with $\psi(x) = |x|^2/2$ and $\|f\|_{L^1(\mathbb{R}^d \times \mathbb{R}^d)} = 1$ has a unique nonnegative stationary solution

$$\mathfrak{M}(x,v) = (2\,\pi)^{-d}\, e^{-\frac{1}{2}(|x|^2+|v|^2)}$$

and $g = f/\mathfrak{M}$ solves the *kinetic Ornstein-Uhlenbeck equation*

$$\frac{\partial g}{\partial t} + \mathsf{T} g = \mathsf{L}\, g$$

with transport operator T and Ornstein-Uhlenbeck operator L

$$\mathsf{T} g := v \cdot \nabla_x g - x \cdot \nabla_v g \quad \text{and} \quad \mathsf{L}\, g := \Delta_v g - v \cdot \nabla_v g$$

The function $h = g^{p/2}$ solves $\frac{\partial h}{\partial t} + \mathsf{T} h = \mathsf{L}\, h + \frac{2-p}{p}\,\frac{|\nabla_v h|^2}{h}$

Sharp rates for the kinetic Fokker-Planck equation

Let $\psi(x) = |x|^2/2$, $d\mu := \mathfrak{M}\,dx\,dv$, $\mathcal{E}[g] := \iint_{\mathbb{R}^d\times\mathbb{R}^d} \varphi_p(g)\,d\mu$

Proposition

Let $p \in [1,2]$ and consider a nonnegative solution of the kinetic Fokker-Planck equation. There is a constant $\mathcal{C} > 0$ such that

$$\mathcal{E}[g(t,\cdot,\cdot)] \le \mathcal{C}\,e^{-t} \quad \forall\,t \ge 0$$

and the rate e^{-t} is sharp as $t \to +\infty$

(Villani), (Arnold, Erb): a twisted *Fisher information* functional

$$\mathcal{I}_\lambda[h] = (1-\lambda)\int_{\mathbb{R}^d} |\nabla_v h|^2\,d\mu + (1-\lambda)\int_{\mathbb{R}^d} |\nabla_x h|^2\,d\mu + \lambda\int_{\mathbb{R}^d} |\nabla_x h + \nabla_v h|^2\,d\mu$$

With $\lambda = 1/2$, we find $\frac{d}{dt}\mathcal{I}_{1/2}[h(t,\cdot)] \le -\,\mathcal{I}_{1/2}[h(t,\cdot)]$

Improved rates (in the large entropy regime)

Rewrite the decay of the *Fisher information* functional as

$$-\frac{1}{2}\,\frac{d}{dt}\int_{\mathbb{R}^d} X^\perp\cdot\mathfrak{M}_0\,X\,d\mu=\int_{\mathbb{R}^d} X^\perp\cdot\mathfrak{M}_1\,X\,d\mu+\int_{\mathbb{R}^d} Y^\perp\cdot\mathfrak{M}_2\,Y\,d\mu$$

where $X=(\nabla_v h,\nabla_x h)$, $\quad Y=(\mathsf{H}_{vv},\mathsf{H}_{xv},Fvv,Fxv)$

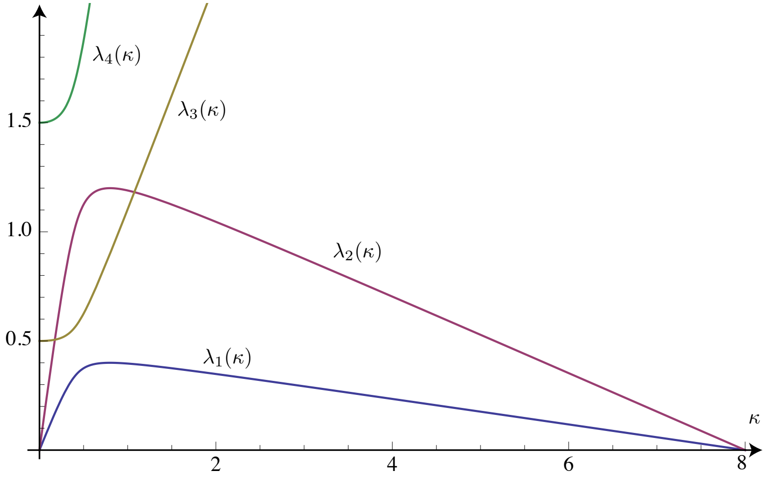
$$\mathfrak{M}_0=\begin{pmatrix}1&\lambda\\ \lambda&\nu\end{pmatrix}\otimes\mathrm{Id}_{\mathbb{R}^d}\,,\quad \mathfrak{M}_1=\begin{pmatrix}1-\lambda&\frac{1+\lambda-\nu}{2}\\ \frac{1+\lambda-\nu}{2}&\lambda\end{pmatrix}\otimes\mathrm{Id}_{\mathbb{R}^d}$$

$$\mathfrak{M}_2=\begin{pmatrix}1&\lambda&-\frac{\kappa}{2}&-\frac{\kappa\,\lambda}{2}\\ \lambda&\nu&-\frac{\kappa\,\lambda}{2}&-\frac{\kappa\,\nu}{2}\\ -\frac{\kappa}{2}&-\frac{\kappa\,\lambda}{2}&2\,\kappa&2\,\kappa\,\lambda\\ -\frac{\kappa\,\lambda}{2}&-\frac{\kappa\,\nu}{2}&2\,\kappa\,\lambda&2\,\kappa\nu\end{pmatrix}\otimes\mathrm{Id}_{\mathbb{R}^d\times\mathbb{R}^d}$$

With constant coefficients

$$\lambda_\star(\lambda,\nu)=\max\left\{\min_X\frac{X^\perp\cdot\mathfrak{M}_1\,X}{X^\perp\cdot\mathfrak{M}_0\,X}\;:\;(\lambda,\nu)\in\mathbb{R}^2\text{ s.t. }\mathfrak{M}_2\ge0\right\}$$

For $(\lambda, \nu) = (1/2)$, $\lambda_\star = 1/2$ and the eigenvalues of $\mathfrak{M}_2(\frac{1}{2}, 1)$ are given as a function of $\kappa = 8\,(2-p)/p \in [0,8]$ are all nonnegative



An improvement based on an adapted

(Arnold, JD)... Improvement for φ-entropies:

$$\mathcal{I} \geq \Phi(\mathcal{E}) \sim \mathcal{E} + \kappa\,\mathcal{E}^2 \quad \text{as} \quad \mathcal{E} \to 0_+$$

with $\Phi(0) = 0$, $\Phi'(0) = 1$, $\Phi'' > 0$. Kinetic analog: (JD, Li, 2018)

Theorem

Let $p \in (1,2)$ and h be a solution of the kinetic Ornstein-Uhlenbeck equation. Then there exists a function $\lambda : \mathbb{R}^+ \to [1/2, 1)$ such that $\lambda(0) = \lim_{t\to+\infty} \lambda(t) = 1/2$ and a function $\rho > 1$ s.t.

$$\frac{d}{dt}\mathcal{J}_{\lambda(t)}[h(t,\cdot)] \leq -\,\rho(t)\,\mathcal{J}_{\lambda(t)}[h(t,\cdot)]$$

As a consequence, for any $t \geq 0$ we have the global estimate

$$\mathcal{J}_{\lambda(t)}[h(t,\cdot)] \leq \mathcal{J}_{1/2}[h_0]\,\exp\left(-\int_0^t \rho(s)\,ds\right)$$

L^2 Hypocoercivity

- ▷ Abstract statement, diffusion limit
- ▷ Mode-by-mode analysis in Fourier variables
- ▷ Refined decay rates in the whole space

Collaboration with...
C. Mouhot and C. Schmeiser
E. Bouin, S. Mischler, C. Mouhot and C. Schmeiser
A. Arnold, C. Schmeiser, and T. Wöhrer

An abstract evolution equation

Let us consider the equation

$$\frac{dF}{dt} + \mathsf{T}F = \mathsf{L}F$$

In the framework of kinetic equations, T and L are respectively the transport and the collision operators

We assume that T and L are respectively anti-Hermitian and Hermitian operators defined on the complex Hilbert space $(\mathcal{H}, \langle\cdot,\cdot\rangle)$
* denotes the adjoint with respect to $\langle\cdot,\cdot\rangle$

Π is the orthogonal projection onto the null space of L

The estimate

$$\frac{1}{2}\,\frac{d}{dt}\|F\|^2 = \langle \mathsf{L}F, F\rangle \le -\,\lambda_m\,\|(1-\Pi)F\|^2$$

is not enough to conclude that $\|F(t,\cdot)\|^2$ decays exponentially
$\Leftarrow$ *microscopic coercivity*

Formal macroscopic / diffusion limit

$F = F(t,x,v)$, $\mathsf{T} = v\cdot\nabla_x$, L good collision operator. Scaled evolution equation

$$\varepsilon\,\frac{dF}{dt} + \mathsf{T}F = \frac{1}{\varepsilon}\,\mathsf{L}F$$

on the Hilbert space $\mathcal{H}$. $F_\varepsilon = F_0 + \varepsilon\,F_1 + \varepsilon^2\,F_2 + \mathcal{O}(\varepsilon^3)$ as $\varepsilon \to 0_+$

$$\begin{aligned}
\varepsilon^{-1} : &\qquad \mathsf{L}F_0 = 0\,,\\
\varepsilon^{0} : &\qquad \mathsf{T}F_0 = \mathsf{L}F_1\,,\\
\varepsilon^{1} : &\qquad \tfrac{dF_0}{dt} + \mathsf{T}F_1 = \mathsf{L}F_2
\end{aligned}$$

The first equation reads as $u = F_0 = \Pi F_0$
The second equation is simply solved by $F_1 = -\,(\mathsf{T}\Pi)\,F_0$
After projection, the $O(\varepsilon)$ equation is

$$\tfrac{d}{dt}\,(\Pi F_0) - \,\Pi\mathsf{T}\,(\mathsf{T}\Pi)\,F_0 = \Pi\mathsf{L}F_2 = 0$$

$$\partial_t u + (\mathsf{T}\Pi)^*\,(\mathsf{T}\Pi)\,u = 0$$

is such that $\frac{d}{dt}\|u\|^2 = -\,2\,\|(\mathsf{T}\Pi)\,u\|^2 \le -\,2\,\lambda_M\,\|u\|^2$
$\Leftarrow$ *Macroscopic coercivity*

The macro part and the Poincaré inequality

▷ Free transport operator: $\mathsf{T}F = v\cdot\nabla_x F$
If $F_0(x,v) = u(x)\,\mathfrak{M}(v)$ with $\mathfrak{M}(v) = (2\pi)^{-d/2}\,e^{-|v|^2/2}$ then

$$(\mathsf{T}\Pi)^*\,(\mathsf{T}\Pi)F_0 = (-\Delta_x u)\,\mathfrak{M}$$

and we obtain the heat equation (*e.g.* on $\mathbb{T}^d$)

$$\partial_t u = \Delta\, u$$

▷ With an external potential ψ so that $\mathsf{T}F = v\cdot\nabla_x F - \nabla_x\psi\cdot\nabla_v F$
we obtain the Fokker-Planck equation

$$\partial_t u = \Delta\, u + \nabla\cdot(u\,\nabla\psi)$$

The operator $\mathsf{A} := \big(1 + (\mathsf{T}\Pi)^*\mathsf{T}\Pi\big)^{-1}(\mathsf{T}\Pi)^*$ is such that

$$\langle \mathsf{A}\mathsf{T}\Pi F, F\rangle \ge \frac{\lambda_M}{1+\lambda_M}\,\|\Pi F\|^2$$

if the Poincaré inequality $\int|\nabla u|^2\,e^{-\psi}\,dx \ge \lambda_M\int|u-\bar u|^2\,e^{-\psi}\,dx$ holds

The assumptions in the compact case

λ_m, λ_M, and C_M are positive constants such that, for any $F \in \mathcal{H}$

▷ *microscopic coercivity:*

$$-\langle \mathsf{L}F, F\rangle \geq \lambda_m \, \|(1-\Pi)F\|^2 \tag{H1}$$

▷ *macroscopic coercivity:*

$$\|\mathsf{T}\Pi F\|^2 \geq \lambda_M \, \|\Pi F\|^2 \tag{H2}$$

▷ *parabolic macroscopic dynamics:*

$$\Pi \mathsf{T} \Pi \, F = 0 \tag{H3}$$

▷ *bounded auxiliary operators:*

$$\|\mathsf{AT}(1-\Pi)F\| + \|\mathsf{AL}F\| \leq C_M \, \|(1-\Pi)F\| \tag{H4}$$

Equivalence and entropy decay

Choice: the limit as $t \to +\infty$ of $F(t,\cdot)$ is zero

The L^2 entropy / Lyapunov functional is defined by

$$\mathsf{H}[F] := \tfrac{1}{2}\,\|F\|^2 + \delta\,\mathrm{Re}\langle \mathsf{A}F, F\rangle$$

▷ *norm equivalence* of $\mathsf{H}[F]$ and $\|F\|^2$

$$\tfrac{2-\delta}{4}\,\|F\|^2 \le \mathsf{H}[F] \le \tfrac{2+\delta}{4}\,\|F\|^2$$

Entropy decay: $\frac{d}{dt}\mathsf{H}[F] = -\,\mathsf{D}[F]$

▷ *entropy decay rate:* for any $\delta > 0$ small enough and $\lambda = \lambda(\delta)$

$$\mathsf{D}[F] \ge \lambda\,\mathsf{H}[F]$$

Theorem

Under (H1)–(H4)*, for any* $t \ge 0$*, there is some some* $\delta > 0$ *such that*

$$\mathsf{H}[F(t,\cdot)] \le \mathsf{H}[F_0]\,e^{-\lambda\,t}$$

$$\|F(t,\cdot)\|^2 \le \mathcal{C}\,\|F_0\|^2\,e^{-\lambda\,t} \quad \textit{with} \quad \mathcal{C} = \tfrac{2+\delta}{2-\delta}$$

Basic example 1: with confinement

Vlasov-Fokker-Planck equation with harmonic potential

$$\frac{\partial f}{\partial t} + v\cdot\nabla_x f - \nabla_x\psi\cdot\nabla_v f = \Delta_v f + \nabla_v\cdot(v\,f)$$

▷ *microscopic coercivity:* Gaussian Poincaré inequality in v

$$\int_{\mathbb{R}^d} |F(v) - \rho\,\mathscr{M}(v)|^2\,\frac{dv}{\mathscr{M}(v)} \le \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} |\nabla_v F(v)|^2\,\frac{dv}{\mathscr{M}(v)} \tag{H1}$$

with $\rho = \int_{\mathbb{R}^d} F(v)\,dv$

▷ *macroscopic coercivity:* Gaussian Poincaré inequality in x

$$\int_{\mathbb{R}^d} \left|\rho(x) - \tfrac{M\,e^{-|x|^2/2}}{(2\pi)^{d/2}}\right|^2 e^{\frac{|x|^2}{2}}\,dx \le \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} |\nabla_x\rho(x)|^2\,e^{\frac{|x|^2}{2}}\,dx \tag{H2}$$

▷ *parabolic macroscopic dynamics* (H3) and *bounded auxiliary operators* (H4) are consequences of elliptic estimates

Basic example 2: without confinement

We consider the Cauchy problem

$$\partial_t f + v \cdot \nabla_x f = \mathsf{L} f \,, \quad f(0,x,v) = f_0(x,v)$$

L is the *Fokker-Planck operator* L_1 or the *linear BGK operator* L_2

$$\mathsf{L}_1 f := \Delta_v f + \nabla_v \cdot (v\,f) \quad \text{and} \quad \mathsf{L}_2 f := \rho_f\,\mathscr{M} - f$$

$\mathscr{M}(v) = \frac{e^{-\frac{1}{2}\,|v|^2}}{(2\,\pi)^{d/2}}$ is the normalized Gaussian function

$\rho_f := \int_{\mathbb{R}^d} f\,dv$ is the spatial density

$$d\gamma := \gamma(v)\,dv \quad \text{where} \quad \gamma := \frac{1}{\mathscr{M}}$$

$$\|f\|^2_{\mathrm{L}^2(dx\,d\gamma)} := \iint_{\mathbb{R}^d\times\mathbb{R}^d} |f(x,v)|^2\,dx\,d\gamma$$

Fourier variables: mode-by-mode hypocoercivity

Let us consider the Fourier transform in x, denote by $\xi \in \mathbb{R}^d$ the Fourier variable, so that $F = \hat{f}$ solves

$$\partial_t F + \mathsf{T} F = \mathsf{L} F\,, \quad F(0,\xi,v) = \hat{f}_0(\xi,v)\,, \quad \mathsf{T} F = i\,(v\cdot\xi) F$$

Goal: apply the abstract method with ξ considered as a parameter

$$\mathcal{H} = \mathrm{L}^2\,(d\gamma)\;, \quad \|F\|^2 = \int_{\mathbb{R}^d} |F|^2\,d\gamma\,, \quad \Pi F = \mathscr{M} \int_{\mathbb{R}^d} F\,dv = \mathscr{M}\,\rho_F$$

The operator A is now defined as

$$(\mathsf{A}F)(v) = \frac{-\,i\,\xi}{1+|\xi|^2} \cdot \int_{\mathbb{R}^d} w\,F(w)\,dw\,\mathscr{M}(v)$$

and, with $X := \|(1-\Pi)F\|$ and $Y := \|\Pi F\|$, we have that

$$|\mathrm{Re}\langle \mathsf{A}F, F\rangle| \le \frac{|\xi|}{1+|\xi|^2}\,X\,Y\,, \quad \|F\|^2 = X^2 + Y^2$$

$$\frac{1}{2}\left(1 - \frac{\delta\,|\xi|}{1+|\xi|^2}\right)(X^2+Y^2) \le \mathsf{H}[F] \le \frac{1}{2}\left(1 + \frac{\delta\,|\xi|}{1+|\xi|^2}\right)(X^2+Y^2)$$

Entropy – entropy production inequality

$$-\langle \mathsf{L}F,F\rangle+\delta\,\langle \mathsf{AT\Pi}F,F\rangle\ge X^2+\frac{\delta\,|\xi|^2}{1+|\xi|^2}\,Y^2$$

$$\mathsf{D}[F]=-\langle \mathsf{L}F,F\rangle+\delta\,\langle \mathsf{AT\Pi}F,F\rangle+\delta\,(\ldots)$$
$$\ge(\lambda_m-\delta)\,X^2+\frac{\delta\,\lambda_M}{1+\lambda_M}\,Y^2-\delta\,C_M\,X\,Y$$

with $\lambda_m=1\,,\quad \Lambda_M=|\xi|^2=:s^2\,,\quad C_M=\dfrac{s\,(1+\sqrt{3}\,s)}{1+s^2}$

$$\mathsf{D}[F]-\lambda\,\mathsf{H}[F]$$
$$\ge\left(1-\tfrac{\delta\,s^2}{1+s^2}-\tfrac{\lambda}{2}\right)X^2-\tfrac{\delta\,s}{1+s^2}\left(1+\sqrt{3}\,s+\lambda\right)X\,Y+\left(\tfrac{\delta\,s^2}{1+s^2}-\tfrac{\lambda}{2}\right)Y^2$$

is (for any $s=|\xi|>0$) a nonnegative quadratic form of X and Y iff...

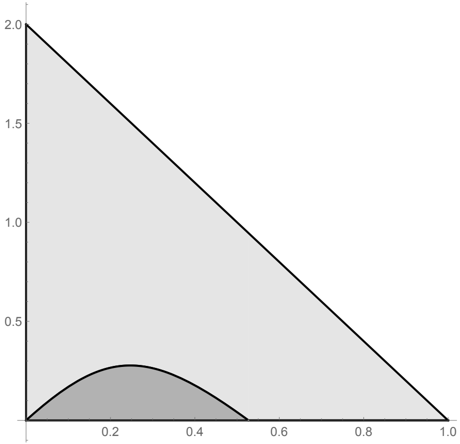


Figure: Horizontal axis: $\delta/2$, vertical axis: λ. Admissible region: grey triangle. Negative discriminant: dark grey area, shown here for $s = 5$

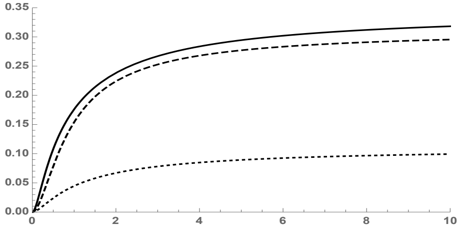


Figure: $s \mapsto \lambda(s)$

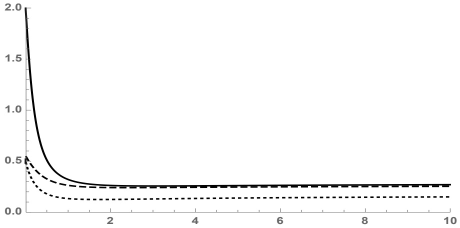


Figure: $s \mapsto \delta(s)$

Results (whole space, no external potential)

On the whole Euclidean space, we can define the entropy

$$\mathsf{H}[f] := \tfrac{1}{2}\,\|f\|^2_{\mathrm{L}^2(dx\,d\gamma)} + \delta\,\langle \mathsf{A}f, f\rangle_{dx\,d\gamma}$$

Replacing the *macroscopic coercivity* condition by *Nash's inequality*

$$\|u\|^2_{\mathrm{L}^2(dx)} \le \mathcal{C}_{\mathrm{Nash}}\,\|u\|^{\frac{4}{d+2}}_{\mathrm{L}^1(dx)}\,\|\nabla u\|^{\frac{2\,d}{d+2}}_{\mathrm{L}^2(dx)}$$

proves that

$$\mathsf{H}[f] \le C\left(\mathsf{H}[f_0] + \|f_0\|^2_{\mathrm{L}^1(dx\,dv)}\right)(1+t)^{-\frac{d}{2}}$$

(Bouin, JD, Mischler, Mouhot, Schmeiser)

Theorem

There exists a constant $C > 0$ such that, for any $t \ge 0$

$$\|f(t,\cdot,\cdot)\|^2_{\mathrm{L}^2(dx\,d\gamma)} \le C\left(\|f_0\|^2_{\mathrm{L}^2(dx\,d\gamma)} + \|f_0\|^2_{\mathrm{L}^2(d\gamma;\,\mathrm{L}^1(dx))}\right)(1+t)^{-\frac{d}{2}}$$

Comments

▷ Use of the *enlargement of the space* method or *factorization method* of (Gualdani, Mischler, Mouhot)

▷ Not limited to Maxwellian local equilibria

▷ Can be compared with spectral methods based on *Lyapunov matrix inequalities* and *twisted Euclidean norms* (Arnold, JD, Schmeiser, Wöhrer, 2021)

▷ Sharper but in most cases still suboptimal estimates can be given with A defined as a pseudo-differential operator (Arnold, JD, Schmeiser, Wöhrer, 2021)

Functional inequalities and hypocoercivity

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In collaboration with Lanoir Addala, Emeric Bouin, Kleber Carrapatoso, Frédéric Hérau, Laurent Lafleche, Xingyu Li, Stéphane Mischler, Clément Mouhot, Christian Schmeiser, Lazhar Tayeb

The global picture: from diffusive to kinetic

- Depending on the local equilibria and on the external potential (H1) and (H2) (which are Poincaré type inequalities) can be replaced by other functional inequalities:

▷ *microscopic coercivity* (H1)

$$-\langle \mathsf{L} F, F\rangle \geq \lambda_m \, \|(1-\Pi)F\|^2$$

$\Longrightarrow$ ***weak Poincaré inequalities*** or ***Hardy-Poincaré inequalities***

▷ *macroscopic coercivity* (H2)

$$\|\mathsf{T}\Pi F\|^2 \geq \lambda_M \, \|\Pi F\|^2$$

$\Longrightarrow$ ***Nash inequality***, ***weighted Nash*** or ***Caffarelli-Kohn-Nirenberg inequalities***

- This can be done at the level of the *diffusion equation* (homogeneous case) or at the level of the *kinetic equation* (non-homogeneous case)

Diffusion (Fokker-Planck) equations

Potential	$V = 0$	$V(x) = \gamma \log \lvert x \rvert$ $\gamma < d$	$V(x) = \lvert x \rvert^\alpha$ $\alpha \in (0,1)$	$V(x) = \lvert x \rvert^\alpha$ $\alpha \geq 1$
Inequality	Nash	Caffarelli-Kohn -Nirenberg	Weak Poincaré or Weighted Poincaré	Poincaré
Asymptotic behavior	$t^{-d/2}$ decay	$t^{-(d-\gamma)/2}$ decay	$t^{-\mu}$ or $t^{-\frac{k}{2(1-\alpha)}}$ convergence	$e^{-\lambda t}$ convergence

Table 1: $\partial_t u = \Delta u + \nabla \cdot (u \nabla V)$

Kinetic Fokker-Planck equations

B = Bouin, L = Lafleche, M = Mouhot, MM = Mischler, Mouhot
S = Schmeiser

Potential	$V = 0$	$V(x) = \gamma \log \lvert x \rvert$ $\gamma < d$	$V(x) = \lvert x \rvert^\alpha$ $\alpha \in (0,1)$	$V(x) = \lvert x \rvert^\alpha$ $\alpha \geq 1$, or $\mathbb{T}^d$ Macro Poincaré
Micro Poincaré $F(v) = e^{-\langle v \rangle^\beta}$, $\beta \geq 1$	BDMMS: $t^{-d/2}$ decay	BDS: $t^{-(d-\gamma)/2}$ decay	Cao: e^{-t^b}, $b < 1$, $\beta = 2$ convergence	DMS, Mischler-Mouhot $e^{-\lambda t}$ convergence
$F(v) = e^{-\langle v \rangle^\beta}$, $\beta \in (0,1)$	BDLS: $t^{-\zeta}$, $\zeta = \min\left\{\frac{d}{2}, \frac{k}{\beta}\right\}$ decay			
$F(v) = \langle v \rangle^{-d-\beta}$	BDLS, fractional			

Table 1: $\partial_t f + v \cdot \nabla_x f = F\, \nabla_v \big(F^{-1}\, \nabla_v f\big)$. Notation: $\langle v \rangle = \sqrt{1 + \lvert v \rvert^2}$

Two additional examples

▷ Linearized Vlasov-Poisson-Fokker-Planck system

▷ Fractional diffusion limits and hypocoercivity

Linearized Vlasov-Poisson-Fokker-Planck system

The *Vlasov-Poisson-Fokker-Planck system* in presence of an external potential ψ is

$$\begin{aligned} &\partial_t f + v\cdot\nabla_x f - (\nabla_x\psi + \nabla_x\phi)\cdot\nabla_v f = \Delta_v f + \nabla_v\cdot(v\,f) \\ &-\Delta_x\phi = \rho_f = \int_{\mathbb{R}^d} f\,dv \end{aligned} \qquad \text{(VPFP)}$$

Linearized problem around $f_\star$: $f = f_\star\,(1+\eta\,h)$, $\iint_{\mathbb{R}^d\times\mathbb{R}^d} h\,f_\star\,dx\,dv = 0$

$$\partial_t h + v\cdot\nabla_x h - (\nabla_x\psi + \nabla_x\phi_\star)\cdot\nabla_v h + v\cdot\nabla_x\psi_h - \Delta_v h + v\cdot\nabla_v h = \eta\,\nabla_x\psi_h\cdot\nabla_v h$$

$$-\Delta_x\psi_h = \int_{\mathbb{R}^d} h\,f_\star\,dv$$

Drop the $\mathcal{O}(\eta)$ term : *linearized Vlasov-Poisson-Fokker-Planck / Ornstein-Uhlenbeck* system

$$\begin{aligned} &\partial_t h + v\cdot\nabla_x h - (\nabla_x\psi + \nabla_x\phi_\star)\cdot\nabla_v h + v\cdot\nabla_x\psi_h - \Delta_v h + v\cdot\nabla_v h = 0 \\ &-\Delta_x\psi_h = \int_{\mathbb{R}^d} h\,f_\star\,dv\,, \quad \iint_{\mathbb{R}^d\times\mathbb{R}^d} h\,f_\star\,dx\,dv = 0 \end{aligned} \qquad \text{(VPFPlin)}$$

Hypocoercivity

Let us define the norm

$$\|h\|^2 := \iint_{\mathbb{R}^d\times\mathbb{R}^d} h^2\, f_\star\, dx\, dv + \int_{\mathbb{R}^d} |\nabla_x \psi_h|^2\, dx$$

(Addala, JD, Li, Tayeb)

Theorem

Let us assume that $d \geq 1$, $\psi(x) = |x|^\alpha$ for some $\alpha > 1$ and $M > 0$. Then there exist two positive constants $\mathcal{C}$ and λ such that any solution h of (VPFPlin) with an initial datum h_0 of zero average with $\|h_0\|^2 < \infty$ is such that

$$\|h(t,\cdot,\cdot)\|^2 \leq \mathcal{C}\, \|h_0\|^2\, e^{-\lambda t} \quad \forall\, t \geq 0$$

Fractional diffusion limits and hypocoercivity

$$\partial_t f + v \cdot \nabla_x f = \mathsf{L} f$$

with *fat tail local equilibrium* $\mathscr{M}$

$$\forall\, v \in \mathbb{R}^d, \quad \mathscr{M}(v) = \frac{c_\gamma}{\langle v \rangle^{d+\gamma}} \quad \text{where} \quad \langle v \rangle := \sqrt{1+|v|^2}\,.$$

▷ *Fokker-Planck* type operator ($\beta = 2$)

$$\mathsf{L}_1 f := \nabla_v \cdot \big(\mathscr{M}\, \nabla_v \big(\mathscr{M}^{-1} f\big)\big)$$

▷ *linear Boltzmann* operator, or *scattering* collision operator

$$\mathsf{L}_2 f := \int_{\mathbb{R}^d} \mathrm{b}(\cdot, v') \Big(f(v')\, \mathscr{M}(\cdot) - f(\cdot)\, \mathscr{M}(v')\Big)\, \mathrm{d}v'$$

with *collision frequency* $\nu(v) := \int_{\mathbb{R}^d} \mathrm{b}(v,v')\, \mathscr{M}(v')\, \mathrm{d}v' \underset{|v|\to+\infty}{\sim} |v|^{-\beta}$

▷ the *fractional Fokker-Planck* operator ($0 < \sigma < 2$, $\beta = \sigma - \gamma$)

$$\mathsf{L}_3 f := \Delta_v^{\sigma/2} f + \nabla_v \cdot (E\, f)$$

+ technical conditions

(Bouin, JD, Lafleche, 2022)

Theorem

Let $d \geq 2$*,* $\beta \in \mathbb{R}$*,* $\gamma > \max\{0, -\beta\}$ *and* $k \in [0, \gamma)$ *such that* $\gamma \neq 2 + \beta$ *or if* $\gamma = 2 + \beta$ *and* $\frac{k}{\beta_+} > \frac{d}{2}$*. If* f *is a solution with initial condition* $f^{\rm in} \in \mathrm{L}^1(\mathrm{d}x\,\mathrm{d}v) \cap \mathrm{L}^2\big(\langle v\rangle^k\,\mathrm{d}x\,\mathscr{M}^{-1}\,\mathrm{d}v\big)$*, then for any* $t \geq 0$*,*

$$\|f(t,\cdot,\cdot)\|^2_{\mathrm{L}^2(\,\mathrm{d}x\,\mathscr{M}^{-1}\,\mathrm{d}v)} \lesssim \frac{\big\|f^{\rm in}\big\|^2_{\mathrm{L}^1(\,\mathrm{d}x\,\mathrm{d}v)} + \big\|f^{\rm in}\big\|^2_{\mathrm{L}^2(\langle v\rangle^k\,\mathrm{d}x\,\mathscr{M}^{-1}\,\mathrm{d}v)}}{(1+t)^\tau}$$

with $\tau = \min\left\{\frac{d}{\alpha}, \frac{k}{\beta_+}\right\}$ *and* $\alpha = \min\left\{\frac{\gamma+\beta}{1+\beta}, 2\right\}$

The exponent α arises from the *fractional diffusion limit*

$$\partial_t u + (-\Delta)^{\alpha/2} u = 0$$

cf. (Mellet, Mischler, Mouhot, 2011), (Jara, Komorowski, Olla), (Bouin, Mouhot)

Special macroscopic modes and hypocoercivity

.

Joint work with with Kleber Carrapatoso, Frédéric Hérau, Stéphane Mischler, Clément Mouhot, Christian Schmeiser

The equation

Consider the kinetic equation

$$\partial_t f = \mathscr{L} f := \mathscr{T} f + \mathscr{C} f \,, \quad f_{|t=0} = f_0$$

with *transport operator* $\mathscr{T}$ given by

$$\mathscr{T} f := -\, v \cdot \nabla_x f + \nabla_x \phi \cdot \nabla_v f$$

where $\phi \in C^2(\mathbb{R}^d, \mathbb{R})$. Let $\rho(x) := e^{-\phi(x)}$ and $\langle \varphi \rangle := \int_{\mathbb{R}^d} \varphi \, \rho \, \mathrm{d}x$

Linear collision operator

$\mathscr{C}$ acts only on $v \in \mathbb{R}^d$, is self-adjoint in $\mathrm{L}^2(\mathscr{M}^{-1})$, with $\mathscr{M}(v) := \frac{e^{-|v|^2/2}}{(2\,\pi)^{d/2}}$ and has the $(d+2)$-dimensional kernel of *collision invariants* given by

$$\operatorname{Ker}\mathscr{C} = \operatorname{Span}\left\{\mathscr{M}, v_1\,\mathscr{M}, \ldots, v_d\,\mathscr{M}, |v|^2\,\mathscr{M}\right\}$$

▷ *Spectral gap property*

$$-\int_{\mathbb{R}^d} f(v)\,\mathscr{C}f(v)\,\frac{dv}{\mathscr{M}(v)} \geq \mathrm{c}_{\mathscr{C}}\,\|f - \Pi f\|^2_{\mathrm{L}^2(\mathscr{M}^{-1})}$$

where Π denotes the $\mathrm{L}^2(\mathscr{M}^{-1})$-orthogonal projection onto $\operatorname{Ker}\mathscr{C}$

▷ For any polynomial function $p(v) : \mathbb{R}^d \to \mathbb{R}$ of degree at most 4, the function $p\,\mathscr{M}$ is in the domain of $\mathscr{C}$ and

$$C(p) := \|\mathscr{C}(p\,\mathscr{M})\|_{\mathrm{L}^2(\mathscr{M}^{-1})} < \infty$$

Other assumptions (1/2)

▷ *Normalization conditions*:

$$\int_{\mathbb{R}^d} \rho(x)\,\mathrm{d}x = 1\,, \quad \int_{\mathbb{R}^d} x\,\rho(x)\,\mathrm{d}x = 0\,, \quad \langle \nabla_x^2 \phi \rangle = \int_{\mathbb{R}^d} \nabla_x^2 \phi\,\rho\,\mathrm{d}x = \mathrm{Id}_{d\times d}$$

▷ *Growth/regularity assumption*

$$|\nabla_x^2 \phi| \le \varepsilon\,|\nabla_x \phi|^2 + C_\varepsilon$$

▷ *Poincaré inequality*

$$c_{\mathrm{P}} \int_{\mathbb{R}^d} |u - \langle u \rangle|^2\,\rho\,\mathrm{d}x \le \int_{\mathbb{R}^d} |\nabla_x u|^2\,\rho\,\mathrm{d}x$$

▷ *Moment bounds on* ρ

$$\int_{\mathbb{R}^d} \Big(|x|^4 + |\phi|^2 + |\nabla_x \phi|^4\Big)\,\rho\,\mathrm{d}x \le \mathrm{C}_\phi$$

Other assumptions (2/2)

▷ *Semi-group property*

$$t \mapsto e^{t\mathscr{L}} \textit{ is a strongly continuous semi-group on } \mathrm{L}^2(\mathfrak{M}^{-1})$$

where $\mathfrak{M}$ is the *global Maxwellian equilibrium*

$$\mathfrak{M}(x,v) := \rho(x)\,\mathscr{M}(v) = \frac{e^{-\frac{1}{2}|v|^2-\phi(x)}}{(2\,\pi)^{d/2}}$$

Special macroscopic modes (1/2)

Special macroscopic modes $\quad \mathscr{C} F = 0\,, \quad \partial_t F = \mathscr{T} F$

$$F = \big(r(t,x) + m(t,x)\cdot v + e(t,x)\,\mathfrak{E}(v)\big)\,\mathfrak{M}\,, \quad \mathfrak{E}(v) := \frac{|v|^2 - d}{\sqrt{2\,d}}$$

▷ *Energy mode* $F = \mathcal{H}\,\mathfrak{M}$ with

$$\mathcal{H}(x,v) := \tfrac12\left(|v|^2 - d\right) + \phi(x) - \langle\phi\rangle$$

▷ The set of *infinitesimal rotations compatible with* ϕ defined as

$$\mathcal{R}_\phi := \left\{x \mapsto A\,x \,:\, A \in \mathfrak{M}^{\rm skew}_{d\times d}(\mathbb{R}) \text{ s.t. } \forall\,x \in \mathbb{R}^d\,,\ \nabla_x\phi(x)\cdot A\,x = 0\right\}$$

gives rise *rotation modes compatible with* ϕ

$$(A\,x\cdot v)\,\mathfrak{M}(x,v)\,, \quad A \in \mathcal{R}_\phi$$

Special macroscopic modes (2/2): harmonic modes

Harmonic directions $E_\phi := \underset{\mathbb{R}^d}{\mathrm{Span}}\{\nabla_x \phi(x) - x\}_{x\in\mathbb{R}^d}\,,\ d_\phi := \dim E_\phi$

▷ the potential is *partially harmonic* if $1 \le d_\phi \le d-1$
harmonic directional modes are defined by

$$(x_i \cos t - v_i \sin t)\,\mathfrak{M}\,,\quad (x_i \sin t + v_i \cos t)\,\mathfrak{M}\,,\quad i \in I_\phi := \{d_\phi+1,\ldots,d\}$$

▷ If $d_\phi = 0$, the potential $\phi(x) = \frac12\,|x|^2 + \frac{d}{2}\,\log(2\,\pi)$ is *fully harmonic*
In addition to the harmonic directional modes, there are *harmonic pulsating modes*

$$\left(\tfrac12\,\left(|x|^2 - |v|^2\right)\cos(2\,t) - x\cdot v\sin(2\,t)\right)\mathfrak{M}$$
$$\left(\tfrac12\,\left(|x|^2 - |v|^2\right)\sin(2\,t) + x\cdot v\cos(2\,t)\right)\mathfrak{M}$$

(Boltzmann, 1876) (Cercignani, 1983) (Uhlenbeck, Ford, 1963)

Theorem (Special macroscopic modes and hypocoercivity)

(1) *All special macroscopic modes are given by*

$$F = \alpha\,\mathfrak{M} + \beta\,\mathcal{H}\,\mathfrak{M} + A\,x\cdot v\,\mathfrak{M} + F_{\mathrm{dir}} + F_{\mathrm{pul}}$$

(2) *There are explicit constants* $C>0$ *and* $\lambda>0$ *such that, for any solution* $f\in\mathcal{C}\big(\mathbb{R}^+;\mathrm{L}^2(\mathfrak{M}^{-1})\big)$ *with initial datum* f_0*, there exists a unique special macroscopic mode* F *such that*

$$\forall\,t\ge0\,,\quad\|f(t)-F(t)\|_{\mathrm{L}^2(\mathfrak{M}^{-1})}\le C\,e^{-\lambda\,t}\,\|f_0-F(0)\|_{\mathrm{L}^2(\mathfrak{M}^{-1})}$$

A micro-macro decomposition

$$\partial_t h = \mathsf{L}\, h := \mathscr{T} h + \mathsf{C} h\,, \quad \mathsf{C} h := \mathfrak{M}^{-1} \mathscr{C}\,(\mathfrak{M}\, h)$$

with $\mathrm{Ker}\,\mathsf{C} = \mathrm{Span}\left\{1, v_1, \ldots, v_d, |v|^2\right\}$ and

$$h := \frac{f - \alpha\,\mathfrak{M} - \beta\,\mathcal{H}\,\mathfrak{M} - F_{\mathrm{rot}} - F_{\mathrm{dir}} - F_{\mathrm{pul}}}{\mathfrak{M}}$$

Micro-macro decomposition

$$h = h^{\parallel} + h^{\perp}\,, \quad h^{\parallel} := r + m \cdot v + e\,\mathfrak{E}(v)$$

$$(r, m, e)(t, x) := \int_{\mathbb{R}^d} \big(1, v, \mathfrak{E}(v)\,\big)\, h(t, x, v)\,\mathfrak{M}(v)\,\mathrm{d}v$$

- f is a special macroscopic modes iff $h^{\perp} = 0$
- all steady states are special macroscopic modes: factorization

(use entropy-dissipation arguments)

Sketch of the proof

The function $h = h^\parallel + h^\perp = r + m \cdot v + e\,\mathfrak{E}(v) + h^\perp$ is such that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|h\|^2 \le -\,2\,\mathrm{c}_{\mathscr{C}}\,\|h^\perp\|^2$$

With the *Witten-Laplace operator* $\Omega := -\,\Delta_x + \nabla_x\phi \cdot \nabla_x + 1$ and

$$E[h] := \int_{\mathbb{R}^d} (v \otimes v - \mathrm{Id}_{d\times d})\,h\,\mathfrak{M}\,\mathrm{d}v\,, \quad \Theta[h] := \int_{\mathbb{R}^d} v\left(\mathfrak{E}(v) - \sqrt{\tfrac{2}{d}}\right) h\,\mathfrak{M}\,\mathrm{d}v$$

we build a *Lyapunov functional*

$$\begin{aligned}\mathcal{F}[h] := \|h\|^2 + \varepsilon\left\langle \Omega^{-1}\,\nabla_x e, \Theta[h]\right\rangle + \varepsilon^{\frac{3}{2}}\left\langle \Omega^{-1}\,\nabla_x^{\mathrm{sym}}\,m_s, E[h] - \sqrt{\tfrac{2}{d}}\,\langle e\rangle\,\mathrm{Id}_{d\times d}\right\rangle \\ + \varepsilon^{\frac{7}{4}}\left\langle \Omega^{-1}\,\nabla_x w_s, m_s\right\rangle + \varepsilon^{\frac{15}{8}}\left\langle -\Omega^{-1}\,\partial_t w_s, w_s\right\rangle \\ -\,\varepsilon^{\frac{61}{32}}\left\langle (X - Y \cdot \nabla_x\phi), \nabla_x\phi \cdot A\,x\right\rangle - \varepsilon^{\frac{62}{32}}\left\langle b, b'\right\rangle - \varepsilon_6\left\langle c', c''\right\rangle\end{aligned}$$

such that, for some $\lambda \ge 0$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F}[h] \le -\,\lambda\,\mathcal{F}[h] \quad \text{and} \quad \|h\|^2 \lesssim \mathcal{F}[h] \lesssim \|h\|^2$$

These slides can be found at

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Thank you for your attention !