Symmetry, symmetry breaking, and phase transitions in interpolation inequalities

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Outline

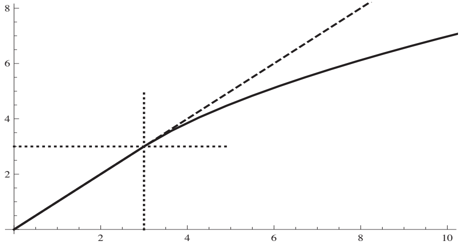
Gagliardo-Nirenberg-Sobolev inequalities on the sphere and phase transitions

▷ Subcritical inequalities and classical bifurcation results

▷ Other mechanisms of phase transition; the *carré du champ* method for the pressure variable

▷ Caarelli-Kohn-Nirenberg inequalities: a proof of symmetry by the parabolic *carré du champ* method

Bifurcation and phase transition in GNS inequalities



$\lambda \mapsto \mu(\lambda)$ *on* $\mathbb{S}^d$ *with* $d = 3$

$$\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{S}^d)} + \frac{\lambda}{p-2}\,\|u\|^2_{\mathrm{L}^2(\mathbb{S}^d)} \ge \frac{\mu(\lambda)}{p-2}\,\|u\|^2_{\mathrm{L}^p(\mathbb{S}^d)}$$

Taylor expansion of $u = 1 + \varepsilon\,\varphi_1$ as $\varepsilon \to 0$ with $-\Delta\varphi_1 = d\,\varphi_1$

$$\mu(\lambda) < \lambda \quad \text{if and only if} \quad \lambda > d$$

▷ The inequality holds with $\mu(\lambda) = \lambda = d$ [Bakry, Emery, 1985] [Beckner, 1993], [Bidaut-Véron, Véron, 1991, Corollary 6.1]

GNS as entropy-entropy production inequalities

- ***(subcritical) Gagliardo-Nirenberg-Sobolev inequality***

$$\|\nabla F\|^2_{\mathrm{L}^2(\mathbb{S}^d)} \geq d\,\mathcal{E}_p[F] := \frac{d}{p-2}\left(\|F\|^2_{\mathrm{L}^p(\mathbb{S}^d)} - \|F\|^2_{\mathrm{L}^2(\mathbb{S}^d)}\right)$$

for any $p \in [1,2) \cup (2,2^*)$
with $2^* := \frac{2\,d}{d-2}$ if $d \geq 3$ and $2^* = +\infty$ if $d = 1$ or 2

- Limit $p \to 2$: the ***logarithmic Sobolev inequality***

$$\int_{\mathbb{S}^d} |\nabla F|^2\,d\mu \geq \frac{d}{2}\,\mathcal{E}_2[F] := \frac{d}{2}\int_{\mathbb{S}^d} F^2 \log\left(\frac{F^2}{\|F\|^2_{\mathrm{L}^2(\mathbb{S}^d)}}\right) d\mu$$

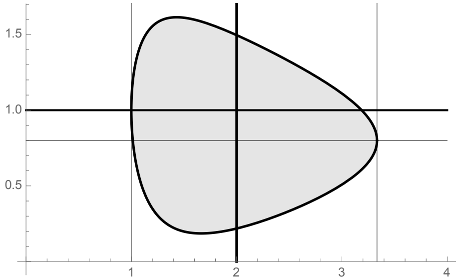
- $p = 1$: ***Poincaré inequality***

$$\|\nabla F\|^2_{\mathrm{L}^2(\mathbb{S}^d)} \geq d\,\mathcal{E}_1[F] := d\left(\|F\|^2_{\mathrm{L}^2(\mathbb{S}^d)} - \|F\|^2_{\mathrm{L}^1(\mathbb{S}^d)}\right)$$

Carré du champ – admissible parameters on $\mathbb{S}^d$

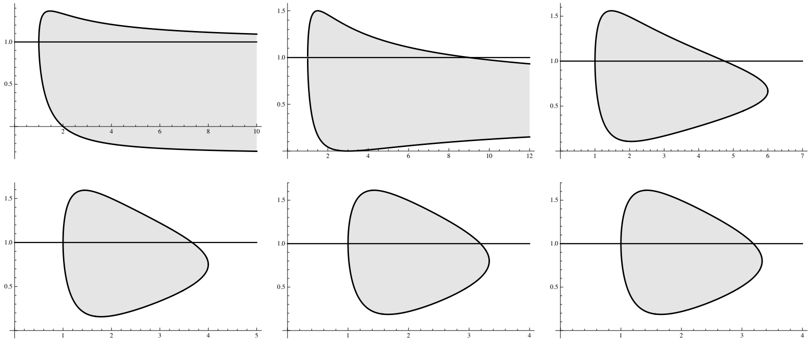
[JD, Esteban, Kowalczyk, Loss] Monotonicity of the deficit along

$$\frac{\partial u}{\partial t} = u^{-p\,(1-m)} \left(\Delta u + (m\,p-1)\,\frac{|\nabla u|^2}{u}\right)$$



Case $d = 5$: admissible parameters $1 \le p \le 2^* = 10/3$ and m (horizontal axis: p, vertical axis: m). Improved inequalities inside !

Admissible parameters



$d = 1, 2, 3$ (first line) and $d = 4, 5$ and 10 (second line)
the curves $p \mapsto m_\pm(p)$ determine the admissible parameters (p, m)
[JD, Esteban, Kowalczyk, Loss 2014] [JD, Esteban, 2019]

$$m_\pm(d,p) := \frac{1}{(d+2)\,p}\left(d\,p + 2 \pm \sqrt{d\,(p-1)\,(2\,d-(d-2)\,p)}\right)$$

Another *Gagliardo-Nirenberg-Sobolev* inequality

[JD, Esteban]

$$\left(\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{S}^d)}+\frac{\lambda}{p-2}\,\|u\|^2_{\mathrm{L}^2(\mathbb{S}^d)}\right)^\theta\|u\|^{2\,(1-\theta)}_{\mathrm{L}^2(\mathbb{S}^d)}\geq\left(\frac{\mu(p,\theta,\lambda)}{p-2}\right)^\theta\,\|u\|^2_{\mathrm{L}^p(\mathbb{S}^d)}$$

- *Symmetry* holds if $\mu(p,\theta,\lambda)=\lambda$, optimal functions are constant
- *Symmetry breaking* if $\lambda>d\,\theta$: take $u_\varepsilon:=1+\varepsilon\,\varphi$, $\Delta\varphi+d\,\varphi=0$

Bakry-Emery exponent : $2^\#:=+\infty$ if $d=1$, $2^\#:=(2\,d^2+1)/(d-1)^2$ if $d\geq 2$

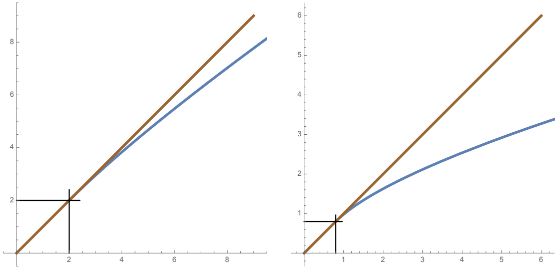
and take $p\in(2,2^\#]$

$$\theta^\#:=3\,\frac{p-2}{4\,p-7}\quad\text{if } d=1\,,\quad\frac{1}{\theta^\#}:=1+\frac{(p-1)\,(2^\#-p)}{p-2}\left(\frac{d-1}{d+2}\right)^2\ \text{if } d\geq 2$$

Proposition

Let $d\geq 1$, $p\in(2,2^\#)$, and $\theta\geq\theta^\#$. The function $\lambda\mapsto\mu(p,\theta,\lambda)$ is monotone increasing, concave and $\mu(p,\theta,\lambda)<\lambda$ if and only if $\lambda>d\,\theta$

Second order phase transition



$d=1$, $p=5$: $\theta=2$ (left) $\theta=0.8$ (right). Bifurcation at $\lambda=\mu=d\,\theta$

Parameter range

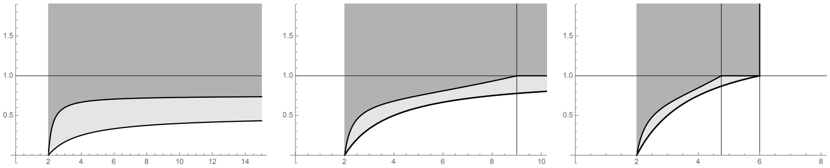
Theorem (Bou Dagher, JD)

Let $d \geq 1$, $p \in (2, 2^*)$ and $\theta > \theta_\star := d\,(p-2)/(2\,p)$
The function $\lambda \mapsto \mu(p, \theta, \lambda)$ is monotone increasing, concave

$$\mu(p, \theta, \lambda) \sim \kappa\,\lambda^{1-\theta_\star/\theta} \quad \text{as} \quad \lambda \to +\infty$$

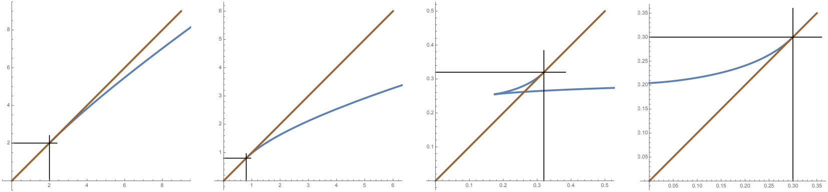
$$\mu(p, \theta, \lambda) \leq \lambda \text{ and } \mu(p, \theta, \lambda) < \lambda \text{ if } \lambda > d\,\theta$$

$$\mu(p, \theta, \lambda) = \lambda \text{ if } \lambda \leq d\,\theta\,,\ \theta \geq \theta^\#\,,\ p \in (2, 2^\#] \text{ or } p > 2 \text{ if } d = 1$$



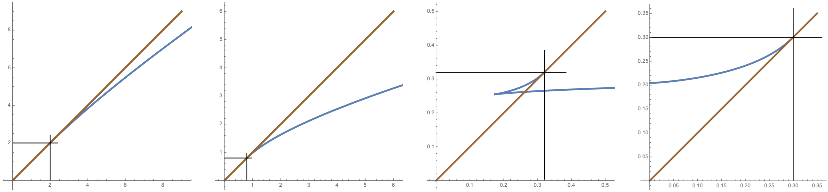
horizontal axis: p, vertical axis: θ
in dimensions $d = 1$, $d = 2$ and $d = 3$ (from left to right)

Second and first order phase transitions

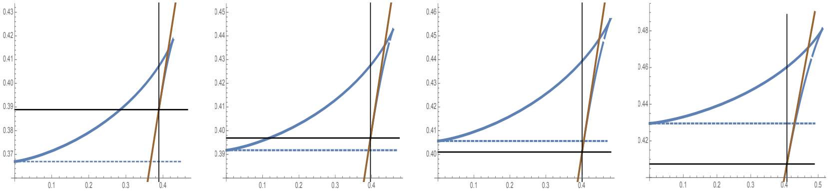


$d=1$, $p=5$, $\theta=2$: $\theta=0.8$, $\theta=0.32$ and $\theta=\theta_\star=0.3$

Second and first order phase transitions



$d = 1$, $p = 5$, $\theta = 2$: $\theta = 0.8$, $\theta = 0.32$ and $\theta = \theta_\star = 0.3$



Critical case: $d = 1$, $\theta = \theta_\star$, for $p = 9.0$, 9.7, 10.1 and 10.8

Reparametrization and consequences

Euler-Lagrange equation for an optimal function (with $\theta = 1$)

$$-\Delta u + \frac{\Lambda}{p-2}\, u = u^{p-1} \qquad (\mathrm{EL}_{1,\Lambda})$$

Theorem (Bou Dagher, JD)

Let $d \geq 1$, $p \in (2, 2^*)$, $\theta \geq \theta_\star$
A solution u of $(\mathrm{EL}_{1,\Lambda})$ also solves $(\mathrm{EL}_{\theta,\lambda})$ for $\lambda = \lambda(\theta, \Lambda)$ with

$$\lambda(\theta, \Lambda) := \tfrac{1}{\theta} \left(\Lambda + (1-\theta)\,(p-2)\, \frac{\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{S}^d)}}{\|u\|^2_{\mathrm{L}^2(\mathbb{S}^d)}} \right)$$

- For $\lambda > 0$ small enough, we have $\mu(\theta, \lambda) = \lambda$
- For $\theta - \theta_\star > 0$ small enough, symmetry breaking occurs for $\lambda < d\,\theta$

Symmetry breaking with $\lambda < d\,\theta$ means *first order phase transition*

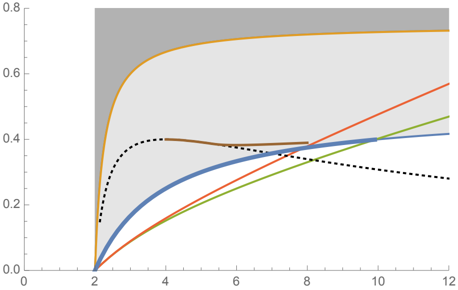
More qualitative properties

Proposition (Bou Dagher, JD)

Let $\theta_0 := \frac{(d+2)\,(d+3)\,(p-2)}{2\,(p^2+2\,p-6)+d\,(p^2+6\,p-12)-d^2\,(p-2)^2}$

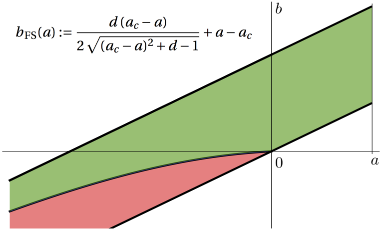
Assuming that the curve $\mathcal{C} : [d, d+\epsilon) \to (\mathbb{R}^+)^2$ *is smooth enough:*

- *If* $\theta \neq \theta_0$, *the curve* $\mathcal{C}$ *bifurcates from* $(d\,\theta, d\,\theta)$ *tangentially to* $\mu = \lambda$
- *The curve* $\mathcal{C}$ *is concave and below* $\mu = \lambda$ *(on the right) if* $\theta > \theta_0$
- *The curve* $\mathcal{C}$ *is convex and above the line* $\mu = \lambda$ *(on the left) if* $\theta < \theta_0$



- blue curve: $p \mapsto \theta_\star(p)$
- yellow curve: if $\theta \geq \theta^\#(p)$, the phase transition is of second order
- red curve: if it is below $p \mapsto \theta_\star(p)$, the phase transition is of first order for $\theta - \theta_\star(p) > 0$ small (Gaussian test functions)
- green curve: if $\kappa(p, \theta_\star) < \theta_\star(p)$, the phase transition is of first order for $\theta - \theta_\star(p) > 0$ small (comparison with GNS on $\mathbb{R}^d$)
- black, dotted curve: $p \mapsto \theta_0(p)$ (at the bifurcation point)
- brown curve $p \mapsto \theta_\bullet(p)$: a numerical approximation of the threshold between first / second order phase transitions

The critical Caffarelli-Kohn-Nirenberg inequality



$$\left(\int_{\mathbb{R}^d}\frac{|v|^p}{|x|^{b\,p}}\,dx\right)^{2/p}\le \mathsf{C}_{a,b}\int_{\mathbb{R}^d}\frac{|\nabla v|^2}{|x|^{2\,a}}\,dx$$

$$a\le b\le a+1\,,\;a<a_c\,,\;d\ge 3$$
$$p=\frac{2\,d}{d-2+2\,(b-a)}>0\,,\;a_c=\tfrac12\,(d-2)$$

▷ *A radial optimal function:*
$$v_\star(x)=\left(1+|x|^{(p-2)\,(a_c-a)}\right)^{-2/(p-2)}$$
among radially symmetric functions

Theorem (JD, Esteban, Loss, 2015)

There is *symmetry*, *i.e.*, $\mathsf{C}_{a,b}=\mathsf{C}^\star_{a,b}$, and all optimal functions are radially symmetric if $b_{\rm FS(a)}\le b<a+1$. If $a<b<b_{\rm FS}(a)$, then there is *symmetry breaking*, $\mathsf{C}_{a,b}>\mathsf{C}^\star_{a,b}$, and optimal functions are not radially symmetric.

[Caffarelli, Kohn, Nirenberg (1984)], [F. Catrina, Z.-Q. Wang (2001)]
[Smets, Willem], [Catrina, Wang], [Felli, Schneider]
[Bonforte, JD, Nazaret, Muratori]

A new proof: rewriting of CKN

1) Change of variables: $v(r,\omega) = u(r^\alpha,\omega)$, $\mathsf{D}_\alpha u = (\alpha\,\partial_r u, \nabla_\omega u)$

$$\int_{\mathbb{R}^d} |\mathsf{D}_\alpha u|^2\,|x|^{n-d}\,dx \geq \mathcal{C}_{\alpha,n}\left(\int_{\mathbb{R}^d} |u|^p\,|x|^{n-d}\,dx\right)^{2/p}$$

with $n = 2\,p/(p-2)$. Symmetry means that the Aubin-Talenti function $u_*(x) := (1+|x|^2)^{-(n-2)/2}$ realizes the equality case

2) Relative measure: with $w = u/u_*$ and $d\mu_q(x) = |u_*(x)|^q\,|x|^{n-d}\,dx$

$$\int_{\mathbb{R}^d} |\mathsf{D}_\alpha w|^2\,d\mu_2\,dx + \tfrac14\,\alpha^2\,n\,(n-2)\int_{\mathbb{R}^d} |w|^2\,d\mu_p\,dx \geq \mathcal{C}_{\alpha,n}\left(\int_{\mathbb{R}^d} |w|^p\,d\mu_p\,dx\right)^{2/p}$$

3) Stereographic projection: $w(x) = f(z,\omega)$ with $z = \frac{1-|x|^2}{1+|x|^2}$, $\omega = \frac{2\,x}{1+|x|^2}$

$$\int_{\mathbb{S}^d}\left(\alpha^2(1-z^2)|f'|^2 + \frac{|\nabla_\omega f|^2}{1-z^2}\right)d\sigma_n + \frac{\alpha^2}{4}\,n\,(n-2)\int_{\mathbb{S}^d}|f|^2\,d\sigma_n \geq \mathcal{K}_{\alpha,n}\left(\int_{\mathbb{S}^d}|f|^p\,d\sigma_n\right)^{2/p}$$

$d\sigma_n = Z_n^{-1}\,(1-z^2)^{(n-2)/2}\,dz\,d\omega$, $z\in[-1,+1]$, $\omega\in\mathbb{S}^{d-1}$

A new proof: fast diffusion equation and *carré du champ*

Let $'$ and ∇ denote the derivatives with respect to $z \in [-1,1]$ and $\omega \in \mathbb{S}^{d-1}$, $\Delta = \nabla \cdot \nabla$ and

$$\mathbf{D} v := \left(\alpha \sqrt{1-z^2}\, v', \frac{1}{\sqrt{1-z^2}} \nabla v \right) , \quad \mathbf{L} v := \mathbf{D} \cdot \mathbf{D} v$$

$$\mathbf{L} v = \alpha^2 \mathcal{L} v + \frac{1}{1-z^2} \Delta v \,, \quad \mathcal{L} v := \left(1-z^2\right) v'' - n\, z\, v'$$

Weighted fast diffusion equation

$$\frac{\partial v}{\partial t} = \mathbf{L} v^m = -\,\mathbf{D} \cdot (v\, \mathbf{D} \mathsf{P}) \,, \quad \mathsf{P} = \frac{m}{1-m}\, v^{m-1} \,, \quad m = \frac{n-1}{n} \,, \quad p = \frac{2\,n}{n-2}$$

$$v = u^p \quad \text{and} \quad \mathcal{D}(t) := \int_{\mathbb{S}^d} |\mathbf{D} u(t,\cdot)|^2 \, d\sigma_n + \frac{n\,\alpha^2}{p-2} \int_{\mathbb{S}^d} |u(t,\cdot)|^2 \, d\sigma_n$$

Proposition (Bou Dagher, JD)

$$\mathcal{D}'(t) \le 0 \;\; \textit{if}\;\; \alpha \le \alpha_{\rm FS} := \sqrt{\tfrac{d-1}{n-1}}$$

- Nonlinear *carré du champ* techniques and Felli & Schneider (FS)

Details

$$\mathcal{D}'(t) = -\frac{8}{(p+2)^2}\int_{\mathbb{S}^d} v^m\left(\mathbf{K}[\mathsf{P}] - m\,n\,\alpha^2\,|\mathbf{D}\mathsf{P}|^2\right)d\sigma_n$$

with $\mathbf{K}[\mathsf{P}] := \frac{1}{2}\,\mathbf{L}\left(|\mathbf{D}\mathsf{P}|^2\right) - \mathbf{D}\mathsf{P}\cdot\mathbf{D}(\mathbf{L}\mathsf{P}) - \frac{1}{n}\,(\mathbf{L}\mathsf{P})^2$

$$\mathbf{K}[\mathsf{P}] = m\left|\alpha^2\left(1-z^2\right)\mathsf{P}'' - \frac{\Delta\mathsf{P}}{(n-1)\left(1-z^2\right)}\right|^2 + 2\,\alpha^2\left|\nabla\mathsf{P}' + \frac{z\,\nabla\mathsf{P}}{1-z^2}\right|^2$$
$$+\,\alpha^2\,(n-1)\,|\mathbf{D}\mathsf{P}|^2$$
$$+\left(1-z^2\right)^{-2}\left(\frac{1}{2}\,\Delta(|\nabla\mathsf{P}|^2) - \nabla\mathsf{P}\cdot\nabla\Delta\mathsf{P} - \frac{(\Delta\mathsf{P})^2}{n-1} - (n-2)\,\alpha^2\,|\nabla\mathsf{P}|^2\right)$$

Corollary (Bou Dagher, JD)

If $n > d \geq 3$, $m = (n-1)/n$ and $p = 2\,n/(n-2)$, then

$$\int_{\mathbb{S}^{d-1}}\rho^q\left(\frac{1}{2}\,\Delta(|\nabla\mathsf{P}|^2) - \nabla\mathsf{P}\cdot\nabla\Delta\mathsf{P} - \frac{(\Delta\mathsf{P})^2}{n-1}\right)d\omega$$
$$= \mathsf{a}\int_{\mathbb{S}^{d-1}}\rho^q\,\left\|\,\mathrm{LP} - \frac{\mathsf{b}}{\mathsf{a}}\,\mathrm{MP}\,\right\|^2 d\omega + \left(\mathsf{c} - \frac{\mathsf{b}^2}{\mathsf{a}}\right)\int_{\mathbb{S}^{d-1}}\rho^q\,\frac{|\nabla\mathsf{P}|^4}{\mathsf{P}^2}\,d\omega$$
$$+\,(n-2)\,\frac{d-1}{n-1}\int_{\mathbb{S}^{d-1}}\rho^q\,|\nabla\mathsf{P}|^2\,d\omega$$

Regularization as in [JD, Zhang]

Stability results based on entropy methods

▷ Subcritical Gagliardo-Nirenberg inequalities on $\mathbb{S}^d$

▷ Sobolev inequality: the Bianchi-Egnell stability estimate made constructive

▷ The Gaussian logarithmic Sobolev inequality seen as an infinite dimensional limit

Gagliardo-Nirenberg inequalities: stability

An improved inequality under orthogonality constraint (Π_1 is a projection on some positive spherical harmonic functions) and the stability inequality arising from the *carré du champ* method can be combined *in the subcritical case* as follows

Theorem (Brigati, JD, Simonov)

Let $d \geq 1$ and $p \in (1, 2^*)$. For any $F \in \mathrm{H}^1(\mathbb{S}^d, d\mu)$, we have

$$\int_{\mathbb{S}^d} |\nabla F|^2 \, d\mu - d\, \mathcal{E}_p[F] \geq \mathscr{S}_{d,p} \left(\frac{\|\nabla \Pi_1 F\|^4_{\mathrm{L}^2(\mathbb{S}^d)}}{\|\nabla F\|^2_{\mathrm{L}^2(\mathbb{S}^d)} + \|F\|^2_{\mathrm{L}^2(\mathbb{S}^d)}} + \|\nabla (\mathrm{Id} - \Pi_1)\, F\|^2_{\mathrm{L}^2(\mathbb{S}^d)} \right)$$

for some explicit stability constant $\mathscr{S}_{d,p} > 0$

▷ The result holds true for the logarithmic Sobolev inequality ($p = 2$), again with an explicit constant $\mathscr{S}_{d,2}$, for any finite dimension d

▷ The ***far away*** regime: use an improved interpolation inequality

If $\|\nabla F\|^2_{\mathrm{L}^2(\mathbb{S}^d)} \,/\, \|F\|^2_{\mathrm{L}^p(\mathbb{S}^d)} \ge \vartheta_0 > 0$, by the convexity of ψ

$$\|\nabla F\|^2_{\mathrm{L}^2(\mathbb{S}^d)} - d\,\mathcal{E}_p[F] \ge d\,\|F\|^2_{\mathrm{L}^p(\mathbb{S}^d)}\,\psi\left(\frac{1}{d}\,\frac{\|\nabla F\|^2_{\mathrm{L}^2(\mathbb{S}^d)}}{\|F\|^2_{\mathrm{L}^p(\mathbb{S}^d)}}\right) \ge \frac{d}{\vartheta_0}\,\psi\left(\frac{\vartheta_0}{d}\right)\,\|\nabla F\|^2_{\mathrm{L}^2(\mathbb{S}^d)}$$

▷ The ***local*** case: $\|\nabla F\|^2_{\mathrm{L}^2(\mathbb{S}^d)} < \vartheta_0\,\|F\|^2_{\mathrm{L}^p(\mathbb{S}^d)}$

Take $\|F\|_{\mathrm{L}^p(\mathbb{S}^d)} = 1$, assume that $\frac{d\,\vartheta_0}{d-(p-2)\,\vartheta_0} > 0$ and deduce from the Poincaré inequality that

$$1 - \frac{\vartheta}{d} < \left(\int_{\mathbb{S}^d} F\,d\mu\right)^2 \le 1$$

+ a Taylor expansion using a partial decomposition on spherical harmonics

Large dimensional limit

... based on the Maxwell-Poincaré lemma [McKean, 1973]
Gagliardo-Nirenberg-Sobolev inequalities on $\mathbb{S}^d$, $p \in [1,2)$

$$\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{S}^d,d\mu_d)} \geq \frac{d}{p-2}\left(\|u\|^2_{\mathrm{L}^p(\mathbb{S}^d,d\mu_d)} - \|u\|^2_{\mathrm{L}^2(\mathbb{S}^d,d\mu_d)}\right)$$

Theorem (Brigati, JD, Simonov)

Let $v \in \mathrm{H}^1(\mathbb{R}^n, dx)$ with compact support, $d \geq n$ and

$$u_d(\omega) = v\left(\omega_1/\sqrt{d}, \omega_2/\sqrt{d}, \ldots, \omega_n/\sqrt{d}\right)$$

where $\omega \in \mathbb{S}^d \subset \mathbb{R}^{d+1}$. With $d\gamma(y) := (2\,\pi)^{-n/2}\,e^{-\frac{1}{2}\,|y|^2}\,dy$,

$$\lim_{d\to+\infty} d\left(\|\nabla u_d\|^2_{\mathrm{L}^2(\mathbb{S}^d,d\mu_d)} - \frac{d}{2-p}\left(\|u_d\|^2_{\mathrm{L}^2(\mathbb{S}^d,d\mu_d)} - \|u_d\|^2_{\mathrm{L}^p(\mathbb{S}^d,d\mu_d)}\right)\right)$$
$$= \|\nabla v\|^2_{\mathrm{L}^2(\mathbb{R}^n,d\gamma)} - \frac{1}{2-p}\left(\|v\|^2_{\mathrm{L}^2(\mathbb{R}^n,d\gamma)} - \|v\|^2_{\mathrm{L}^p(\mathbb{R}^n,d\gamma)}\right)$$

An explicit stability result for the Sobolev inequality

Sobolev inequality on $\mathbb{R}^d$ with $d \geq 3$, $2^* = \frac{2\,d}{d-2}$ and sharp constant S_d

$$\|\nabla f\|^2_{\mathrm{L}^2(\mathbb{R}^d)} \geq \mathsf{S}_d\,\|f\|^2_{\mathrm{L}^{2^*}(\mathbb{R}^d)} \quad \forall\, f \in \dot{\mathrm{H}}^1(\mathbb{R}^d) = \mathscr{D}^{1,2}(\mathbb{R}^d)$$

with equality on the manifold $\mathcal{M}$ of the Aubin–Talenti functions

$$g_{a,b,c}(x) = c\left(a + |x-b|^2\right)^{-\frac{d-2}{2}}\,, \quad a \in (0,\infty)\,, \quad b \in \mathbb{R}^d\,, \quad c \in \mathbb{R}$$

Theorem (JD, Esteban, Figalli, Frank, Loss)

There is a constant $\beta > 0$ with an explicit lower estimate which does not depend on d such that for all $d \geq 3$ and all $f \in \mathrm{H}^1(\mathbb{R}^d) \setminus \mathcal{M}$ we have

$$\|\nabla f\|^2_{\mathrm{L}^2(\mathbb{R}^d)} - \mathsf{S}_d\,\|f\|^2_{\mathrm{L}^{2^*}(\mathbb{R}^d)} \geq \frac{\beta}{d}\,\inf_{g \in \mathcal{M}} \|\nabla f - \nabla g\|^2_{\mathrm{L}^2(\mathbb{R}^d)}$$

- No compactness argument
- The (estimate of the) constant β is explicit
- The decay rate β/d is optimal as $d \to +\infty$

Stability for the Sobolev inequality: the history

▷ [Rodemich, 1969], [Aubin, 1976], [Talenti, 1976]
In the inequality $\|\nabla f\|^2_{\mathrm{L}^2(\mathbb{R}^d)} \geq \mathsf{S}_d \, \|f\|^2_{\mathrm{L}^{2^*}(\mathbb{R}^d)}$, the optimal constant is

$$\mathsf{S}_d = \tfrac{1}{4}\, d\,(d-2)\,|\mathbb{S}^d|^{1-2/d}$$

with equality on the manifold $\mathcal{M} = \{g_{a,b,c}\}$ of the *Aubin-Talenti functions*

▷ [Lions] a qualitative stability result

$$\textit{if} \;\lim_{n\to\infty} \|\nabla f_n\|_2^2/\|f_n\|_{2^*}^2 = \mathsf{S}_d \,,\; \textit{then} \;\lim_{n\to\infty} \inf_{g\in\mathcal{M}} \|\nabla f_n - \nabla g\|_2^2/\|\nabla f_n\|_2^2 = 0$$

▷ [Brezis, Lieb, 1985] a quantitative stability result ?

▷ [Bianchi, Egnell, 1991] there is some non-explicit $c_{\mathrm{BE}} > 0$ such that

$$\|\nabla f\|_2^2 \geq \mathsf{S}_d \,\|f\|_{2^*}^2 + c_{\mathrm{BE}} \inf_{g\in\mathcal{M}} \|\nabla f - \nabla g\|_2^2$$

- The strategy of Bianchi & Egnell involves two steps:
  – a local (spectral) analysis: the *neighbourhood* of $\mathcal{M}$
  – a local-to-global extension based on concentration-compactness : the *far away regime*
- The constant c_{BE} is not explicit

A stability result for the logarithmic Sobolev inequality

- Use the inverse stereographic projection to rewrite the result on $\mathbb{S}^d$

$$\|\nabla F\|^2_{\mathrm{L}^2(\mathbb{S}^d)} - \frac{1}{4}\,d\,(d-2)\left(\|F\|^2_{\mathrm{L}^{2^*}(\mathbb{S}^d)} - \|F\|^2_{\mathrm{L}^2(\mathbb{S}^d)}\right)$$
$$\geq \frac{\beta}{d} \inf_{G\in\mathcal{M}(\mathbb{S}^d)}\left(\|\nabla F - \nabla G\|^2_{\mathrm{L}^2(\mathbb{S}^d)} + \frac{1}{4}\,d\,(d-2)\,\|F-G\|^2_{\mathrm{L}^2(\mathbb{S}^d)}\right)$$

- Rescale by $\sqrt{d}$, consider a function depending only on n coordinates and take the limit as $d\to+\infty$ to approximate the Gaussian measure $d\gamma = e^{-\pi\,|x|^2}\,dx$

Corollary (JD, Esteban, Figalli, Frank, Loss)

With $\beta>0$ as in the result for the Sobolev inequality

$$\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{R}^n,d\gamma)} - \pi\int_{\mathbb{R}^n} u^2\log\left(\frac{|u|^2}{\|u\|^2_{\mathrm{L}^2(\mathbb{R}^n,d\gamma)}}\right)d\gamma$$
$$\geq \frac{\beta\,\pi}{2}\inf_{a\in\mathbb{R}^n,\,c\in\mathbb{R}}\int_{\mathbb{R}^n}|u-c\,e^{a\cdot x}|^2\,d\gamma$$

Stability for the logarithmic Sobolev inequality

▷ [Gross, 1975] *Gaussian logarithmic Sobolev inequality* for $n \geq 1$

$$\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{R}^n,d\gamma)} \geq \pi \int_{\mathbb{R}^n} u^2 \log\left(\frac{|u|^2}{\|u\|^2_{\mathrm{L}^2(\mathbb{R}^n,d\gamma)}}\right) d\gamma$$

▷ [Weissler, 1979] scale invariant (but dimension-dependent) version of the Euclidean form of the inequality

▷ [Stam, 1959], [Federbush, 1969], [Costa, 1985] *Cf.* [Villani, 2008]

▷ [Bakry, Emery, 1984], [Carlen, 1991] equality iff

$$u \in \mathscr{M} := \left\{ w_{a,c} \,:\, (a,c) \in \mathbb{R}^d \times \mathbb{R} \right\} \quad \text{where} \quad w_{a,c}(x) = c\, e^{a\cdot x} \quad \forall\, x \in \mathbb{R}^n$$

[Carlen, 1991] reinforcement of the inequality (Wiener transform)

▷ [McKean, 1973], [Beckner, 92] (LSI) as a large d limit of Sobolev

▷ [Bobkov, Gozlan, Roberto, Samson, 2014], [Indrei et al., 2014-23] stability in Wasserstein distance, in $\mathrm{W}^{1,1}$, *etc.*

▷ [JD, Toscani, 2016] Comparison with Weissler's form, a (dimension dependent) improved inequality

▷ [Fathi, Indrei, Ledoux, 2016] improved inequality assuming a Poincaré inequality (Mehler formula)

The global and the local problem

$$\mathsf{d}(u,v)^2 := \mathsf{q}[u-v] \quad \text{where} \quad \mathsf{q}[w] := \|\nabla w\|^2_{\mathrm{L}^2(\mathbb{S}^d)} + \tfrac{d}{p-2}\,\|w\|^2_{\mathrm{L}^2(\mathbb{S}^d)}$$

- *deficit* : $\delta[u] := \|\nabla u\|^2_{\mathrm{L}^2(\mathbb{S}^d)} + \frac{d}{p-2}\left(\|u\|^2_{\mathrm{L}^2(\mathbb{S}^d)} - \|u\|^2_{\mathrm{L}^p(\mathbb{S}^d)}\right)$, $\quad p = \frac{2\,d}{d-2}$
- *distance* to the set $\mathscr{M}$ of the Aubin-Talenti (optimal) functions

$$\mathsf{d}(u,\mathscr{M}) := \inf_{v\in\mathscr{M}} \mathsf{d}(u,v)$$

$\lim_{t\to+\infty} \mathsf{d}\big(u(t,\cdot),\mathscr{M}\big) = 0$ and $\delta[u(t,\cdot)]$ is monotone non-increasing if

$$\frac{\partial u}{\partial t} = m\,u^{(m-1)\,p}\left(\Delta u + (m\,p-1)\,\frac{|\nabla u|^2}{u}\right)$$

For a given $\varepsilon \in (0,1)$, u is in the ***far away*** regime if

$$\mathsf{d}(u,\mathscr{M})^2 > \varepsilon\,\mathsf{q}[u]$$

and in the neighbourhood of $\mathscr{M}$ if $\mathsf{d}(u,\mathscr{M})^2 \le \varepsilon\,\mathsf{q}[u]$

local stability : $\mathcal{I}(\varepsilon) := \inf\left\{\frac{\delta[u]}{\mathsf{d}(u,\mathscr{M})^2} \,:\, u\in \mathrm{H}^1(\mathbb{S}^d,d\sigma)\,,\ \mathsf{d}(u,\mathscr{M})^2 \le \varepsilon\,\mathsf{q}[u]\right\}$

A new proof for the global to local reduction

[Bonforte, JD, Esteban, Figalli, Frank, Loss] on an idea by Christ. If we start in the ***far away*** regime, which means

$$\mathsf{d}(u_{|t=0},\mathscr{M})^2 > \varepsilon\,\mathsf{q}[u_{|t=0}]$$

using $\mathsf{d}(u_{|t=0},\mathscr{M}) \le \mathsf{d}(u_{|t=0},0) = \mathsf{q}[u_{|t=0}]$, $\|u(t,\cdot)\|_{\mathrm{L}^p(\mathbb{S}^d)} = 1$ we obtain

$$\frac{\delta[u_{|t=0}]}{\mathsf{d}(u_{|t=0},\mathscr{M})^2} \ge \frac{\mathsf{q}[u_{|t=0}] - \frac{d}{p-2}}{\mathsf{q}[u_{|t=0}]} \ge 1 - \frac{\frac{d}{p-2}}{\mathsf{q}[u(t,\cdot)]} = \frac{\delta[u(t,\cdot)]}{\mathsf{q}[u(t,\cdot)]}$$

We know that

$$\lim_{t\to+\infty}\mathsf{q}[u(t,\cdot)] = \frac{d}{p-2} \quad\text{and}\quad \lim_{t\to+\infty}\mathsf{d}\big(u(t,\cdot),\mathscr{M}\big)^2 = 0$$

so that for some $t_* > 0$ we have

$$\mathsf{q}[u(t_*,\cdot)] = \tfrac{1}{\varepsilon}\,\mathsf{d}\big(u(t_*,\cdot),\mathscr{M}\big)^2$$

$$\frac{\delta[u_{|t=0}]}{\mathsf{d}(u_{|t=0},\mathscr{M})^2} \ge \frac{\delta[u(t_*,\cdot)]}{\mathsf{q}[u(t_*,\cdot)]} = \varepsilon\,\frac{\delta[u(t_*,\cdot)]}{\mathsf{d}\big(u(t_*,\cdot),\mathscr{M}\big)^2} \ge \varepsilon\,\mathcal{I}(\varepsilon)$$

Further results (and conjectures) on symmetry

▷ Caffarelli-Kohn-Nirenberg inequalities for spinor (complex) valued functions in dimension $d = 3$

▷ Caffarelli-Kohn-Nirenberg inequalities for spinor (complex) valued functions in dimension $d = 2$

▷ A Sobolev inequality for Dirac operators

Symmetry results for spinors in dimension $d=3$

We consider *2-spinors*, which are $\mathbb{C}^2$-valued function

$$\mathbb{R}^3 \ni x \mapsto \psi(x) = \begin{pmatrix} \psi_1(x) \\ \psi_2(x) \end{pmatrix} \in \mathbb{C}^2$$

Caffarelli-Kohn-Nirenberg inequalities for spinors

$$\int_{\mathbb{R}^3} \frac{|\sigma \cdot \nabla \psi(x)|^2}{|x|^{2\,\alpha}}\,dx \ge \mathcal{C}_{\alpha,\beta} \left(\int_{\mathbb{R}^3} \frac{|\psi(x)|^p}{|x|^{\beta\,p}}\,dx \right)^{2/p} \qquad \text{(SCKN)}$$

where $\partial_j = \partial_{x_j}$ and the gradient term is defined by

$$\sigma \cdot \nabla \psi = \sum_{j=1}^{3} \sigma_j\,\partial_j \psi$$

and $\sigma = (\sigma_j)_{j=1,2,3}$ is the family of the *Pauli matrices*

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$\alpha \le \beta \le \alpha + 1$, $p = 6/(1 - 2\,\alpha + 2\,\beta)$, $\mathcal{C}_{\alpha,\beta} \ge 0$ is the best constant

Symmetry for spinors

Proposition (JD, Esteban, Frank, Loss)

Let $\Lambda := \left\{ k - \frac{1}{2} \,:\, k \in \mathbb{Z} \setminus \{0\} \right\}$
If $\alpha \in \Lambda$*, then* $\mathcal{C}_{\alpha,\beta} = 0$ *for all* $\alpha \le \beta \le \alpha + 1$
If $\alpha \notin \Lambda$*, then* $\mathcal{C}_{\alpha,\beta} > 0$ *for all* $\alpha \le \beta \le \alpha + 1$

Angular decomposition in eigenspaces of $\sigma \cdot L$

$$\mathrm{L}^2\left(\mathbb{S}, \mathbb{C}^2; d\omega\right) = \bigoplus_{k \in \mathbb{Z} \setminus \{-1\}} \mathcal{H}_k$$

where $L := \omega \wedge (-i\,\nabla)$ is the *angular momentum* operator

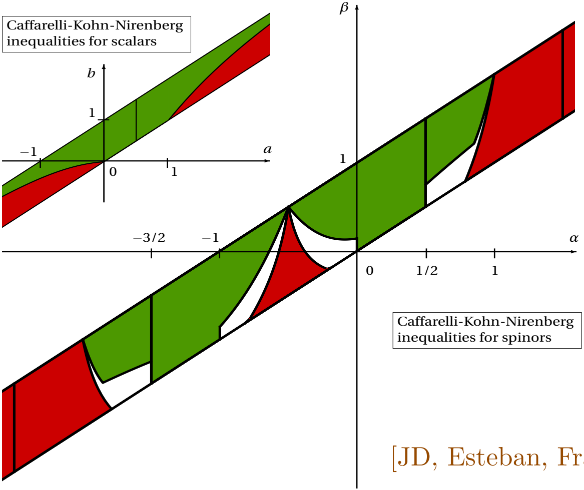
Definition

A spinor ψ on $\mathbb{R}^3$ is *symmetric* if there is a constant $\chi_0 \in \mathbb{C}^2$ and a complex-valued function f on $\mathbb{R}_+$ such that

$$\psi(x) = f(r)\,\chi_0 \quad \text{or} \quad \psi(x) = f(r)\,\sigma \cdot \omega\,\chi_0\,, \quad r = |x|\,, \quad \omega = x/r$$

i.e., $\psi \in \mathcal{H}_0$ or $\mathcal{H}_{-2}$

Results



[JD, Esteban, Frank, Loss]

Symmetry regions: green; symmetry breaking regions: red

The ingredients of the proof

- Existence of optimizers
- A Hardy inequality case: $\mathcal{C}_{\alpha,\alpha+1} = \min_{k\in\mathbb{Z}\setminus\{-1\}} \left(k-\alpha+\frac{1}{2}\right)^2$
- Passing to logarithmic variables ▷ see slide +1

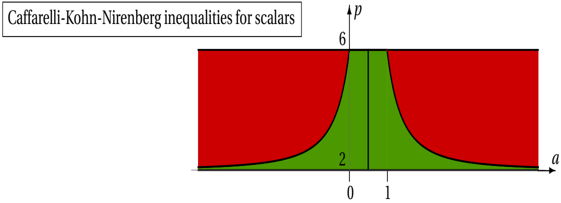
$$\iint_{\mathbb{R}\times\mathbb{S}} \left(|\partial_s\phi|^2 + \left|\left(\sigma\cdot L - \alpha + \tfrac{1}{2}\right)\phi\right|^2\right) ds\,d\omega \ge C_{\alpha,p}\left(\iint_{\mathbb{R}\times\mathbb{S}} |\phi|^p\,ds\,d\omega\right)^{2/p}$$

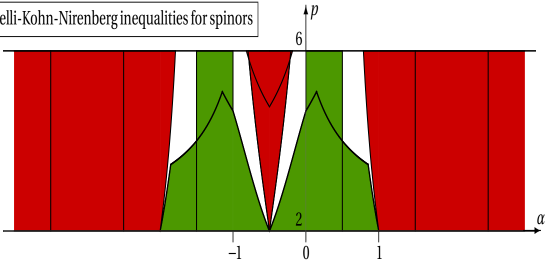
- Monotonicity properties: for some $\alpha_\star : (2,6) \to [-1/2, 0]$

$$C_{\alpha,p} < C^\star_{\alpha,p} \quad \text{if} \quad -1/2 \le \alpha < \alpha_\star(p)$$
$$C_{\alpha,p} = C^\star_{\alpha,p} \quad \text{if} \quad \alpha_\star(p) \le \alpha < 1/2$$

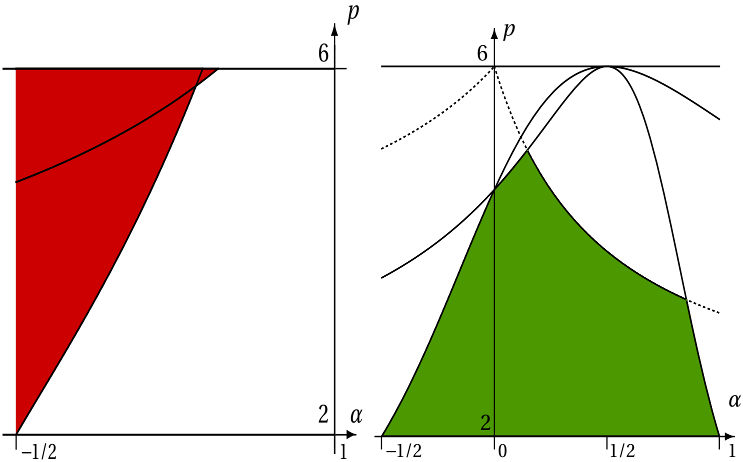
- A Gagliardo-Nirenberg interpolation inequality for spinors on the sphere based on tools of harmonic analysis ▷ see slide +2
- A Keller-Lieb-Thirring estimate
- A chain of (optimal) estimates
- Instability: study of the quadratic form obtained by linearization and representation using spherical harmonics ▷ see slide +2

Logarithmic variables





Symmetry *versus* symmetry breaking (details)



Symmetry results for spinors in dimension $d = 2$

- the $d = 2$ *spinorial Caffarelli-Kohn-Nirenberg inequality*

$$\int_{\mathbb{R}^2} \frac{|\sigma \cdot \nabla \psi|^2}{|x|^{2\alpha}}\, dx \geq C_{\alpha,p} \left(\int_{\mathbb{R}^2} \frac{|\psi|^p}{|x|^{\beta p}}\, dx \right)^{2/p} \qquad \text{(SCKN)}$$

for spinor valued functions $\psi : \mathbb{R}^2 \to \mathbb{C}^2$

▷ the *logarithmic Caffarelli-Kohn-Nirenberg inequality*

$$\int_{\mathbb{R}} \int_{\mathbb{S}^1} \left(|\partial_s \phi(s,\theta)|^2 + |(\alpha - i\sigma_3 \partial_\theta)\phi(s,\theta)|^2 \right) ds\, d\theta \geq C_{\alpha,p} \left(\int_{\mathbb{R}} \int_{\mathbb{S}^1} |\phi(s,\theta)|^p\, ds\, d\theta \right)^{2/p}$$

- *Interpolation inequalities for Aharonov–Bohm magnetic fields* $A(x) = (x_2, -x_1)/|x|^2$

$$\int_{\mathbb{R}^2} |(-i\nabla - \alpha A)\psi|^2\, dx \geq C^{\mathrm{AB}}_{\alpha,p} \left(\int_{\mathbb{R}^2} \frac{|\psi|^p}{|x|^2}\, dx \right)^{2/p} \qquad \text{(AB)}$$

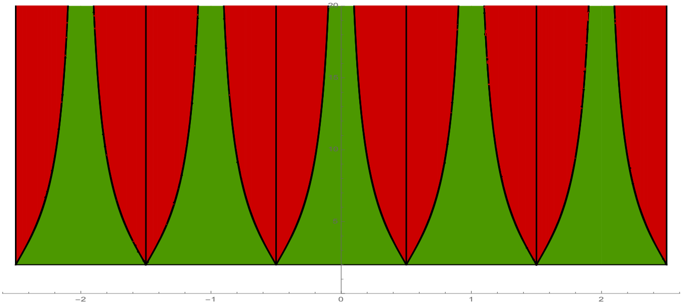
Theorem (JD, Frank, Weixler)

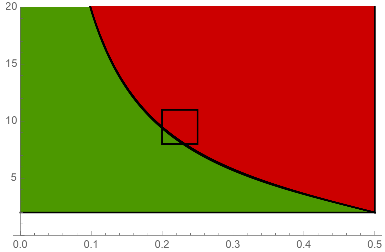
$C_{\alpha,p} = C^{\mathrm{AB}}_{\alpha,p}$ for any $(\alpha, p) \in (0, 1/2) \times (2, +\infty)$

Symmetry *versus* symmetry breaking

Theorem (JD, Frank, Weixler)

- For every $\alpha \in (0, 1/2)$ and $p > 2$, there is an optimizer with $C_{\alpha,p} > 0$ and $\lim_{\alpha \to 0_+} C_{\alpha,p} = 0$. Symmetry holds if and only if $\alpha \in \big(0, \alpha(p)\big]$ for some function $p \mapsto \alpha(p) : (2, \infty) \to (0, 1/2)$
- The symmetry and symmetry breaking regions are symmetric with respect to $\alpha = 0$ and 1-periodic





(SCKN) *with* $d = 2$. *Horizontal axis:* $\alpha \in (0, 1/2)$. *Vertical axis:* $p \in (2, \infty)$

- *Symmetry range: green, by the equivalence with Aharonov-Bohm problem and entropy methods for flows associated to (CKN) inequalities*
- *Symmetry breaking range: red and blue; Undecided in the tiny white gap*

- magnetic ring: an interpolation inequality on $\mathbb{S}^1$ [JD, Esteban, Laptev, Loss]
- Aharonov-Bohm and Caffarelli-Kohn-Nirenberg inequalities [Bonheure, JD, Esteban, Laptev, Loss]
- a Gegenbauer polynomial basis to study linear instability numerically

The Keller-Lieb-Thirring inequality (Schrödinger)

- With $q < 2^* := 2\,d/(d-2)$ if $d \geq 3$, and $\vartheta = d\,(q-2)/(2\,q)$

The *Gagliardo-Nirenberg-Sobolev inequality*

$$\|\nabla u\|_{\mathrm{L}^2(\mathbb{R}^d)}^{\vartheta}\,\|u\|_{\mathrm{L}^2(\mathbb{R}^d)}^{1-\vartheta} \geq \mathscr{C}_q\,\|u\|_{\mathrm{L}^q(\mathbb{R}^d)}$$

can be rewritten as

$$\|\nabla u\|_{\mathrm{L}^2(\mathbb{R}^d)}^2 + \lambda\,\|u\|_{\mathrm{L}^2(\mathbb{R}^d)}^2 \geq \mathsf{C}_q\,\lambda^{1-\vartheta}\,\|u\|_{\mathrm{L}^q(\mathbb{R}^d)}^2$$

for any $(\lambda, u) \in (0, +\infty) \times \mathrm{H}^1(\mathbb{R}^d)$, with $\mathscr{C}_q^2 = \vartheta^\vartheta (1-\vartheta)^{1-\vartheta}\,\mathsf{C}_q$

- Let λ be such that $\mathsf{C}_q\,\lambda^{1-\vartheta} = \|V\|_{\mathrm{L}^p(\mathbb{R}^d)}$. The Schrödinger energy is

$$\int_{\mathbb{R}^d} |\nabla u|^2\,dx - \int_{\mathbb{R}^d} V\,|u|^2\,dx \geq \|\nabla u\|_{\mathrm{L}^2(\mathbb{R}^d)}^2 - \|V\|_{\mathrm{L}^p(\mathbb{R}^d)}\,\|u\|_{\mathrm{L}^q(\mathbb{R}^d)}^2$$

$$\geq -\left(\mathsf{C}_q^{-1}\,\|V\|_{\mathrm{L}^p(\mathbb{R}^d)}\right)^{1/(1-\vartheta)}\,\|u\|_{\mathrm{L}^2(\mathbb{R}^d)}^2$$

Keller-Lieb-Thirring inequality : with $\eta = 1/(1-\vartheta) = 2\,p/(2\,p-d)$

$$\forall\, V \in \mathrm{L}^p(\mathbb{R}^d)\,, \quad \big(\lambda_1 - (-\Delta - V)\big)_- \geq -\,\mathsf{K}_p\,\|V\|_{\mathrm{L}^p(\mathbb{R}^d)}^{\eta}$$

The ***free Dirac operator*** in dimension d is defined by

$$\not{D}_m := \sum_{j=1}^{d} \alpha_j \left(-\,\mathrm{i}\,\partial_j\right) + m\,\beta = \boldsymbol{\alpha}\cdot\left(-\,\mathrm{i}\,\nabla\right) + m\,\beta$$

- $d=1$, $\boldsymbol{\alpha}=\sigma_2$ and $\beta=\sigma_3$: $\not{D}_m=\sigma_2\left(-\,\mathrm{i}\,\partial_1\right)+m\,\sigma_3$
- $d=2$, $\boldsymbol{\alpha}=(\sigma_j)_{j=1,2}$ and $\beta=\sigma_3$: $\not{D}_m=\sum_{j=1}^{2}\sigma_j(-\,\mathrm{i}\,\partial_j)+m\,\sigma_3$
- $d=3$, $\boldsymbol{\alpha}=(\alpha_k)_{k=1,2,3}$ and β such that

$$\alpha_k := \begin{pmatrix} 0 & \sigma_k \\ \sigma_k & 0 \end{pmatrix} \quad \text{and} \quad \beta := \begin{pmatrix} \mathbb{I}_2 & 0 \\ 0 & -\mathbb{I}_2 \end{pmatrix}$$

Here $(\sigma_j)_{j=1,2,3}$ are the Pauli matrices

Ground state : $\lambda_D(V)$ is the lowest eigenvalue in $(-\,m,m)$ of $\not{D}_m - V$

$$\Lambda_D(\alpha,p) := \inf\left\{\lambda_D(V)\ :\ V\in \mathrm{L}^p(\mathbb{R}^d,\mathbb{R}^+)\ \text{and}\ \|V\|_{\mathrm{L}^p(\mathbb{R}^d)}=\alpha\right\}$$

A Sobolev/Keller inequality for a Dirac operator

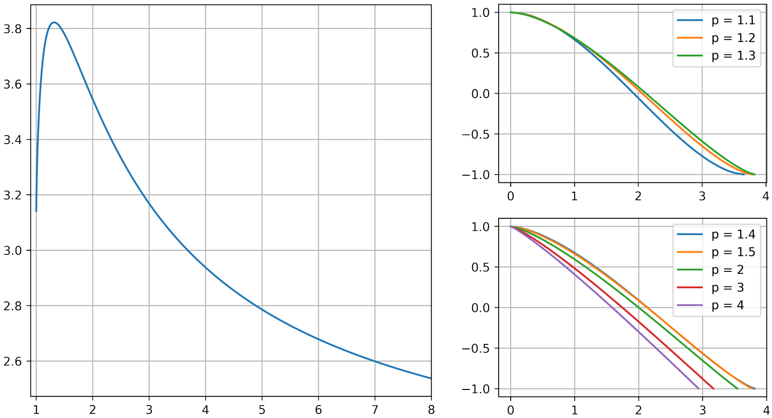
Theorem (JD, Gontier, Pizzichillo, van den Bosch)

Let $p \geq d \geq 1$. There exists $\alpha_\star(p) > 0$ such that the map $\alpha \mapsto \Lambda_D(\alpha, p)$ defined on $\big[0, \alpha_\star(p)\big)$ is continuous, strictly decreasing, takes values in $(-m, m]$, and such that

$$\lim_{\alpha \to 0_+} \Lambda_D(\alpha, p) = m \quad \text{and} \quad \lim_{\alpha \to \alpha_\star(p)} \Lambda_D(\alpha, p) = -m$$

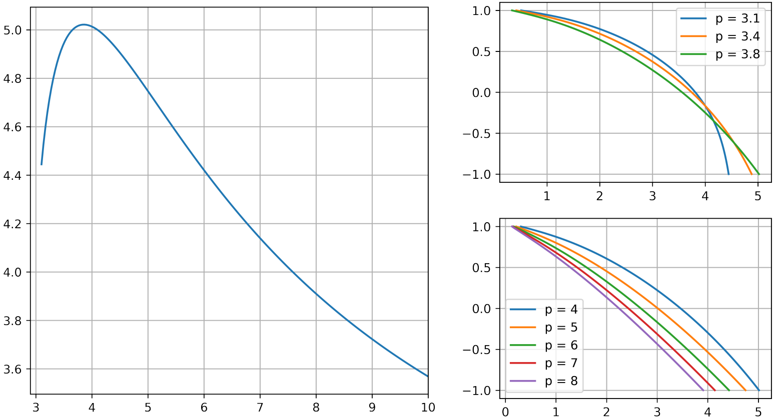
If $(p, d) \neq (1, 1)$, then $\Lambda_D(\alpha, p)$ is attained on $\big(0, \alpha_\star(p)\big)$

The case of dimension $d = 1$



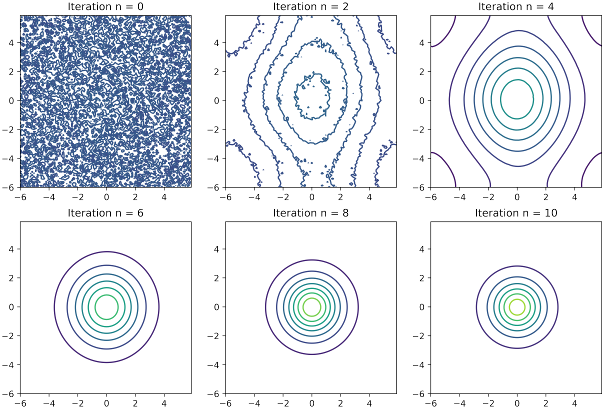
With $m = 1$ *The function* $p \mapsto \alpha_\star(p)$ *(left), has a maximum at* $p \approx 1.32$ *and* $\lim_{p\to 1_+} \alpha_\star(p) = \pi$ *and* $\lim_{p\to+\infty} \alpha_\star(p) = 2$
The function (right) $\Lambda_D(\alpha, p)$ *for various* p *is such that* $\Lambda_D\big(\alpha_\star(p), p\big) = -1$

The radial case in dimension $d = 3$



Radial case with $d = 3$ *and* $m = 1$
(Left) The function $p \mapsto \alpha_\star^{\rm rad}(p)$ *reaches its maximum at* $p \approx 3.86$
(Right) The maps $\alpha \mapsto \Lambda_D^{{\rm rad},(\kappa=1)}(\alpha, p)$

Is the optimal potential radial?



A numerical answer... Contour lines of the potential (by a fixed point method) for $p = 3$ *and* $\lambda = 1/2$*, for some initial potential chosen at random*

Conjecture. The optimal potential at $\alpha = \alpha_\star(p)$ is an ***Aubin-Talenti*** profile up to an angular spinor

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Thank you for your attention !